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Toroid, altermagnetic, and noncentrosymmetric ordering in metals

V.P. Mineev

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Contents

Abstract. This article is dedicated to the 60th anniversary of the Landau Institute for Theoretical Physics and presents a review of normal and superconducting properties of toroidal, altermagnetic, and noncentrosymmetric metals. Metals with toroidal order are compounds not possessing symmetry in respect of space and time inversion but are symmetric in respect of the product of these operations. An electric current propagating through samples of such a material causes its magnetization. Superconducting states in toroidal metals are a mixture of singlet and triplet states. Superconductivity is gapless even in ideal crystals without impurities. Altermagnets are antiferromagnetic metals that have a specific spin splitting of electron bands determined by time inversion in combinations with rotations and reflections of a crystal lattice. Similar splitting takes place in metals whose symmetry does not have a spatial inversion operation. Both of these types of materials have an anomalous Hall effect. A current propagating through a noncentrosymmetric metal causes magnetization, but this is not the case in altermagnets. On the other hand, in altermagnets, there is a specific piezomagnetic Hall effect. Superconducting pairing in noncentrosymmetric metals occurs between electrons occupying states in one zone, whereas, in altermagnets, we are dealing with interband pairing, which is unfavorable for the formation of a superconducting state.

Keywords: magnetism, superconductivity, strongly correlated electronic systems

V.P. Mineev
Landau Institute for Theoretical Physics, Russian Academy of Sciences,
prosp. Akademika Semenova 1A, 142432 Chernogolovka,
Moscow region, Russian Federation
E-mail: mineev@itp.ac.ru

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1. Introduction

Piezomagnetism and the magnetoelectric effect in dielectric antiferromagnetic materials are well-known phenomena closely related to magnetic symmetry [1]. New interest in these phenomena has arisen recently in connection with the discovery of the first examples of metallic compounds with the same magnetic symmetry, but possessing new, sometimes unexpected, physical properties. And, as is typical of the modern commercial style of writing scientific papers, a new sonorous terminology has appeared, designed to emphasize the significance of the authors' achievements. Thus, magnetoelectric metals began to be called *metals with a toroidal order*. In turn, piezomagnetic metals were called *altermagnets*. Somewhat earlier, the first metallic compounds were discovered whose symmetry does not contain the space inversion operation. They were called *noncentrosymmetric metals*. This article presents an overview of the normal and superconducting properties of these three types of materials.

2. Metals with toroid order

Substances with crystal symmetry which do not contain the operation of time reversal R or space inversion I but are invariant with respect to their product IR are called magnetoelectrics. Landau and Lifshitz [1] showed that, if a crystal with such symmetry is placed in a constant magnetic (or electric) field, an electric (or magnetic) moment proportional to the field is produced in the crystal. I.E. Dzyaloshinskii [2] gave the first example of magnetoelectric antiferromagnetic Cr_2O_3. It has the point symmetry group

$$\mathbf{D}_{3d}(\mathbf{D}_3) = (E, C_3, C_3^2, 3u_2, 3\sigma_d R, 2S_6 R, IR) \qquad (1)$$

containing the product of time and space inversion, but does not include these operations separately. The corresponding thermodynamic potential invariant with respect to operations

(1) is

$$\Phi_{\rm em} = -\alpha_\perp (E_x H_x + E_y H_y) - \alpha_\parallel E_z H_z \,. \tag{2}$$

So, this material in an external electric field acquires magnetization

$$M_y = -\frac{\partial \Phi_{\rm em}}{\partial H_y} = \alpha_\perp E_y \,. \tag{3}$$

The magnetoelectric effect in the antiferromagnet Cr_2O_3 was discovered by D.N. Astrov [3]. Despite the absence of a magnetic moment, this material exhibits the magneto-electric Kerr effect, that is, rotation of polarization of light reflected from a crystal with respect to incident light polarization. This birefringence has the opposite sign for magnetic domains related to each other by time reversal and can be used to observe antiferromagnetic domains. The corresponding symmetry considerations were developed in the elegant paper by W.F. Brown et al [4], although the microscopic theory of this phenomenon [5] and complete phenomenological treatment [6] had already appeared after the effect was discovered experimentally [7].

2.1 Electron spectrum

Cr_2O_3 is an antiferromagnetic dielectric. A metal with the same symmetry as Cr_2O_3 also possesses magnetoelectric properties. The electron spectrum of such a metal invariant with respect to all operations of the group $\mathbf{D}_{3d}(\mathbf{D}_3)$,

$$\varepsilon_{\mathbf{k}} = \varepsilon^{\rm e}_{\mathbf{k}} + \varepsilon^{\rm o}_{\mathbf{k}} \,, \qquad \varepsilon^{\rm e}_{\mathbf{k}} = f(k_x^2 + k_y^2, k_z^2) \,, \tag{4}$$

$$\varepsilon^{\rm o}_{\mathbf{k}} = \gamma (3k_y^2 k_x - k_x^3) \,, \tag{5}$$

consists of two parts, even and odd regarding its argument $\mathbf{k}$. This is a general property of a metal with a symmetry that does not include the operations of time inversion R or space inversion I separately, but is invariant with respect to its product IR. They are called *metals with toroidal order* or simply *toroids*. There is a vast amount of literature devoted to substances with toroidal order (see, for instance, [8, 9]). Normal properties and superconducting states in toroids were discussed in article [10].

Recently, the metallic compound Mn_2Au with toroidal magnetic order was discovered [11]. Mn_2Au is a collinear antiferromagnet with a Néel vector parallel or antiparallel to the [110] or $[1\bar{1}0]$ directions. In Fig. 1 is shown the magnetic structure of the antiferromagnetic domain of this compound with the Néel vector parallel to the $[1\bar{1}0]$ direction.

Its symmetry group is

$$\mathbf{D}_{2h}(\mathbf{C}_{2v}) = (E, U_{xy}, \sigma_h, \sigma_{x\bar{y}}, RU_{x\bar{y}}, R\sigma_{xy}, RC_{2z}, RI) \,. \tag{6}$$

Here, the operations $(E, U_{xy}, \sigma_h, \sigma_{x\bar{y}})$ forming group $\mathbf{C}_{2v}$ are the operation of rotation at an angle π around the axis [110] and reflections in the planes passing through it and perpendicular to each other. The electron spectrum invariant vis-à-vis all operations of the group $\mathbf{D}_{2h}(\mathbf{C}_{2v})$ is

$$\varepsilon_{\mathbf{k}} = \varepsilon^{\rm e}_{\mathbf{k}} + \varepsilon^{\rm o}_{\mathbf{k}} \,, \qquad \varepsilon^{\rm e}_{\mathbf{k}} = f(k_x^2 + k_y^2, k_z^2) \,, \tag{7}$$

$$\varepsilon^{\rm o}_{\mathbf{k}} = \gamma (k_x + k_y) \,. \tag{8}$$

The Fermi surface determined by the equation

$$\varepsilon_{\mathbf{k}} = \varepsilon_{\rm F} \tag{9}$$

is asymmetrical, because $\varepsilon_{\mathbf{k}} \neq \varepsilon_{-\mathbf{k}}$.

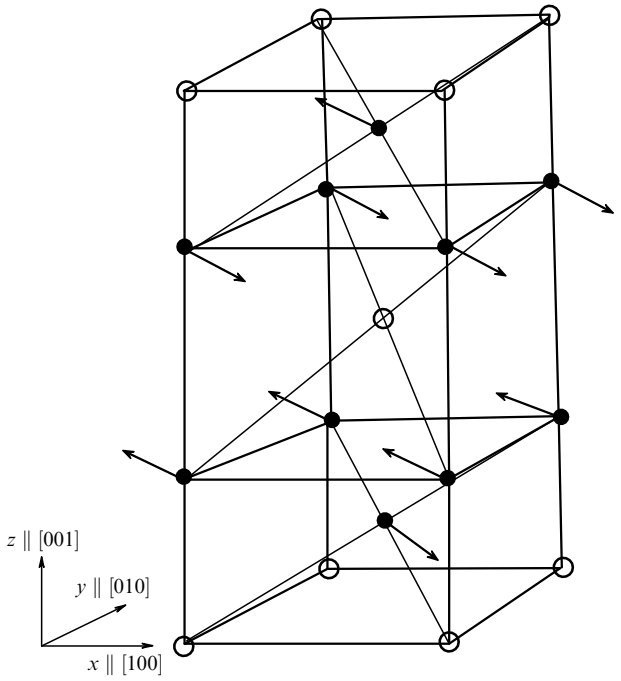

Figure 1. Magnetic structure of Mn_2Au showing order and orientation of Mn ion magnetic moments (see text). Dots/circles correspond to gold sites.

