Physics – Uspekhi **67** (9) 906 – 911 (2024)

PHYSICS OF OUR DAYS

PACS numbers: 03.67.Ac, 03.67.Lx, 42.50.Dv

ON THE 90TH ANNIVERSARY OF THE P.N. LEBEDEV PHYSICAL INSTITUTE (LPI)

Computable and noncomputable in the quantum domain: statements and conjectures

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DOI: https://doi.org/10.3367/UFNe.2024.07.039721

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Abstract. Significant advances in the development of computing devices based on quantum effects and the demonstration of their use to solve various problems have rekindled interest in the nature of the "quantum computational advantage." Although various attempts to quantify and characterize the nature of the quantum computational advantage have previously been made, this question largely remains open. Indeed, there is no universal approach that allows determining the scope of problems whose solution can be accelerated by quantum computers, in theory of in practice. In this paper, we consider an approach to this question based on the concept of complexity and reachability of quantum states. On the one hand, the class of quantum states that are of interest for quantum computing must be complex, i.e., not amenable to simulation by classical computers with less than exponential resources. On the other hand, such quantum states must be reachable on a practically feasible quantum computer. This means that the unitary operation that transforms the initial quantum state into the desired one must be decomposable into a sequence of one- and two-qubit gates of a length that is at most polynomial in the number of qubits. By formulating several statements and conjectures, we discuss the question of describing a class of problems whose solution can be accelerated by a quantum computer.

Keywords: quantum computing, quantum complexity, quantum algorithms

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Received 17 May 2024, revised 19 July 2024
Uspekhi Fizicheskikh Nauk **194** (9) 960 – 966 (2024)
Translated by S Alekseev

1. Introduction

The development of the technology for fabricating processors based on semiconductor microelectronics [1] has allowed computing power to constantly increase over the last decades; as a result, computing devices are now used virtually everywhere and on a daily basis. However, some computation problems retain an extremely high computational complexity for the available devices. Examples of such problems are, first, the factorization of integers into prime factors, where, given a composite number N, one has to find its nontrivial factors p and q (such that $N = p \times q$). This problem has direct applications in cryptography, because the analysis of the security of public-key cryptography algorithms is based on the assumption of the difficulty of solving the large-N factorization problem [2]. A second example is provided by the modeling of complex quantum systems, in particular, the calculation of the energy states of large molecules. The study of the properties of molecules and chemical reactions is important for various applications, for example, the creation of materials with specified properties and the design of drugs. Third, finally, of interest are combinatorial optimization problems, where the best solution has to be found among a large set of possible candidates, e.g., in logistics and scheduling tasks. Although classical algorithms and computing devices continue to evolve, and the demise of Moore's law [3] is a considerable overstatement [4], the above-mentioned computation problems apparently cannot be solved efficiently with existing devices and known algorithmic methods, even assuming the predicted growth of their power.

One approach to expanding computation power is to build fundamentally different types of computing devices: *quantum computers*, which use phenomena that manifest themselves at the level of individual quantum objects, such as individual atoms, ions, and photons, as well as macroscopic nonlinear superconducting circuits that exhibit the properties of single atoms [5]. Quantum computers, also often called *quantum processors* or *quantum coprocessors* (which is perhaps the most accurate way to describe them, because quantum devices work in conjunction with classical ones and also have classical input and output interfaces), are regarded as a pathway to solving classically hard computational problems. For example, for the already mentioned factorization problems [6] and modeling of complex quantum systems [7], quantum algorithms are known that are capable of solving such a problem in polynomial time.

Historically, several concepts have led to the emergence of quantum computers. Feynman proposed a quantum computer as a tool for modeling other quantum systems [8, 9], which are believed to be difficult to model using classical methods [10]. More precisely, for a class of many-body interacting quantum systems, the resources for modeling, i.e., calculating the measurement probabilities ('strong modeling') or generating a finite sample of measurement results (''weak modeling''), are assumed to grow exponentially with the system dimension. This phenomenon is known as the 'quantum entanglement threshold' [11]. Then, indeed, starting from a certain size of the system, its classical modeling becomes impossible, and therefore alternative approaches are needed, with quantum entanglement playing a key role.

