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Relativistic-nonlinear resonant absorption and generation of harmonics of electromagnetic radiation in an inhomogeneous plasma

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Contents

Abstract. Since the pioneering work of V L Ginzburg and N G Denisov, who discovered the effect of linear absorption of electromagnetic radiation in an inhomogeneous plasma due to strong plasma resonance at near-critical density, the theory of this phenomenon has not been developed steadily in application to strong electromagnetic fields of practical interest today, primarily laser ones. This review is devoted to a systematic presentation of the results of an analytic relativistic-nonlinear theory of resonant absorption and generation of laser radiation harmonics in an inhomogeneous plasma with a strongly manifested plasma electron nonlinearity up to laser light intensities that make the motion of electrons in the vicinity of the critical density relativistic. Using the methods of the modern theory of transformation groups, we describe the structure of the nonlinear electromagnetic field in a plasma resonance, nonlinear effects of the suppression of the resonant absorption coefficient and the angle shift of the resonance absorption curve with increasing laser intensity, the formation of sufficiently smooth spectra of laser light harmonics, decreasing in accordance with a power law, the spectrum of laser light harmonics emitted by plasma near the breaking threshold of a nonlinear plasma field, and the basic properties of the electrostatic field generated in the vicinity of the critical density.

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1. Introduction

About seventy years ago, a systematic study of the phenomenon of *plasma resonance* was initiated in the theoretical work of Denisov and Ginzburg [1–3]. Since then, the classic effect of a *linear transformation* of an electromagnetic wave into a plasma wave, observed under oblique incidence of p-polarized radiation with a frequency ω_0 on a weakly inhomogeneous plasma, has become well known in the theory of the interaction of electromagnetic (laser) radiation with an inhomogeneous plasma.[1] It occurs in the region where the laser frequency ω_0 is comparable to the natural oscillation frequency of the plasma ω_L. The resonance of electron oscillations in the laser radiation field with Langmuir electron oscillations leads to an increase in electrostatic oscillations, i.e., to plasma resonance, which manifests itself as a sharp increase on the potential electric field in the vicinity of the critical plasma density [2–5]. The study of plasma resonance is of fundamental and applied interest because it underlies the development of processes such as *resonance absorption* [6, 7], the *generation of harmonics* [8–16], the *generation of quasistatic fields* near the critical density [14, 17], and the formation of *fast particles* in laser plasma [18–26]. These processes, in turn, are of special importance in light of the global problem of inertial confinement fusion (ICF) [27–36], which remain relevant.

Resonance absorption (RA) is one of the most important channels to transfer laser radiation energy into an inhomogeneous plasma. It makes a significant, and sometimes decisive, contribution to the absorption coefficient. From a practical standpoint, RA studies deal with problems of the interaction of laser radiation with plasma formed as a result of laser ablation of solid targets, and mainly with ICF experiments [31–34], as well as the problem of harmonic generation [10, 11, 14]. The linear theory of RA, developed in the 1960s–1970s [6, 37] based on the model proposed in [1], still underlies many calculations and estimates performed in the framework of ICF experiments [31]. However, the natural logic of the development of laser technologies has made it routine to use lasers with an intensity that makes the applicability of the linear theory of RA doubtful.

Modern experiments on ICF with direct irradiation, including the concept of shock ignition, are characterized by large plasma inhomogeneity scales (tens and hundreds of laser wavelengths) and laser intensities up to 10^{17} W cm^{-2}. Under such conditions, manifestations of strong nonlinear effects at a near-critical plasma density are possible [14, 38–44], and hence there is a need to significantly improve the theory of absorption of intense laser radiation in the critical density domain of inhomogeneous plasma, where the phenomenon of *relativistic-nonlinear plasma resonance* occurs [43, 44]. The fact is that, at a plasma resonance under ICF conditions, an increase in the plasma field can give rise to the electric field strength reaching relativistic magnitudes, although the intensity of the incident laser field remains significantly lower than the relativistic one [40, 41, 43, 44]. Such a strong nonlinear resonance has a significant impact on the absorption process [44-46], which means that the task of constructing the theory of RA under conditions of relativistic plasma resonance is very relevant.

The strong charge separation that occurs in the vicinity of the plasma resonance generates a powerful *quasistatic electric field* localized near the critical density [14, 17]. Already in the second half of the 1970s, a number of experimental facts were known that indicated the generation of strong quasistatic electric fields in plasma under the effect of high-power laser radiation [47, 48]; studying them was of interest from the standpoint of the processes of transport [49] and acceleration of particles in plasma [21]. In particular, the formation of *fast electrons* in the critical density region deserves attention as an undesirable effect in the ICF context, because it provides preliminary heating of the target, which hampers its optimal compression and hence further initiation of the thermonuclear reaction [20, 23, 24, 30, 33, 50, 51]. Therefore, studying the spatial structure of quasistatic electric fields excited in the vicinity of the critical plasma density is of interest and can serve as the basis for a theoretical interpretation of the observed generation of fast electrons.

Harmonic generation (HG) of laser radiation is another important process that develops under plasma resonance conditions and has attracted the attention of researchers as a tool for obtaining high-frequency sources of secondary radiation [15] and for diagnosing the laser plasma corona (including in ICF experiments), i.e., the possibility of determining the plasma density, temperature gradient, and corona velocity from the spectral composition of secondary radiation [52, 53]. Since the 1970s, numerous experimental and theoretical studies have been carried out around the world on the generation of harmonics whose characteristics can be used to determine the indicated parameters in localized regions of the plasma: near the critical and quarter-critical density [12, 52, 53]. For example, recent experiments at the NIF (National Ignition Facility) carried out within the direct compression scheme [34] used the spectra of the $\omega_0/2$ subharmonic to determine the electron temperature of dense laser plasma. From a theoretical standpoint, the description of HG in the regime of a strongly nonlinear plasma resonance, observed in classical experiments [54, 55], is of particular interest. The slowly decaying spectra of secondary radiation obtained in [54, 55] are not described by the perturbation theory [11], which predicts an exponential decrease in the intensities of harmonics with an increase in the harmonic number. Going beyond the perturbation theory was first achieved in [14], where the analytically found coefficients of conversion to higher harmonics demonstrated a rich spectral composition of secondary radiation, with a strong nonrelativistic nonlinearity of electron motion in the vicinity of the critical density. Study [14] was continued in the last decade [44, 56, 57]; there, the problem of the relativistic effects exerted on HG by the dynamics of the electron component of plasma in the vicinity of plasma resonance was solved.

