METHODOLOGICAL NOTES

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On 'Schrödinger's cat' fringes

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Contents

Abstract. **Non-Gaussian quantum states whose Wigner functions can take negative values are important both for fundamental tests of quantum physics and for the quantum information technologies that have been under active development recently. A typical example of a non-Gaussian state is the so-called Schrödinger's cat state. Its very interesting feature is that its 'classical' part (two Gaussian maxima) is geometrically separated from the 'nonclassical' part (interference fringes). In this paper, several methodological issues related to these fringes are considered.**

Keywords: Schrödinger's cat, Wigner functions, non-Gaussian states

1. Introduction

The famous thought experiment formulated by Schrödinger in [1] is perhaps the most vivid demonstration of the main methodological problem of quantum physics, namely the problem of the absence of a smooth limiting transition from the quantum description of the world to the classical one. In this thought experiment, a method is proposed that is completely correct from the point of view of quantum mechanics for transforming the superposition state of a microscopic object (an atom) into the superposition state of a macroscopic object (a 'cat,' or more precisely, a composite system, 'atom + cat'). But it is 'obvious' that macroscopic objects cannot be in a state of superposition, so is there something wrong with quantum mechanics?

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All existing interpretations of quantum mechanics somehow solve this problem. In particular, in the Copenhagen interpretation, 'extra' branches of the wave function are randomly cut off by means of quantum state reduction. In the Everett many-worlds interpretation, all branches of evolution coexist, but cannot communicate with each other due to the linearity of the Schrödinger equation.

In the last two or three decades, interest in quantum states that are superpositions of two well-distinguishable wave functions has moved into the practical realm due to the rapid progress of quantum information processing and transmission technologies. Of course, we are not yet dealing with the states of 'Schrödinger's cat' (SC) for really macroscopic objects. The bulk of work in this area deals with optical field states that are superpositions of the form

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\left(|\alpha\rangle + \exp(\mathrm{i}\theta)|-\alpha\rangle\right), \qquad (1)$$

where $|\alpha\rangle$ and $|-\alpha\rangle$ are coherent states, and N is a normalization factor (see review Ref. [2] and references therein). At present, it is possible to generate SC states with values $\alpha^2 \sim 3$ [3, 4]. Methods have also been proposed for obtaining SC states with amplitudes up to $\alpha \sim 4-5$ [5, 6]. Recent years have seen the demonstration of similar states for translational mechanical degrees of freedom (see, for example, Ref. [7]).

The SC states are typical examples of so-called non-Gaussian quantum states, i.e., those whose Wigner functions [8] are different in shape from two-dimensional Gaussian bells. It is well known that non-Gaussian states are 'more quantum' than Gaussian (e.g., coherent) states. In particular, Gaussian states cannot be orthogonal to each other; they allow a classical description in terms of local hidden variables [9]; quantum information processing protocols using only Gaussian states can be efficiently simulated by classical computers [10].

It is also known that the orthogonality properties of SC states, which stem from their non-Gaussian nature, allow them to be effectively used in optical interferometric measurements [11].

A very interesting feature of the Wigner functions of SC states is that the 'classical' part, i.e., the two Gaussian maxima corresponding to the 'alive' and 'dead' statuses of the cat, is

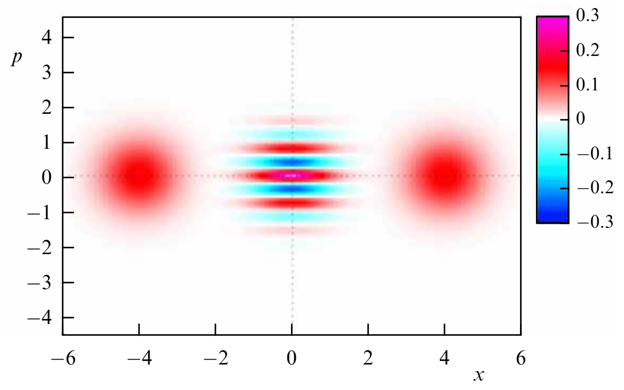

Figure 1. Wigner function of state (1) for $\alpha = 2\sqrt{2}$ and $\theta = 0$.

geometrically separated from the 'nonclassical' part, known as SC 'interference fringes.' An example of the Wigner function of state (1) is shown in Fig. 1. The three parts mentioned are clearly discernible in it. This feature makes the SC states a useful subject of study, for example, the effect of dissipation on non-Gaussian quantum states.

