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Schwinger–DeWitt technique in quantum gravity

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Contents

Abstract. This paper consists of two parts. The first is a review of the classical Schwinger–DeWitt technique for calculating the effective action in quantum field theory and quantum gravity. We systematically present the background field and heat kernel methods and the calculation of heat kernel coefficients for minimal second-order operators, and apply these methods to calculations of the divergent part of the one-loop effective action. We then discuss the method of universal functional traces, which is also applicable to higher-order minimal and nonminimal operators. In the second part, we discuss new results obtained recently on off-diagonal heat kernel expansions for higher-order minimal operators. These expansions, which generalize the standard DeWitt ansatz, are shown to have the form of a double functional series in some new special functions, which we call the generalized exponential functions. The properties of these functions and expansions constructed from them are discussed in detail, including the presence of terms with arbitrarily large negative powers of the proper time. Finally, we discuss two different covariant methods for calculating the coefficients of off-diagonal expansions: a generalized Fourier transform and the perturbation theory.

Keywords: Schwinger–DeWitt technique, proper-time method, heat kernel, effective action, higher-order theories, off-diagonal expansions

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1. Introduction

The importance of functional methods in modern quantum field theory (QFT) has increased enormously [1, 2]. The functional approach within QFT is based on the study of generating functionals for quantum field correlators, which encode all information about the field theory model under consideration. They are functionals of background fields, i.e., of either external sources or average fields of a general type. The use of functional methods has largely directed progress in studying the renormalizability of various QFT models, their renormalization group properties, anomalies, and so on.

Moreover, because the geometry of space–time itself can be regarded as such a background field, this opens up the possibility of developing an approach where QFT is first constructed on a fixed classical space–time background and at the next stage the back reaction of quantum fields (both matter fields and gravitons) on the underlying classical curved background is considered [1, 3–5]. Although this approach obviously loses its applicability on the Planck scale, it is important from both practical and general theoretical standpoints. From the practical standpoint, it is about studying phenomena where both quantum and gravitational effects are significant, but which are still far from the Planck scale (for example, the physics of massive black holes and the early stages of cosmological evolution). From the general theoretical standpoint, it is a necessary step towards the construction of complete quantum gravity.

The application of functional methods in QFT is based on a combination of two main ideas: the background field method and the heat kernel method, which we address in Sections 2.1 and 2.2 below. Anticipating the treatment in what follows, we say that the heat kernel method allows effectively describing the features of quantum field propagators and, further, regularizing and renormalizing Feynman integrals. The convenience and power of this method underlie its utmost importance in the analysis of gauge theories and

modified (super)gravity models: it underlies most of the results on their (non)renormalizability, renormalization group behavior, anomalies, and so on.

However, the applicability range of the heat kernel method actually extends far beyond the scope of QFT. We can say that this method is currently one of the key and most commonly used tools in all modern mathematical physics. From the standpoint of pure mathematics, it has deep relations to the theory of pseudodifferential operators [6], index theorems [7], K-theory, spectral and noncommutative geometries, etc. The range of its practical applications is equally wide, from solid state physics to the analysis of markets.

Before proceeding with a detailed description of the heat kernel method and its QFT application, a few historical remarks are in order. From its very inception, the method developed along two intertwined but still distinct directions, which can be conventionally called 'physical' and 'mathematical.' The foundation was laid by Hadamard [8] in his work on hyperbolic linear equations, where he first obtained an expansion for the Green's function, and by the work of Fock [9], who noticed that many quantities in quantum theory can be conveniently represented as integrals with respect to an auxiliary 'proper time' variable τ. The mathematical direction was developed in the work of Meenakshisundaram on the Laplace operator on a Riemannian manifold [10, 11]. In turn, Schwinger [12] developed Fock's proper time method for renormalizing divergent integrals in QFT.

But a true revolution occurred in the early 1960s when DeWitt [1], by an extremely successful combination of the ideas of Schwinger and Hadamard, showed that the one-loop effective action of a theory can be expressed in terms of the coefficients of the heat kernel of its wave operator and developed his classical method for their calculation of the minimal second-order operator on a curved space–time. DeWitt's method served as the basis for all subsequent additions to QFT; we discuss it in detail in Section 2.3.

At about the same time, Seeley's important work [13] on complex powers of elliptic operators appeared, which determined the subsequent development of the mathematical direction. It was followed by a stream of studies of the heat kernel, still ongoing, centered on Gilkey's work [14–20]. The mathematical approach to the heat kernel is essentially based on the well-developed theory of pseudodifferential operators [6] and the use of so-called 'functoriality properties.' Despite the importance of this area, it has a number of significant disadvantages from physicists' standpoint. First, the mathematical approach is fine for proving general theorems and estimates, but it seems too abstract and not entirely convenient for applied calculations. Second, from the very beginning, it was limited to the case of only compact manifolds, while, for a physicist, naturally, noncompact physical space–time is of greatest interest. Finally, mathematicians are usually only interested in the trace of the heat kernel, which may also be insufficient for physics applications. Therefore, we do not discuss the work in the mathematical field in any more detail, but concentrate instead on the computational physical approach and its applications to QFT.

The Schwinger–DeWitt method has been successfully applied to lower-spin fields and Yang–Mills gauge theories [21–23], which underlie the modern Standard Model of particle physics, as well as to gravity and supergravity theories [24–26]. Ordinary Einstein gravity with the action linear in curvature is known [1] to be nonrenormalizable. This problem is solved by introducing higher-derivative terms into the Lagrangian theory: in the simplest case, adding terms to the action that are quadratic in curvature [27]. Such modified models have also been analyzed within the framework of the general Schwinger–DeWitt approach. In particular, their asymptotic freedom was investigated [28, 29]. The Schwinger–DeWitt technique has been used to study the general properties of dimensional and zeta-functional regularizations [24, 30] and the conformal anomaly of various conformally invariant models at the classical level in curved space [31–34], and to calculate the quantum average of the energy–momentum tensor for a general metric and in spaces with various types of symmetries [35–39], including the effective potential on the de Sitter space [40]. Renormalizable and, in particular, Weyl-anomaly-free conformal supergravity was studied in [41, 42].

Although such gravity, quadratic in curvature, serves as the basis for the cosmological inflation model proposed by Starobinsky [43], the presence of higher derivatives gives rise to Ostrogradsky ghosts and a unitarity violation. To overcome this problem, much attention has recently been attracted to Hořava–Lifshitz-type models [44], which allow preserving the renormalizability and unitarity simultaneously at the cost of breaking Lorentz invariance at high energies, and which have also been analyzed by the Schwinger–DeWitt method and its generalizations [45–47].

But this class of models is plagued by an essential difficulty: the method proposed by DeWitt for calculating the heat kernel coefficients is directly applicable only to Laplace-type operators (so-called minimal second-order operators). Therefore, the analysis of models with higher derivatives or a nonminimal operator required developing indirect calculation methods that allow a more complex case to be somehow reduced to the already known DeWitt case. Their presentation and application to quantum field models, including the so-called method of universal functional traces, which we briefly discuss in Section 2.5, is contained in Barvinsky and Vilkovsky's work [48]. We also note a series of works by Gusynin et al. [49–54], who calculated the heat kernel coefficients using the Fourier transform. An important development of the DeWitt method was the covariant perturbation theory, where the local DeWitt series in powers of dimensional tensor structures is partially resummed into a series in powers of curvature with nonlocal form factors [55–60]. Further references can be found in review papers [61–63].

This review consists of two parts. The first (Section 2) is devoted to the presentation of the results, which have already become classical, on the use of the heat kernel method in QFT and quantum gravity. We sequentially consider the background field (Section 2.1) and the heat kernel (Section 2.2) methods and the calculation of heat kernel coefficients for Laplace-type operators (Section 2.3); we then proceed with the application of these methods to the calculation of the one-loop effective action (Section 2.4) and, finally, briefly discuss indirect methods (Section 2.5). The second half of the review (Section 3) is devoted to the method of off-diagonal expansions of the heat kernel developed by the authors in recent years, which is applicable to higher-order minimal operators and to a wide class of nonminimal (causal) operators. In contrast to indirect methods, this is a direct generalization of the DeWitt technique for Laplace-type operators, but at the same time it is quite distinct from it in a

number of unusual features, which offer a new look at the classic results expounded in the first part.

2. Classic Schwinger–DeWitt method and its generalizations

We first go into some details about the physical situations that we discuss in what follows and introduce the necessary notation.

Any field theory lives on some space–time $\mathcal{M}$, by which we invariably mean a d-dimensional (pseudo-)Riemannian manifold with a metric g_{ab} defined on it. Although the real physical metric is Lorentzian, i.e., has the signature $(-+\cdots+)$, in practice all calculations are usually carried out for the Euclidean metric with the signature $(++\cdots+)$, after which the so-called 'Wick rotation' is performed, providing a transition to the physical case by analytic continuation in the complex time variable x^0. Accordingly, we also consider the Euclidean metric throughout this paper. Associated with the metric g_{ab} is the usual Levi-Civita connection ∇_a for which the metric is covariantly constant, $\nabla_a g_{bc} = 0$, and which has no torsion. Although the methods discussed below can be relatively easily extended to more general cases involving torsion and nonmetricity, we do not discuss these issues in this review.

Next, a certain set of fields $\varphi(x) = \varphi^A(x)$ are assumed to live on space–time $\mathcal{M}$, which, from a mathematical standpoint, are sections of a vector bundle over $\mathcal{M}$. Their indices A can be of any nature: space–time, spinor, internal, or combined. We omit them whenever possible in what follows, using hats to denote matrices in the space of fields. In particular, $\hat{1} = \delta^A_B$ simply denotes the identity matrix. The Riemann tensor $R^c{}_{dab}$ and the curvature $\hat{\mathcal{R}}_{ab}$ in the bundle are defined standardly via commutators of the covariant derivatives ∇_a:

$$[\nabla_a, \nabla_b] v^c = R^c_{dab} v^d , \tag{2.1}$$

$$[\nabla_a, \nabla_b] \varphi = \hat{\mathcal{R}}_{ab} \varphi . \tag{2.2}$$

Finally, at the classical level, the fields must satisfy certain equations of motion. It is assumed that such equations are determined by the least action principle. It is extremely convenient to additionally introduce auxiliary external sources of the fields $J(x) = J_A(x)$, such that the equations of motion of the theory become

$$\frac{\delta S}{\delta \varphi} = -J , \tag{2.3}$$

where $S[\varphi]$ is the classical action functional of the theory.

2.1 Background field method

The background field method is based on decomposing the total quantum field $\varphi(x)$ into two parts, which respectively correspond to the background field $\Phi(x) = \langle \varphi(x) \rangle$ (where $\langle \ldots \rangle$ is the quantum average in the presence of external field sources $J(x)$) and small quantum fluctuations $\phi(x)$:

$$\varphi = \Phi + \phi . \tag{2.4}$$

It is convenient to expand the classical action in a functional Taylor series with respect to small perturbations:

$$S[\Phi + \phi] = \sum_{n=0}^{\infty} \frac{1}{n!} S_n[\Phi] \, \phi^n , \tag{2.5}$$

$$S_n[\Phi | x_1, \ldots, x_n] = \left. \frac{\delta^n S[\varphi]}{\delta\varphi(x_1) \ldots \delta\varphi(x_n)} \right|_{\varphi=\Phi} , \tag{2.6}$$

$$S_n[\Phi] \, \phi^n = \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, S_n[\Phi | x_1, \ldots, x_n] \phi(x_1) \ldots \phi(x_n) . \tag{2.7}$$

The linearized classical equation for the small perturbations $\phi(x)$ propagating on a given background $\Phi(x)$ is then determined by the second variational derivatives of the action,

$$\hat{F}(\nabla)\phi = 0, \quad \text{where} \quad \hat{F}(\nabla)\delta(x,y) = S_2[\Phi | x, y] , \tag{2.8}$$

and the 'classical vertices' $S[\Phi x_1, \ldots, x_n]$ are n-point functions that determine the nonlinear coupling of fluctuations. Importantly, both the operator $\hat{F}(\nabla)$ and the vertices $S_n[\Phi]$ are functionals of the average background field $\Phi(x)$.

At the quantum level, all vacuum correlators of quantum fields $\langle \varphi(x_1) \ldots \varphi(x_n) \rangle$ are encoded in a single object, the generating functional:

$$\frac{Z[J]}{Z[0]} = \big\langle \exp(\varphi J) \big\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \, \big\langle \varphi(x_1) \ldots \varphi(x_n) \big\rangle J(x_1) \ldots J(x_n) , \tag{2.9}$$

$$\big\langle \varphi(x_1) \ldots \varphi(x_n) \big\rangle = \frac{1}{Z[J]} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0} , \tag{2.10}$$

which is, in turn, determined by the Feynman functional integral[1]

$$Z[J] = \int D\varphi \, \exp \frac{1}{\hbar} \big(-S[\varphi] - \varphi J \big) . \tag{2.11}$$

Next, we successively introduce the generating functional of connected correlation functions

$$W[J] = -\hbar \ln Z[J] \tag{2.12}$$

and its Legendre transformation — the *effective action*

$$\Gamma[\Phi] = \big(W[J] - \Phi J \big)_{J=J[\Phi]} , \tag{2.13}$$

where $J[\Phi]$ is the inversion of the functional dependence $\Phi[J] = \delta W[J]/\delta J$. Then, the equation describing the back reaction of quantum corrections on the classical background $\Phi(x)$ takes the form

$$\Gamma_1[\Phi | x] = \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} = -J(x) . \tag{2.14}$$

Therefore, the average field $\Phi(x)$, the effective action $\Gamma[\Phi]$, and Eqn (2.14) are the respective quantum analogues of the classical field $\varphi(x)$, the action $S[\varphi]$, and equations of motion (2.3).