For example, we consider the state of n qubits (of two-level quantum systems)

$$|\psi\rangle = \bigotimes_i \left(\alpha_i|0\rangle + \beta_i|1\rangle\right), \tag{1}$$

where α_i and β_i are complex numbers satisfying the normalization condition $|\alpha_i|^2 + |\beta_i|^2 = 1$ for $i = 1, \ldots, n$. Describing such a state requires up to $2n$ real numbers, for example, if angles on the Bloch sphere are used for the parameterization. But, if a sufficiently long sequence consisting of single-qubit operations and a two-qubit entanglement operation acting on different pairs of qubits (e.g., the controlled NOT gate CNOT [12]) are applied to a quantum state, then the resultant state becomes entangled, i.e., not representable as a product of subsystem states. There is then apparently no straightforward way to simulate such a state using linear resources. We can assume that the required resources would grow as 2^n, i.e., exponentially, as the system dimension increases. For $n = 100$, direct simulation would require storing a 2^{100}-dimensional complex vector in memory and computing the results of rotations in a 2^{100}-dimensional space, which seems impossible with any computing hardware. Therefore, the presence of any entangled superposition state should be considered a prerequisite for quantum computational advantage.

However, the Gottesman–Knill theorem [13–15] demonstrates that this is not so in a number of cases. A many-particle entangled state prepared by using only a set of gates from the Clifford group, applied to a computational-basis state (so-called stabilizer state), can be simulated with polynomial resources with respect to any Pauli measurements, including measurements in the computational basis. An example of Clifford operations is given by the CNOT gate. Examples of a many-particle entangled quantum state belonging to this class are the Greenberger–Horne–Zeilinger (GHZ) state [16] and graph states [17]. This demonstrates the naivety of the argument that the computational advantage of quantum computers comes solely from the superposition and entangled nature of the quantum states being processed. Another example is given by quantum circuits consisting of match gates, which are also known to allow efficient simulation on a classical computer [18].

The question of which classes of quantum states can be modeled classically plays an important role in achieving quantum computational advantage, i.e., demonstrating that a quantum computer can solve a problem faster than classical computing devices. However, this question is related not only to the degree of entanglement but also to the type of measurements and operations applied. Indeed, applying a layer of non-Clifford operations to an entangled stabilizer state just before computational-basis measurements (which is equivalent to implementing some common local measurement) makes the Gottesman–Knill theorem inapplicable, and the corresponding state is difficult to simulate classically. Also, adding non-Clifford operations, such as the T-gate, to a quantum circuit makes it unmodelable using classical resources. These examples show that the space of possible states of a quantum system — its Hilbert space — is nonuniform in terms of modeling complexity (relative to measurements in the computational basis): an n-qubit separable state requires linear resources, an n-qubit entangled state prepared only by Clifford operations is polynomially complex in modeling, and n-qubit entangled states are known that may require exponential resources for modeling (for example, prepared by non-Clifford operations).

On the other hand, although a quantum processor is regarded as a universal quantum device, which means that any unitary operation can be implemented in principle (with a specified accuracy), not all unitary operations can be effectively decomposed into sequences of one- and two-qubit quantum operations (gates), which are the operations implemented by actual quantum processors. Indeed, the decomposition of an arbitrary $2^n \times 2^n$ unitary matrix into $2{\times}2$ and $4{\times}4$ unitary matrices (the respective matrices of one- and two-qubit operations) is exponentially long in n. In a number of exceptional cases, such sequences can be linear or polynomial (as in the case of Shor's factorization algorithm [6]). Manin's observation [19] regarding ''a larger capacity of the quantum state space'' is based on the assumption that such states are *achievable* using a practically feasible quantum computer.