Modern experiments on the interaction of laser radiation with plasma at high laser intensities deal with the strong nonlinearity of electromagnetic fields. The key role of nonlinearity was noted in the first theoretical studies on the interaction of powerful electromagnetic radiation with inhomogeneous plasma (see, e.g., [58, 59]) and was subsequently confirmed in laboratory experiments and studies of ionospheric plasma [60–63]. Plasma inhomogeneity can be caused both by the action of a strong electromagnetic field during self-focusing of laser radiation [58, 60] and by natural plasma inhomogeneity, e.g., during ionospheric experiments [61–63], and can also result from the combination of these two effects [63]. Because strong nonlinearity significantly complicates the task of giving an analytic description, nonlinear processes, for example, RA, are typically studied numerically [39, 40, 64–66] or using semi-analytic models [38, 41] in which equations that are simplified compared with the original ones are first obtained by analytic methods and then solved numerically. Fully numerical approaches often involve a kinetic description based on the particle-in-cell (PIC) method [39, 40, 64, 65]. However, despite the rapid development of numerical methods widely used in high-energy laser physics, analytic approaches can still be an

[1] Below, we clarify the condition for weak plasma inhomogeneity in terms of the laser–plasma parameters.

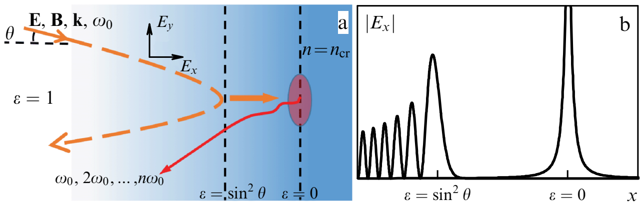

Figure 1. (a) Schematic representation of oblique incidence of a p-polarized plane electromagnetic wave at an angle θ onto a plasma that is inhomogeneous along the x-axis. At the reflection point, the dielectric constant takes the value $\varepsilon(x) = \sin^2\theta$; at the point of plasma resonance, plasma density n is equal to critical value $n_{\rm cr}$ and the dielectric constant vanishes: $\varepsilon(x) = 0$. (b) Qualitative picture of spatial distribution of longitudinal electric field amplitude in an inhomogeneous plasma whose density varies along the x coordinate, with an oblique incidence of a p-polarized electromagnetic wave without taking small dissipation effects in the vicinity of the critical density into account. At the reflection point, $\varepsilon(x) = \sin^2\theta$, and at the plasma resonance point, $\varepsilon(x) = 0$.

effective way to study processes associated with strong nonlinearity.

We note that numerical calculations of the absorption and generation of laser radiation harmonics by an inhomogeneous plasma face a problem associated with the different scales of modeling domains when calculating the self-consistent electromagnetic field in a plasma. On the one hand, to resolve a narrow resonance region, which is tenths of the laser wavelength in width, a small step on a space-time grid is needed. On the other hand, when calculating fields in the bulk of a plasma target, determined by the density inhomogeneity scale, which is tens and hundreds of laser wavelengths, a large number of such steps are necessary, which critically increases the resource consumption of the computation. In addition, the use of numerical methods makes it difficult to obtain practically necessary scalings from laser–plasma parameters due to the multiparameter dependence of numerical calculations, because nonlinear absorption depends on the laser pulse intensity, the angle of incidence on the plasma, the geometry and density of the plasma target, the expansion velocity, and temperature [40, 67]. In this regard, theoretical approaches are important, allowing one to obtain analytic solutions whose value lies not only in the fact that they provide a visual description of processes dependent on numerous parameters of the problem but also in the fact that they can be used in constructing and justifying numerical and analytic methods as model solutions that highlight the key properties of the system under consideration.

In this review, we present the results of using the method of renormalization group (RG) symmetries based on the modern theory of transformation groups [68–70]; the method emerged from developing ideas on functional self-similarity [71] and Bogoliubov's quantum-field RG [72–75] considered as a group of continuous Lie transformations. The use of this method has allowed constructing a self-consistent analytic solution of a highly nonlinear problem of the interaction of laser radiation with an inhomogeneous plasma. This solution has been used to describe the structure of the singularity of the nonlinear electromagnetic field in plasma resonance [42, 43, 76], to solve the problem of nonlinear RA [44–46], to find the intensities of higher harmonics generated in a resonance region and emitted into a vacuum [44, 56, 57], and to characterize the properties of a quasistatic electric field [45] generated in the vicinity of the critical density of an inhomogeneous laser plasma.

The review is divided into five parts, including three more sections in addition to the introduction and conclusions. In Section 2, the structure of nonlinear plasma resonance is found and studied in detail, and the mechanism of breaking relativistic plasma oscillations in the vicinity of the critical plasma density and the spatial properties of the longitudinal electric field of plasma resonance are considered. Section 3 is devoted to finding the relativistic-nonlinear RA coefficient, studying the applicability limits of the hydrodynamical model of plasma resonance in terms of the physical parameters of the laser–plasma system, and discussing the properties of the quasistatic electric field. In Section 4, we find the conversion coefficients to higher harmonics of laser radiation, study the spectral composition of secondary radiation, and discuss the relation to previous theories of HG by the plasma resonance mechanism.