Finally, the semiclassical (loop) expansion of the effective action in powers of the Planck constant $\hbar$,

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \hbar^n \Gamma^{(n)}[\Phi] , \quad \text{where} \quad \Gamma^{(0)}[\Phi] = S[\Phi] , \tag{2.15}$$

[1] As noted above, we write all expressions for a Euclidean QFT, and they therefore differ from the perhaps more familiar Lorentzian notation by the absence of additional imaginary units i.

allows obtaining expressions for each order $\Gamma^{(n)}[\Phi]$ in the form of space–time integrals of the classical propagator $\hat{G}(x,y)$ defined as the kernel of the operator inverse to (2.8),

$$\hat{F}(\nabla)\hat{G}(x,y) = \hat{1}\delta(x,y)\,, \tag{2.16}$$

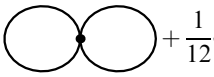
and the classical vertices $S_n[\Phi]$ in (2.6). For this, we rewrite expression (2.11) in the form

$$\exp\left(-\frac{\Gamma[\Phi]}{\hbar}\right) = \int D\phi\, \exp\frac{1}{\hbar}\left(-S[\Phi+\phi] + \Gamma_1[\Phi]\,\phi\right), \tag{2.17}$$

substitute expansions (2.5) and (2.15), and equate terms with the same powers of $\hbar$. In the resulting expressions, the path integrals are already Gaussian and can therefore be easily evaluated using Wick's theorem.

It is convenient to present the results obtained in this way graphically in the form of (space–time) Feynman diagrams, where the classical vertex $S_n[\Phi]$ is shown with a thick dot with n outgoing lines , and the classical propagator $\hat{G}(x,y)$, with a line connecting the corresponding vertices. It turns out that the term $\Gamma_1[\Phi]\,\phi$ in (2.17) ensures a systematic subtraction of the 'tadpoles' (i.e., subdiagrams connected to the rest of the diagram by a single line). Therefore, each n-loop contribution is given by a set of one-particle irreducible (OPI) vacuum diagrams with n loops. In particular, it can be shown that

$$\Gamma^{(1)}[\Phi] = \frac{1}{2}\ln\,\mathrm{Det}\,\hat{F}(\nabla) = \frac{1}{2}\,\mathrm{Tr}\ln\hat{F}(\nabla) = \frac{1}{2}$$  , (2.18)

$$\Gamma^{(2)}[\Phi] = \frac{1}{8}$$  $$+\,\frac{1}{12}$$  , (2.19)

where Det and Tr denote the functional determinant and the functional trace.

We emphasize once again that the expressions obtained in this way are not functions of the particle momenta and coupling constants but are *functionals* of all average background fields of the model under consideration (which can include scalar, spinor, and vector fields, a metric, and so on: $\varphi = \varphi(x), \psi(x), A_a(x), g_{ab}(x), \ldots$). Therefore, this form of the diagram technique is defined not on the trivial background of flat space–time and vanishing average fields but on an arbitrary fixed background.

2.2 Heat kernel method

Not unexpectedly, however, the space–time integrals corresponding to Feynman diagrams obtained in the background field method are divergent and therefore require a regularization and renormalization procedure. As noted in the Introduction, a universal tool for this is the heat kernel (or proper time) method. Its main idea is the observation that any power of an operator $\hat{F}$ can be written in the form of a so-called Schwinger representation, given by an integral of the operator exponential $\exp(-\tau\hat{F})$ with respect to the auxiliary parameter τ called the 'proper time,'

$$\hat{F}^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}\tau\,\tau^{s-1}\exp(-\tau\hat{F})\,, \tag{2.20}$$

where $\Gamma(s)$ is the standard Euler gamma function. Here, $\hat{F}$ can be an arbitrary positive-definite differential (or even pseudo-differential) operator, and the exponent can take arbitrary complex values, except negative integers: $s \neq 0, -1, -2, \ldots$. For $s \in \mathbb{N}$, this relation can be directly verified by alternating the action of $\hat{F}$ and the integration by parts s times. Moreover, from representation (2.20), using the properties of the gamma functions, it is easy to obtain properties that would naturally be expected of a complex power, for example, $\hat{F}^a\,\hat{F}^b = \hat{F}^{a+b}$.

The inverse transformation is given by

$$\exp(-\tau\hat{F}) = \frac{1}{2\pi\mathrm{i}}\int_{w-\mathrm{i}\infty}^{w+\mathrm{i}\infty}\frac{\tau^{-s}\Gamma(s)}{\hat{F}^s}\,\mathrm{d}s\,, \tag{2.21}$$

where the integration contour is parallel to the imaginary axis for a sufficiently large positive w. The last expression is easy to understand as follows: the gamma function $\Gamma(s)$ has simple poles at $s_n = -n$ with the residues $(-1)^n/n!$. The integral over s then reduces to the sum of residues at these poles, which exactly reproduces the standard Taylor series expansion for the operator exponential $\exp(-\tau\hat{F})$.

Next, it turns out to be convenient to pass from operators to their kernels. Actually, the heat kernel $\hat{K}_F(\tau|x,x')$ of an operator $\hat{F}(\nabla)$ is typically understood as the kernel of its operator exponential $\exp(-\tau\hat{F})$:

$$\hat{K}_F(\tau|x,x') = \exp\left(-\tau\hat{F}(\nabla)\right)\frac{1}{\sqrt{g(x)}}\,\delta(x,x')\,. \tag{2.22}$$

This is a two-point (i.e., depending on x and x') matrix-valued function. Obviously, it is a solution of the differential heat equation

$$(\partial_\tau + \hat{F}_x)\hat{K}_F(\tau|x,x') = 0 \tag{2.23}$$

with the initial condition

$$\hat{K}_F(0|x,x') = \frac{\hat{1}}{\sqrt{g(x)}}\,\delta(x,x')\,. \tag{2.24}$$

The terminology established in the literature ('heat equation,' 'heat kernel,' etc.) is historically determined by the fact that, in the case of three-dimensional flat space $\mathcal{M} = \mathbb{R}^3$, a single scalar field $\varphi(x)$, and the covariant Laplacian taken as the operator $F(\nabla) = -\Box = -g^{ab}\nabla_a\nabla_b$, Eqn (2.23) indeed coincides with the standard heat conductance equation.[2]

Now, passing from operators to their kernels in relations (2.20) and (2.21), we find that the heat kernel $\hat{K}_F(\tau|x,x')$ and the Green's function $\hat{G}_{F^s}(x,x') = \hat{F}^{-s}\delta(x,x')$ are related by the direct and inverse Mellin transformations:

$$\hat{G}_{F^s}(x,x') = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}\tau\,\tau^{s-1}\hat{K}_F(\tau|x,x')\,, \tag{2.25}$$

$$\hat{K}_F(\tau|x,x') = \frac{1}{2\pi\mathrm{i}}\int_{w-\mathrm{i}\infty}^{w+\mathrm{i}\infty}\mathrm{d}s\,\tau^{-s}\Gamma(s)\,\hat{G}_{F^s}(x,x')\,. \tag{2.26}$$

These transformations are a convenient tool for regularizing various divergent quantities. It is remarkable that it is then possible to equally successfully apply and combine various methods of covariant regularization: dimensional (when the space–time dimension is formally assumed to differ from the physical one by a small amount, $d = 4 + \varepsilon$), ζ-functional

[2] We note that, for the covariant Laplacian, instead of Δ accepted in the Euclidean case, we use the notation $\Box$, which is more suitable in light of the applications we are interested in; in addition, it allows the symbol Δ to be used for the Pauli–Van Vleck–Morette determinant (2.47).

(based on some special analytic continuation) [30], point splitting, etc. (a detailed discussion and comparison of various regularization methods can be found, e.g., in [4]).

To show more specifically how exactly this is done, we suppose that the inverse propagator of the theory, $\hat{F}(\nabla)$, is a minimal operator of order $2v$, i.e., its leading term can be represented as the covariant Laplacian raised to the vth power,

$$\hat{F}(\nabla) = \hat{1}(-\Box)^v + \hat{P}(\nabla)\,, \tag{2.27}$$

where

$$\hat{P}(\nabla) = \sum_{k=0}^{2v-1} \hat{P}_k^{a_1\ldots a_k} \nabla_{a_1} \ldots \nabla_{a_k} \tag{2.28}$$

includes all terms of lower orders in derivatives. It is well known that, for $s < d/2v$, the Green's function $\hat{G}_{F^s}(x, x')$ tends to infinity in the coincidence limit $x' \to x$. Accordingly, the functional trace

$$\operatorname{Tr} \hat{F}^{-s} = \int \sqrt{g}\; \mathrm{d}^d x\; \operatorname{tr} \hat{G}_{F^s}(x, x) \tag{2.29}$$

is also undefined for sufficiently small s. But the functional trace of the heat kernel

$$\operatorname{Tr} \exp\left(-\tau \hat{F}\right) = \int \sqrt{g}\; \mathrm{d}^d x\; \operatorname{tr} \hat{K}_F(\tau|x, x) \tag{2.30}$$

is guaranteed to be well defined for $\tau > 0$. Its Mellin transform gives the so-called *operator zeta function*

$$\zeta_F(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}\tau\, \tau^{s-1} \operatorname{Tr} \exp\left(-\tau \hat{F}\right). \tag{2.31}$$

The last integral converges everywhere in the domain $\operatorname{Re} s > d/2v$ and defines an analytic function in it; to the domain $\operatorname{Re} s < d/2v$, the function $\zeta_F(s)$ can be continued analytically. Then, the expression for $\operatorname{Tr} \hat{F}^{-s}$ can be regularized as follows: instead of divergent expression (2.29), we set

$$\operatorname{Tr} \hat{F}^{-s} \stackrel{\text{reg}}{=} \zeta_F(s)\,. \tag{2.32}$$

Further, the functional determinant of $\hat{F}(\nabla)$ can also be naturally regularized using the operator zeta function

$$\operatorname{Det} \hat{F}(\nabla) \stackrel{\text{reg}}{=} \exp\left(-\zeta_F'(0)\right) \tag{2.33}$$

(in mathematics, this relation is often promoted to a definition [64]), and we then obtain the one-loop effective action of the theory in the form

$$\Gamma^{(1)} = \frac{1}{2} \operatorname{Tr} \ln \hat{F}(\nabla) \stackrel{\text{reg}}{=} -\frac{1}{2} \int_0^\infty \frac{\mathrm{d}\tau}{\tau} \operatorname{Tr} \exp\left(-\tau \hat{F}\right). \tag{2.34}$$

Thus, the one-loop effective action for the theory with the inverse propagator $\hat{F}(\nabla)$ can be expressed in terms of only the diagonal elements $x = x'$ of the heat kernel $\hat{K}_F(\tau|x, x')$ (we note that, if we want to calculate the higher-loop contributions similarly, then we encounter nondiagonal elements with $x \neq x'$ in general). In the case of minimal operator (2.27) and a manifold without boundary, $\partial\mathcal{M} = \varnothing$, it is well known that, in the coincidence limit $x = x'$, the heat kernel has a power-law asymptotic expression for small values of the proper time $\tau \to 0$,

$$\hat{K}_F(\tau|x, x) = \tau^{-d/2v} \sum_{m=0}^{\infty} \tau^{m/v} \hat{A}_m(F|x)\,, \tag{2.35}$$

where the heat kernel coefficients $\hat{A}_m(F|x)$ are constructed from the coefficients of the operator $\hat{P}_k^{a_1\ldots a_k}$, the Riemann tensor $R^a{}_{bcd}$, and the curvature of the bundle $\hat{\mathcal{R}}_{ab}$.[3]

The general form of asymptotic expansion (2.35) of the heat kernel diagonal is determined by dimensional considerations. Indeed, in the coincidence limit, the only negative-dimension quantity is the proper time τ. For a minimal operator of order $2v$, we then have

$$\dim \tau = -2v\,. \tag{2.36}$$

In expansion (2.35), the general prefactor $\tau^{-d/2v}$ is responsible for the total dimension of the heat kernel, $\dim \hat{K}_F(\tau|x, x') = d$, while each product in the summand is dimensionless, whence we have

$$\dim \hat{A}_m(F|x) = 2m\,. \tag{2.37}$$

In addition, the derivatives of the background fields have positive dimensions in the coincidence limit:

$$\mathfrak{R} = \{R^a{}_{bcd}, \hat{\mathcal{R}}_{ab}, \hat{P}_k^{a_1\ldots a_k}\}\,, \quad \text{where} \quad \dim \nabla = 1\,, \tag{2.38}$$

$$\dim R^a{}_{bcd} = \dim \hat{\mathcal{R}}_{ab} = 2\,, \quad \dim \hat{P}_k^{a_1\ldots a_k} = 2v - k\,. \tag{2.39}$$

Then, the heat kernel coefficients in the coincidence limit are constructed from contractions of the derivatives of the background fields with due account for the overall background dimension:

$$\hat{A}_m(F|x) \propto \sum \nabla^l \mathfrak{R}^k\,, \quad \text{where} \quad \dim\left(\nabla^l \mathfrak{R}^k\right) = 2m\,. \tag{2.40}$$

Thus, from the standpoint of dimensional analysis, expansion (2.35) can be regarded as an expansion in increasing powers of the background dimension, and the proper time τ is then simply a parameter responsible for the grading by the dimension of the local terms $\nabla^l \mathfrak{R}^k$.