Thus, quantum computers are useful for analyzing classically unmodelable and quantum-achievable states. How vast is the class of such states? What is the Hilbert-space structure of this class? Such questions motivate us to take the first steps towards classifying quantum states from the above standpoint. Finding relations between different classes of states can shed new light on the nature of quantum computational advantage, which in this context is supervenient on the size and structure of the set of complex quantum states that cannot be modeled classically with less-than-exponential resources. We present a first version of a complexity diagram of quantum states, in which certain classes are not yet clearly delineated. The proposed classification reflects the existing approaches to assessing the complexity of quantum states from the standpoint of many-body physics, condensed matter physics, and quantum information theory.

It is worth noting that we are here working with ideal qubits, also called logical qubits. In real physical systems, noise exerts a significant effect on the computation process. However, either this noise can be suppressed to an acceptable level, or, alternatively, the effect can be eliminated using error correction codes. Today, experimental demonstration of the feasibility of error correction is one of the key areas in the development of quantum computing. Among the most striking results is the implementation of a "logical quantum algorithm," i.e., the action of logical (error-corrected) operations on logical qubits using the Quantinuum H1-1 ion quantum processor [20].

Today, issues associated with the implementation of quantum computing devices have become the subject of work across a wide scientific community. In Russia, this area is being developed within the framework of the Quantum Computing Roadmap coordinated by the Rosatom State Corporation; a review of the development of quantum technologies in Russia as of 2019 (the Roadmap start) is presented in [21]. One of the most dynamically developing platforms for quantum computing is represented by trapped ions. The creation of quantum computing devices on this platform requires the development of research subjects that are traditionally thoroughly pursued at the Lebedev Physical Institute [22], such as laser cooling, ultrastable lasers, and frequency standards [23], as well as the development of new areas, for example, in the field of multilevel quantum logic for the implementation of quantum algorithms [24]. In 2020–2024, this allowed building quantum computing devices with several dozen qubits and demonstrating the operation of quantum algorithms [24–26], including for the modeling of phase transitions [27]. The quantum computing devices being developed allow practical tests of hypotheses regarding the relation between complexity classes of quantum states.

2. Quantum-state picture and universal gate model of quantum computing

We focus on the n-qubit quantum state $|\Psi_n\rangle$ that appears after applying all unitary operations U_n in a quantum circuit to the state $|0\rangle^{\otimes n}$; in other words, this quantum state $|\Psi_n\rangle = U|0\rangle^{\otimes n}$ appears prior to measurement in the computational basis. Why are we interested in states, although we could speak not only about complex quantum states but also about complex quantum processes U and complex Hamiltonians H that define nontrivial quantum dynamics $U = \exp(-\mathrm{i}Ht)$ depending on time t or define a complex ground state, or about complex measurements?

First, the Choi–Jamiolkowski isomorphism allows all quantum processes (such as U) corresponding to quantum circuits to be considered quantum states in the extended Hilbert space $\mathcal{H}_{\text{in}} \otimes \mathcal{H}_{\text{out}}$ (where $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$ are the Hilbert spaces of input and output states). As regards Hamiltonians and measurements, the equivalence of the adiabatic [28], measurement-based [17], and gate [12] models of quantum computing allows all the complexity to be discussed in terms of quantum states without loss of generality. We note that arbitrary output measurements can always be reduced to measurements in the computational basis by appending suitable unitary operations to the quantum circuit. A similar approach can be resorted to when considering operations built on direct feedback from the results of intermediate measurements, which can be replaced by control unitary operations and deferred measurements. Thus, our approach to the complexity of quantum states is universal due to the universality of the gate model of quantum computing.

3. Classification of states

Let us recall that, in order to achieve a quantum computational advantage, we consider quantum states that are exponentially difficult to model, i.e., difficult to predict the probabilities of the implementation of measurement outcomes or at least to imitate the corresponding procedure of their random playout. If the relevant states are amenable to classical modeling, expecting the advantages of quantum computing would seem unrealistic. Below, we introduce certain classes of states and formulate several statements and conjectures about them.