In this paper, several methodological issues related to the SC fringes are considered. In Section 2, a convenient tool for subsequent analysis is introduced, namely, the generalized SC state (4), and a simple universal formula (7) is obtained expressing the Wigner function of such a state in terms of the Wigner function of the corresponding basis state $\psi(x)$. Discussed in Section 3 is the question of what, in fact, interferes in the SC states to form the fringes. Next, in Section 4, the feasibility of direct (nontomographic) observation of these fringes is discussed. Finally, Section 5 briefly summarizes the results of this paper.

2. Wigner function of generalized state of Schrödinger's cat

Following the convention adopted in quantum optics, we will use the dimensionless normalized coordinate and momentum defined by the relations

$$\hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2}}\,, \quad [\hat{x}, \hat{p}] = i\,, \tag{2}$$

where $\hat{a}$ is the annihilation operator. In the case of a mechanical harmonic oscillator, x and p are related to the 'usual' dimensional coordinate and momentum by the formulas

$$x = \sqrt{\frac{m\Omega_0}{\hbar}}\,X\,, \quad p = \frac{P}{\sqrt{\hbar m \Omega_0}}\,, \tag{3}$$

where m is the mass and Ω_0 is the oscillator's eigen frequency.

Assume for simplicity that the parameter α in formula (1) is real. The generality is not limited in this case, since any complex α can be transformed into a real one by a unitary rotation in the phase plane. Next, we consider a generalization of such a state defined by a wave function that has the following form in the coordinate representation:

$$\Psi(x) = \langle x|\Psi\rangle = \frac{1}{\sqrt{N}}\left[\psi(x-\xi) + \psi(x+\xi)\exp(i\theta)\right], \tag{4}$$

where $\psi(x)$ is an arbitrary basis wave function and ξ is a real parameter. The normalization factor N in this case is

$$N = 2(1+\varkappa)\,, \tag{5}$$

where

$$\varkappa = \mathrm{Re}\int_{-\infty}^{\infty} \psi^*(x-\xi)\psi(x+\xi)\exp(i\theta)\,dx\,. \tag{6}$$

For the purposes of this methodological note, the advantage of state (4) is that its basis function ψ can be so chosen that the overlap of components $\psi(x-\xi)$ and $\psi(x+\xi)$ is strictly zero (see Section 3).

Proceeding from definition (4), we can obtain a universal closed expression for the Wigner function of the generalized SC state. Directly using the definition of the Wigner function (see, for example, monograph [12]), it is easily shown that, for any states of the form (4), it can be represented as

$$W(x,p) = \frac{1}{N}\left[W_0(x-\xi, p) + W_0(x+\xi, p) + 2W_1(x,p)\right], \tag{7}$$

where

$$W_0(x,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi^*\left(\frac{x-y}{2}\right)\psi\left(\frac{x+y}{2}\right)\exp(-ipy)\,dy \tag{8}$$

is the Wigner function of the basis state $\psi(x)$ and

$$W_1(x,p) = W_0(x,p)\cos(2p\xi + \theta)\,. \tag{9}$$

Note also that

$$\int_{-\infty}^{\infty} W_1(x,p)\,dx\,dp = \varkappa\,. \tag{10}$$

The structure of formula (7) generalizes that shown in Fig. 1. It comprises two shifted and scaled copies of the Wigner function W_0 of the basis state ψ_0, as well as interference fringes produced by another (unshifted) copy of W_0 modulated by the oscillating factor $\cos(2p\xi + \theta)$.

3. Interference fringes

One can see from formula (7) that the interference term W_1 does not vanish even if the functions $\psi(x+\xi)$ and $\psi(x-\xi)$ do not overlap, i.e., if for any x

$$\psi^*(x-\xi)\psi(x+\xi) = 0\,. \tag{11}$$

Specifically, it follows (contrary to popular belief) that the interference fringes do not disappear with increasing distance 2ξ between the 'live' and 'dead' components of the cat; in this case, only the modulation frequency of the function W_1 increases.