To summarize, calculating the one-loop effective action of the theory using the heat kernel method amounts to calculating the coefficients $\operatorname{tr} \hat{A}_m(F|x)$ as functions of the background fields $\mathfrak{R}$. If $\hat{F}(\nabla)$ is a Laplace-type operator, i.e., a second-order minimum operator, a simple and elegant method for calculating the heat kernel coefficients was proposed by DeWitt, as is described in the next section. But this method is not directly applicable to those theories where $\hat{F}(\nabla)$ is a higher-order operator or is not minimal. Section 2.5 is devoted to this more general case.

2.3 Classic DeWitt method

The method proposed by DeWitt allows calculating the coefficients of the heat kernel for a Laplace-type operator (i.e., a minimal second-order operator (2.27) with $v = 1$). This

[3] Heat kernel coefficients are known in the literature under a variety of names associated with the names of DeWitt [1], Hadamard [8], Meenakshisundaram [10, 11], Schwinger [12], Seeley [13], and Gilkey [14] in various combinations.

operator is standardly written in the form

$$\hat{F}(\nabla) = -\hat{1}\Box + \hat{P} + \frac{\hat{1}}{6}\,R\,, \tag{2.41}$$

where the 'potential term' $\hat{P}(x)$ does not contain derivatives, i.e., is just a matrix (because the term with first-order derivatives can always be eliminated by redefining the covariant derivative ∇_a), and the term with $R/6$ is traditionally added for convenience.

But, before we go on to discuss the DeWitt method, we introduce several more definitions that are standard in differential geometry, to be consistently used in what follows. We systematically use square brackets to denote the coincidence limit $x = x'$ for a two-point function $f(x, x')$:

$$\bigl[f(x, x')\bigr] = f(x, x)\,. \tag{2.42}$$

We assume that the points x and x' are close enough that there is a unique geodesic between them. Then, the important Synge's world function $\sigma(x, x')$ is defined as half the square of the geodesic distance between the points. We introduce the shorthand notation for the derivatives of the world function:

$$\sigma_{a_1\ldots a_n} = \nabla_{a_n}\ldots\nabla_{a_1}\sigma\,. \tag{2.43}$$

In particular, $\sigma^a(x, x') = \nabla^a\sigma$ is simply a vector at a point x, tangent to the geodesic connecting x and x', whose length is proportional to the geodesic distance between the points. Then, the definition of the world function can be rewritten as

$$\sigma_a\sigma^a = 2\sigma\,. \tag{2.44}$$

By differentiating this master relation, we obtain another important property:

$$\sigma^b\sigma^a{}_b = \sigma^a\,. \tag{2.45}$$

In the coincidence limit, obviously, we have

$$[\sigma] = 0\,, \qquad [\sigma^a] = 0\,, \qquad [\sigma^a{}_b] = \delta^a_b\,. \tag{2.46}$$

It can be shown that the Pauli–Van Vleck–Morette determinant

$$\Delta(x, x') = \frac{\det\left(-\nabla_a\nabla_{b'}\sigma\right)}{\sqrt{g(x)\,g(x')}} \tag{2.47}$$

is related to the rate of the divergence of geodesics $\sigma^a{}_a$ as

$$\sigma^a\nabla_a\Delta = \Delta(d - \sigma^a{}_a)\,. \tag{2.48}$$

We similarly define the parallel translation tensor $\hat{\mathcal{I}}(x, x')$ along a geodesic connecting points x and x' by the relations

$$\sigma^a\nabla_a\hat{\mathcal{I}} = 0\,, \qquad \bigl[\hat{\mathcal{I}}\bigr] = \hat{1}\,, \tag{2.49}$$

and introduce the shorthand notation for its derivatives:

$$\hat{\mathcal{I}}_{a_1\ldots a_n} = \nabla_{a_n}\ldots\nabla_{a_1}\hat{\mathcal{I}}\,. \tag{2.50}$$

The coincidence limits $[\sigma_{a_1\ldots a_n}]$ and $[\hat{\mathcal{I}}_{a_1\ldots a_n}]$ can be calculated using only the rules for commuting covariant derivatives (2.1) and (2.2).

With all the necessary definitions given, we now proceed directly to the DeWitt method. Its main idea is very simple: in the case of a Laplace-type operator (2.41), we use the ansatz

$$\hat{K}_F(\tau|x, x') = \frac{\Delta^{1/2}(x, x')}{(4\pi\tau)^{d/2}}\exp\left(-\frac{\sigma(x, x')}{2\tau}\right) \times\sum_{m=0}^{\infty}\tau^m\,\hat{a}_m(F|x, x')\,, \tag{2.51}$$

where we call the two-point matrix-valued functions $\hat{a}_m(F|x, x')$ (independent of the proper time τ) the HaMiDeW coefficients or simply the heat kernel coefficients. The specific form of this ansatz and, in particular, the appearance of the Pauli–Van Vleck–Morette determinant $\Delta^{1/2}(x, x')$ in it raised to a certain power is due to the similarity with the semiclassical approximation, where the proper time τ plays a role similar to that of Planck's constant $\hbar$. We note, however, that the factor $\Delta^{1/2}$ can, in principle, be changed or removed altogether by redefining the coefficients $\hat{a}_m(F|x, x')$.[4]

We now substitute ansatz (2.51) in heat equation (2.23) for operator (2.41), require that the terms at different powers of the proper time τ vanish independently, and simplify the resulting equations using relations (2.45) and (2.48). We then obtain an infinite chain of recursive relations

$$(m + \sigma^a\nabla_a)\hat{a}_m = -\Delta^{-1/2}\hat{F}(\nabla)\Delta^{1/2}\hat{a}_{m-1}\,, \tag{2.52}$$

with the initial condition

$$\hat{a}_0 = \hat{\mathcal{I}}\,. \tag{2.53}$$

If we differentiate recursive relations (2.52) and then take the coincidence limit $x = x'$, we can express the coincidence limit $[\nabla^k\hat{a}_{m+1}]$ in terms of $[\nabla^l\hat{a}_m]$, $l \leqslant k + 2$, and the limits $[\sigma_{a_1\ldots a_n}]$ and $[\hat{\mathcal{I}}_{a_1\ldots a_n}]$. By sequentially calculating $[\nabla^k\hat{a}_m]$, we obtain exact local expressions for them in the form of combinations of contractions of covariant derivatives of the background fields $\mathfrak{R}$ in (2.38).

In particular, for the simplest coincidence limits, this method allows obtaining the well-known expressions (which can be found in [1] or [48])

$$[\hat{a}_0] = \hat{1}\,, \quad [\nabla_a\hat{a}_0] = 0\,, \quad [\nabla_a\nabla_b\hat{a}_0] = \frac{1}{2}\,\hat{\mathcal{R}}_{ab}\,, \tag{2.54}$$

$$[\hat{a}_1] = -\hat{P}\,, \quad [\nabla_a\hat{a}_1] = -\frac{1}{2}\,\nabla_a\hat{P} - \frac{1}{6}\,\nabla^b\hat{\mathcal{R}}_{ba}\,, \tag{2.55}$$

$$[\hat{a}_2] = \frac{1}{180}\bigl(R_{abcd}R^{abcd} - R_{ab}R^{ab} + \Box R\bigr)\hat{1} + \frac{1}{2}\,\hat{P}^2 + \frac{1}{12}\,\hat{\mathcal{R}}_{ab}\hat{\mathcal{R}}^{ab} - \frac{1}{6}\,\Box\hat{P}\,. \tag{2.56}$$

DeWitt's method does not allow obtaining closed expressions for the coefficients $\hat{a}_m(x, x')$ beyond the coincidence limit. However, they can be reconstructed from the coincidence limits $[\nabla^k\hat{a}_m]$ using a covariant Taylor series expansion. That this procedure can be carried out consistently confirms the validity of the initial choice of ansatz (2.51).

In the coincidence limit $x = x'$, the exponential and determinant $\Delta^{1/2}$ disappear from expansion (2.51), and we obtain the well-known asymptotic form for the heat kernel

[4] As we see in what follows, the generalized Fourier transform method does generate expansion (3.43), where the determinant Δ is raised to a different power.

diagonal, Eqn (2.35), with the coefficients

$$\hat{A}_m(F|x) = (4\pi)^{-d/2}\left[\hat{a}_m(F|x,x')\right]. \tag{2.57}$$

2.4 Ultraviolet divergences and anomalies

We use dimensional regularization by formally setting the dimension of space equal to 2ω, and the even physical dimension d, not necessarily equal to 4, to be fixed. Then, the regularization removal corresponds to the limit $\omega \to d/2$. To also eliminate infrared divergences, we introduce an additional constant mass term $\hat{1}\, m^2$ into operator (2.27) of the theory. Taking the integral with respect to the proper time τ in (2.34) and using the asymptotic expansion for the heat kernel diagonal, Eqns (2.35) and (2.57), yields the one-loop effective action decomposed into divergent, logarithmic, and finite parts [1, 48, 65],

$$\Gamma^{(1)} = \Gamma_{\rm div} + \Gamma_{\log} + \Gamma_{\rm fin}\,, \tag{2.58}$$

each of which is representable as an expansion in inverse powers of m^2:

$$\Gamma_{\rm div} = \frac{1}{2(4\pi)^{d/2}} \sum_{n=0}^{d/2} \left[\frac{1}{\omega - d/2} - \psi\left(\frac{d}{2} - n + 1\right)\right] \times \frac{(-m^2)^{d/2-n}}{(d/2-n)!} \int \sqrt{g}\, {\rm d}^d x\ {\rm tr}\,[\hat{a}_n]\,, \tag{2.59}$$

$$\Gamma_{\log} = \frac{1}{2(4\pi)^{d/2}} \ln\frac{m^2}{\mu^2} \sum_{n=0}^{d/2} \frac{(-m^2)^{d/2-n}}{(d/2-n)!} \int \sqrt{g}\, {\rm d}^d x\ {\rm tr}\,[\hat{a}_n]\,, \tag{2.60}$$

$$\Gamma_{\rm fin} = -\frac{m^d}{2(4\pi)^{d/2}} \sum_{n=d/2+1}^{\infty} \frac{\Gamma(n-d/2)}{m^{2n}} \int \sqrt{g}\, {\rm d}^d x\ {\rm tr}\,[\hat{a}_n]\,. \tag{2.61}$$

In the expression for $\Gamma_{\rm div}$, $\psi(z)$ is the digamma function (the logarithmic derivative of the gamma function), and in the expression for $\Gamma_{\log}$, μ^2 is some mass parameter reflecting the ambiguity of the renormalization procedure.

Obviously, this asymptotic expansion of the effective action makes sense only when $[\hat{a}_n] \ll m^{2n}$, i.e., if, in view of (2.40), the background fields and their derivatives are small compared with the corresponding power of the mass parameter: $\nabla^l \mathfrak{R}^n \ll m^{2n+l}$.

The simplest example of this local expansion is the effective Coleman–Weinberg potential [66] for a 4-dimensional scalar field with the $\lambda\varphi^4/12$ self-coupling. In the case where the average field is constant and effectively plays the role of a mass parameter $m^2 = \lambda\varphi^2$, logarithmic part (2.60) is represented by a single nonzero HaMiDeW coefficient $a_0 = 1$ and is given by the space–time integral of the effective potential:

$$\Gamma_{\rm CW} = \int {\rm d}^4 x\, \frac{\lambda^2\varphi^4}{64\pi^2} \ln\frac{\lambda\varphi^2}{\mu^2}\,. \tag{2.62}$$

An important application of the local expansion is the calculation of the anomaly in the trace of the energy–momentum tensor associated with the local Weyl invariance violation when renormalizing ultraviolet divergences. In an even dimension d, the divergent part of the action and the trace anomaly are determined by the coefficient ${\rm tr}\,[\hat{a}_{d/2}]$. In particular, for $d = 4$, we have

$$\Gamma_{\rm div} = -\frac{1}{32\pi^2}\frac{1}{2-\omega}\int \sqrt{g}\ {\rm d}^4 x\ {\rm tr}\,[\hat{a}_2]\,, \tag{2.63}$$

$$\langle T^a_a \rangle = \frac{2g_{ab}}{\sqrt{g}}\frac{\delta\Gamma}{\delta g_{ab}} = -\frac{1}{(4\pi)^2}\,{\rm tr}\,[\hat{a}_2]\,. \tag{2.64}$$

An exact calculation [4, 48] shows that, for a conformally invariant theory with N_0 real scalar fields, $N_{1/2}$ Dirac fermions, and N_1 vector multiplets (including the corresponding contributions from the Faddeev–Popov ghosts), the sought coefficient can be written in the form

$$\frac{1}{(4\pi)^2}\,{\rm tr}\,[\hat{a}_2] = cW^2 - aE - b\Box R\,, \tag{2.65}$$

where $W^2 = W_{abcd}W^{abcd}$ is the square of the Weyl conformal tensor and $E = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ is the Gauss–Bonnet density. The coefficients a and c are given by simple formulas (demonstrating that the contribution of each particle to the anomaly is only determined by its type),

$$a = \frac{1}{360(4\pi^2)}\left(N_0 + 11N_{1/2} + 62N_1\right), \tag{2.66}$$

$$c = \frac{1}{120(4\pi^2)}\left(N_0 + 6N_{1/2} + 12N_1\right), \tag{2.67}$$

and the coefficient b at $\Box R$ turns out to be scheme dependent (it is equal to $-2c/3$ for dimensional regularization and to $-c$ for zeta-functional regularization).