2.2 Kramers degeneracy

The Hamiltonian in the Schrödinger equation for an electron in such a metal commutes with the product of time and space inversion operations RI. This means that two spinor eigenfunctions $\psi_\alpha(\mathbf{r})$ and $RI\psi_\alpha(\mathbf{r})$ correspond to each energy $\varepsilon_{\mathbf{k}}$. They are orthogonal to each other. Indeed, the operation of the time reversal is $R_{\alpha\beta} = -{\rm i}\sigma^y_{\alpha\beta} K_0$, where $\sigma^y_{\alpha\beta}$ is the Pauli matrix, K_0 is the operation of complex conjugation, and

$$\begin{aligned}
\mathcal{I} &= \int {\rm d}^3\mathbf{r} \left[\psi^\star_\alpha(\mathbf{r}) I R_{\alpha\beta} \psi_\beta(\mathbf{r})\right] \\
&= \int {\rm d}^3\mathbf{r} \left[\psi^\star_\alpha(\mathbf{r})(-{\rm i})\sigma^y_{\alpha\beta} \psi^\star_\beta(-\mathbf{r})\right] \\
&= -\int {\rm d}^3\mathbf{r} \left[\psi^\star_\alpha(\mathbf{r})(-{\rm i})\sigma^y_{\beta\alpha} \psi^\star_\beta(-\mathbf{r})\right] \\
&= -\int {\rm d}^3\mathbf{r} \left[\psi^\star_\beta(\mathbf{r}) I R_{\beta\alpha} \psi_\alpha(\mathbf{r})\right] = -\mathcal{I} \,.
\end{aligned} \tag{10}$$

Thus, $\mathcal{I} = 0$. Hence, Kramers degeneracy of each energy level takes place.

2.3 Current in thermodynamic equilibrium

Due to the asymmetry of the energy spectrum, toroid metals possess nonzero electric current in thermodynamic equilibrium,

$$\mathbf{j} = 2e \int \frac{{\rm d}^3 k}{(2\pi)^3} \frac{\partial \varepsilon_{\mathbf{k}}}{\partial \mathbf{k}} f(\varepsilon_{\mathbf{k}}) \,, \tag{11}$$

where $f(\varepsilon_{\mathbf{k}}) = [\exp((\varepsilon_{\mathbf{k}} - \mu)/T) + 1]^{-1}$ is the Fermi distribution function. These currents are reminiscent of dissipationless diamagnetic currents flowing in metals in a magnetic field. A similar property is also present in noncentrosymmetric metals supporting dissipationless spin currents in thermodynamic equilibrium. We will discuss this phenomenon later. In real specimens with many antiferromagnetic

domains, the currents and corresponding magnetic moment space distribution acquire a complex structure.

2.4 Zero-field current induced Hall effect

Toroid metals are magnetoelectrics. An electric field applied to such a metal induces magnetization. For example, in the case of mono-domain antiferromagnet Mn_2Au with the structure shown in Fig. 1, the thermodynamic potential invariant with respect to all operations enumerated in (6) is

$$\Phi_{\rm em} = -\alpha(E_{x\bar{y}}H_z + E_zH_{x\bar{y}})\,. \tag{12}$$

An electric field directed along the z-axis causes magnetization parallel or antiparallel to the direction of the Néel vector:

$$M_{x\bar{y}} = \alpha E_z\,. \tag{13}$$

One can also say that an electric current $j_z = \rho_z^{-1}E_z$ along the z-axis causes magnetization

$$M_{x\bar{y}} = \alpha\rho_z j_z\,. \tag{14}$$

As a result, an electric field arises in such a sample that is perpendicular to both the current and the induced magnetic moment:

$$E_{xy} = \frac{1}{nec}\,\alpha\rho_z j_z^2\,. \tag{15}$$

This is the current induced Hall effect in a zero magnetic field. In multi-domain specimens, the current induced magnetization will have complex space distribution.

The effect of bulk magnetization induced by an electric current has been observed in semiconducting tellurium [12] and then in the antiferromagnetic metallic compound UNi_4B [13, 14], where the zero-field Hall effect was also registered [14].

3. Superconducting states in toroid metals

3.1 Order parameter

Superconducting compounds with toroid symmetry are currently unknown. The theory of superconductivity for this type of substance will be presented here with the hope that possible applications will appear in the future. In the absence of symmetry in relation to space inversion, the superconducting order parameters in toroid metals consist of the sum of singlet and triplet parts:

$$\Delta_{\mathbf{k},\alpha\beta} = \Delta\Phi_{\alpha\beta}(\mathbf{k}) = \Delta\left[\phi^{\rm s}_{\mathbf{k}}\,{\rm i}\sigma^y_{\alpha\beta} + (\boldsymbol{\phi}^{\rm t}_{\mathbf{k}}\boldsymbol{\sigma}_{\alpha\gamma})\,{\rm i}\sigma^y_{\gamma\beta}\right]. \tag{16}$$

Here, Δ is the coordinate dependent complex amplitude, and $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}^x, \hat{\sigma}^y, \hat{\sigma}^z)$ are the Pauli spin matrices. The functions $\phi^{\rm s}_{\mathbf{k}}$ and $\boldsymbol{\phi}^{\rm t}_{\mathbf{k}}$ correspond to representations of the symmetry group of concrete toroidal metals. For instance, in the case of a single domain antiferromagnet with symmetry group (6), the functions of irreducible representations $\Gamma = A, B, C, D$ are presented in the table.

3.2 Bardeen–Cooper–Schrieffer theory

The Bardeen–Cooper–Schrieffer (BCS) Hamiltonian has the standard form

$$H = H_0 + H_{\rm int} = \sum_{\mathbf{k}}(\xi_{\mathbf{k}} + \varepsilon^{\rm o}_{\mathbf{k}})a^+_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha} + \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}'} V_{\alpha\beta,\lambda\mu}(\mathbf{k},\mathbf{k}')a^+_{-\mathbf{k}\alpha}a^+_{\mathbf{k}\beta}a_{\mathbf{k}'\lambda}a_{-\mathbf{k}'\mu} \tag{17}$$

Table

Γ	$\phi^{\rm s}_{\mathbf{k}}$	$\boldsymbol{\phi}^{\rm t}_{\mathbf{k}}$
A	$a_1(\hat{k}_x + \hat{k}_y)^2 + a_2\hat{k}_z^2$	${\rm i}a_3(\hat{k}_x - \hat{k}_y)\hat{z}$
B	$b_1(\hat{k}_x + \hat{k}_y)\hat{k}_z$	${\rm i}b_2(\hat{k}_x\hat{y} - \hat{k}_y\hat{x})$
C	$c_1(\hat{k}_x - \hat{k}_y)\hat{k}_z$	${\rm i}c_2(\hat{k}_x+\hat{k}_y)(\hat{x}+\hat{y}) + {\rm i}c_3(\hat{k}_x-\hat{k}_y)(\hat{x}-\hat{y}) + {\rm i}c_4\hat{k}_z\hat{z}$
D	$d_1(\hat{k}_x+\hat{k}_y)(\hat{k}_x-\hat{k}_y)$	${\rm i}d_2(\hat{k}_x + \hat{k}_y)\hat{z}$

Here, $\hat{k}_x, \hat{k}_y, \hat{k}_z$ are components of unit vector of momentum $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$, and $\hat{x}$, $\hat{y}$, $\hat{z}$ are unit vectors of directions in spin space.

with the only difference being that the kinetic energy now has even

$$\xi_{\mathbf{k}} = \varepsilon^{\rm e}_{\mathbf{k}} - \mu \tag{18}$$

and odd $\varepsilon^{\rm o}_{\mathbf{k}}$ with respect to momentum parts. In the pairing interaction

$$V_{\alpha\beta,\lambda\mu}(\mathbf{k},\mathbf{k}') = -V_\Gamma\Phi_{\alpha\beta}(\mathbf{k})\Phi^\dagger_{\lambda\mu}(\mathbf{k}')\,, \tag{19}$$

only the term related to irreducible representation Γ corresponding to the superconducting state with maximal critical temperature was left. After the usual mean field transformation, the Hamiltonian acquires the following form:

$$H = \frac{1}{2}\sum_{\mathbf{k}}(\xi_{\mathbf{k}} + \varepsilon^{\rm o}_{\mathbf{k}})a^+_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha} - \frac{1}{2}\sum_{\mathbf{k}}(\xi_{-\mathbf{k}} + \varepsilon^{\rm o}_{-\mathbf{k}})a_{-\mathbf{k}\alpha}a^+_{-\mathbf{k}\alpha} + \frac{1}{2}\sum_{\mathbf{k}}\Delta_{\mathbf{k},\alpha\beta}a^+_{\mathbf{k}\alpha}a^+_{-\mathbf{k}\beta} + \frac{1}{2}\sum_{\mathbf{k}}\Delta^\dagger_{\mathbf{k},\alpha\beta}a_{-\mathbf{k}\alpha}a_{\mathbf{k}\beta} + \frac{1}{2}\sum_{\mathbf{k}\alpha}(\xi_{-\mathbf{k}} + \varepsilon^{\rm o}_{-\mathbf{k}}) + \frac{1}{2}\sum_{\mathbf{k}}\Delta_{\mathbf{k},\alpha\beta}F^+_{\mathbf{k},\beta\alpha}\,, \tag{20}$$

where the matrix of the order parameter

$$\Delta_{\mathbf{k},\alpha\beta} = -\sum_{\mathbf{k}'} V_{\beta\alpha,\lambda\mu}(\mathbf{k},\mathbf{k}')\langle a_{\mathbf{k}\lambda}a_{-\mathbf{k}\mu}\rangle \tag{21}$$

is expressed through the 'anomalous average':

$$F_{\mathbf{k},\alpha\beta} = \langle a_{\mathbf{k}\alpha}a_{-\mathbf{k}\beta}\rangle\,. \tag{22}$$

Here, $\langle\ldots\rangle$ means subsequent quantum mechanical and thermal averaging.