We consider the following set of n-qubit quantum states:

- Stab: the set of stabilizer ('Clifford') states, i.e., quantum states obtained by applying quantum circuits of Clifford gates to a computational-basis state;
- ClassSimMeas: the set of states for which a classical algorithm exists whose complexity is no greater than a polynomial in n and which is capable of reproducing the results of measuring such states in the computational basis, at least in the weak sense;
- ClassNonSimMeas: the set of states for which no classical algorithm exists with complexity of at most a polynomial in n that would reproduce the results of measuring such states in the computational basis, at least in the weak sense;
- $\text{QuantPrep}_{1,2}$: the set of states that can be prepared on a quantum computer using a quantum circuit consisting of a number, at most a polynomial in n, of one- and two-qubit gates applied to some initial computational-basis state;
- $\text{NotQuantPrep}_{1,2}$: the set of states that cannot be prepared on a quantum computer using a quantum circuit consisting of a number, at most a polynomial in n, of one- and two-qubit gates applied to some initial computational-basis state;
- AreaLaw (VolLaw): the set of states for which the entanglement entropy of a region of space has a tendency to grow as the size of the boundary (volume) of the region for sufficiently large regions;
- QuantCompAdv: the set of states that arise before a measurement in the computational basis in quantum algorithms with explicit circuits (in particular, without oracles such as 'black boxes'), having an advantage greater than a polynomial in n compared to the best (known or theoretically conceivable) classical algorithm.

We note that ClassNonSimMeas is the complement of ClassSimMeas, and $\text{QuantPrep}_{1,2}$ is the complement of $\text{NotQuantPrep}_{1,2}$. We also comment on the class $\text{QuantPrep}_{1,2}$. Its definition emphasizes the possibility of expressing the general unitary transformation in terms of practically available one- and two-qubit gates. This class corresponds to the set of quantum states that can be obtained using a realistic quantum computer, while, as is known, the decomposition of an arbitrary n-qubit unitary U_n into a sequence of one- and two-qubit gates typically requires an exponentially long sequence. This class can be extended, say, to $\text{QuantPrep}_{1,2,\ldots,m}$ if the quantum hardware supports up to m-qubit gates, which is usually not yet available in practice. However, even if such gates exist for some fixed m, this does not affect the asymptotic behavior of the gate sequence length. The same is true for $\text{NotQuantPrep}_{1,2}$.

Recent studies in many-body quantum physics and condensed matter physics, which incorporate the concept of modeling quantum systems within classical approaches, have shown that a decisive role is played by the behavior of entanglement when partitioning the entire system into two parts. We note that such a partitioning is usually implemented with regard to the topology of the corresponding physical system, for example, within a one-dimensional circuit or a two-dimensional or three-dimensional array of quantum objects. For example, we can talk about spin systems. There are effective classical tools for modeling entangled quantum many-body systems, where the entanglement of a region of space has a tendency to scale (for sufficiently large regions) as the size of the region boundary. However, if the entanglement grows as the volume of one of the regions, such methods may no longer work efficiently. According to our definition, we call these classes AreaLaw and VolLaw. In particular, tensor networks such as matrix product states (MPSs) are good for approximating AreaLaw quantum states [29], while neural network quantum states (NNQSs) are considered useful for describing certain VolLaw states [30–32]. We also note that AreaLaw and VolLaw are not each other's complements, because states can exist whose entanglement scales as neither volume nor area (e.g., can be constant or scale as some nontrivial power of area).

4. Relations between classes of states

Let us consider a number of statements and conjectures concerning the relations between the introduced classes of quantum states.

By definition, we have

Statement 1. QuantCompAdv belongs to $\text{QuantPrep}_{1,2}$ and ClassNonSimMeas.