2.5 Method of universal functional traces

The DeWitt technique described above, however, essentially relies on the choice of ansatz (2.51) and is directly applicable solely to Laplace-type operators (2.41) but not to minimal operators (2.27) of a higher order $v > 1$, and certainly not to nonminimal operators, i.e., those that cannot be represented in form (2.27).

Indeed, it is straightforward to verify that a naive substitution of DeWitt ansatz (2.51) into heat equation (2.23) for minimal higher-order operator (2.27) leads to inconsistent equations. Attempts to modify ansatz (2.51) undertaken from time to time (see, e.g., [67]) based on the semiclassical approximation or other considerations eventually fail.[5]

But these more general operators are also extremely important in applications. Higher-order minimal operators arise in higher-derivative gravity theories, as well as in other cases. A typical example is given by so-called 'conformally covariant differential operators,' such as the Paneitz fourth-order operator

$$\Delta_4 = \Box^2 + 2R^{ab}\nabla_a\nabla_b - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^a R)\nabla_a \tag{2.68}$$

and its analogues; an extensive amount of the literature is devoted to the study of operators of this type (see, e.g., [68–70]). Nonminimal operators already arise in the simplest case of electrodynamics considered in the general Lorentz-covar-

[5] As we see in the second part of this paper, recent results on terms with arbitrarily large negative powers of τ being present in this case make it clear why these attempts were initially doomed to failure.

iant gauge, with the operator

$$\hat{F}(\nabla) = F_b^a(\nabla) = -\delta_b^a \Box + \lambda \nabla^a \nabla_b + \dots, \tag{2.69}$$

for a wide class of such gauges in ordinary Einstein's gravity [48], and in modified gravity models [28, 29].

The importance of higher-order minimal operators and nonminimal operators has led to the development of indirect methods, which in one way or another reduce the problem of calculating the heat kernel coefficients for these classes of operators to a simpler problem for Laplace-type operators, which can already be solved by the DeWitt method. The most common of these indirect methods is the so-called 'universal functional trace method' [48, 71].

The essence of this method can be explained using the example of a higher-derivative theory where the inverse propagator is given by operator (2.27). The one-loop functional determinant of this theory,

$$\operatorname{Tr}\ln\left(\hat{1}(-\Box)^v + \hat{P}(\nabla)\right) = v\,\operatorname{Tr}\ln\Box + \operatorname{Tr}\ln\left(\hat{1} + \frac{\hat{P}(\nabla)}{(-\Box)^N}\right), \tag{2.70}$$

can be expanded in a series in powers of the nonlocal perturbation $\hat{P}(\nabla)/(-\Box)^N$. If we then commute all powers of $P(\nabla)$ to the left and all inverse powers of $\Box$ to the right, we express the result as an infinite series of coincidence limits

$$\left[\nabla_{a_1}\dots\nabla_{a_n}\,\frac{1}{(-\Box)^m}\,\delta(x,x')\right], \tag{2.71}$$

contracted with some tensors of increasing dimension. Universal functional traces (2.71) can already be calculated using the DeWitt method, because an mth power of the inverse Laplacian can easily be represented as an integral over the proper time of the heat kernel of the operator $\Box$ (see Eqn (2.25)).

The method of universal functional traces has proved to be extremely effective, in particular, in calculating the beta functions of a $(3+1)$-dimensional Hořava-type projective gravity model [46, 47]. This problem is currently impossible to solve using conventional diagram techniques in momentum space on a flat background, because it requires calculating hundreds of thousands of Feynman diagrams.

3. Off-diagonal heat kernel expansions for higher-order minimal operators

Although the method of universal functional traces, at least in the one-loop approximation, in principle allows calculating everything that we need, it has one very significant drawback: as can be seen from (2.71), it directly operates only with coincidence limits. But the structure of the heat kernel $\hat{K}_F(\tau|x,x')$ outside its diagonal, at $x \neq x'$, may also be important (for example, if we want to calculate higher-loop contributions).

In addition, as is clear from considering the DeWitt method, the coefficients $\hat{a}_m(x,x')$ enter recursive relations (2.52) precisely as two-point functions, and not as the coincidence limits $[\hat{a}_m]$. Calculating just the coincidence limits $[\hat{a}_m]$ already requires knowing the coincidence limits of the derivatives of the lower coefficients $[\nabla^k \hat{a}_l]$, $l < m$. Thus, the diagonal of the heat kernel is determined by its overall structure outside the diagonal, at $x \neq x'$. It is therefore natural to expect that clarifying the details of this structure and taking them into account can help in developing new, more efficient methods, even for calculating the coincidence limits $x = x'$. The indirect methods developed to date simply ignore the structure of the heat kernel outside the diagonal. This is exactly the gap that the study of off-diagonal expansions is aimed at filling.

3.1 Dimensional analysis

The nature of the difficulties we encounter when trying to generalize DeWitt's method to the case of higher-order operators can be clarified somewhat by using dimensional analysis. As we noted in the discussion after formula (2.35), the general form of the asymptotic expansion of the heat kernel diagonal is determined by dimensional considerations.

But beyond the coincidence limit, at $x \neq x'$, the situation becomes much more involved, because, in that case, in addition to the proper time τ, two other quantities of negative dimension appear,

$$\dim\sigma = -2 \quad \text{and} \quad \dim\sigma^a = -1\,, \tag{3.1}$$

which can be used to construct dimensionless combinations

$$\frac{\sigma}{\tau^{1/v}} \quad \text{and} \quad \frac{\sigma^a}{\tau^{1/2v}}\,. \tag{3.2}$$

Their appearance is not forbidden by dimensional considerations, and therefore the expansion of the heat kernel outside the diagonal can, in principle, include terms with arbitrarily high powers of these dimensionless combinations. But this would obviously lead to the appearance of arbitrarily large negative powers of the proper time τ.

If we return to DeWitt's ansatz (2.51) in the case $v = 1$ and apply a similar argument, it becomes clear that there is in fact nothing strange about the appearance of negative powers of τ outside the coincidence limit: to see that they do actually occur, it suffices to expand the exponential factor $\exp(-\sigma/2\tau)$ in a series. A miracle manifests itself in just the opposite: in the fact that, for Laplace-type operators, the heat kernel expansion outside the coincidence limit is remarkably structured such that all negative powers of τ sum up into a single exponential factor $\exp(-\sigma/2\tau)$, which is regular in the coincidence limit $x = x'$ (and hence the negative powers of τ disappear from it, as they should), but which develops an essential singularity as $\tau \to 0$. In any case, the existence of the DeWitt ansatz does not follow from dimensional considerations, and therefore looks accidental from the standpoint of dimensional analysis alone.

Therefore, we cannot *a priori* expect a similar resummation to necessarily take place for higher-order minimal operators. This leaves room for a wide range of different opinions. On the one hand, an optimist would claim that the possibility of such a resummation must follow from the semiclassical approximation. This would mean that the strategy behind DeWitt's method can be transferred to the case of higher-order operators without significant changes. On the other hand, a pessimist could just as well express doubts about the existence of any observable structure beyond the coincidence limit, pointing out that the ratios $\sigma^a/\tau^{1/2v}$ can give rise to increasingly complex tensor expressions at increasingly more negative powers of τ.

This imaginary dispute can be resolved only by a dedicated study of the heat kernel structure beyond the coincidence limit, which underscores the importance of the problem under consideration. Our recent results show that the truth, as is usually the case, is exactly half-way: on the one

hand, a resummation of dimensionless combinations $\sigma/\tau^{1/v}$ into some new special functions similar to the exponential in the DeWitt ansatz does happen, and this leads to a beautiful structure, in its own way, of the off-diagonal expansions of the heat kernel. On the other hand, however, such a resummation is no longer complete: the expansion is now realized in the form of a functional series in special functions, which leaves terms with negative powers of τ as free factors.[6]

Thus, a generalization of DeWitt expansion (2.51) to the case of higher-order minimal operators is achievable. Moreover, it can be regarded as a gradient expansion in the background dimension, which is exactly what is needed for effective quantum field theory. However, the presence of terms with arbitrarily large negative powers of τ in it makes it impossible to construct a system of recursive relations for their coefficients, similar to (2.52). Nevertheless, we have developed two algorithms, which are consistent with each other, for calculating such generalized coefficients: one based on the so-called 'generalized Fourier transform' and the other based on the perturbation theory. In a sense, the yield of our methods exceeds the DeWitt method results, because the latter is limited to answers in the form of coincidence limits $[\nabla^k \hat{a}_m]$, while we are able to immediately obtain closed expressions for the coefficients in the form of two-point functions at $x \neq x'$.

3.2 Generalized exponentials

We start by presenting a very simple idea, which nevertheless became the starting point for all subsequent research. First, in DeWitt ansatz (2.51), we redefine the coefficients $\hat{a}_m(F|x,x')$ so as to absorb the common prefactor $\Delta^{1/2}(x,x')$, and then write the expansion in the form of a functional series:

$$\hat{K}_F(\tau|x,x') = \sum_{m=0}^{\infty} \mathbb{K}_m(\sigma,\tau)\, \hat{a}_m(F|x,x')\,, \tag{3.3}$$

$$\mathbb{K}_m(\sigma,\tau) = \frac{\tau^m}{(4\pi\tau)^{d/2}} \exp\left(-\frac{\sigma}{2\tau}\right). \tag{3.4}$$

At this point, this is just a rewriting, containing nothing new and apparently rather unprepossessing.

But we can now observe that, first, successive actions of the covariant derivative ∇_a reduce the functions $\mathbb{K}_m(\sigma,\tau)$ to a single basic function $\mathbb{K}_0(\sigma,t)$ and, second, that this function is just the heat kernel of the covariant Laplacian $-\Box = -g^{ab}\nabla_a\nabla_b$ in flat d-dimensional space.

Therefore, if we want to generalize DeWitt expansion (2.51) to the case of higher-order minimal operators, it is natural to first find the heat kernel for its top-order term, i.e., for a power of the Laplacian $(-\Box)^v$, also in the flat d-dimensional space.

In [74], we found that the corresponding heat kernel has the form

$$\mathbb{K}_0^{(v,d)}(\sigma,\tau) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \exp\left(-\tau k^{2v} + \mathrm{i} k_a \sigma^a\right)$$

$$= \frac{1}{(4\pi\tau^{1/v})^{d/2}}\, \mathcal{E}_{v,\,d/2}\left(-\frac{\sigma}{2\tau^{1/v}}\right), \tag{3.5}$$

[6] As regards the semiclassical approximation, we discussed its inapplicability in the case of higher-order operators in detail in [72]. The point is that the asymptotic behavior of the heat kernel can be obtained by the saddle point method [73] in the limit $z = \sigma/2\tau^{1/v} \to \infty$. But, for $v > 1$, the corresponding functions turn out to be singular in the coincidence limit $\sigma \to 0$ and cannot therefore be used as an ansatz.

where the functions $\mathcal{E}_{v,\alpha}(z)$ are specially chosen to play the same role as the exponential in the standard DeWitt expansion (2.51). This is why we use the name 'generalized exponentials.'

These generalized exponentials $\mathcal{E}_{v,\alpha}(z)$ play a key role in the rest of this study. We therefore provide a brief summary of their properties that we need in what follows. Some properties, along with their derivation and detailed discussion, can be found in [72].

Generalized exponentials can be defined via the Mellin–Barnes integral

$$\mathcal{E}_{v,\alpha}(-z) = \frac{1}{2\pi\mathrm{i}} \int_C \varepsilon_{v,\alpha}(s) z^{-s}\, \mathrm{d}s\,, \tag{3.6}$$

$$\varepsilon_{v,\alpha}(s) = \frac{\Gamma(s)\Gamma\big((\alpha-s)/v\big)}{v\Gamma(\alpha-s)} = \int_0^\infty z^{s-1} \mathcal{E}_{v,\alpha}(-z)\, \mathrm{d}z\,, \tag{3.7}$$

where the integration contour C is drawn in the complex plane such that it separates the poles of $\Gamma(s)$ that go to the left from the poles of $\Gamma((\alpha-s)/v)$ that go to the right.