A more compact form of Eqn (20) is

$$H = \frac{1}{2}\sum_{\mathbf{k}}\varepsilon_{\mathbf{k},ij}A^+_{\mathbf{k},i}A_{\mathbf{k},j} + \frac{1}{2}\sum_{\mathbf{k}\alpha}(\xi_{\mathbf{k}} - \varepsilon^{\rm o}_{\mathbf{k}}) + \frac{1}{2}\sum_{\mathbf{k}}\Delta_{\mathbf{k},\alpha\beta}F^+_{\mathbf{k},\beta\alpha}\,. \tag{23}$$

Here, the operators

$$A^+_{\mathbf{k},i} = (a^+_{\mathbf{k}\alpha}a_{-\mathbf{k}\alpha})\,,\quad A_{\mathbf{k},i} = \begin{pmatrix} a_{\mathbf{k}\alpha} \\ a^+_{-\mathbf{k}\alpha}\end{pmatrix} \tag{24}$$

and

$$\varepsilon_{\mathbf{k},ij} = \begin{pmatrix} (\xi_{\mathbf{k}} + \varepsilon^{\rm o}_{\mathbf{k}})\delta_{\alpha\beta} & \Delta_{\mathbf{k},\alpha\beta} \\ \Delta^\dagger_{\mathbf{k},\alpha\beta} & (-\xi_{\mathbf{k}} + \varepsilon^{\rm o}_{\mathbf{k}})\delta_{\alpha\beta}\end{pmatrix}. \tag{25}$$

Diagonalizing the Hamiltonian by means the Bogoliubov transformation,

$$A_{\mathbf{k},i} = U_{ij} B_{\mathbf{k},j}\,, \quad B_{\mathbf{k},j} = \begin{pmatrix} b_{\mathbf{k}\alpha} \\ b^{+}_{-\mathbf{k}\alpha} \end{pmatrix},$$

$$U_{ij} = \begin{pmatrix} u_{\mathbf{k},\alpha\beta} & v_{\mathbf{k},\alpha\beta} \\ v^{\dagger}_{\mathbf{k},\alpha\beta} & -u_{\mathbf{k},\alpha\beta} \end{pmatrix}, \tag{26}$$

$$u_{\mathbf{k},\alpha\beta} = \frac{\xi_{\mathbf{k}} + E^{\mathrm{e}}_{\mathbf{k}}}{\sqrt{(\xi_{\mathbf{k}} + E^{\mathrm{e}}_{\mathbf{k}})^2 + \Delta^2_{\mathbf{k}}}}\,\delta_{\alpha\beta}\,, \tag{27}$$

$$v_{\mathbf{k},\alpha\beta} = \frac{\Delta_{\alpha\beta}(\mathbf{k})}{\sqrt{(\xi_{\mathbf{k}} + E^{\mathrm{e}}_{\mathbf{k}})^2 + \Delta^2_{\mathbf{k}}}}\,, \tag{28}$$

$$E^{\mathrm{e}}_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + \Delta^2_{\mathbf{k}}}\,, \quad \Delta^2_{\mathbf{k}} = \frac{1}{2}\,\Delta^{\dagger}_{\mathbf{k},\alpha\beta}\Delta_{\mathbf{k},\beta\alpha}\,, \tag{29}$$

we obtain

$$\frac{1}{2}\sum_{\mathbf{k}} \varepsilon_{\mathbf{k},ij} A^{+}_{\mathbf{k},i} A_{\mathbf{k},j} = \frac{1}{2}\sum_{\mathbf{k}} E_{\mathbf{k},ij} B^{+}_{\mathbf{k},i} B_{\mathbf{k},j}\,, \tag{30}$$

where

$$E_{\mathbf{k},ij} = \begin{pmatrix} (\varepsilon^{\mathrm{o}}_{\mathbf{k}} + E^{\mathrm{e}}_{\mathbf{k}})\delta_{\alpha\beta} & 0 \\ 0 & (\varepsilon^{\mathrm{o}}_{\mathbf{k}} - E^{\mathrm{e}}_{\mathbf{k}})\delta_{\alpha\beta} \end{pmatrix}. \tag{31}$$

Thus, the energy of excitations is

$$E_{\mathbf{k}} = \varepsilon^{\mathrm{o}}_{\mathbf{k}} + E^{\mathrm{e}}_{\mathbf{k}}\,. \tag{32}$$

The corresponding density of states is

$$N(E) = 2\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\delta(E - E_{\mathbf{k}})\,. \tag{33}$$

We see that, near the surface determined by the equation $\xi_{\mathbf{k}} = 0$, there is a vast region where $E_{\mathbf{k}} < 0$; hence, a superconducting state is proved to be *gapless*:

$$N(E = 0) \neq 0\,.$$

This property of superconducting states in superconductors with toroidal order, in particular, has a nonzero specific heat ratio $(C(T)/T)_{T\to 0} \neq 0$ in completely pure metal without impurities and crystal imperfections.

The order parameter is determined by Eqn (21). By applying to this expression the Bogoliubov transformation, we obtain

$$\begin{aligned}\Delta_{\mathbf{k},\alpha\beta} &= -\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\,V_{\beta\alpha,\lambda\mu}(\mathbf{k},\mathbf{k}')\,\frac{1 - f_{\mathbf{k}'} - f_{-\mathbf{k}'}}{2E^{\mathrm{e}}_{\mathbf{k}'}}\,\Delta_{\mathbf{k}',\lambda\mu} \\ &= -\int \frac{\mathrm{d}^3\mathbf{k}'}{(2\pi)^3}\,V_{\beta\alpha,\lambda\mu}(\mathbf{k},\mathbf{k}') \\ &\quad\times \frac{\tanh(E_{\mathbf{k}'}/2T) + \tanh(E_{-\mathbf{k}'}/2T)}{4E^{\mathrm{e}}_{\mathbf{k}'}}\,\Delta_{\mathbf{k}',\lambda\mu}\,.\end{aligned} \tag{34}$$

Here, we used the commutation rules of the operators $b_{\mathbf{k}\alpha}, b^{+}_{\mathbf{k}\alpha}$, and the symmetry property

$$v_{\mathbf{k},\alpha\beta} = -v_{-\mathbf{k},\beta\alpha} \tag{35}$$

and expressed the average $\langle b^{+}_{\mathbf{k}\alpha} b_{\mathbf{k}\beta}\rangle = f_{\mathbf{k}}\delta_{\alpha\beta}$ through the Fermi distribution function

$$f_{\mathbf{k}} = f(E_{\mathbf{k}}) = \frac{1}{\exp\left((\varepsilon^{\mathrm{o}}_{\mathbf{k}} + E^{\mathrm{e}}_{\mathbf{k}})/T\right) + 1}\,. \tag{36}$$

At $T \to T_{\mathrm{c}}$ one can neglect $\Delta^2_{\mathbf{k}}$ in $E^{\mathrm{e}}_{\mathbf{k}}$ in Eqn (34). Estimating the integral with logarithmic accuracy, we come to the expression for critical temperature similar to the usual BCS formula,

$$T_{\mathrm{c}} \approx \varepsilon_0 \exp\left(-\frac{1}{\tilde{N}_0 V_{\Gamma}}\right), \tag{37}$$

where ε_0 is the cut-off for energy of pairing interaction and $\tilde{N}_0$ is the normal density of states averaged over the Fermi surface with a weight corresponding to the angular dependent functions of a given irreducible representation.

3.3 Free energy linear in order parameter gradients

Let us now discuss a possible peculiar property of the inhomogeneous state in superconductors with toroidal symmetry. The expression for the superconducting current

$$\mathbf{j} = -\frac{2e}{\hbar}\,K\left[\Delta^{\star}\left(-\mathrm{i}\nabla + \frac{2e}{\hbar c}\,\mathbf{A}\right)\Delta + \mathrm{c.c.}\right] \tag{38}$$

changes its sign under the time reversal R as well under the space inversion I, but it is invariant with respect to the product of this operations IR. Thus, the current has toroidal symmetry. Hence, one can expect, as has been pointed out by various authors (see, for instance, [8]), the existence of a term linear in gradients

$$F_{\nabla} = C_i\, j_i \tag{39}$$

in the superconducting free energy density for metals with toroid symmetry. The direction of vector $\mathbf{C}$ is determined by the direction of the Néel vector of the toroid antiferromagnet.