Here, the question may arise: what actually interferes if the functions $\psi(x-\xi)$ and $\psi(x+\xi)$ do not overlap? The answer is quite obvious. The Wigner wave function is defined in the phase space (x,p), and the wave functions in the momentum representation are as important for it as in the coordinate representation. Function (4) in the momentum representation is of the form

$$\Phi(p) = \langle p|\Psi\rangle = \frac{\phi(p)}{\sqrt{N}}\left[\exp\left(-\mathrm{i}p\xi\right) + \exp\left(\mathrm{i}(p\xi+\theta)\right)\right], \tag{12}$$

where

$$\phi(p) = \int_{-\infty}^{\infty} \psi(x)\exp\left(-\mathrm{i}px\right)\frac{\mathrm{d}x}{\sqrt{2\pi}} \tag{13}$$

is the wave function of the basis state in the momentum representation. For the corresponding probability distribution, we obtain

$$\left|\Phi(p)\right|^2 = \frac{2\left|\phi(p)\right|^2}{N}\left[1+\cos\left(2p\xi+\theta\right)\right] \tag{14}$$

(compare with formula (7)). In this case, a narrow coordinate wave function of width $\Delta x \ll \xi$ yields a wide momentum wave function of width

$$\Delta p \sim \frac{1}{\Delta x} \gg \frac{1}{\xi}, \tag{15}$$

in accordance with the fact that many modulation periods π/ξ fit into the interval Δp.

By way of example, consider the case of a 'squeezed' Schrödinger's cat, whose basis wave function is

$$\psi(x) = \frac{1}{(\pi s)^{1/4}}\exp\left(-\frac{x^2}{2s}\right), \tag{16}$$

where s is the squeeze factor. In this case,

$$W(x,p) = \frac{1}{\pi s N}\left[\exp\left(-\frac{(x-\xi)^2}{s}\right) + \exp\left(-\frac{(x+\xi)^2}{s}\right) + 2\exp\left(-\frac{x^2}{s}\right)\cos\left(2p\xi+\theta\right)\right]\exp\left(-sp^2\right). \tag{17}$$

On the one hand, such a basis state is Gaussian and therefore does not complicate the consideration by additional (apart from the cat fringes) negative-valued regions. On the other hand, by decreasing the factor s, it is possible to provide exponentially small values of the overlap coefficient (11) for any given ξ.

The plot of the Wigner function (17) is shown in Fig. 2 for characteristic values of parameters s and ξ. The shortening of the interference fringes associated with the narrowing of the function $W_1(x,p)$ in the coordinate direction is clearly visible.

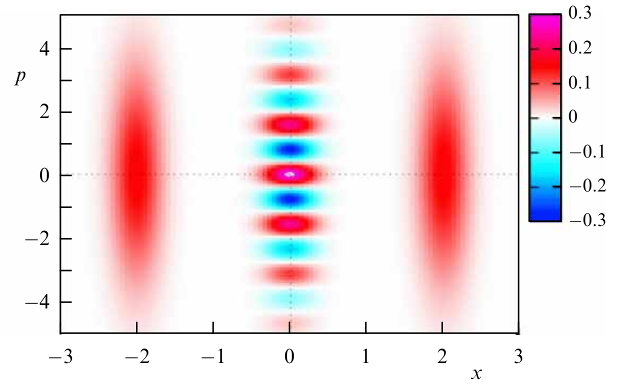

Figure 2. Wigner function (17) for $s = 0.1$, $\xi = 2$, and $\theta = 0$.

4. On the feasibility of observing Schrödinger's cat fringes

Is it possible to experimentally observe the SC fringes? It would seem that the obvious answer is yes, it is possible, since the quantum tomographic procedure [13, 14] allows one, in principle, to reconstruct the Wigner function of any quantum state. However, quantum tomography is, an essentially ensemble procedure, which, in fact, makes it possible to accurately (in the ideal case) reconstruct any quantum state without any restrictions associated with the Heisenberg uncertainty principle. However, for the same reason, quantum tomography cannot be employed, for example, to detect an external effect on a single object. Therefore, let us pose the question differently: can the structure of the fringes manifest itself in the results of an 'ordinary' direct measurement limited by the uncertainty principle?