2. Nonlinear plasma resonance in inhomogeneous plasma

Solutions of the equations describing oblique incidence of electromagnetic waves on a weakly inhomogeneous plane-layered isotropic plasma were first studied in detail by Zhekulin [77, 78] in relation to the problem of radio wave propagation in the ionosphere. In such a medium, with the density and the dielectric constant dependent on one spatial coordinate, waves with different polarizations of the electric vector (perpendicular and parallel to the plane of incidence) propagate independently of one another. Moreover, the problem of reflection of an s-polarized wave, whose electric vector is perpendicular to the plane of incidence, is not fundamentally different from the simplest problem of normal incidence [3, 79]. In front of the reflection (turning) point, an oscillatory structure of a standing electromagnetic wave then forms as a result of the superposition of the incident and reflected waves; behind the reflection point, the field decays exponentially into the plasma. The only difference is the shift in the point of reflection of the incident wave. A different picture is observed in the case of p-polarization of the incident electromagnetic wave, whose electric vector lies in the plane of incidence (Fig. 1). In that case, a standing wave is again formed, which, after passing the reflection point, first attenuates in the bulk of the plasma, but the point where the medium dielectric constant ε vanishes is singular (Fig. 1b): Zhekulin showed that the requirement for the solution of the wave equation to vanish at infinity (in the region of negative values of ε) leads to the field becoming infinite at the point where $\varepsilon = 0$. He did not, however, suggest a suitable strategy to eliminate the singularity.

Later, Försterling and Wüster [80, 81] gave a more detailed analysis of this singularity of the field occurring in the course of the propagation of an electromagnetic wave in an inhomogeneous plasma. Using approximate solutions applicable in a small neighborhood of a zero of ε, they found that the longitudinal component (the one along the plasma inhomogeneity gradient) of the electric field has a $1/\varepsilon$ singularity, and the transverse component has a logarithmic singularity. Hence, it was concluded that a sharp increase in the electric field strength in the region where $\varepsilon \to 0$ makes it impossible to describe the field using the standard dielectric constant, because the motion of electrons in the field with a strong spatial inhomogeneity is no longer harmonic. Försterling and Wüster were the first to demonstrate that the equations for the field become nonlinear under such conditions, and whenever a wave of a certain frequency propagates in an inhomogeneous plasma, waves of other (multiple) frequencies — *higher harmonics* — arise [81].

In 1956, an article by Denisov, "On a singularity of the field of an electromagnetic wave propagating in an inhomogeneous plasma" [1], was published, which in many ways became fundamental in the theory of radio wave propagation in plasma and to the ICF problem. Developing the ideas of predecessors [77, 78, 80, 81], the structure of the electromagnetic field with a singularity was studied in [1] in detail within a linear theory (in the incident wave amplitude) based on Maxwell's equations and the equations of collisionless hydrodynamics for electrons. The physical nature of the electric field singularity was clarified for the first time. The crucial point is that longitudinal plasma oscillations can build up in an inhomogeneous plasma due to a *resonance* of electron oscillations in the field of an electromagnetic wave with a frequency ω_0 and natural oscillations of the plasma with the Langmuir frequency $\omega_{\rm L}$. A *plasma resonance* occurring at $\varepsilon = 0$, i.e., at the coincident frequencies ω_0 and $\omega_{\rm L}$, leads to an increase in electrostatic oscillations and to a sharp increase in the potential electric field in the vicinity of the critical plasma density (Fig. 1b). Thus, in [1], a qualitative description was given of the role of plasma resonance in the specific process of absorption of electromagnetic radiation by an inhomogeneous plasma, which was subsequently called *resonance absorption*. In addition, the field amplification coefficient was calculated for the first time and a finite value of the electric field amplitude of the plasma resonance was shown to be ensured either by collisional dissipation of plasma oscillations in the case of cold plasma or by the escape of plasma waves when thermal effects are taken into account in the case of hot plasma [1–3]. Subsequently, the results in [1] served as the basis for the development of various theories pertaining to ionospheric and laser plasmas. In ionospheric plasma, these results underlie research (both theoretical and experimental) on the linear transformation and absorption of waves [82], on the formation of artificial ionospheric irregularities elongated along the magnetic field [83, 84], on the creation of an artificially ionized region in the stratosphere [85], and on nonlinear phenomena in plasma resonance in the ionosphere [63], including the kinetic theory of plasma turbulence with the formation of cavitons — plasma density pits that cause the acceleration of electrons. In laser plasma, study [1] contributed to the development of the theories of HG [8–16], quasistatic fields [14, 17], RA [6, 7], and the generation of fast particles from the critical density region of inhomogeneous laser plasma [18–26], which is the subject of many studies by Russian and foreign researchers. Thanks to steady improvement in the technology of laser-physics experiments, a progressively greater intensity of laser radiation was being attained, and it hence became necessary to take nonlinear plasma oscillation effects into account in the vicinity of the resonance.

Strongly nonlinear nonrelativistic plasma oscillations were first studied by Akhiezer and Lyubarskii [86] in analyzing stationary nonlinear Langmuir oscillations in the form of traveling waves that are solutions of the equations of cold collisionless hydrodynamics for a spatially homogeneous electron plasma with a neutralizing ion background. When the finite (but still essentially nonrelativistic) speed of the electrons is taken into account, the frequency of nonlinear oscillations is independent of the velocity amplitude and is determined by the classical Langmuir formula. Generalizing the method used in [86] to the case of arbitrary electron velocities, Akhiezer and Polovin [87] studied various regimes of longitudinal and transverse electron oscillations in plasma in detail. They found, in particular, that the relativistic motion of electrons gives rise to a dependence of the frequency of longitudinal oscillations on the velocity amplitude, namely, a decrease in frequency with increasing velocity due to a relativistic change in the particle 'mass.'[2] Thus, as the speed of a traveling wave approaches the speed of light, $v \to c$, the frequency tends to zero, because the electron mass tends to infinity.