To verify this property, let us consider the superconducting free energy quadratic in terms of the order parameter:

$$\begin{aligned}\mathcal{F} = \frac{1}{2V}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,\Delta^{\star}(\mathbf{q})\Delta(\mathbf{q}) - \frac{T}{2}\sum_{\omega}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \\ \times \Delta^{\star}_{\mathbf{k},\alpha\beta}(\mathbf{q}) G\left(-\mathbf{k} + \frac{\mathbf{q}}{2}, -\omega_n\right) G\left(\mathbf{k} + \frac{\mathbf{q}}{2}, \omega_n\right)\Delta_{\mathbf{k},\beta\alpha}(\mathbf{q})\,,\end{aligned} \tag{40}$$

where

$$G(\mathbf{k},\omega_n) = \frac{1}{\mathrm{i}\omega_n - \xi_{\mathbf{k}} - \varepsilon^{\mathrm{o}}_{\mathbf{k}}} \tag{41}$$

is the normal state electron Green's function, and $\omega_n = 2\pi T(n + 1/2)$ is the Matsubara frequency. Omitting simple but cumbersome calculations, we only indicate that, after performing summation over frequencies followed by decomposing the sub-integral expression in powers of $(\partial\xi_{\mathbf{k}}/\partial\mathbf{k})\mathbf{q}$ and $(\partial\varepsilon^{\mathrm{o}}_{\mathbf{k}}/\partial\mathbf{k})\mathbf{q}$, the integral over angles of momentum $\mathbf{k}$ of the part linear in $\mathbf{q}$ turns out to be equal to zero. This means that term (39) vanishes identically.

4. Altermagnetic and noncentrosymmetric metals

There is another type of magnetic structure in which the magnetic symmetry group does not contain the time reversal

R by itself, but this operation enters only in combination with rotations or reflections or else is not present at all. Consequently, such substances in general, are capable of possessing piezomagnetic properties [1, 15, 16]. Piezomagnetism was discovered in antiferromagnetic fluorides of cobalt CoF_2 and manganese MnF_2 by A.S. Borovik-Romanov [17]. These substances have a simple tetragonal lattice and the symmetry of space group $\mathbf{D}_{4h}^{14}$. In their unit cell, there are two metallic ions in positions (000) and $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. The magnetic structure has been determined neutronographically by R.A. Erickson [18] (Fig. 2).

The symmetry group of CoF_2 and MnF_2 is

$$\mathbf{D}_{4h}(\mathbf{D}_{2h}) = (E, C_2, 2U_2t, \sigma_h, 2\sigma_v t, I, 2C_{4z}Rt, 2U_2'R, 2\sigma_v'R, 2C_4\sigma_h Rt)\,. \tag{42}$$

Here, we use the same notations for the operations of rotations and reflections as in textbook [19]. For example, $2U_2'R$ — rotations at angle π around the [110] or $[1\bar{1}0]$ axis accompanied by operation of time reversal R. The crystal symmetry of these substances is nonsymmorphic and some of the operations enumerated in (42) are accompanied by a shift at half period $t = t_{1/2} = (a, a, c)/2$ along the prism diagonal.

On a large scale, in comparison with interatomic distances, the operation t-shift plays no role and the essential symmetry is only with respect to rotations and reflections in combination with time reversal R. The piezomagnetic thermodynamic potential invariant for all these operations is

$$\Phi_{pm} = -\lambda_1(\sigma_{xz}H_y + \sigma_{yz}H_x) - \lambda_2\sigma_{xy}H_z\,, \tag{43}$$

and corresponding additional magnetization arising with the application of shear stress σ_{xz} is

$$M_y = -\frac{\partial\Phi_{pm}}{\partial H_y} = \lambda_1\sigma_{xz}\,. \tag{44}$$

This effect was measured and reported in [17].

Both CoF_2 and MnF_2 are dielectric antiferromagnets. The same crystallographic structure and antiferromagnetic order has metallic compound RuO_2 determined by Z.H. Zhu et al. [20] by means of resonant X-ray scattering. The energy of an electron as a function of momentum in a metal with a structure symmetric with respect to all the operations in Eqn (42) has the following form:

$$\varepsilon_{\alpha\beta} = \varepsilon_{\mathbf{k}}\delta_{\alpha\beta} + \boldsymbol{\gamma}_{\mathbf{k}}\boldsymbol{\sigma}_{\alpha\beta}\,, \tag{45}$$

$$\boldsymbol{\gamma}_{\mathbf{k}} = \gamma_1 \sin(k_z b)\left[\sin(k_y a)\hat{x} + \sin(k_x a)\hat{y}\right] + \gamma_2 \sin(k_x a)\sin(k_y a)\hat{z}\,, \tag{46}$$

where $\varepsilon = \varepsilon(\mathbf{k})$ is the translation invariant even function with symmetry $\mathbf{D}_{4h}(\mathbf{D}_{2h})$ and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. Here, we have taken into account that the shift $\pi(1/a, 1/a, 1/b)$ on the half basis vector in the reciprocal space corresponds to the operation $t_{1/2}$ in coordinate space. Equation (46) defining the vector $\boldsymbol{\gamma}_{\mathbf{k}}$ is the simplest possible expression that has the necessary symmetry properties.

In general, the electron spectrum of a metal such that its group of symmetry G (magnetic class) contains the operation of time reversal only in combination with rotations or reflections has the form of Eqn (45) invariant regarding all operations of the group G. There is a subclass of these types of metals such that the angular average

$$\int \frac{d\Omega_{\mathbf{k}}}{4\pi}\,\boldsymbol{\gamma}_{\mathbf{k}} = 0\,. \tag{47}$$

These types of metals that look like antiferromagnets in reciprocal space are called *altermagnets*. The electron spin ordering in altermagnets determined by exchange interaction is in general noncollinear.

In some cases, one must take into account interband spin-orbit interaction and work with a more general 4×4 matrix electron spectrum:

$$\hat{\mathcal{E}}_{\mathbf{k}} = \frac{(\varepsilon_{1\mathbf{k}} + \boldsymbol{\gamma}_{1\mathbf{k}}\boldsymbol{\sigma})(\tau_0 + \tau_3)}{2} + i\tau_2\boldsymbol{\varphi}_{\mathbf{k}}\boldsymbol{\sigma} + \frac{(\varepsilon_{2\mathbf{k}} + \boldsymbol{\gamma}_{2\mathbf{k}}\boldsymbol{\sigma})(\tau_0 - \tau_3)}{2}\,. \tag{48}$$

Here, τ_0, τ_1, τ_2, τ_3 are the band Pauli matrices. This form of spectrum is important in the study of the anomalous Hall effect in altermagnets (see below).

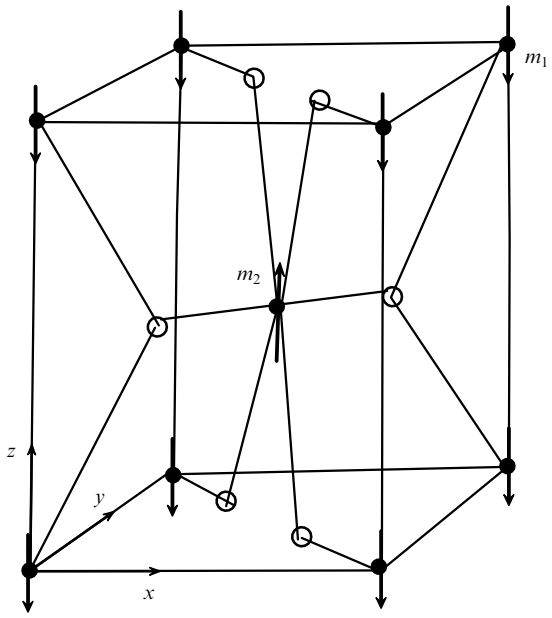

Figure 2. Magnetic structure of MnF_2 showing order and orientation of Mn ion magnetic moments. Dots/circles correspond to fluorine sites.

4.1 Electronic states

In the subsequent text, we will work with a simple 2×2 matrix spectrum (45) which has the same form as in noncentrosymmetric metals:

$$\hat{\varepsilon}(\mathbf{k}) = \varepsilon_{\mathbf{k}}\sigma_0 + \boldsymbol{\gamma}_{\mathbf{k}}\boldsymbol{\sigma}\,. \tag{49}$$

(See for example [21] and references therein.) Thus, all the calculations for these different types of metals look identical, but one must remember that, in altermagnets, vector $\boldsymbol{\gamma}_{-\mathbf{k}} = \boldsymbol{\gamma}_{\mathbf{k}}$ is an even function of $\mathbf{k}$, whereas in noncentrosymmetric metals, it is odd: $\boldsymbol{\gamma}_{-\mathbf{k}} = -\boldsymbol{\gamma}_{\mathbf{k}}$. The scalar part of spectrum $\varepsilon_{\mathbf{k}} = \varepsilon_{-\mathbf{k}}$ is even in both cases.