It is obvious that the required procedure should provide information about both the coordinate and the momentum of the object. A canonical example is the so-called coherent measurement [15, 16], characterized by the same measurement errors for the coordinate and momentum, $\Delta x = \Delta p = 1/\sqrt{2}$, and described by the positive operator-valued measure (POVM) [17] of the form

$$\hat{\Pi}(\tilde{x},\tilde{p}) = \frac{1}{2\pi}|\tilde{\alpha}\rangle\langle\tilde{\alpha}|. \tag{18}$$

Here, $|\tilde{\alpha}\rangle$ is the ket-vector of the coherent state, with parameter $\tilde{\alpha}$ equal to

$$\tilde{\alpha} = \frac{\tilde{x}+\mathrm{i}\tilde{p}}{\sqrt{2}}, \tag{19}$$

and $\tilde{x}$, $\tilde{p}$ are the measurement results. The statistics of the coherent measurement results are given by the Husimi function [18]:

$$Q(\tilde{x},\tilde{p}) = \left\langle\Psi|\hat{\Pi}\left(\tilde{x},\tilde{p}\right)|\Psi\right\rangle, \tag{20}$$

which is the Wigner function [19] smeared symmetrically in all directions in the phase plane.

For an optical wave, this procedure can be implemented by dividing it with a symmetric ($R = T$) beam splitter with subsequent measurement of two orthogonal quadratures of the output beams using two homodyne detectors placed in the output ports of the beam splitter (Fig. 3).

It is obvious, however, that to resolve the SC fringes, the measurement errors of the momentum and coordinate must be consistent with the geometry of these fringes and, notably, can differ from each other:

$$\Delta x = \sqrt{\frac{s_m}{2}}, \quad \Delta p = \frac{1}{\sqrt{2s_m}}, \tag{21}$$

where s_m is the measure of asymmetry of Δx and Δp (the measurement squeeze factor). Such a measurement, which we will call squeezed coherent, can also be implemented using the setup shown in Fig. 3, but the beam splitter in it must now be asymmetric, $R \neq T$. In this case, the POVM will be of the following form:

$$\hat{\Pi}(\tilde{x},\tilde{p},s_m) = \frac{1}{2\pi}|\tilde{\alpha},s_m\rangle\langle\tilde{\alpha},s_m|, \tag{22}$$

where $|\tilde{\alpha},s_m\rangle$ is the displaced squeezed state:

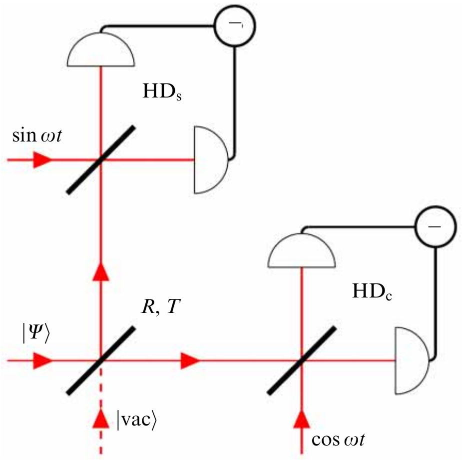

Figure 3. Possible implementation of POVMs (18) and (22). Here, the quantum state $|\Psi\rangle$ under study is mixed with a vacuum field on a beam splitter. Two homodyne detectors HD$_c$ and HD$_s$ are located at output ports of beam splitter, measuring cosine and sine quadrature of output beams, respectively. R, T are reflection and transmission coefficients of beam splitter.

$$\langle x|\tilde{\alpha}, s_m\rangle = \frac{1}{(\pi s_m)^{1/4}} \exp\left[-\frac{(x-\tilde{x})^2}{2s_m} + \mathrm{i}\tilde{p}x\right], \tag{23}$$

(compare with formula (16)).

The corresponding probability distribution for the measurement results will be of the form

$$Q(\tilde{x}, \tilde{p}, s_m) = \langle\Psi|\hat{\Pi}(\tilde{x}, \tilde{p}, s_m)|\Psi\rangle\,. \tag{24}$$

When $s_m = 1$, it coincides with the usual Husimi function (20). When $s_m \neq 1$, we obtain a ‘squeezed’ Husimi function, smeared predominantly either by the coordinate (when $s_m > 1$) or by the momentum (when $s_m < 1$). For $s_m \ll 1$, it degenerates into a one-dimensional coordinate probability distribution, and, for $s_m \gg 1$, into a momentum distribution. In all cases, function (24) remains nonnegative everywhere.