This definition makes it straightforward that, for $v = 1$, the situation reduces to the one that is already known,

$$\mathcal{E}_{1,\alpha}(z) = \exp z\,, \tag{3.8}$$

and heat kernels (3.5) reduce to the function $\mathbb{K}_0(\sigma,t)$.

The generalized exponentials $\mathcal{E}_{v,\alpha}(z)$ belong to the class of so-called Fox H-functions or, more precisely, the Fox–Wright Ψ-functions. In our study of the properties of $\mathcal{E}_{v,\alpha}(z)$, we essentially relied on the well-developed theory of these special functions [72].

For example, if we close the contour C in (3.6) on the left and evaluate the integral using residues, we obtain the power series

$$\mathcal{E}_{v,\alpha}(z) = \frac{1}{v} \sum_{m=0}^{\infty} \frac{\Gamma\big((\alpha+m)/v\big)}{\Gamma(\alpha+m)}\, \frac{z^m}{m!}\,. \tag{3.9}$$

This series is everywhere convergent for $v > 1/2$, and is therefore the Taylor series of an entire function $\mathcal{E}_{v,\alpha}(z)$. Conversely, if we close the contour on the right, then, for noninteger v, we obtain a power-law asymptotic form of the function $\mathcal{E}_{v,\alpha}(z)$ as $z \to \infty$ (for integer v, the poles of the gamma functions in the numerator and denominator cancel each other out, and the power-law asymptotic regime is superseded with an exponential one). It is interesting to note that, for $v < 1/2$, the situation is just the opposite: the series in powers of z^{-1} is convergent everywhere, and the series in powers of z becomes asymptotic (and at the critical value $v = 1/2$, the function can be found exactly).

Another key property of generalized exponentials is the remarkable differentiation rule

$$\frac{\mathrm{d}^\beta}{\mathrm{d}z^\beta}\, \mathcal{E}_{v,\alpha}(z) = \mathcal{E}_{v,\alpha+\beta}(z)\,. \tag{3.10}$$

The properties of generalized exponentials were studied in great detail in [72]. In particular, several potentially useful integral representations for such functions were obtained, their connection with the Bessel and Bessel–Clifford functions was noted, exponential asymptotic formulas as $z \to \infty$ were obtained at integer v (which is a rather subtle problem), and their consistency with the answer suggested by the more familiar saddle point method was shown. Among the interesting properties of these functions, we note that, as

$z \to \infty$, in contrast to the exponential, $\mathcal{E}_{v,\alpha}(-z)$ decreases not monotonically but in an oscillatory manner. For full details, we refer the interested reader to the paper cited.

Among the new properties not present in [72], of importance for us are additional relations among the generalized exponentials at integer v. In this case, as is easy to verify, they satisfy the equation

$$\prod_{k=1}^{v-1}\left(z\,\frac{\mathrm{d}}{\mathrm{d}z}+\alpha+k\right)\mathcal{E}_{v,\alpha+v}(z)=\frac{1}{v}\,\mathcal{E}_{v,\alpha}(z)\,. \tag{3.11}$$

3.3 Powers of Laplace-type operators

The question may nevertheless arise: even if we know that the generalized exponentials arise in the heat kernel of a power of the Laplacian in the case of flat space, how can we be confident that these functions play an important role in the general case of minimal operators on a curved background? A partial answer to this can be deduced from the problem of finding the heat kernel expansion for a power of a Laplace-type operator (which, in addition, underlies the method developed in Section 3.5).

We therefore assume that a Laplace-type operator $\hat{H}(\nabla)$ in (2.41) is given, whose heat kernel expansion has form (3.3). We want to obtain a similar heat kernel expansion for a power of this operator, $\hat{F}(\nabla)=\hat{H}^v(\nabla)$. This can be done quite easily using the direct and inverse Mellin transformations (2.25) and (2.26).

First, substituting expansion (3.3) into transformation (2.25) and integrating the series term-wise, we obtain the expansion for the Green's function

$$\hat{G}_{H^s}(x,x')=\sum_{m=0}^{\infty}\mathbb{G}_m(s,\sigma)\,\hat{a}_m(H|x,x')\,, \tag{3.12}$$

where the basis Green's functions $\mathbb{G}_m(s,\sigma)$ are given by the integral

$$\begin{aligned}\mathbb{G}_m(s,\,\sigma)&=\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}\tau\,\tau^{s-1}\,\mathbb{K}_m(\tau,\,\sigma)\\&=\frac{\Gamma(d/2-m-s)}{(4\pi)^{d/2}\Gamma(s)}\left(\frac{\sigma}{2}\right)^{s+m-d/2},\end{aligned} \tag{3.13}$$

which is easy to calculate using the substitution $z=\sigma/2\tau$ and the definition of the gamma function.

We next substitute expansion (3.12) into transformation (2.26) for the operator power $\hat{H}^v$. Having again integrated the series term-wise, we find the heat kernel expansion

$$\hat{K}_{H^v}(\tau|x,x')=\sum_{m=0}^{\infty}\mathbb{K}_m^{(v,d)}(\tau,\sigma)\,\hat{a}_m(H|x,x')\,, \tag{3.14}$$

where the new basis kernels $\mathbb{K}_m^{(v,d)}(\tau,\sigma)$ are given by the integral

$$\begin{aligned}\mathbb{K}_m^{(v,d)}(\tau,\sigma)&=\frac{1}{2\pi\mathrm{i}}\int_{w-\mathrm{i}\infty}^{w+\mathrm{i}\infty}\mathrm{d}s\,\tau^{-s}\Gamma(s)\,\mathbb{G}_m(vs,\sigma)\\&=\frac{\tau^{(m-d/2)/v}}{(4\pi)^{d/2}}\,\mathcal{E}_{v,d/2-m}\left(-\frac{\sigma}{2\tau^{1/v}}\right),\end{aligned} \tag{3.15}$$

which can be reduced to the definition of the generalized exponential in (3.6) by setting $z=\sigma/2\tau^{1/v}$ and $\mu=d/2-m-vs$.

Thus, we have verified that the heat kernel for a power of a Laplace-type operator, $\hat{H}^v$, has expansion (3.14) at $x\neq x'$, which is similar in form to DeWitt expansion (3.3) for the original operator. Moreover, the coefficients in the expansion for a power of the operator are the same HaMiDeW coefficients $\hat{a}_m(H|x,x')$ for the original operator $\hat{H}(\nabla)$.

The only difference is that new basis kernels (3.15) replace the original basis kernels (3.4). Moreover, all original basis kernels $\mathbb{K}_m(\sigma,\tau)$ included the same exponential $\exp(-\sigma/2\tau)$, which could therefore be pulled out of the sum, making expansion (3.3) a power series in the proper time τ, but each new basis kernel $\mathbb{K}_m^{(v,d)}(\tau,\sigma)$ has its own generalized exponential $\mathcal{E}_{v,d/2-m}(-\sigma/2\tau^{1/v})$. These different functions can no longer be pulled out of the sum, and therefore expansion (3.14) is not a power series but a functional series.

If we now move to the coincidence limit $x=x'$ in the expansion for the operator power $\hat{H}^v$ and substitute the values of the generalized exponentials at zero, $\mathcal{E}_{v,\alpha}(0)=\Gamma(\alpha/v)/v\Gamma(\alpha)$, we obtain

$$\hat{K}_{H^v}(t|x,x)=t^{-d/2v}\sum_{m=0}^{\infty}t^{m/v}\,\hat{A}_m(H^v|x)\,, \tag{3.16}$$

$$\hat{A}_m(H^v|x)=\frac{\Gamma\big((d/2-m)/v\big)}{v\Gamma(d/2-m)}\,\hat{A}_m(H|x)\,. \tag{3.17}$$

This is the well-known Fegan–Gilkey formula obtained in a new way [17], and the property of the heat kernel coefficients to preserve their form when the operator is raised to a power is precisely what is called the 'functoriality property' in the mathematical literature.

The analysis of the heat kernel expansions outside the diagonal, at $x\neq x'$, significantly simplifies the proof of the corresponding properties compared with the approach usually accepted in the mathematical literature, which deals with operator ζ-functions only in the coincidence limit $x=x'$. At the same time, our approach does not suffer from anything similar to the appearance of mysterious terms that are logarithmic in the proper time τ. There has been a long debate in the literature about such terms (see, e.g., [75]), but, in light of our approach, it seems that they are likely to be a mere artifact of the method used. Moreover, the above transformations are only the simplest example of a general scheme of reasoning, which can be extended without much difficulty to a considerably wider area: from Laplace-type operators to general minimal operators of higher orders and even some types of nonminimal (causal) operators; from the heat kernel of an operator power $\exp(-t\hat{H}^v)$, to functions of a more general form, for example, $\hat{H}^{-\mu}\exp(-t\hat{H}^v)$ or $(\hat{H}^\mu+\lambda)^{-1}$, etc. Relevant issues related to such 'generalized functoriality' will be addressed in [76], a paper currently being prepared for publication.

The Laplace type operator $\hat{F}(\nabla)=\hat{H}^v(\nabla)$ raised to a power is the simplest minimal operator of a higher order $2v$. Therefore, it is quite tempting to use expansion (3.14) obtained for it as a replacement for the DeWitt ansatz in the case of minimal differential operators of general form (2.27). However, it can be easily verified directly that substituting expansion (3.14) into heat conduction equation (2.23) does not lead to a consistent chain of recursive relations. We hence draw the conclusion that, for a minimal operator that cannot be represented as a power, the expansion actually has a more complex structure, even if it somehow reduces to (3.14) in the special case of a Laplace-type operator raised to a certain power. In what follows, we substantiate this preliminary conclusion by deriving an expansion for the general minimal operator using two methods simultaneously: the so-called 'generalized Fourier transform' and the perturbation theory

(with a power of an operator representing the unperturbed case), building upon the results in this section. In addition to the two computation algorithms, which are interesting to compare with each other as well as with the method of universal functional traces, this allows us to answer the questions as to why a generalization of the DeWitt method fails and regarding the impossibility of constructing generalized recursive relations in the coordinate representation.

3.4 Generalized Fourier transform

The use of the Fourier transform underlies the study of (pseudo)differential operators and their spectral geometry (see, e.g., [13, 14]). However, in most mathematical texts, it is used in a somewhat inconvenient coordinate form. In a fully covariant form, the Fourier transform method on a curved space–time was developed in the form of 'symbol calculus' by Widom [77–79]. In this form, it was successfully applied to the calculation of the coincidence limits of the heat kernel coefficients by Gusynin and coauthors [49–54].

In contrast to these studies, we use the generalized Fourier transform to infer the structure of the heat kernel outside the diagonal, at $x \neq x'$. The corresponding algorithm for the generalized exponentials was discussed in detail in [80]. In this subsection, we reproduce the main stages of its derivation, omitting technical details and referring the interested reader to the cited work.

Before proceeding to the method itself, we introduce an abbreviated notation that proves convenient in what follows. We systematically omit contractions of repeated indices, instead writing

$$\hat{F}_k * \nabla^k = \hat{F}_k^{a_1 \dots a_k} \nabla_{a_1} \dots \nabla_{a_k} . \tag{3.18}$$

An arbitrary order-N differential operator (acting on the set of fields $\varphi(x) = \varphi^A(x)$, as the hats remind us) can be written in the form

$$\hat{F}(\nabla) = \sum_{k=0}^{N} \hat{F}_k(x) * \nabla^k . \tag{3.19}$$

We note that we originally restricted ourselves exclusively to the case of a Riemannian space–time in this paper, but the method of the generalized Fourier transform and further considerations based on it are applicable in a much broader context of theories with torsion and nonmetricity, when the connection ∇_a is no longer defined by the metric g_{ab}. A discussion of the details can be found in [80] and we do not dwell on the subject here.

The vector $\sigma^{a'}(x, x')$ can be regarded as a generalization of the vector $x' - x$ connecting points in flat space. The generalized Fourier transform method in curved space is based on the covariant integral representation of the delta function in terms of plane waves $\exp(\mathrm{i} k_{b'} \sigma^{b'})$, where $k_{b'}$ is the cotangent vector at x'. This representation has the form [7]

$$\hat{1}\delta(x, x') = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \exp(\mathrm{i} k_{a'} \sigma^{a'}) \hat{\mathcal{I}}(x, x') . \tag{3.20}$$

[7] Alternatively, we could just as well have used the expansion in the functions $\exp(\mathrm{i} k_b \sigma^b)$. Although we prefer (3.20) for certain reasons, and although these two possible choices lead to different expressions for the coefficients outside the coincidence limit, both give the same local expressions after passing to the coincidence limit. Therefore, comparing the results obtained by these two methods can even be used as an error-detection tool.

In fact, instead of the parallel translation tensor $\hat{\mathcal{I}}(x, x')$, we could of course have chosen any other matrix-valued two-point function that gives the identity matrix $\hat{1}$ in the coincidence limit and in addition has the correct transformation properties with respect to both arguments x and x' (for this reason, the constant matrix $\hat{1}$ is not suitable for the covariant formalism). This method has some inherent ambiguity, but it should not affect the final answers obtained by integrating over momenta. In this regard, our choice of the tensor $\hat{\mathcal{I}}(x, x')$ seems to be the simplest and most natural.