The eigenvalues of matrix (49) are

$$\varepsilon_+(\mathbf{k}) = \varepsilon + \gamma\,, \quad \varepsilon_-(\mathbf{k}) = \varepsilon - \gamma\,, \tag{50}$$

where $\gamma = |\boldsymbol{\gamma}_{\mathbf{k}}|$. The corresponding eigenfunctions are given by

$$\Psi_{\alpha}^{+}(\mathbf{k}) = \frac{1}{\sqrt{2\gamma(\gamma+\gamma_z)}}\begin{pmatrix}\gamma+\gamma_z\\ \gamma_+\end{pmatrix},$$
$$\Psi_{\alpha}^{-}(\mathbf{k}) = \frac{t_+^{\star}}{\sqrt{2\gamma(\gamma+\gamma_z)}}\begin{pmatrix}-\gamma_-\\ \gamma+\gamma_z\end{pmatrix}, \tag{51}$$

where $\gamma_{\pm} = \gamma_x \pm \mathrm{i}\gamma_y$ and $t_+^{\star} = -\gamma_+/\sqrt{\gamma_+\gamma_-}$. The eigenfunctions obey the orthogonality conditions

$$\Psi_{\alpha}^{\lambda_1\star}(\mathbf{k})\Psi_{\alpha}^{\lambda_2}(\mathbf{k}) = \delta_{\lambda_1\lambda_2}\,, \quad \Psi_{\alpha}^{\lambda}(\mathbf{k})\Psi_{\beta}^{\lambda\star}(\mathbf{k}) = \delta_{\alpha\beta}\,. \tag{52}$$

Here, a summation over the repeating spin $\alpha = \uparrow, \downarrow$ or band $\lambda = +, -$ indices is implied.

In altermagnets, as in noncentrosymmetric metals, the eigenfunctions from different bands transform into each other by an operation of time inversion,

$$-\mathrm{i}(\sigma_y)_{\alpha\beta}K_0\Psi_{\beta}^{+}(\mathbf{k}) \propto \Psi_{\alpha}^{-}(\mathbf{k})\,,$$

where K_0 is the operation of complex conjugation. Thus, the Kramers degeneracy is lifted.

There are two Fermi surfaces with different Fermi momenta $\mathbf{k}_{\mathrm{F}\pm}$. They are determined by the equations

$$\varepsilon_{\pm}(\mathbf{k}) = \mu\,, \tag{53}$$

and the Fermi velocities are given by the derivatives

$$\mathbf{v}_{\mathrm{F}\pm} = \left.\frac{\partial\bigl(\varepsilon_{\pm}(\mathbf{k})\bigr)}{\partial\mathbf{k}}\right|_{k=k_{\mathrm{F}\pm}}. \tag{54}$$

4.2 Spin current in thermodynamic equilibrium

The density of spin current is

$$\mathbf{j}_i = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\boldsymbol{\sigma}_{\alpha\beta}\,\frac{\partial\varepsilon_{\beta\gamma}(\mathbf{k})}{\partial k_i}\,n_{\gamma\alpha}(\mathbf{k},\omega)\,. \tag{55}$$

The matrix of the equilibrium electron distribution function is

$$n_{\alpha\beta} = \frac{n_+ + n_-}{2}\,\delta_{\alpha\beta} + \frac{n_+ - n_-}{2\gamma}\,\boldsymbol{\gamma}\boldsymbol{\sigma}_{\alpha\beta}\,, \tag{56}$$

where $n_{\lambda} = \left[\exp\left((\varepsilon_{\lambda}-\mu)/T\right)+1\right]^{-1}$ is the Fermi distribution function.

The integral (55) in altermagnets is equal to zero. However, in noncentrosymmetric metals, there is nonzero spin current density in thermodynamic equilibrium [22, 23]:

$$\mathbf{j}_i = \int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{\partial\boldsymbol{\gamma}}{\partial k_i}(n_+ + n_-) + \frac{\partial\varepsilon_{\mathbf{k}}}{\partial k_i}(n_+ - n_-)\,\frac{\boldsymbol{\gamma}}{\gamma}\right]. \tag{57}$$

The presence of dissipationless spin currents is a property of noncentrosymmetric metals similar to the presence of electric currents in thermodynamic equilibrium in a metal with toroid order given by Eqn (11).

4.3 Spin susceptibility

The spin quantization axis is given by the unit vector $\hat{\boldsymbol{\gamma}} = \boldsymbol{\gamma}/|\boldsymbol{\gamma}|$. The projections of the electron spins in two bands in the $\hat{\boldsymbol{\gamma}}$ direction have opposite signs:

$$(\hat{\boldsymbol{\gamma}}_{\mathbf{k}}\boldsymbol{\sigma}_{\alpha\beta})\Psi_{\beta}^{\pm}(\mathbf{k}) = \pm\Psi_{\alpha}^{\pm}(\mathbf{k})\,. \tag{58}$$

In an external magnetic field, the matrix of the electron energy is

$$\hat{\varepsilon}(\mathbf{k}) = \varepsilon_{\mathbf{k}}\sigma_0 + \boldsymbol{\gamma}_{\mathbf{k}}\boldsymbol{\sigma} - \mathbf{h}\boldsymbol{\sigma}\,. \tag{59}$$

The field is written here as $\mathbf{h} = \mu_{\mathrm{B}}\mathbf{H}$. The band energies are now given by

$$\varepsilon_{\lambda,\mathbf{h}}(\mathbf{k}) = \varepsilon_{\mathbf{k}} + \lambda|\boldsymbol{\gamma}_{\mathbf{k}} - \mathbf{h}|\,, \quad \lambda = \pm\,. \tag{60}$$

Along with the changes in band energies, the spin quantization axis also deviates from its zero field direction:

$$\hat{\boldsymbol{\gamma}}_{\mathbf{k}} \to \hat{\boldsymbol{\gamma}}_{\mathbf{h}}(\mathbf{k}) = \frac{\boldsymbol{\gamma}_{\mathbf{k}} - \mathbf{h}}{|\boldsymbol{\gamma}_{\mathbf{k}} - \mathbf{h}|}\,. \tag{61}$$

The magnetic moment is written as

$$\mathbf{M} = \mu_{\mathrm{B}}\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\hat{\boldsymbol{\gamma}}_{\mathbf{h}}(\mathbf{k})\bigl[n\bigl(\varepsilon_{+,\mathbf{h}}(\mathbf{k})\bigr) - n\bigl(\varepsilon_{-,\mathbf{h}}(\mathbf{k})\bigr)\bigr]\,, \tag{62}$$

where $n(\varepsilon_{\lambda}) = \left[\exp\left((\varepsilon_{\lambda}-\mu)/T\right)+1\right]^{-1}$ is the Fermi distribution function. Taking the term of the first order in the magnetic field, we obtain for the magnetic susceptibility

$$\chi_{ij} = -\mu_{\mathrm{B}}^2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left\{\hat{\gamma}_i\hat{\gamma}_j\left[\frac{\partial n(\varepsilon_+)}{\partial\varepsilon_+} + \frac{\partial n(\varepsilon_-)}{\partial\varepsilon_-}\right]\right.$$
$$\left. + (\delta_{ij} - \hat{\gamma}_i\hat{\gamma}_j)\,\frac{n(\varepsilon_+) - n(\varepsilon_-)}{|\boldsymbol{\gamma}|}\right\}. \tag{63}$$

The first term under the sign of integration contains the derivatives of the jumps in the Fermi distributions $\partial n(\varepsilon_{\pm})/\partial\varepsilon_{\pm} = -\delta(\varepsilon_{\pm} - \mu)$. The second one originates from the deviation in the spin quantization direction for the quasiparticles filling the states between the Fermi surfaces of two bands. Thus, the magnetic moment arising in altermagnets in an external magnetic field is determined by the same formula as in noncentrosymmetric metals [24].