Only such states can provide quantum computational advantage using realistic quantum computing devices.

Statement 2. In accordance with the Gottesman–Knill theorem [13–15], Stab belongs to ClassSimMeas.

This is obvious due to the possibility of modeling quantum stabilizer states using polynomial resources [13–15].

Statement 3. By virtue of [15, 33, 34], Stab belongs to $\text{QuantPrep}_{1,2}$. Specifically, the complexity of preparing a given stabilizer state $|\Psi_n\rangle$ scales as $\mathcal{O}(n^2/\log n)$.

This is because stabilizer states can be efficiently prepared using one- and two-qubit gates, for example, via a set of Pauli rotations and CNOT gates. An example of such a state is the multiqubit GHZ state

$$|\text{GHZ}_n\rangle = \frac{|0\rangle^{\otimes n} + |1\rangle^{\otimes n}}{\sqrt{2}}, \tag{2}$$

where n is the number of qubits.

Statement 4. Stab intersects AreaLaw and VolLaw.

Examples of states at the intersection of these classes are given by cluster states used in measurement-based quantum computing [17].

Statement 5. AreaLaw intersects ClassSimMeas.

This is based on the fact that tensor networks in the MPS form of the state can effectively describe entangled quantum states whose entanglement grows in accordance with the area law [29]. MPSs are particularly well suited for describing one-dimensional quantum lattice systems with a gap and with local interactions [35].

Conjecture 1. AreaLaw belongs to ClassSimMeas.

This conjecture is a strengthening of Statement 1. We here assume that *all quantum states whose entanglement grows as the subsystem partition area* can be simulated on a classical computer with at most polynomial resources (e.g., using MPSs or generalizations).

Statement 6. ClassSimMeas intersects VolLaw.

This is because some states from VolLaw can be effectively described using NNQSs [30–32]; for example, there are states of one-dimensional systems that cannot be effectively described using MPSs, but can be described using NNQSs [30, 32].

Statement 7. QuantPrep intersects AreaLaw and VolLaw.

An AreaLaw state can be prepared by applying $\sim n$ random two-qubit gates acting between neighboring qubits (within a given topology). A VolLaw state, accordingly, can be prepared by applying $\sim n^2$ random two-qubit gates to all possible pairs of qubits. We note that an arbitrary two-qubit gate can be implemented using at most three CNOT gates [36].

Conjecture 2. There are VolLaw states outside ClassSimMeas.

Examples of such states are presumably the quantum states produced by random circuits (in the absence of noise) that were used to demonstrate quantum computational advantage (this was done for noisy circuits in [37, 38]). In fact, quantum circuits were chosen for such demonstrations so as to ensure the preparation of VolLaw states and thereby eliminate the possibility of their simulation by tensor-network ansatzes and other known classical methods. However, we are unaware of a rigorous proof for this intuitively natural conjecture. We also note that, in the noisy case, a classical polynomial algorithm was proposed in [39] (which, however, does not concern the above-mentioned experiments on the quantum advantage of a finite-size circuit).

Conjecture 3. ClassSimMeas intersects $\text{NotQuantPrep}_{1,2}$.

This conjecture is related to the fact that all known algorithms for preparing an arbitrary state

$$|\Psi_n\rangle = \sum_{\mathbf{x}\in\{0,1\}^n} C_{\mathbf{x}}|\mathbf{x}\rangle \tag{3}$$

with probability amplitudes $\{C_{\mathbf{x}}\}$ have exponential complexity in the number of operations required (see a recent review of results in [40]). On the other hand, we can imagine a situation where the function $\mathbf{x}\mapsto|C_{\mathbf{x}}|^2$ is constructed such that it can be efficiently calculated on a classical computer, but at the same time does not respect the structure of the tensor product of the physical space of qubits. A similar situation is realized, in particular, in NNQSs, where the $C_{\mathbf{x}}$ are essentially the output signals from the neural network for a given input $\mathbf{x}$.