The results in [86, 87] implicitly indicated a limit, determined by the nonlinearity itself, on the amplitude of oscillations of the electron component of the plasma. This was pointed out by Dawson in [88], where, in the language of the derivative of the Lagrangian displacement of plasma electrons relative to their initial positions, a wavebreaking criterion for the profile of the nonlinear plasma wave was found when the oscillation amplitude exceeds a certain critical value. Later, in [89], the stationary structure of longitudinal electron oscillations and the conditions for their breaking in plasma were studied based on solving the system of Vlasov–Poisson equations using a simple 'water bag' type of model distribution of electron velocities on the background of stationary ions. It was shown that the limit amplitude of oscillations corresponding to breaking monotonically decreases with an increase in the ratio of the thermal velocity of electrons to the phase velocity of the wave.

Stationary (quasistationary) regimes of nonrelativistic plasma oscillations in the vicinity of a plasma resonance were studied, based on the Ginzburg–Denisov plasma resonance model, in work on the generation of harmonics of p-polarized laser radiation incident on an inhomogeneous plasma [8, 10, 11, 13, 14]. In contrast to studies [8, 10, 11, 13] carried out within the weakly nonlinear theory, Kovalev and Pustovalov [14] developed a nonperturbative approach that allowed taking the strong nonlinearity (in the electric field amplitude) of the motion of electrons into account, albeit with the relativistic effects ignored. Using the method of RG symmetries [68], a stationary nonlinear structure of the electromagnetic field in the plasma resonance region was found and used to calculate the amplitudes of the harmonics of the electromagnetic wave incident on inhomogeneous plasma. A generalization of the breaking condition [88] to the strongly nonlinear nonrelativistic case of Langmuir oscillations in the vicinity of the critical density was also formulated. The approach to describing nonlinear plasma oscillations beyond the perturbation theory proposed in [14] was used in [42, 43, 76] to reconstruct the structure of the electric field with the relativistic effects of electron motion at a near-critical plasma density taken into account. In presenting the results of the theory of nonlinear plasma resonance, we mainly follow [42, 43, 76].

In relation to the problems of particle acceleration and plasma heating, the nonrelativistic dynamics of the electron component of the plasma in the vicinity of the critical density was discussed in [18, 21, 22] in a different setting, the one

[2] The use of quotation marks here and below emphasizes that we are talking about the changes in the laws of dynamics at high speeds, but with the stipulation that mass, being a four-dimensional scalar, is a relativistic invariant.

corresponding to the so-called 'capacitor model.' In this model, forced oscillations of an inhomogeneous plasma layer placed in an external uniform high-frequency electric field are considered. Due to plasma inhomogeneity [21, 88], the process of electron oscillations is nonstationary, giving rise to the breaking of the plasma wave profile in a finite time. The calculation of the maximum amplitude of forced Langmuir oscillations corresponding to the breaking instant, carried out in [18] for a cold plasma, was continued in [90] for a plasma with a finite temperature of electrons whose distribution function has a 'water bag' shape.

Taking the relativistic nonlinearity of electron dynamics into account significantly complicates the problem of analytic research, and relativistic plasma oscillations in inhomogeneous plasmas are therefore studied mainly by numerical modeling methods, e.g., as in [40, 91, 92] or in the framework of approximate and semianalytic models [41, 93–95]. In the numerical modeling of plasmas, both the hydrodynamic description using the Lagrangian formalism [91, 92] and the kinetic approach based on the PIC method [40] are used. When studying the nonlinear evolution of relativistic plasma waves [93], the solution to the initial value problem exhibits a steepening of the wave front with time, resulting in its breaking.

2.1 Construction of strongly nonlinear solutions of plasma resonance theory equations using the method of renormalization group symmetries

To describe the dynamics of electrons and the structure of the electromagnetic field near the critical density and, in subsequent sections, also the processes of nonlinear absorption and generation of harmonics of a p-polarized electromagnetic wave incident on a weakly inhomogeneous plasma along the x coordinate at an angle θ and characterized by the electric $\mathbf{E}$ and magnetic $\mathbf{B}$ fields with a frequency ω_0,

$$\mathbf{B} = \frac{1}{2}\left\{0, 0, B_0(x)\right\}\exp\left(\mathrm{i}k_y y - \mathrm{i}\omega_0 t\right) + \text{c.c.}\,,$$
$$\mathbf{E} = \frac{1}{2}\left\{E_{0x}(x), E_{0y}(x), 0\right\}\exp\left(\mathrm{i}k_y y - \mathrm{i}\omega_0 t\right) + \text{c.c.}\,, \tag{2.1}$$
$$k_y = k_0\sin\theta\,, \quad k_0 = \frac{\omega_0}{c}\,,$$

we take the initial equations to be the equations of collisionless hydrodynamics of cold relativistic electron plasma and Maxwell's equations:

$$\partial_t\mathbf{p} + (\mathbf{v}\partial_{\mathbf{r}})\mathbf{p} = e\left(\mathbf{E} + \frac{1}{c}\,[\mathbf{vB}]\right), \quad \partial_t n_\mathrm{e} + \operatorname{div}(n_\mathrm{e}\mathbf{v}) = 0\,,$$
$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\,\partial_t\mathbf{B}\,, \quad \operatorname{rot}\mathbf{B} = \frac{1}{c}\,\partial_t\mathbf{E} + \frac{4\pi}{c}\,en_\mathrm{e}\mathbf{v}\,, \quad \operatorname{div}\mathbf{B} = 0\,,$$
$$\operatorname{div}\mathbf{E} = 4\pi(en_\mathrm{e} + e_\mathrm{i}n_\mathrm{i})\,, \quad \mathbf{p} \equiv m\mathbf{v}\gamma = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}\,. \tag{2.2}$$

Here, m and e are the electron mass and charge, c is the speed of light in a vacuum, n_e, $\mathbf{v}$, and $\mathbf{p}$ are the density, speed, and momentum of plasma electrons, and $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field strengths of p-polarized electromagnetic radiation with the electric field component in (2.1) directed along the plasma inhomogeneity gradient. Ions with a fixed density n_i are considered a stationary neutralizing background that corresponds to the electron plasma approximation used in this paper, and the effects of thermal motion and collisions of electrons that are not included in Eqns (2.2) are assumed to be small, although their role in the regularization of the plasma resonance singularity is taken into account, in accordance with the procedure for constructing a nonlinear solution.