4.4 Kinetic equation

In the band representation, the equilibrium distribution function (56) is given by the diagonal matrix

$$n_{\lambda_1\lambda_2} = \Psi_{\alpha}^{\lambda_1\star}(\mathbf{k})n_{\alpha\beta}\Psi_{\beta}^{\lambda_2}(\mathbf{k}) = \begin{pmatrix}n(\varepsilon_+) & 0\\ 0 & n(\varepsilon_-)\end{pmatrix}_{\lambda_1\lambda_2}. \tag{64}$$

The Hermitian matrices of the nonequilibrium distribution functions in the band and spin representations are related by

$$f_{\lambda_1\lambda_2}(\mathbf{k}) = \Psi_{\alpha}^{\lambda_1\star}(\mathbf{k})\,f_{\alpha\beta}\Psi_{\beta}^{\lambda_2}(\mathbf{k})\,. \tag{65}$$

The kinetic equation for the electron distribution function in noncentrosymmetric metals was obtained in [25] from the general matrix quasi-classic kinetic equation derived by V.P. Silin [26]. In presence of an electric field $\mathbf{E}$, the linearized matrix kinetic equation for the Fourier amplitudes of deviation of the distribution function from equilibrium

$$g_{\lambda_1\lambda_2}(\mathbf{k},\omega) = f_{\lambda_1\lambda_2}(\mathbf{k}) - n_{\lambda_1\lambda_2}$$

is

$$e\begin{pmatrix} (\mathbf{v}_+\mathbf{E})\,\dfrac{\partial n_+}{\partial\varepsilon_+} & (\mathbf{w}_\pm\mathbf{E})(n_- - n_+) \\ (\mathbf{w}_\mp\mathbf{E})(n_+ - n_-) & (\mathbf{v}_-\mathbf{E})\,\dfrac{\partial n_-}{\partial\varepsilon_-} \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{i}(\varepsilon_- - \varepsilon_+)g_\pm(\mathbf{k}) \\ \mathrm{i}(\varepsilon_+ - \varepsilon_-)g_\mp(\mathbf{k}) & 0 \end{pmatrix} = \hat{I}. \tag{66}$$

Here, we put, for brevity, $n(\varepsilon_+) = n_+, n(\varepsilon_-) = n_-$. The quantities

$$\mathbf{w}_\pm(\mathbf{k}) = \Psi_\alpha^{+\star}(\mathbf{k})\,\frac{\partial\Psi_\alpha^-(\mathbf{k})}{\partial\mathbf{k}} = \frac{t_+^\star}{2\gamma}\left(-\frac{\partial\gamma_-}{\partial\mathbf{k}} + \frac{\gamma_-}{\gamma+\gamma_z}\,\frac{\partial(\gamma+\gamma_z)}{\partial\mathbf{k}}\right) \tag{67}$$

are the *interband* Berry connections,

$$\mathbf{w}_\mp = -\mathbf{w}_\pm^\star\,.$$

Unlike group velocities $\mathbf{v}_+, \mathbf{v}_-$, the dimensionality of the Berry connections $\mathbf{w}_\pm$ and $\mathbf{w}_\mp$ is $1/k$.

$\hat{I}$ is the matrix integral of scattering. In the Born approximation, the collision integral $I_{\lambda_1\lambda_2}$ for electron scattering on impurities (see Appendix A in paper [25]) is

$$\begin{aligned} I^{\mathrm{i}}_{\lambda_1\lambda_2}(\mathbf{k}) = 2\pi n_{\mathrm{i}}\int\frac{\mathrm{d}^3k'}{(2\pi)^3}\left|V(\mathbf{k}-\mathbf{k}')\right|^2 \\ \times\Big\{O_{\lambda_1\nu}(\mathbf{k},\mathbf{k}')\big[g_{\nu\mu}(\mathbf{k}')O_{\mu\lambda_2}(\mathbf{k}',\mathbf{k}) \\ - O_{\nu\mu}(\mathbf{k}',\mathbf{k})g_{\mu\lambda_2}(\mathbf{k})\big]\delta(\varepsilon'_\nu - \varepsilon_{\lambda_2}) \\ + \big[O_{\lambda_1\nu}(\mathbf{k},\mathbf{k}')g_{\nu\mu}(\mathbf{k}') \\ - g_{\lambda_1\nu}(\mathbf{k})O_{\nu\mu}(\mathbf{k},\mathbf{k}')\big]O_{\mu\lambda_2}(\mathbf{k}',\mathbf{k})\delta(\varepsilon'_\mu - \varepsilon_{\lambda_1})\Big\}. \end{aligned} \tag{68}$$

Here, we introduced the notations $\varepsilon_{\lambda_1} = \varepsilon_{\lambda_1}(\mathbf{k}), \varepsilon'_\mu = \varepsilon_\mu(\mathbf{k}')$, etc.,

$$O_{\lambda_1\lambda_2}(\mathbf{k},\mathbf{k}') = \Psi_\sigma^{\lambda_1\star}(\mathbf{k})\Psi_\sigma^{\lambda_2}(\mathbf{k}')\,, \tag{69}$$

such that

$$O_{\lambda_1\lambda_2}(\mathbf{k},\mathbf{k}') = O^\star_{\lambda_2\lambda_1}(\mathbf{k}',\mathbf{k})\,.$$

The expression for the collision integral for electro-electron scattering can be found in Appendix B in paper [25].

4.5 Conductivity

If the energy of band splitting exceeds the electron-impurity scattering rate

$$v_{\mathrm{F}}(k_{\mathrm{F}-} - k_{\mathrm{F}+}) \gg \frac{1}{\tau_i}\,, \tag{70}$$

one can disregard the collision integrals in the off-diagonal terms of matrix kinetic equation (66) and use the collisionless solution for the off-diagonal terms of the matrix distribution function:

$$g_\pm = e(\mathbf{w}_\pm\mathbf{E}) = \frac{e(\mathbf{w}_\pm\mathbf{E})(n_- - n_+)}{-\mathrm{i}(\varepsilon_- - \varepsilon_+)}\,, \tag{71}$$

$$g_\mp = e(\mathbf{w}_\mp\mathbf{E}) = \frac{e(\mathbf{w}_\mp\mathbf{E})(n_+ - n_-)}{-\mathrm{i}(\varepsilon_+ - \varepsilon_-)}\,. \tag{72}$$

It has been shown that, in the stationary case, this type of off-diagonal term does not make a contribution to the electric current [25]. On the other hand, substituting these expressions into the diagonal parts of collision-integral matrices (68) allows us to ignore them in all the terms containing off-diagonal elements of the distribution function. These terms are $v_{\mathrm{F}}(k_{\mathrm{F}-} - k_{\mathrm{F}+})\tau_i \gg 1$ times smaller than those with diagonal elements.

Then, the system Eqn (66) for

$$g_{\alpha\beta}(\mathbf{k}) = \begin{pmatrix} g_+(\mathbf{k}) & 0 \\ 0 & g_-(\mathbf{k}) \end{pmatrix}_{\alpha\beta} \tag{73}$$

acquires the following form:

$$(\mathbf{v}_+\mathbf{E})\,\frac{\partial n(\varepsilon_+)}{\partial\varepsilon_+} = I^{\mathrm{i}}_+\,, \tag{74}$$

$$(\mathbf{v}_-\mathbf{E})\,\frac{\partial n(\varepsilon_-)}{\partial\varepsilon_-} = I^{\mathrm{i}}_-\,, \tag{75}$$

where

$$\begin{aligned} I^{\mathrm{i}}_+ = 4\pi n_{\mathrm{i}}\int\frac{\mathrm{d}^3k}{2\pi^3}\left|V(\mathbf{k}-\mathbf{k}')\right|^2 \\ \times\Big\{O_{++}(\mathbf{k}\mathbf{k}')O_{++}(\mathbf{k}'\mathbf{k})\big[g_+(\mathbf{k}') - g_+(\mathbf{k})\big]\delta(\varepsilon'_+ - \varepsilon_+) \\ + O_{+-}(\mathbf{k}\mathbf{k}')O_{-+}(\mathbf{k}'\mathbf{k})\big[g_-(\mathbf{k}') - g_+(\mathbf{k})\big]\delta(\varepsilon'_- - \varepsilon_+)\Big\}, \end{aligned} \tag{76}$$

$$\begin{aligned} I^{\mathrm{i}}_- = 4\pi n_{\mathrm{i}}\int\frac{\mathrm{d}^3k}{2\pi^3}\left|V(\mathbf{k}-\mathbf{k}')\right|^2 \\ \times\Big\{O_{--}(\mathbf{k}\mathbf{k}')O_{--}(\mathbf{k}'\mathbf{k})\big[g_-(\mathbf{k}') - g_-(\mathbf{k})\big]\delta(\varepsilon'_- - \varepsilon_-) \\ + O_{-+}(\mathbf{k}\mathbf{k}')O_{+-}(\mathbf{k}'\mathbf{k})\big[g_+(\mathbf{k}') - g_-(\mathbf{k})\big]\delta(\varepsilon'_+ - \varepsilon_-)\Big\}. \end{aligned} \tag{77}$$

Thus, we come to a system of two equations coupled through the collision integrals containing intraband as well as interband electron scattering terms. We can search for the solution of Eqns (76), (77) in the following form:

$$g_+ = -e\tau_+\,\frac{\partial n_+}{\partial\xi_+}(\mathbf{v}_+\mathbf{E})\,,\quad g_- = -e\tau_-\,\frac{\partial n_-}{\partial\xi_-}(\mathbf{v}_-\mathbf{E})\,, \tag{78}$$

where the scattering times τ_+, τ_- are even functions of the wave vector. They should be found as the solution of equations (76), (77).