We revert to the case of the SC state (4), assuming that the basis state is squeezed (see formula (16)). In this case, it follows from formula (24) that

$$Q(\tilde{x},\, \tilde{p},\, s_m) = \frac{1}{N}\Bigg[Q_0(\tilde{x}-\xi,\, \tilde{p}, s_m) + Q_0(\tilde{x}+\xi,\, \tilde{p}, s_m) + 2Q_0(\tilde{x}, \tilde{p}, s_m)\exp\left(-\frac{\xi^2}{s+s_m}\right)\cos\left(\frac{2s_m\xi\tilde{p}}{s+s_m}+\theta\right)\Bigg], \tag{25}$$

where

$$Q_0(\tilde{x}, \tilde{p}, s_m) = \frac{\sqrt{s s_m}}{\pi(s+s_m)} \exp\left[-\frac{\tilde{x}^2 + s s_m \tilde{p}^2}{s+s_m}\right] \tag{26}$$

is the ‘compressed’ Husimi function of the basis state. It is easily seen that, in general, the structure of formula (25) coincides with the structure of the corresponding Wigner function (17), with one significant difference: in the case of formula (25), the interference term can be suppressed by the factor $\exp\left[-\xi^2/(s+s_m)\right]$. To avoid this, the factor s_m must be large enough. On the other hand, if it significantly exceeds the distance 2ξ between the ‘live’ and ‘dead’ components of the SC, then the double-humped structure of the coordinate distribution vanishes in the function Q.

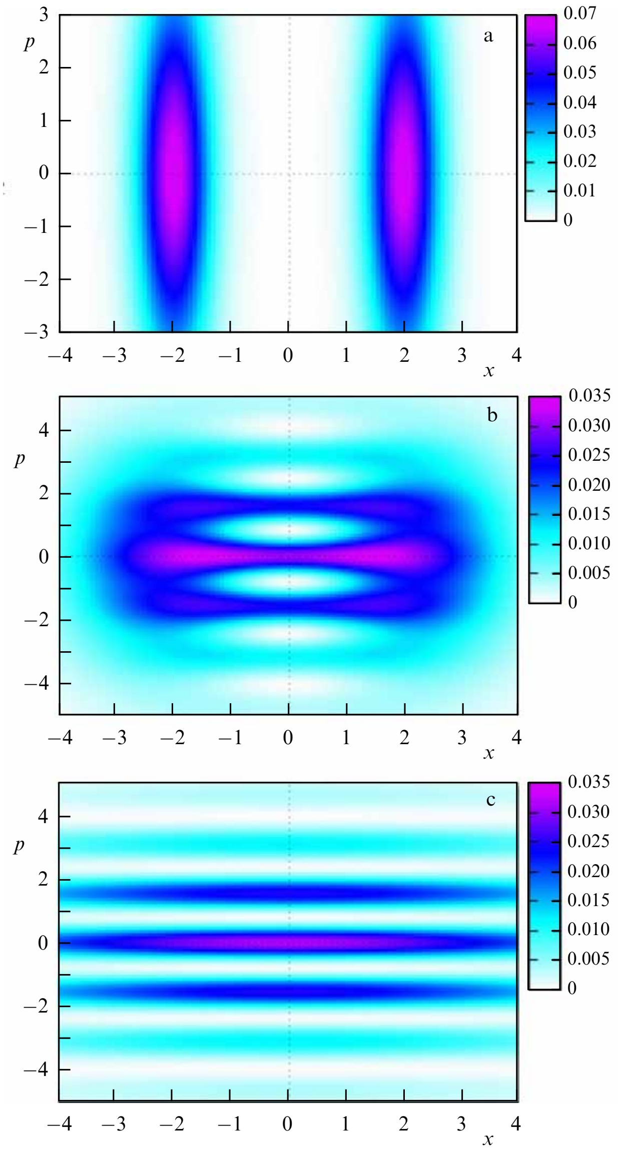

Figure 4. Probability distributions (25) for (a) $s_m = 0.3$, (b) $s_m = 3$, and (c) $s_m = 30$. Remaining parameters: $s = 0.1$, $\xi = 2$, and $\theta = 0$.

The three characteristic cases described are shown in Fig. 4. One can see from the figure that the value of s_m can be selected so that both the fringes and the two classical maxima are clearly discernable.

5. Conclusion

Using a convenient general expression for the Wigner function of ‘Schrödinger’s cat’ quantum states, this paper discusses several methodologically interesting properties of such states, in particular, the reason for the appearance of their characteristic interference fringes. It is also shown that there is a nontomographic measurement procedure, namely squeezed coherent measurement, which allows visualization of the ‘Schrödinger’s cat’ fringes.

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