Assuming the x and y components of the electron velocity and the electric field, as well as the z component of the magnetic field, to be nonzero, after eliminating the electron density, we obtain the following system of equations from (2.2):

$$\begin{gathered}
\gamma(\partial_t v + av\partial_x v + au\partial_y v) + \frac{a^2\gamma^3}{c^2}\left[vu(\partial_t u + av\partial_x u + au\partial_y u)\right.\\
\left. + v^2(\partial_t v + av\partial_x v + au\partial_y v)\right] = P + \frac{au}{c}\,R\,,\\
\gamma(\partial_t u + av\partial_x u + au\partial_y u) + \frac{a^2\gamma^3}{c^2}\left[vu(\partial_t v + av\partial_x v + au\partial_y v)\right.\\
\left. + u^2(\partial_t u + av\partial_x u + au\partial_y u)\right] = Q - \frac{av}{c}\,R\,,\\
\partial_t P + av\partial_x P + av\partial_y Q - c\partial_y R + \omega_\mathrm{L}^2 v = 0\,,\\
\partial_t Q + au\partial_x P + au\partial_y Q + c\partial_x R + \omega_\mathrm{L}^2 u = 0\,,\\
\partial_t R + c\partial_x Q - c\partial_y P = 0\,.
\end{gathered} \tag{2.3}$$

Here and hereafter, we use the notation $\partial_{x^i} \equiv \partial/\partial x^i$ for the partial derivative with respect to x^i $(i = 1, \ldots, n)$. The functions $v = v_x/a$ and $u = v_y/a$ describe the normalized values of the electron velocity components; $\gamma = 1/\sqrt{1 - (a^2/c^2)(v^2 + u^2)}$; $P = eE_x/ma$, $Q = eE_y/ma$, and $R = eB_z/ma$ are normalized values of the components of the electric $\{E_x, E_y, 0\}$ and magnetic $\{0, 0, B_z\}$ fields, where $a = -2e|B_1(0)|\sin\theta/m\omega_0^2 L$ is a dimensionless constant proportional to the amplitude of the magnetic field $|B_1(0)|$ at the plasma resonance point $x = 0$; and $B_1(0)$ is a complex amplitude of the Fourier component of the magnetic field at $x = 0$ at the laser frequency ω_0:

$$B_1(0) = \left|B_1(0)\right|\exp\left[\mathrm{i}\arg B_1(0)\right] = \frac{m\omega_0^2 La}{2|e|\sin\theta}\exp\left[\mathrm{i}\arg B_1(0)\right]. \tag{2.4}$$

The electron Langmuir frequency $\omega_\mathrm{L} \equiv \omega_\mathrm{L}(x) = (4\pi e^2 n_0/m)^{1/2}$ of a plasma with the density $n_0(x) = n_\mathrm{i}e_\mathrm{i}/|e|$ is approximated in the vicinity of the plasma resonance point $x = 0$ by a linear dependence on the x coordinate, $n_0(x) = (1 + x/L)n_\mathrm{c}$, where $n_\mathrm{c} = m\omega_0^2/4\pi e^2$ is the critical density and L is the characteristic density inhomogeneity scale, defined as the ratio of the electron density n_e to its gradient calculated at the critical density: $L = |n_\mathrm{e}/\nabla n_\mathrm{e}|_{n_\mathrm{e}=n_\mathrm{c}}$. In the case of a smooth gradient of plasma inhomogeneity, $L \gg \delta, 1/k_0$ (where δ is the plasma resonance width), the linear dependence on the x coordinate is valid in the vicinity of the resonance for any monotonic density profile. The condition of weak plasma inhomogeneity $k_0 L \gg 1$ is also critically important for the effective RA.

The p-polarization of an electromagnetic wave incident on the plasma considered here corresponds to a nonzero projection of the electric field onto the direction of the spatial inhomogeneity gradient in the plasma. In this case, in the vicinity of $x = 0$, where the frequency of electron oscillations in the laser radiation field is equal to the frequency of Langmuir electron oscillations of the plasma, $\omega_0 = \omega_\mathrm{L}$, a resonant increase in the longitudinal electric field occurs, and part of the energy of the incident electromagnetic wave is transferred to longitudinal plasma fluctuations. Therefore, repeating the reasoning in [8, 14], we take into account that the largest contribution to the nonlinear effects of plasma resonance is made by the x component of the electric field and electron velocity:

$$v_x \gg v_y \,, \quad E_x \gg E_y \gg B_z \,. \tag{2.5}$$

To obtain equations for the longitudinal resonantly amplified plasma field, we select two equations involving the x-components of the electric field and the electron velocity from system (2.3), using the hierarchy of fields and velocities (2.5) in the vicinity of the critical density. We then arrive at a pair of nonlinear first-order partial differential equations for the x components of the normalized electric field P and the electron velocity v near the plasma resonance,

$$\begin{aligned} &\partial_t v + av\partial_x v = P\gamma_0^{-3} \,, \\ &\partial_t P + av\partial_x P = -\omega_0^2 v \,, \end{aligned} \tag{2.6}$$

where $\gamma_0 = (1-\beta v^2)^{-1/2}$ and $\beta = a^2/c^2$. The equations for the transverse components of velocity u and electric field Q are found similarly:

$$\begin{aligned} &\partial_t(u\gamma_0) + av\partial_x(u\gamma_0) = Q \,, \\ &\omega_0\partial_x Q + k_y\partial_t P = 0 \,. \end{aligned} \tag{2.7}$$

It is assumed in Eqns (2.6) and (2.7) that $\omega_{\rm L} = \omega_0$, which means that we neglect the dependence of the frequency $\omega_{\rm L}$ on the x coordinate, which is justified in the case of a weakly inhomogeneous plasma. We are interested in analytic solutions of Eqns (2.6) and (2.7) in an arbitrary order of nonlinearity, i.e., beyond the perturbation theory.