The ensuing expected relations among the classes of states are shown in Fig. 1. We further discuss the question of the size of the classes of states for a fixed n. It is known that the set of stabilizer states Stab contains

$$2^n \prod_{i=1}^{n} (2^i + 1) \tag{4}$$

states [41]. At the same time, $\text{QuantPrep}_{1,2}$, AreaLaw, VolLaw, and $\text{NotQuantPrep}_{1,2}$ contain an infinite number of states, because adding arbitrary continuous local operations to their preparation schemes leaves states inside these classes. ClassSimMeas contains an infinite number of states because AreaLaw and $\text{QuantPrep}_{1,2}$ intersect. An infinite subset inside QuantCompAdv contains states arising in random schemes, for example, with single-qubit gates distributed uniformly randomly with respect to the Haar measure. It seems interesting to ask whether a class of problems can be specified in the definition of the quantum computational advantage within QuantCompAdv such that the corresponding set of

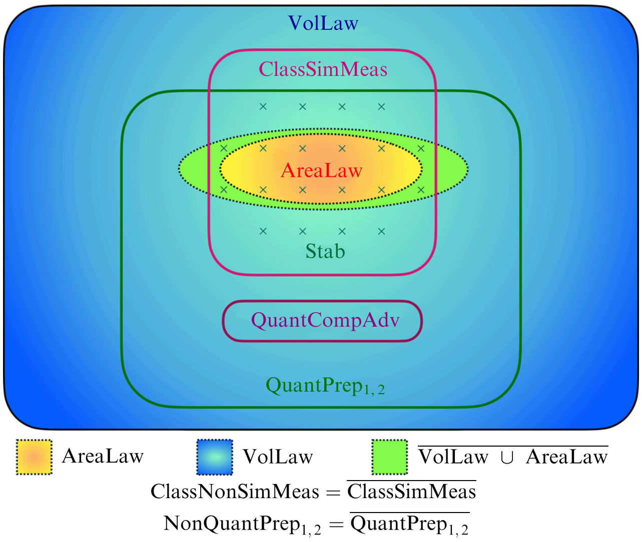

Figure 1. Schematic illustration of presented relations among classes of states. $\overline{\text{Class}}$ denotes the complement set of Class. $\times$ symbols in the notation for the Stab class (which belongs to the intersection of ClassSimMeas and QuantPrep$_{1,2}$, but does not exhaust it) are used to emphasize finiteness of this class for a fixed n.

states, or more precisely, the set of corresponding readout probability distributions, becomes finite.

5. Conclusions

In this paper, we proposed an approach to drawing connections between different classes of quantum states. We note that the "simplicity" of the states does not mean that they are generally impractical for quantum information technologies. A good example is given by the BB84 quantum key distribution protocol (see [42] for a review), where stabilizer states and Pauli measurements are used to solve the problem of information-theoretic secure generation of cryptographic keys.

Another interesting area comprises oracle-invoking algorithms that can provide a provable advantage (see, e.g., the single-run Bernstein–Vazirani algorithm [43]). However, we believe that the potential speedup in such algorithms can be attributed to the field of quantum communication rather than quantum computing, at least from a practical standpoint.

Third, a pressing issue is the construction of complexity classes taking the impact of errors on quantum computing processes into account.

Finally, we note specific attempts to propose classes of quantum states for studying 'complexity transitions' [44, 45], i.e., sets of quantum states that (being achievable) can be driven more or less complex by changing their parameters. An example of such states is provided by the class of sign-alternating Dicke states, in which the transition to the VolLaw class is observed for a certain set of parameters [45]. Such states can serve as a basis for experimental verification of the conjectures formulated above.

Acknowledgments. The authors express their gratitude to B I Bantysh and I V Ermakov for the fruitful discussions. This work was supported by the Priority 2030 program at the National University of Science and Technology MISiS (project K1-2022-027).

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