The electric current density is

$$\mathbf{j} = e\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\frac{\partial\varepsilon_{\alpha\beta}(\mathbf{k})}{\partial\mathbf{k}}\,g_{\beta\alpha}(\mathbf{k},\omega)\,. \tag{79}$$

Transforming it into the band representation, we obtain

$$\begin{aligned} \mathbf{j} = e\int\frac{\mathrm{d}^3k}{(2\pi)^3}\,\Psi_\alpha^{\lambda_1\star}(\mathbf{k})\,\frac{\partial\varepsilon_{\alpha\beta}(\mathbf{k})}{\partial\mathbf{k}}\,\Psi_\beta^{\lambda_2}(\mathbf{k}) \\ \times\Psi_\gamma^{\lambda_2\star}(\mathbf{k})g_{\gamma\delta}(\mathbf{k},\omega)\Psi_\delta^{\lambda_1}(\mathbf{k}) \\ = e\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left\{\frac{\partial\varepsilon_{\lambda_1\lambda_2}(\mathbf{k})}{\partial\mathbf{k}} + \big[\mathbf{w}_{\lambda_1\lambda_3},\varepsilon_{\lambda_3\lambda_2}\big]\right\}g_{\lambda_2\lambda_1}(\mathbf{k})\,, \end{aligned} \tag{80}$$

where $[\ldots,\ldots]$ is the commutator. Performing matrix multiplication, we obtain

$$\mathbf{j} = e\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[\mathbf{v}_+g_+ + \mathbf{v}_-g_- + (\mathbf{w}_\pm g_\mp - \mathbf{w}_\mp g_\pm)(\varepsilon_- - \varepsilon_+)\right]. \tag{81}$$

Ignoring off-diagonal terms of the distribution function and substituting solutions (78), we obtain the expression

$$\mathbf{j} = -e^2 \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[\tau_+ \frac{\partial n_+}{\partial \xi_+} \mathbf{v}_+ (\mathbf{v}_+ \mathbf{E}) + \tau_- \frac{\partial n_-}{\partial \xi_-} \mathbf{v}_- (\mathbf{v}_- \mathbf{E}) \right], \quad (82)$$

determining the conductivity due to electron scattering on impurities. The corresponding derivation of conductivity determined by joint processes of scattering on impurities and electron–electron scattering is derived in paper [27].

4.6 Magnetoelectric effect

Altermagnets are invariant with respect to space inversion; hence, the external electric field does not cause magnetization to appear in them. On the contrary, noncentrosymmetric metals placed in an electric field are magnetized. In semiconductors, this effect was predicted long ago by E.L. Ivchenko and G.E. Pikus [28] and was reviewed in the recently published paper [29]. The magnetoelecricity in a 2D metal with the Rashba spin-orbit interaction was considered first by V.M. Edelstein [30]. A more general treatment has been developed recently in paper [31].

The density of magnetization

$$\mathbf{M} = \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} \boldsymbol{\sigma}_{\alpha\beta} g_{\beta\alpha} = \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} \boldsymbol{\sigma}_{\lambda_1\lambda_2} g_{\lambda_2\lambda_1} \quad (83)$$

is determined by the integral from the product of the distribution function and Pauli matrices in the band representation. When disregarding off-diagonal terms of the distribution function, we obtain

$$\mathbf{M} = \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} \left[\frac{\boldsymbol{\gamma}_\mathbf{k}}{\gamma} (g_+ - g_-) \right]. \quad (84)$$

Substituting Eqn (78), we obtain

$$\mathbf{M} = -e \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} \frac{\boldsymbol{\gamma}_\mathbf{k}}{\gamma} \left[\tau_+ \frac{\partial n_+}{\partial \xi_+} (\mathbf{v}_+ \mathbf{E}) - \tau_- \frac{\partial n_-}{\partial \xi_-} (\mathbf{v}_- \mathbf{E}) \right]. \quad (85)$$

Thus, an application of an electric field to a noncentrosymmetric metal causes the appearance of specimen magnetization.

4.7 Anomalous Hall effect

The Hall conductivity is the antisymmetric nondissipative part of the conductivity tensor $\sigma_{ij} = -\sigma_{ji}$, which determines the relationship between the current and the Hall electric field arising in a magnetic field in a direction perpendicular to both the current and the magnetic field:

$$j_x = \sigma_{xy} E_y^{\mathrm{H}}. \quad (86)$$

The anomalous Hall effect arises because, in general, electron states adapt to the presence of an electric field, and the velocity of an electron in a state with momentum $\mathbf{k}$ acquires an additional term [32, 33]

$$v_i^n = \frac{1}{\hbar} \frac{\partial \varepsilon_\mathbf{k}^n}{\partial k_i} + \frac{e}{\hbar} \Omega_{ij}^n E_j, \quad (87)$$

where Ω_{ij}^n is the Berry curvature tensor of the nth band with energy $\varepsilon_\mathbf{k}^n$. The corresponding Hall conductivity is

$$\sigma_{ij} = \frac{e^2}{\hbar} \sum_n \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} n(\varepsilon_n) \Omega_{ij}^n. \quad (88)$$

Here, $n(\varepsilon_n) = \{\exp((\varepsilon_n - \mu)/T) + 1\}^{-1}$ is the Fermi–Dirac distribution function.

The antisymmetric tensor of Berry curvature for the band $\lambda = +$ is

$$\Omega_{ij}^+ = \mathrm{i} \left(\frac{\partial \Psi_\alpha^{+\star}}{\partial k_i} \frac{\partial \Psi_\alpha^+}{\partial k_j} - \frac{\partial \Psi_\alpha^{+\star}}{\partial k_j} \frac{\partial \Psi_\alpha^+}{\partial k_i} \right). \quad (89)$$

The corresponding Berry curvature for the band $\lambda = -$ is $\Omega_{ij}^- = -\Omega_{ij}^+$; hence, the Hall conductivity is

$$\sigma_{ij} = \frac{e^2}{\hbar} \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} \left[n(\varepsilon_+) - n(\varepsilon_-) \right] \Omega_{ij}^+. \quad (90)$$

Let us calculate the Berry curvature for an altermagnet with spectrum (46) in the presence of a magnetic field along the $\hat{z}$-direction such that

$$\boldsymbol{\gamma}_\mathbf{k} = \gamma_1 \sin(k_z b) \left[\sin(k_y a) \hat{x} + \sin(k_x a) \hat{y} \right] + \left(\gamma_2 \sin(k_x a) \sin(k_y a) - \mu_\mathrm{B} H \right) \hat{z}. \quad (91)$$

Substituting eigenfunctions (51) into equation (89) and performing differentiation, we obtain

$$\Omega_{xy}^+ = -\frac{\gamma_1^2 a^2}{2\gamma^3} \cos(k_x a) \cos(k_y a) \times \left[\gamma_2 \sin(k_x a) \sin(k_y a) - \mu_\mathrm{B} H \right]. \quad (92)$$

Substitution of this expression into Eqn (90) yields the anomalous Hall conductivity,

$$\sigma_{xy} = \frac{e^2 \mu_\mathrm{B} \gamma_1^2 a^2}{2\hbar} H \times \int \frac{\mathrm{d}^3 \mathbf{k}}{(2\pi)^3} \left[n(\varepsilon_+) - n(\varepsilon_-) \right] \frac{\cos(k_x a) \cos(k_y a)}{\gamma^3}. \quad (93)$$

A similar expression for the Hall conductivity can also be found for noncentrosymmetric metals where vector $\boldsymbol{\gamma}_\mathbf{k}$ is an odd function of the wave vector.

The field independent part of Ω_{xy}^+ vanishes with integration and does not contribute to the Hall conductivity. This is also true for the Hall conductivity determined by the interband Berry curvature considered in paper [34]. However, in general, one can expect the existence of the Hall conductivity even in the absence of a magnetic field. The possibility of the Hall effect in noncollinear antiferromagnetic materials in the absence of an external magnetic field was predicted for Mn_3Ir about a decade ago [35, 36]. Recently, the existence of the same phenomenon in the collinear antiferromagnet RuO_2 was pointed out [37]. Numerical estimates of the Hall conductivity in [35, 36] as well as in [37] were made using first-principles calculations of the electronic structure. At the moment, the corresponding phenomenological derivation is absent. One can only assume that this is achievable by making use of the 4×4 phenomenological spectrum given by Eqn (48).