We first consider Eqns (2.6) and use their group properties to find a solution. We note that these equations involve two parameters, a and β, corresponding to the contributions of the electron nonrelativistic (or convective, associated with the term $av\partial_x$) and relativistic (associated with the term $P\gamma_0^{-3} = P(1-\beta v^2)^{3/2}$) nonlinearities. Generally speaking, the measure of relativistic nonlinearity is the parameter $\beta \sim 1/c^2$, which tends to zero in the nonrelativistic limit $c \to \infty$. However, the choice of the relativism parameter in the form $\beta = a^2/c^2$ (which arises naturally after the transition to dimensionless variables) is dictated by the significant simplification of subsequent calculations in that case. At $a = 0$ and $\beta = 0$, Eqns (2.6) transform into equations that describe linear plasma resonance and have well-known solutions [1, 3, 37], corrections to which can be found by constructing the perturbation theory series in a and β. Using the symmetry properties of the equations, solutions found using the perturbation theory can be related into solutions for finite a and β. This approach is based on the method of RG symmetries, well known in theoretical physics [68]; the symmetries are sought using algorithms of modern group-theory analysis [96, 97].

The gist of the method is as follows. At the first stage, we calculate the widest group of point transformations allowed by the original equations (in our case, Eqns (2.6)) in the space of all dependent and independent variables and the parameters included in the equations. Next, using the procedure of group restriction to a particular solution, a finite-dimensional subgroup of the admitted group is identified, under which the solution of the original system obtained within the perturbation theory with respect to the chosen parameters remains invariant. Finally, the use of finite transformations specified by this subgroup allows relating solutions for small values of parameters, i.e., perturbative solutions, to highly nonlinear solutions that correspond to finite values of these parameters.

The infinite group of continuous point transformations in the space of six variables, a, β, t, x, v, and P, allowed by Eqns (2.6) is defined by the infinitesimal operator (the group generator)

$$X = \xi^1\partial_t + \xi^2\partial_x + \xi^3\partial_a + \xi^4\partial_\beta + \eta^1\partial_v + \eta^2\partial_P \tag{2.8}$$

with coordinates ξ^i, $i = 1, \ldots, 4$, and η^j, $j = 1, 2$, which are functions of the variables t, x, v, and P and the parameters a and β, and which are determined in accordance with the standard procedure of group theory analysis [96, 97] (see Appendix A and [42, 43, 76]). A perturbative solution of system (2.6) can be constructed using two parameters a and β, and the procedure for restricting[3] the infinite group leads to RG transformations that extend this solution to the region of finite values of a and β. Omitting cumbersome calculations, we present the resultant RG transformations defined by two infinitesimal operators

$$\begin{aligned} &R_1 = \xi^2_{\rm RG}\partial_x + \partial_a \,, \\ &R_2 = \xi^1_{\rm RG}\partial_t + \partial_\beta + \eta^1_{\rm RG}\partial_v \,, \end{aligned} \tag{2.9}$$

with the coordinates $\xi^1_{\rm RG}$, $\xi^2_{\rm RG}$, and $\eta^1_{\rm RG}$ given by

$$\begin{aligned} \xi^1_{\rm RG} = \frac{1}{\beta}\Bigg[&\frac{(6+\beta I_1)\,{\rm E}(\mu,\sigma)}{2\sqrt{4+\beta I_1}} - \frac{{\rm F}(\mu,\sigma)}{\sqrt{4+\beta I_1}} \\ &+ \frac{\sqrt{\beta}P/\omega_0}{4+\beta I_1}\sqrt{\frac{1/\sqrt{z}-1}{1+1/\sqrt{z}}}\Bigg]\,, \\ \xi^2_{\rm RG} = -\frac{P}{\omega_0^2}\,,\quad &\eta^1_{\rm RG} = -\frac{z^{3/2}}{2v\beta^2}\left(2 + \frac{1}{z^{3/2}} - \frac{3}{\sqrt{z}}\right), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} &\mu = \arcsin\frac{P/\omega_0}{\sqrt{I_1}}\,, \quad \sigma = \sqrt{\frac{\beta I_1}{4+\beta I_1}}\,, \\ &I_1 = \frac{2}{\beta}\left(\frac{1}{\sqrt{z}} - 1\right) + \frac{P^2}{\omega_0^2}\,. \end{aligned} \tag{2.11}$$

Here, $z \equiv 1 - \beta v^2$, and ${\rm F}(\mu,\sigma)$ and ${\rm E}(\mu,\sigma)$ are incomplete elliptic integrals of the respective first and the second kind [98, 99],

$$\begin{aligned} &{\rm F}(\mu,\sigma) = \int_0^\mu \frac{{\rm d}x}{\sqrt{(1-\sigma x^2)(1-x^2)}}\,, \\ &{\rm E}(\mu,\sigma) = \int_0^\mu \frac{\sqrt{1-\sigma x^2}}{\sqrt{1-x^2}}\,{\rm d}x\,. \end{aligned} \tag{2.12}$$

Calculating the commutator of the operators R_1 and R_2, we can easily verify that $[R_1, R_2] = 0$, i.e., the group generated by them is Abelian. Therefore, the finite group transformations corresponding to R_1 and R_2 on the plane $\{a, \beta\}$, given by integrating the Lie equations, can be regarded as a sequence of two independent steps with the respective group transformation parameters a and β, as shown in Fig. 2. The arrow

[3] See [68] for details on the procedure of restriction.