We apply the operator exponential $\exp(-\tau \hat{F}(\nabla))$ to representation (3.20) and move the plane waves past the operator $\hat{F}(\nabla)$ in accordance with the commutation relation

$$\exp(-\mathrm{i} k_{b'} \sigma^{b'}) \hat{F}(\nabla_a) \exp(\mathrm{i} k_{b'} \sigma^{b'}) = \hat{F}(\nabla_a + \mathrm{i} k_{b'} \sigma_a^{b'}) , \tag{3.21}$$

where $\sigma_a^{b'}(x, x') = \nabla_a \sigma^{b'}(x, x')$. As a result, we obtain the integral representation for the heat kernel:

$$\hat{K}(\tau | x, x') = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \exp(\mathrm{i} k_{a'} \sigma^{a'}) \hat{\mathbf{K}}(\tau, \mathbf{k} | x, x') , \tag{3.22}$$

$$\hat{\mathbf{K}}(\tau, \mathbf{k} | x, x') = \exp\left[-\tau \hat{F}(\nabla_a + \mathrm{i} k_{b'} \sigma_a^{b'})\right] \hat{\mathcal{I}}(x, x') \tag{3.23}$$

(here and hereafter, we use boldface for Fourier transforms).

It may seem that expression (3.23) in and of itself is already the sought Fourier transform of the heat kernel in closed form, and we could simply expand the corresponding operator exponential in a power series. This strategy, however, would not lead us to the final goal — an expansion in powers of the background dimension $1/l$, because the leading term of the operator $\hat{F}(\nabla_a + \mathrm{i} k_{b'} \sigma_a^{b'})$ has zero background dimension: $\dim \hat{F}_N^{a_1 \dots a_N}(x) = 0$. Therefore, our task is to explicitly extract $O(1/l^0)$ terms from the operator exponential, find a zeroth-order solution explicitly, and only then construct a perturbation theory in $1/l$ over that solution.

Hence, instead of expression (3.23), we must use an ansatz for the Fourier transform of the heat kernel,

$$\hat{\mathbf{K}}(\tau, \mathbf{k}) = \exp\left[-\tau (\mathrm{i}\mathbf{k})^N * \hat{\mathcal{F}}\right] \hat{\mathbf{T}}(\nabla) \hat{\mathcal{I}} , \tag{3.24}$$

where

$$\hat{\mathcal{F}} \equiv \hat{\mathcal{F}}^{b'_1 \dots b'_N} = \hat{F}_N^{a_1 \dots a_N} \sigma_{a_1}^{b'_1} \dots \sigma_{a_N}^{b'_N} \tag{3.25}$$

is a matrix-valued two-point function, which is a scalar with respect to x and a tensor with N upper indices with respect to x' (it plays the role of the principal symbol of the operator $\hat{F}(\nabla)$ typically used in Fourier analysis of (pseudo)differential operators) and $\hat{\mathbf{T}}(\nabla) \equiv \hat{\mathbf{T}}(\nabla, \tau, \mathbf{k} | x, x')$ is an operator unknown at this stage.

After a careful expansion in powers of the momentum $\mathbf{k}$ and all the necessary commutations (see the details in [80]), we obtain the following problem for the operator $\hat{\mathbf{T}}(\nabla, \tau, \mathbf{k})$:

$$(\partial_\tau + \hat{\mathbf{F}}) \hat{\mathbf{T}}(\nabla, \tau, \mathbf{k}) = 0 , \tag{3.26}$$

$$\hat{\mathbf{T}}(\nabla, 0, \mathbf{k}) = \hat{1} . \tag{3.27}$$

Here, new operators are introduced as

$$\hat{\mathbf{F}}(\nabla) = \hat{\mathbf{F}}(\nabla, \tau, \mathbf{k} | x, x') = \sum_{m=0}^{N-1} \sum_{n=0}^{N-m} \tau^n (\mathrm{i}\mathbf{k})^{m+Nn} * \{\hat{F}\}_{m,n} , \tag{3.28}$$

$$\{\hat{F}\}_{m,n} = \sum_{k=m+n}^{N} \hat{F}_k(x) * \{\nabla^k\}_{m,n} , \tag{3.29}$$

and the action of the operation $\{\ldots\}_{m,n}$ on a set of covariant derivatives $\nabla^k = \nabla_{a_1}\ldots\nabla_{a_k}$ produces $\binom{k}{m,n} = k!/(m!n!(k-m-n)!)$ terms, in each of which m derivatives are replaced by functions of the form $\sigma_a^{b'}$, and n derivatives, by functions of the form $-\nabla_a\hat{\mathcal{F}}$, with the order being strictly preserved.[8] Thus, $\{\hat{F}\}_{m,n} = \{\hat{F}\}_{m,n}^{b'_1\ldots b'_m c'_1\ldots c'_{Nn}}$ are operators of the order $N-m-n$ and dimension $N-m$ with $m+Nn$ upper primed indices, and $\{\hat{F}\}_{N,0} = \hat{\mathcal{F}}$.

We now expand the operator $\hat{\mathbf{T}}(\tau,\mathbf{k})$ in a double series in powers of τ and $\mathbf{k}$:

$$\hat{\mathbf{T}}(\nabla,\tau,\mathbf{k}) = \sum_{n=0}^{\infty}\sum_{l=0}^{L_n}\tau^n(\mathrm{i}\mathbf{k})^l * \hat{T}_{n,l}(\nabla)\,, \tag{3.30}$$

where $\hat{T}_{n,l}(\nabla) = \hat{T}_{n,l}^{b'_1\ldots b'_l}(\nabla|x,x')$ are some unknown operator-valued coefficients which are tensors with l upper primed indices, and L_n is the summation limit, which is still unknown. Substituting expansions (3.28) and (3.30) into Eqn (3.26) leads to a system of recursive relations for $\hat{T}_{n,l}(\nabla)$:

$$(n+1)\hat{T}_{n+1,l} = -\sum_{p=0}^{N-1}\sum_{q=0}^{N-p}\{\hat{F}\}_{p,q}\hat{T}_{n-q,l-p-Nq}\,, \tag{3.31}$$

$$\hat{T}_{0,0} = \hat{1}\,,\quad \hat{T}_{0,l} = 0 \quad \text{for} \quad l>0\,. \tag{3.32}$$

(It is assumed here that $\hat{T}_{n,l}=0$ for $n<0$ or $l<0$ and, as it should be, composing a tensor $\{\hat{F}\}_{p,q}$ of valence $p+Nq$ with a tensor $\hat{T}_{n-q,l-p-Nq}$ of valence $l-p-Nq$ on the right-hand side of the equation generates a valence-l tensor on the left-hand side.) The initial values (3.32) follow from initial condition (3.27) and allow the recursive relations to be sequentially solved for all $\hat{T}_{n,l}$. Also, from system (3.31), (3.32), it is easy to obtain the summation limit in (3.30):

$$L_n(N) = \left(N-\frac{1}{2}\right)n\,. \tag{3.33}$$

An example of calculating the operators $\hat{T}_{n,l}(\nabla)$ in terms of lower-order operators is shown in Fig. 1 for $\hat{T}_{6,5}(\nabla)$ and $N=2$.

We note that the recursive relations allow closed expressions to be obtained for some operators $\hat{T}_{n,l}(\nabla)$, for example,

$$\hat{T}_{n,l} = \frac{1}{n!}\{(-\hat{F})^n\}_l\,,\quad \text{for}\quad l<N\,, \tag{3.34}$$

$$\hat{T}_{2k,(2N-1)k} = \frac{1}{2^k k!}\left(-\{\hat{F}\}_{N-1,1}\right)^k. \tag{3.35}$$

However, an arbitrary operator $\hat{T}_{n,l}(\nabla)$ cannot be obtained using such simple formulas. The most we can say in the general case is that the operator can be represented as a sum of terms (with some coefficients) of the form

$$\{\hat{F}^{k_1}\}_{m_1,n_1}\ldots\{\hat{F}^{k_p}\}_{m_p,n_p}\,, \tag{3.36}$$

where

$$\sum_{i=1}^{p}(k_i+n_i) = n \quad\text{and}\quad \sum_{i=1}^{p}(Nn_i+m_i) = l\,. \tag{3.37}$$

[8] We note that, compared to [80], we changed the notation: in that paper, this operation was denoted by double square brackets $[\![\ldots]\!]_{m,n}$, but here we use curly brackets $\{\ldots\}_{m,n}$ for it, and the double square brackets are reserved for nested commutators in what follows.

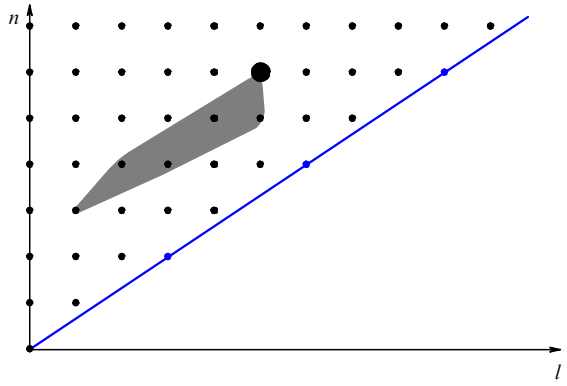


Figure 1. Operators $\hat{T}_{n,l}$ for $N=2$. Blue line shows the maximum value of l, $L_n(N) = (N-1/2)n$. Each point on the plane at position (n,l) denotes operator $\hat{T}_{n,l}$. Shaded area shows means to calculate operator on the left-hand side of recursive relations (3.31) (in this example, $\hat{T}_{6,5}$).

With the expansion of the Fourier transform $\hat{\mathbf{K}}(\nabla,\tau,\mathbf{k})$ obtained, finding the sought heat kernel expansion only requires integrating this expansion with respect to the momenta $\mathbf{k}$. Substituting (3.24) and (3.30) into integral (3.22), we obtain our main result for the heat kernel of a general positive-definite differential operator of order N,

$$\hat{K}(\tau|x,x') = \sum_{n=0}^{\infty}\tau^n\sum_{l=0}^{L_n}\hat{S}_l(\tau) * \hat{T}_{n,l}(\nabla)\,\hat{\mathcal{I}}(x,x')\,, \tag{3.38}$$

where

$$\begin{aligned}\hat{S}_l(\tau) &= \hat{S}_{l,b'_1\ldots b'_l}(\tau|x,x')\\ &= \int\frac{\mathrm{d}^d k}{(2\pi)^d}(\mathrm{i}\mathbf{k})^l\exp\left[-\tau(\mathrm{i}\mathbf{k})^N * \hat{\mathcal{F}} + \mathrm{i}k_{a'}\sigma^{a'}\right]\end{aligned} \tag{3.39}$$

are two-point matrix-valued tensors with l primed lower indices.

Importantly, expansion (3.38) is effective from the QFT standpoint, as an expansion in the dimension of background fields, because, as n increases, the background dimension of the corresponding term also increases monotonically:

$$\dim\hat{T}_{n,l}(\nabla) = nN - l \geqslant \frac{n}{2}\,. \tag{3.40}$$

In the particular case of minimal operator (2.27) of a higher order $2v$, integrals (3.39) reduce to a single basic integral (3.5) and are therefore easy to calculate in terms of generalized exponentials. Again referring to [80] for simple but somewhat cumbersome technicalities, we give the final answer:

$$\begin{aligned}\hat{S}_{l,a'_1\ldots a'_l}(\tau|x,x') &= \frac{\Delta^{-1}(x,x')}{(4\pi\tau^{1/v})^{d/2}}\\ &\times\sum_{r\geqslant l/2}^{l}\frac{S_{r,l,a'_1\ldots a'_l}}{(-2\tau^{1/v})^r}\,\mathcal{E}_{v,d/2+r}\left(-\frac{\sigma}{2\tau^{1/v}}\right)\hat{1}\,,\end{aligned} \tag{3.41}$$

where the new totally symmetric tensor coefficients $S_{r,l} = S_{r,l,a'_1\ldots a'_l}(x,x')$ with $r\geqslant 0$ and $l=r,\ldots,2r$ have l primed lower indices and no longer depend on τ. Each $S_{r,l}$ consists of $2r-l$ factors of the form $\sigma_{a'}$ and $l-r$ factors of the form $\gamma_{a'b'} = \bar{\sigma}^c_{a'}g_{cd}\bar{\sigma}^d_{b'}$, where $\bar{\sigma}^a_{b'}(x,x')$ is the inverse tensor to $\sigma_a^{b'}$, with the combinatorial coefficients equal to

the number of different terms appearing in symmetrizing over l indices. For example,

$$S_{1,1} = \sigma_{a'}\,, \qquad S_{1,2} = \gamma_{a'b'}\,, \qquad S_{2,2} = \sigma_{a'}\sigma_{b'}\,,$$
$$S_{2,3} = 3\gamma_{(a'b'}\sigma_{c')}\,, \quad S_{2,4} = 3\gamma_{(a'b'}\gamma_{c'd')}\,, \tag{3.42}$$
$$S_{3,3} = \sigma_{a'}\sigma_{b'}\sigma_{c'}\,, \quad S_{3,4} = 6\gamma_{(a'b'}\sigma_{c'}\sigma_{d')}\,,$$

and so on. These coefficients have an important property: because $[\sigma_{a'}] = 0$ and $[\gamma_{a'b'}] = g_{ab}$, it follows that, in the coincidence limit, they are nonvanishing only for $l = 2r$.