4.8 Piezomagnetic Hall effect

Altermagnets possess piezomagnetism. For instance, under stress along the diagonal xy-direction, an altermagnet with symmetry (42) acquires magnetization along the z-axis:

$$M_z = \lambda_2 \sigma_{xy}. \quad (94)$$

Hence, an electric current in the basal plane of such an altermagnet in the presence of σ_{xy} stress will induce the appearance of an electric field in the direction perpendicular to the current,

$$j_x = \sigma_{xy}^{\mathrm{H}} E_y \,, \tag{95}$$

where the Hall conductivity

$$\sigma_{xy}^{\mathrm{H}} = \frac{e^2 \mu_{\mathrm{B}} \gamma_1^2 a^2}{2\hbar}\, \lambda_2 \sigma_{xy} \times \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \left[n(\varepsilon_+) - n(\varepsilon_-) \right] \frac{\cos(k_x a)\cos(k_y a)}{\gamma^3} \tag{96}$$

is proportional to the stress.

5. Superconducting states in altermagnets

The BCS Hamiltonian for singlet pairing in the spinor representation has the following form:

$$\hat{H} = \sum_{\mathbf{k}\alpha\beta} (\varepsilon_{\mathbf{k}} \delta_{\alpha\beta} + \boldsymbol{\gamma}_{\mathbf{k}} \boldsymbol{\sigma}_{\alpha\beta} - \mu \delta_{\alpha\beta}) a^{\dagger}_{\mathbf{k}\alpha} a_{\mathbf{k}\beta} + \frac{1}{2} \sum_{\mathbf{k}\mathbf{k}'} \sum_{\alpha\beta\gamma\delta} V(\mathbf{k},\mathbf{k}') (\mathrm{i}\sigma_y)_{\alpha\beta} (\sigma_y)^{\dagger}_{\gamma\delta} a^{\dagger}_{-\mathbf{k}\alpha} a^{\dagger}_{\mathbf{k}\beta} a_{\mathbf{k}'\gamma} a_{-\mathbf{k}'\delta} \,. \tag{97}$$

Here,

$$V(\mathbf{k},\mathbf{k}') = -V_0 \varphi_i^{\Gamma}(\mathbf{k}) \varphi_i^{\Gamma\star}(\mathbf{k}') \tag{98}$$

is the paring potential decomposed over the basis of even $\varphi_i^{\Gamma}(\mathbf{k}) = \varphi_i^{\Gamma}(-\mathbf{k})$ functions of a given irreducible representation Γ of the crystal symmetry group. For example, for an altermagnet with symmetry group $\mathbf{D}_{4h}(\mathbf{D}_{2h})$ consisting of operations enumerated in Eqn (42), the function transforming according to the unit representation is

$$\varphi(\mathbf{k}) \propto \mathrm{i}(\hat{k}_x^2 - \hat{k}_y^2) \,. \tag{99}$$

Here, $\hat{k}_x$, $\hat{k}_y$ are the components of unit vector $\mathbf{k}/k_{\mathrm{F}}$.

Transforming into the band representation

$$a_{\mathbf{k}\alpha} = \Psi_{\alpha}^{\lambda}(\mathbf{k}) c_{\mathbf{k}\lambda} \,, \tag{100}$$

we obtain

$$\hat{H} = \sum_{\mathbf{k}\lambda} \left(\varepsilon_{\lambda}(\mathbf{k}) - \mu \right) c^{\dagger}_{\mathbf{k}\lambda} c_{\mathbf{k}\lambda} + \frac{1}{2} \sum_{\mathbf{k}\mathbf{k}'} \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} V_{\lambda_1\lambda_2\lambda_3\lambda_4}(\mathbf{k},\mathbf{k}') c^{\dagger}_{-\mathbf{k}\lambda_1} c^{\dagger}_{\mathbf{k}\lambda_2} c_{\mathbf{k}'\lambda_3} c_{-\mathbf{k}'\lambda_4} \,. \tag{101}$$

Here,

$$V_{\lambda_1\lambda_2\lambda_3\lambda_4}(\mathbf{k},\mathbf{k}') = V(\mathbf{k},\mathbf{k}') t_{\lambda_2}(\mathbf{k}) t^{\star}_{\lambda_4}(\mathbf{k}') \sigma^x_{\lambda_1\lambda_2} \sigma^x_{\lambda_3\lambda_4} \,, \tag{102}$$

and $t_{\lambda}(\mathbf{k}) = -\lambda\gamma_-/\sqrt{\gamma_+\gamma_-}$ is the phase factor.

It is obvious from this expression that pairing in altermagnets is the pairing of electrons from different bands. This distinguishes them from noncentrosymmetric metals where

$$V_{\lambda_1\lambda_2\lambda_3\lambda_4}(\mathbf{k},\mathbf{k}') = V(\mathbf{k},\mathbf{k}') t_{\lambda_2}(\mathbf{k}) t^{\star}_{\lambda_4}(\mathbf{k}') \delta_{\lambda_1\lambda_2} \delta_{\lambda_3\lambda_4} \,, \tag{103}$$

and the pairing mostly occurs between electrons from the same band [21].

The situation in altermagnets is reminiscent of pairing in conventional superconductors with singlet pairing in a magnetic field, which splits the Fermi surfaces for electrons with opposite spins. That leads to paramagnetic suppression of superconductivity. In altermagnets, the same effect takes place in the absence of a field, which leads to an effective reduction in the temperature of transition to the superconducting state or even to complete suppression of superconductivity. Thus, the possibility of the existence of superconducting altermagnets raises doubts. Nevertheless, for completeness, we present here the theoretical description of superconductivity in altermagnets.

The Gor'kov equations are

$$\begin{pmatrix} \mathrm{i}\omega\delta_{\lambda_1\lambda_2} - H_{\lambda_1\lambda_2} & -\tilde{\Delta}_{\lambda_1\lambda_2} \\ -\tilde{\Delta}^{\dagger}_{\lambda_1\lambda_2} & \mathrm{i}\omega\delta_{\lambda_1\lambda_2} + H_{\lambda_1\lambda_2} \end{pmatrix} \times \begin{pmatrix} G_{\lambda_2\lambda_3} & -\tilde{F}_{\lambda_2\lambda_3} \\ -\tilde{F}^{\dagger}_{\lambda_2\lambda_3} & -G_{\lambda_2\lambda_3} \end{pmatrix} = \delta_{\lambda_1\lambda_3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{104}$$

where

$$\mathrm{i}\omega\delta_{\lambda_1\lambda_2} - H_{\lambda_1\lambda_2} = \begin{pmatrix} \mathrm{i}\omega - \varepsilon_+ + \mu & 0 \\ 0 & \mathrm{i}\omega + \varepsilon_- - \mu \end{pmatrix}, \tag{105}$$

and the phase factor is absorbed in the expressions for the order parameter and the Gor'kov function:

$$\Delta_{\lambda_1\lambda_2}(\mathbf{k}) = t_{\lambda_2}(\mathbf{k}) \tilde{\Delta}_{\lambda_1\lambda_2}(\mathbf{k}) \,, \tag{106}$$

$$\tilde{\Delta}_{\lambda_1\lambda_2}(\mathbf{k}) = (\sigma_x)_{\lambda_1\lambda_2} \Delta(\mathbf{k}) \,, \tag{107}$$

$$F_{\lambda_1\lambda_2}(\mathbf{k},\omega_n) = t_{\lambda_2}(\mathbf{k}) \tilde{F}_{\lambda_1\lambda_2}(\mathbf{k},\omega_n) \,, \tag{108}$$

where $\omega_n = \pi T(2n+1)$ is the Matsubara frequency. The self-consistency equation is

$$\tilde{\Delta}_{\lambda_1\lambda_2}(\mathbf{k}) = -\frac{T}{2} \sum_n \sum_{\mathbf{k}'} V(\mathbf{k},\mathbf{k}') (\sigma_x)_{\lambda_2\lambda_1} (\sigma_x)_{\lambda_3\lambda_4} \tilde{F}_{\lambda_3\lambda_4}(\mathbf{k}',\omega_n) \,, \tag{109}$$

where

$$\tilde{F}_{\lambda_1\lambda_2}(\mathbf{k},\omega_n) = \Delta \begin{pmatrix} 0 & G^n_+(\mathbf{k},\omega_n) G_-(\mathbf{k},-\omega_n) \\ G^n_-(\mathbf{k},\omega_n) G_+(\mathbf{k},-\omega_n) & 0 \end{pmatrix} \tag{110}$$

is the matrix Gor'kov function and

$$G_{\pm}(\mathbf{k},\omega_n) = -\frac{\mathrm{i}\omega_n + \varepsilon_{\pm} - \mu}{\omega_n^2 + (\varepsilon_{\pm} - \mu)^2 + \Delta^2} \,, \tag{111}$$

$$G^n_{\pm}(\mathbf{k},\omega_n) = \frac{1}{\mathrm{i}\omega_n - \varepsilon_{\pm} + \mu} \tag{112}$$

are the band Green's functions in the superconducting and normal state, respectively. The order parameters in the spin and band representations are related to each other as

$$\Delta_{\alpha\beta}(\mathbf{k}) = (\mathrm{i}\sigma_y)_{\alpha\beta} \Delta(\mathbf{k}) \,. \tag{113}$$

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