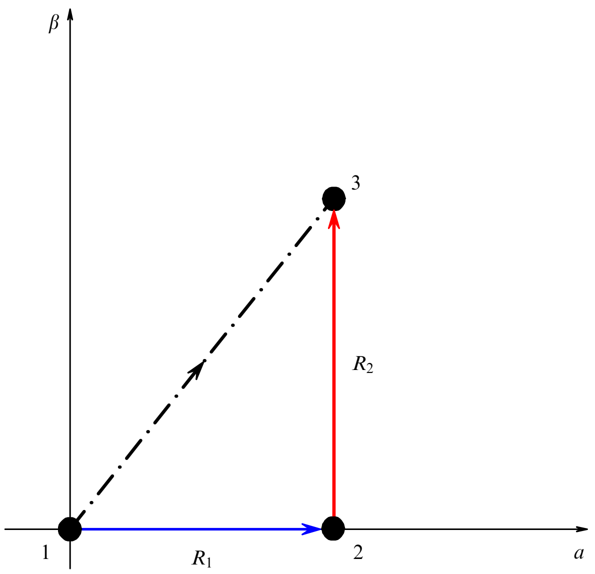

Figure 2. RG transformation using a two-parameter RG on the plane of transformation parameters $\{a, \beta\}$, implementing the transition between theories of plasma resonance. Point 1 corresponds to the linear theory, point 2, to the highly nonlinear nonrelativistic theory, and point 3, to the nonlinear relativistic theory.

from point 1 to point 2 denotes a transformation relating the solution of a linear problem with a nonlinear nonrelativistic solution, and the arrow from point 2 to point 3 corresponds to the transition from a nonlinear nonrelativistic to a relativistic solution. Because R_1 does not contain differentiation operators with respect to the time t, the speed v, and the field P, whereas R_2 does not contain differentiation operators with respect to the coordinate x and the electric field P, it follows that these quantities are invariants of the transformations corresponding to R_1 and R_2. At the first step, determined by the action of R_1, i.e., by solving the corresponding Lie equation

$$\frac{\mathrm{d}x}{\mathrm{d}a} = \xi^2_{\mathrm{RG}} , \quad x\big|_{a=0} = \eta , \tag{2.13}$$

we find the finite transformations for the time t, coordinate x, field P, and speed v,

$$\begin{aligned} &t_{\mathrm{I}} = t_{\mathrm{linear}} , \quad x_{\mathrm{I}} = \eta - a\omega_0^{-2} P_{\mathrm{I}} , \\ &P_{\mathrm{I}} = P_{\mathrm{linear}}(\chi, \eta) , \quad v_{\mathrm{I}} = v_{\mathrm{linear}}(\chi, \eta) , \end{aligned} \tag{2.14}$$

where $\chi = \omega_0 t_{\mathrm{linear}}$ and $\eta = x(0, \eta)$ denotes the coordinate $x = x(a, \eta)$ at a zero value of the parameter a. Formulas (2.14) determine the transition from a solution obtained in the linear theory to one that takes electron nonlinearity into account, but without relativistic effects. The invariance of the time t, speed v, and field P is manifested in the fact that they are determined by the linear theory formulas (which is indicated by the subscript 'linear'). Thus, the nonlinear structure of the field and the electron speed depends on the form of the electric field P_{linear}, which is a solution of a linearized system of equations corresponding to (2.6). When choosing a linear solution that can be extended by the RG transformation to the region of finite values of the nonlinearity parameters, we use the result obtained for cold electron plasma with a linear density profile [1, 37]. In this case, the field at the point $x = 0$ has a $1/\varepsilon$ singularity (with $\varepsilon = 1 - \omega_{\mathrm{L}}^2/\omega_0^2$ being the dielectric constant of the plasma, which vanishes at the resonance $\omega_0 = \omega_{\mathrm{L}}$), which is eliminated, however, when collisions or buildup of longitudinal plasma oscillations are taken into account, thereby determining the final width Δ of the linear plasma resonance. For a particular type of linear structure of the field, expressions for the field and speed corresponding to the first step of the transformation can be written as

$$\begin{aligned} P_{\mathrm{I}} &= -\frac{\omega_0^2 L^2}{\Delta^2 + \eta^2} (\eta \cos\chi + \Delta \sin\chi) , \\ v_{\mathrm{I}} &= -\frac{\omega_0 L^2}{\Delta^2 + \eta^2} (\eta \sin\chi - \Delta \cos\chi) , \\ x_{\mathrm{I}} &= \eta + \frac{a L^2}{\Delta^2 + \eta^2} (\eta \cos\chi + \Delta \sin\chi) , \end{aligned} \tag{2.15}$$

where Δ is determined either by the thermal motion of electrons with a thermal speed V_{T} or by the low collision frequency ν of particles in the plasma:

$$\Delta = \max\left\{ \frac{\nu L}{\omega_0} ; \left(\frac{3 V_{\mathrm{T}}^2 L}{\omega_0^2} \right)^{1/3} \right\} . \tag{2.16}$$

We now take the second step, the one associated with finite transformations with respect to the relativism parameter β, which follow from the Lie equations corresponding to the generator R_2:

$$\begin{aligned} &\frac{\mathrm{d}t}{\mathrm{d}\beta} = \xi^1_{\mathrm{RG}} , \quad t|_{\beta=0} = t_{\mathrm{I}} , \\ &\frac{\mathrm{d}v}{\mathrm{d}\beta} = \eta^1_{\mathrm{RG}} , \quad v|_{\beta=0} = v_{\mathrm{I}} . \end{aligned} \tag{2.17}$$

Integrating Eqns (2.17) gives the transition from the nonlinear nonrelativistic solution in (2.15) to the relativistic solution. The invariance of the electric field P ($P_{\mathrm{II}} = P_{\mathrm{I}}$) and the coordinate x ($x_{\mathrm{II}} = x_{\mathrm{I}}$) under the transformation associated with R_2 allows using the corresponding expressions from (2.15), and the calculation, based on (2.17), of the final transformations of the velocity $v \equiv v_{\mathrm{II}}$ and the 'time' $\tau \equiv \omega_0 t_{\mathrm{II}}$ then leads to the formulas

$$\begin{aligned} v &= v_{\mathrm{I}} \frac{\left(1 + (1/4)\beta v_{\mathrm{I}}^2\right)^{1/2}}{1 + (1/2)\beta v_{\mathrm{I}}^2} , \\ \tau &= \chi - \left(\sqrt{4 + \beta I_1}\, \mathrm{E}(\varphi, k) - \frac{2\,\mathrm{F}(\varphi, k)}{\sqrt{4 + \beta I_1}} - \varphi \right) , \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} &\varphi = \arcsin \frac{P_{\mathrm{I}}/\omega_0}{\sqrt{I_1}} , \quad k = \sqrt{\frac{\beta I_1}{4 + \beta I_1}} , \\ &I_1 = \frac{2}{\beta} \left(\frac{1}{\sqrt{z}} - 1 \right) + \frac{P_{\mathrm{I}}^2}{\omega_0^2} . \end{aligned} \tag{2.19}$$