After the substitution of (3.41) into expansion (3.38) and resummation, we obtain our main result for the heat kernel expansion of the general minimal differential operator of order $2v$,

$$\hat{K}_F(\tau|x,x') = \Delta^{-1}\sum_{m=-\infty}^{\infty}\sum_{n\geqslant N_m}^{\infty}\mathbb{K}^{(v,d)}_{m,n}(\sigma,\tau)\,\hat{b}_{m,n}(x,x')\,, \tag{3.43}$$

where we introduce new basis kernels

$$\mathbb{K}^{(v,d)}_{m,n}(\sigma,\tau) = \frac{\tau^{(m-d/2)/v}}{(4\pi)^{d/2}}\,\mathcal{E}_{v,\,d/2+nv-m}\left(-\frac{\sigma}{2\tau^{1/v}}\right). \tag{3.44}$$

Here, the lower summation limit for n in the mth order of expansion in $\tau^{1/v}$ is given by

$$N_m(v) = \begin{cases} \dfrac{m}{v}\,, & m > 0\,, \\[2ex] \dfrac{2|m|}{2v-1}\,, & m < 0\,. \end{cases} \tag{3.45}$$

We call the two-point matrix-valued functions $\hat{b}_{m,n}(x,x')$ the generalized HaMiDeW coefficients. They are given by finite contractions of the differential operators $\hat{T}_{n,l}(\nabla)$ introduced above with the tensor coefficients $S_{p,l}(x,x')$,

$$\hat{b}_{m,n}(x,x') = \frac{1}{(-2)^{vn-m}} \times \sum_{l=vn-m}^{\lfloor L_{m,n}\rfloor} S_{vn-m,\,l}(x,x') * \hat{T}_{n,l}(\nabla)\,\hat{\mathcal{I}}(x,x')\,, \tag{3.46}$$

where the upper summation limit for l is the integer part of

$$L_{m,n} = 2vn - \max\left\{2m, \frac{n}{2}\right\}. \tag{3.47}$$

In the special case $m = vn$, as follows from (3.35), the coefficients acquire an especially simple form:

$$\hat{b}_{vn,n}(x,x') = \hat{T}_{n,0}(\nabla)\hat{\mathcal{I}} = \frac{1}{n!}(-\hat{F})^n\,\hat{\mathcal{I}}\,. \tag{3.48}$$

The most significant difference between expansions (3.43) and the expansion (2.51) for a second-order operator is the appearance of arbitrarily large negative powers of τ and the absence of a common exponential factor in front of the power series in proper time. Instead, each generalized HaMiDeW coefficient $\hat{b}_{m,n}$ at $\tau^{m/v}$ is multiplied by its own generalized exponential, which explicitly depends on both indices m and n of the double infinite series.

Due to the appearance of arbitrarily large negative powers of τ, expansion (3.43) can no longer be interpreted as an expansion in the small proper time parameter $\tau \to 0$. Instead, it should be understood as an expansion in the background dimension, and this is exactly what we need for renormalization and for effective field theory. Indeed, as follows from (3.40), the dimensions of $\hat{b}_{m,n}(x,x')$ are always positive and increase with the indices m and n:

$$\dim\,\hat{b}_{m,n} \geqslant \min_{\{l\}}(\dim\,\hat{T}_{n,l}) = \max\left\{2m, \frac{n}{2}\right\}. \tag{3.49}$$

We note a significant difference between the generalized Fourier transform method and the classical DeWitt method. In the latter, to calculate the coincidence limit $[\hat{a}_m]$, we must know the coincidence limits for all lower coefficients, as well as for a certain number of their derivatives. By contrast, in our method, the recursive procedure involves not the generalized coefficients $\hat{b}_{m,n}$ themselves but the operators $\hat{T}_{n,l}(\nabla)$; the coefficients are then obtained independently of each other in each order in the form of contractions of these operators with the tensors $S_{r,l}$ in (3.46). Moreover, this procedure generates not the coincidence limits $[\hat{b}_{m,n}]$ but the coefficients $\hat{b}_{m,n}(x,x')$ as exact two-point functions with $x \neq x'$.

We have implemented the algorithm described in this section in the symbolic computing system *Wolfram Mathematica* using the packages *xAct* and *xTras* and applied it to Laplace-type operator (2.41) and the fourth-order minimal operator of the general form

$$\hat{F}(\nabla) = \hat{1}\Box^2 + \hat{\Omega}^{abc}\nabla_a\nabla_b\nabla_c + \hat{D}^{ab}\nabla_a\nabla_b + H^a\nabla_a + \hat{P}\,. \tag{3.50}$$

Here, we do not go into the details of these calculations, which allowed not only reproducing the coincidence limits previously obtained by other authors but also obtaining results that have not yet been reported in the literature (in particular, contributions of third-order terms involving $\hat{\Omega}^{abc}$). For the details, we again refer the reader to [80].

An important circumstance that we do discuss here instead is the apparent contradiction between the results generated by the Fourier transform and the standard DeWitt method. Indeed, if we set $v = 1$ in formulas (3.43) and (3.46), we obtain the following expansion for the Laplace-type operator:

$$\hat{K}_F(\tau|x,x') = \frac{\Delta^{-1}}{(4\pi\tau)^{d/2}}\exp\left(-\frac{\sigma}{2\tau}\right)\sum_{m=-\infty}^{\infty}\tau^m\hat{b}_m(x,x')\,, \tag{3.51}$$

$$\hat{b}_m(x,x') = \sum_{n\geqslant N_m}^{\infty}\hat{b}_{m,n}(x,x')\,. \tag{3.52}$$

As we see, terms with negative powers of the proper time persist in this expansion, whereas there are simply no such terms in DeWitt's ansatz. How is this possible?

The apparent paradox can be resolved by calculations. It turns out that, in the case of a Laplace-type operator, all the generalized coefficients $\hat{b}_{m,n}(x,x')$ that we were able to calculate for $m < 0$, even if this is not obvious at first glance, vanish identically due to special relations that exist between the tensors $\sigma^b{}_{a_1\ldots a_n}$ and $\hat{\mathcal{I}}_{a_1\ldots a_n}$. An example of such relations can be obtained by acting on (2.45) and (2.49) with the operator of differentiation along the geodesic $\sigma^a\nabla_a$ and then simplifying the resultant expressions using (2.45). This leads to the relations

$$\sigma^{a_1}\ldots\sigma^{a_n}\sigma^b{}_{a_1\ldots a_n} = 0 \quad \text{for} \quad n > 1\,, \tag{3.53}$$

$$\sigma^{a_1}\ldots\sigma^{a_n}\hat{\mathcal{I}}_{a_1\ldots a_n} = 0\,. \tag{3.54}$$

There are other, more complex relations of this kind.

We say that these types of coefficients that vanish identically outside the coincidence limit are *spurious*. A natural question arises as to whether the same happens for higher-order operators. Are all coefficients at negative powers of τ for them also spurious? Symbolic calculations give a negative answer to this question: although some coefficients of a fourth-order operator are spurious, there certainly are those that do not vanish identically. For example, the simplest coefficient $\hat{b}_{-1,1}$ contains a term of the form $\hat{\Omega}^{abc}\sigma_a\sigma_b\sigma_c\hat{\mathcal{I}}(x,x')$. This shows that the appearance of coefficients with negative powers of τ is not an artifact of the method used but is an essential feature of the heat kernel expansions for higher-order operators.

On the other hand, although there are nontrivial coefficients at negative powers of τ for higher-order operators, it follows from the well-known expansion for heat kernel diagonal (2.35) that they must at least vanish in the coincidence limit. Indeed, the remark after formula (3.42) that $[S_{l,p}]\neq 0$ only for $l=2p$ implies that a nonvanishing contribution to the sum over l in (3.46) can only be made if $l=2(Mn-m)\leqslant L_n(2M)=(2M-1/2)n$, whence we have the constraint

$$[\hat{b}_{m,n}]=0 \quad \text{for} \quad n>4m\,. \tag{3.55}$$

(The coincidence limits actually start vanishing much earlier, which is consistent with a stronger estimate in (3.80), which follows from the perturbation theory.) In the case $v=2$, the generalized coefficients are schematically shown in Fig. 2.

Moving to the coincidence limit $x=x'$ in expansion (3.43) with constraint (3.55) taken into account, we return to the well-known expansion of the heat kernel diagonal in (2.35), where the coefficients $\hat{A}_m(F|x)$ are expressed in terms of the coincidence limits of the two-point generalized coefficients $[\hat{b}_{m,n}(F|x,x')]$ as

$$\hat{A}_m=\frac{1}{(4\pi)^{d/2}}\sum_{n\geqslant m/v}^{4m}\frac{\Gamma\bigl((d/2-m)/v+n\bigr)}{v\Gamma(d/2-m+vn)}\,[\hat{b}_{m,n}]\,. \tag{3.56}$$

As we see, the coincidence limit leads to a natural truncation of expansion (3.43) because the coefficient at each power of the proper time τ is given by only a finite number of terms of the nontruncated expansion outside the diagonal.

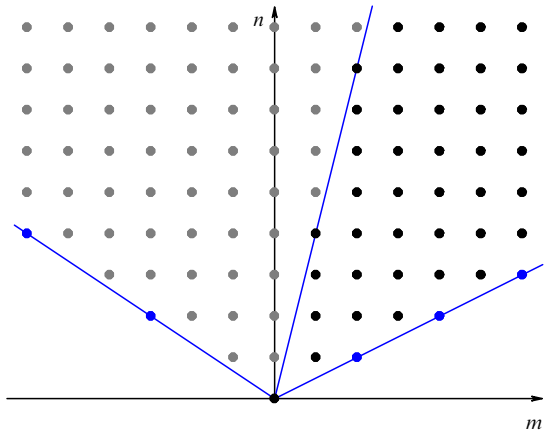


Figure 2. Generalized coefficients $\hat{b}_{m,n}$ in the case $v=2$. Blue lines on left and right represent expression (3.45) for $N_m(2)$. Coefficients shown with gray dots are certain to vanish in the coincidence limit, and those shown with black dots can make a nonzero contribution; line separating them is defined by condition (3.55).

3.5 Perturbation theory

We next discuss the second method of calculating the heat kernel coefficients for minimal higher-order operators. As we saw in Section 3.3, if $\hat{H}(\nabla)$ is an arbitrary Laplace-type operator (2.41), then the technique involving the direct/inverse Mellin transformation allows heat kernel expansion (3.14) to be immediately obtained for an arbitrary power $\hat{H}^v(\nabla)$. Because an order-$2v$ minimal differential operator $\hat{F}(\nabla)$ of the general form, Eqn (2.27), cannot be represented as the vth power of a second-order operator, the heat kernel expansion for it does not have to be of form (3.14). Indeed, although expansion (3.43) obtained using the generalized Fourier transform still contains the generalized exponentials $\mathcal{E}_{v,\alpha}(-\sigma/2\tau^{1/v})$, it nevertheless has a somewhat more complex form: it involves a double summation, and an infinite series of terms with negative powers of the proper time appear.

However, an arbitrary order-$2v$ minimal operator $\hat{F}(\nabla)$ in (2.27) can always be represented in the form

$$\hat{F}(\nabla)=\hat{H}^v(\nabla)+\hat{W}(\nabla)\,, \tag{3.57}$$

where $\hat{H}(\nabla)$ is a Laplace-type operator (2.41) and the addition $\hat{W}(\nabla)$ is an arbitrary differential operator of the lower order $2v-1$. The main idea of our second method is very simple and natural: in expansion (3.57), we consider the term $\hat{H}^v(\nabla)$ as the 'unperturbed part' of the operator, and the term $\hat{W}(\nabla)$ as a 'perturbation,' and then seek the expansion of the heat kernel $\hat{K}_F(\tau|x,x')$ of the total operator $\hat{F}(\nabla)$ within the perturbation theory in powers of $\hat{W}(\nabla)$ over the already known 'unperturbed kernel' of $\hat{H}^v(\nabla)$.

Thus, using the analogy with the perturbation theory in quantum mechanics, we pass to the 'interaction picture.' There, the perturbation acquires a dependence on the proper time τ,

$$\hat{W}_\tau=\exp\left(\tau\hat{H}^v\right)\hat{W}(\nabla)\exp\left(-\tau\hat{H}^v\right), \tag{3.58}$$

and the 'evolution operator'

$$\hat{U}_\tau=\exp\left(\tau\hat{H}^v\right)\exp\left(-\tau\hat{F}\right) \tag{3.59}$$

satisfies the equation

$$(\partial_\tau+\hat{W}_\tau)\hat{U}_\tau=0\,,\qquad \hat{U}_0=\hat{1}\,. \tag{3.60}$$

We can then expand the evolution operator in a series in powers of the perturbation $\hat{W}_\tau$,

$$\begin{aligned}\hat{U}_\tau&=\hat{1}-\int_0^\tau \mathrm{d}t\,\hat{W}_t\hat{U}_t=\bar{T}\exp\left(-\int_0^\tau \mathrm{d}t\,\hat{W}_t\right)\\&=\sum_{n=0}^{\infty}(-1)^n\int_{\{\tau\}}\mathrm{d}^n t\,\hat{W}_{t_n}\ldots\hat{W}_{t_1}\,,\end{aligned} \tag{3.61}$$

where $\bar{T}$ is the operation of antichronological ordering, and integration is carried out over an n-dimensional hyperpyramid $0<t_1<\ldots<t_n<\tau$,

$$\int_{\{\tau\}}\mathrm{d}^n t=\int_0^\tau \mathrm{d}t_n\int_0^{t_n}\mathrm{d}t_{n-1}\ldots\int_0^{t_2}\mathrm{d}t_1\,. \tag{3.62}$$

Next, we note that, as is clear from definition (3.58), commuting the operators $\hat{W}_t$ through $\exp(-\tau\hat{H}^v)$ simply reduces to a shift of the proper time parameter:

$$\exp(-\tau\hat{H}^v)\,\hat{W}_t = \hat{W}_{t-\tau}\exp(-\tau\hat{H}^v)\,. \tag{3.63}$$

With (3.59), it hence follows that

$$\exp(-\tau\hat{F}) = \exp(-\tau\hat{H}^v)\,\hat{U}_\tau = \hat{U}'_\tau\exp(-\tau\hat{H}^v)\,, \tag{3.64}$$

where antichronological ordering is replaced with the chronological one in the new operator $\hat{U}'_\tau$:

$$\begin{aligned}\hat{U}'_\tau &= \bar{T}\exp\left(-\int_0^\tau \mathrm{d}t\,\hat{W}_{t-\tau}\right) = T\exp\left(-\int_0^\tau \mathrm{d}t\,\hat{W}_{-t}\right)\\ &= \sum_{n=0}^{\infty}(-1)^n\int_{\{\tau\}}\mathrm{d}^n t\,\hat{W}_{-t_1}\ldots\hat{W}_{-t_n}\,.\end{aligned} \tag{3.65}$$

In the language of kernels, relation (3.64) takes the form

$$\hat{K}_F(\tau|x,x') = \hat{U}'_\tau\hat{K}_{H^v}(\tau|x,x')\,. \tag{3.66}$$

Thus, the operator $\hat{U}'_\tau$ implements the heat kernel transformation caused by introducing the perturbation $\hat{W}(\nabla)$ into the operator. The operators $\hat{W}_{-t}$ and $\hat{U}'_\tau$ are integral, but if we expand them in power series in τ, the coefficients of this expansion are differential operators.

We first do this for $\hat{W}_{-t}$. We use the well-known relation

$$\exp(-tB)\,A\exp(tB) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\,[\![A,B]\!]_k\,, \tag{3.67}$$

where $[\![\ldots,\ldots]\!]_k$ denotes a nested kth order commutator defined as

$$[\![A,B]\!]_0 = A\,, \tag{3.68}$$

$$\begin{aligned}[\![A,B]\!]_k &\equiv \big[[\![A,B]\!]_{k-1},B\big]\\ &= \Big[\ldots\big[[A,\underbrace{B],B}_{k}\big],\ldots,B\Big]\,,\quad k>0\,.\end{aligned} \tag{3.69}$$

It immediately follows from relation (3.67) that

$$\hat{W}_{-t} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\,\hat{V}_k(\nabla)\,,\quad\text{where}\quad \hat{V}_k(\nabla) = [\![\hat{W},\hat{H}^v]\!]_k \tag{3.70}$$

are differential operators of the order $(2v-1)(k+1)$ and dimension $2v(k+1)$.

We now substitute expansion (3.70) into representation (3.65). The proper time integral is evaluated straightforwardly,[9]

$$\int_{\{t\}}\mathrm{d}t^n\,t^\alpha = \frac{\tau^{n+|\alpha|}}{c(\alpha)}\,, \tag{3.71}$$

$$c(\alpha) = (\alpha_1+1)(\alpha_1+\alpha_2+2)\ldots(|\alpha|+n)\,, \tag{3.72}$$

and we hence obtain an expansion of the heat kernel deformation operator $\hat{U}'_\tau$ in a series in powers in τ,

$$\hat{U}'_\tau = \sum_{k=0}^{\infty}\tau^k\,\hat{U}_k(\nabla)\,, \tag{3.73}$$

[9] Here, we use the standard multi-index notation $\alpha=(\alpha_1,\ldots,\alpha_n)$, $t^\alpha = t_1^{\alpha_1}\ldots t_n^{\alpha_n}$, $|\alpha| = \alpha_1+\ldots+\alpha_n$, and $\alpha! = \alpha_1!\ldots\alpha_n!$.

where

$$\hat{U}_k(\nabla) = \sum_{n+|\alpha|=k}\frac{(-1)^n}{\alpha!c(\alpha)}\,\hat{V}_{\alpha_n}(\nabla)\ldots\hat{V}_{\alpha_1}(\nabla) \tag{3.74}$$

are differential operators of the order $(2v-1)k$ and dimension $2vk$.

We now want to represent the heat kernel $\hat{K}_F(\tau|x,x')$ of the perturbed operator $\hat{F}(\nabla)$ as a functional series in generalized exponentials. For this, we substitute the expansions of unperturbed kernel (3.14) and kernel deformation operator (3.73) into formula (3.66) and commute the operators $\hat{U}_k(\nabla)$ with the generalized exponentials $\mathcal{E}_{v,\,d/2-m}(-\sigma/2\tau^{1/v})$. Taking the rule for differentiating generalized exponentials (3.10) into account, we introduce the following notation for the result of commuting an order-N differential operator through a generalized exponential:

$$\begin{aligned}&\hat{F}(\nabla)\,\mathcal{E}_{v,\,\alpha}\left(-\frac{\sigma}{2\tau^{1/v}}\right)\\ &\qquad= \sum_{n=0}^{N}\tau^{-n/v}\,\mathcal{E}_{v,\,\alpha+n}\left(-\frac{\sigma}{2\tau^{1/v}}\right)\langle\hat{F}\rangle_n\,.\end{aligned} \tag{3.75}$$

The operation $\langle\ldots\rangle_n$ is very similar to the curly bracket operation $\{\ldots\}_{n,0}$ introduced in (3.29), with the only difference being that the n covariant derivatives ∇_a are replaced not with $\sigma_a^{b'}$ but with the functions $-\sigma_a/2$. Otherwise, their properties and calculation rules are the same.

After commuting all the generalized exponentials to the left and performing a necessary resummation (the details of which we omit as well), we obtain the sought heat kernel expansion outside the diagonal,

$$\hat{K}_F(\tau|x,x') = \sum_{m=-\infty}^{\infty}\sum_{n\geqslant K_m}^{\infty}\mathbb{K}_{m,n}^{(v,d)}(\sigma,\tau)\,\hat{a}_{m,n}(F|x,x')\,, \tag{3.76}$$

which involves a double sum of the same basic kernels (3.44) as in expansion (3.43). The new generalized coefficients $\hat{a}_{m,n}(F|x,x')$ follow by acting with the deformation operators $\langle\hat{U}_n(\nabla)\rangle_k$ on the HaMiDeW coefficients of the original Laplace-type operator $\hat{H}(\nabla)$,

$$\hat{a}_{m,n}(F|x,x') = \sum_{l\geqslant L_{m,n}}^{m+(v-1)n}\langle\hat{U}_n(\nabla)\rangle_k\hat{a}_l(H|x,x')\,, \tag{3.77}$$

where $k = vn+l-m$. The lower summation limits for n and l are

$$K_m = \max\left\{0,\,-\frac{m}{v-1}\right\}, \tag{3.78}$$

$$L_{m,n} = \max\{0, m-vn\}\,. \tag{3.79}$$

The index n can be interpreted as the perturbation theory order in powers of $\hat{W}(\nabla)$.

Just as in the case of expansion (3.43), the coincidence limits of the coefficients at negative powers of τ vanish. Indeed, because $[\sigma_a]=0$ and $[\sigma_{ab}]=g_{ab}$, the operator $\langle\hat{U}_n(\nabla)\rangle_k$ makes a nonzero contribution to the coincidence limit only if each of the k functions σ_a contained in it is further differentiated at least one more time. Because the operator $\hat{U}_n(\nabla)$ has the order $(2v-1)n$, only terms with $k\leqslant(v-1/2)n$ can survive. And because $k=vn+l-m$ and $l\geqslant 0$, this in

turn leads to the constraint

$$[\hat{a}_{m,n}] = 0 \quad \text{for} \quad n > 2m\,, \tag{3.80}$$

which is much stronger than a similar constraint in (3.55).

We note that the summation limits in (3.43) and (3.76) differ, as do the generalized coefficients $\hat{b}_{m,n}$ and $\hat{a}_{m,n}$ entering them. The reason is that neither of the algorithms we use is very 'smart,' in the sense that they generate mechanistic expressions that can be significantly improved. An example is the appearance of spurious coefficients: obviously, a more careful consideration of relations like (3.53) and (3.54) among the tensors $\sigma^{b}{}_{a_1\ldots a_n}$ and $\hat{\mathcal{I}}_{a_1\ldots a_n}$ would allow simplifying many generalized coefficients, and all spurious coefficients would then vanish.[10] Another example is the presence of terms where the world function $\sigma(x,x')$ enters as a factor. It is easy to see that, using equation (3.11), which relates various generalized exponentials $\mathcal{E}_{v,\alpha}(z)$, we can systematically eliminate such terms, bringing the expansion to some simple canonical form.

4. Conclusions

To conclude, we note that, along with presenting the well-known Schwinger–DeWitt technique in quantum field theory and gravity, we have also obtained a number of new results in developing this method and opened up new prospects for its extensions.

For minimal operators of a higher order $2v$, Eqn (2.27), we first established the structure of the heat kernel *outside the diagonal* (i.e., for $x \neq x'$) in the form of double functional series (3.43) (or (3.76)). This expansion is a direct generalization of DeWitt's ansatz (2.51) for Laplace-type operators, although it differs in two significant points.

First, instead of the standard exponential factor $\exp(-\sigma/2\tau)$, the expansion is carried out in terms of the 'generalized exponentials' $\mathcal{E}_{v,\alpha}(-z)$ of the dimensionless ratio $z = \sigma/2\tau^{1/v}$. Their remarkable properties, including differentiation rule (3.10) and Mellin transformation (3.7), allow effectively manipulating these new functions.

Second, the off-diagonal expansions involve terms with arbitrarily large negative powers of the proper time $\tau^{m/v}$. As it should be, all the corresponding coefficients vanish in the coincidence limit $x = x'$, which guarantees that the well-known asymptotic expansion of the heat kernel diagonal in (2.35) is reproduced. However, the absence of a 'bottom' in the system of off-diagonal coefficients $\hat{b}_{m,n}(F|x,x')$ makes it impossible to construct recursive relations for them.

We have nevertheless developed and implemented two algorithms for obtaining off-diagonal coefficients with the symbolic computing system *Wolfram Mathematica*; the algorithms are based on the Fourier transform (Section 3.4) and on the perturbation theory (Section 3.5). Both algorithms provide more than the classical DeWitt method does in the sense that they allow obtaining not only the coincidence limits $[\hat{b}_{m,n}]$ but also closed expressions for the coefficients with $x \neq x'$. Of course, the act of going beyond the diagonal in and of itself does not provide any gain in computational efficiency and in fact leads to a significant increase in complexity. However, it clarifies the general off-diagonal structure of the heat kernel, whose careful consideration should allow the calculation of coincidence limits to be streamlined.

Another advantage that comes from going off the diagonal is the possibility of a more flexible use of integral transformations, which generalize the technique used in Section 3.3 with the term-by-term application of the direct and inverse Mellin transformations. The 'generalized functoriality' statement can be loosely formulated as follows: for a wide class of 'good' operators $\hat{F}$ and a wide class of 'good' functions f, the off-diagonal expansion of the kernel of the operator function $f(\hat{F})$ has the form

$$f(\hat{F})\,\frac{\hat{1}}{\sqrt{g}}\,\delta(x,x') = \sum_m \mathbb{K}_m(f|\sigma)\,\hat{a}_m(F|x,x')\,,$$

where the matrix-valued two-point off-diagonal coefficients $\hat{a}_m(F|x,x')$ depend only on the operator $\hat{F}$ and the space–time geometry but not on the function f; the scalar basis kernels $\mathbb{K}_m(f|\sigma)$, conversely, depend only on the form of f but not on the operator $\hat{F}$ or the geometry. We plan to publish our results in this area in a separate paper [76], which is currently in preparation.

In addition, both algorithms, the Fourier transform and the perturbation theory, can be relatively easily generalized to a much wider class of so-called causal operators (which can be defined, for example, as those that give a minimal operator when multiplied by some other operator). These new results will be presented in a forthcoming paper [81].

Although the method of off-diagonal expansions, which generalizes the classical DeWitt technique, is still taking its very first steps, the results obtained to date are sufficient to conclude, without overstatement, that its appearance opens up new prospects both in the study of the heat kernel and in a wide range of its potential applications, including QFT and studies of modified gravity models.

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[10] It seems that the perturbation theory method is better in this regard than the Fourier transform, because it is based on the DeWitt expansion for a Laplace-type operator $\hat{H}(\nabla)$, in which case no spurious coefficients appear.

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