Formulas (2.15), considered together with (2.18), implicitly describe the nonlinear structure of the longitudinal components of the field and electron velocity in terms of the parametric variables η and χ, taking two types of nonlinearity into account: nonrelativistic and relativistic. The explicit dependence of the electric field and the electron velocity on the true coordinate x and time t is determined by eliminating η

and χ from (2.15) and (2.18). Expressions (2.15) and (2.18) are the result of continuing the corresponding solutions of equations linearized with respect to (2.6) to finite values of the parameters $a \neq 0$ and $\beta \neq 0$ using the RG transformation. We note that the coordinate transformation law in (2.14) completely coincides with what was found in [14] when constructing a nonlinear nonrelativistic solution. A significant difference from the nonrelativistic case [14] is that, in addition to the x coordinate, the RG transformations also involve the velocity v and time t. The electric field P remains an invariant of RG transformations, which corresponds to the vanishing of the coordinate $\eta^2_{\rm RG} = 0$. A similar result follows for the transverse components of the electric field Q and electron velocity u when the action of the transformation group by (2.8) is extended to Eqns (2.7). Supplementing generator (2.8) with the terms $\eta^3 \partial_u$ and $\eta^4 \partial_Q$, we obtain [56], at which, with an accuracy not exceeding the accuracy of calculating the nonlinear currents, the group generator coordinates $\eta^3_{\rm RG}$ and $\eta^4_{\rm RG}$ can be assumed equal to zero, and the functions Q and u, to be invariants of the RG transformations. This last fact allows using the functions from [14] for the amplitudes of the transverse components of the electric field Q and velocity u:

$$u = k_y \omega_0 L^2 \left[\frac{1}{2} \ln \left(\frac{k_y^2 e^{2\rm C}}{4} (\varDelta^2 + \eta^2) \right) \cos\chi - \arccos \frac{\eta}{\sqrt{\eta^2 + \varDelta^2}} \sin\chi \right],$$
$$Q = k_y \omega_0^2 L^2 \left[-\frac{1}{2} \ln \left(\frac{k_y^2 e^{2\rm C}}{4} (\varDelta^2 + \eta^2) \right) \sin\chi - \arccos \frac{\eta}{\sqrt{\eta^2 + \varDelta^2}} \cos\chi \right]. \qquad (2.20)$$

Here, $\rm C = 0.5772156$ is the Euler–Mascheroni constant. The difference from the nonrelativistic case in [14] for u and Q is that expressions (2.20) must be considered together with the finite 'time' transformation from (2.18).

It thus follows that expressions (2.15), (2.18), and (2.20) implicitly define the space–time structure of the electric field and the electron velocity, namely, their longitudinal (P and v) and transverse (Q and u) components with both nonrelativistic and relativistic nonlinearities of electron motion near the plasma resonance taken into account. Combining formulas (2.15) and (2.18) gives an exact solution of the system of equations (2.6), as can be verified by substitution, and expressions (2.20) and (2.18) describe the space–time dependences of the amplitudes of the transverse components of the electric field and the electron velocity with an accuracy $\sim k_y$. In Sections 3 and 4, we obtain general relativistic expressions for the currents in the plasma resonance region that are sources of secondary radiation at the main frequency ω_0 and its multiple frequencies $n\omega_0$, $n \geqslant 2$. These expressions are constructed from the electric field and electron velocity functions on a plasma resonance and, according to the construction conditions, have an accuracy not exceeding k_y. Consequently, when finding the structure of the transverse components of the electric field and electron velocity, we can limit ourselves to the accuracy $\sim k_y$. In the nonrelativistic limit $c \to \infty$, the found expressions transform into the formulas of the nonlinear nonrelativistic theory in [14].

Putting Eqns (2.15), (2.18), and (2.20) together and moving to the normalized functions and variables

$$P_0 = \frac{a}{\varDelta \omega_0^2} P, \quad Q_0 = \frac{a}{\varDelta \omega_0^2} Q, \quad v_1 = \frac{a}{\varDelta \omega_0} v,$$
$$u_0 = \frac{a}{\varDelta \omega_0} u, \quad x_0 = \frac{x}{\varDelta}, \quad l = \frac{\eta}{\varDelta},$$

we write the resulting expressions for the nonlinear structure of the electric field and electron velocity in the vicinity of the critical plasma density:

$$P_0 = -\frac{A}{1 + l^2} (l \cos\chi + \sin\chi),$$
$$v_0 = -\frac{A}{1 + l^2} (l \sin\chi - \cos\chi), \quad x_0 = l - P_0,$$
$$u_0 = -AB \sin\theta \left[\arccos \frac{l}{\sqrt{1 + l^2}} \sin\chi - \frac{1}{2} \ln \left(\frac{k_y^2 e^{2\rm C}}{4} (1 + l^2) \right) \cos\chi \right],$$
$$Q_0 = -AB \sin\theta \left[\arccos \frac{l}{\sqrt{1 + l^2}} \cos\chi + \frac{1}{2} \ln \left(\frac{k_y^2 e^{2\rm C}}{4} (1 + l^2) \right) \sin\chi \right], \qquad (2.21)$$
$$v_1 = v_0 \, \frac{\left(1 + (1/4) B^2 v_0^2\right)^{1/2}}{1 + (1/2) B^2 v_0^2},$$
$$\tau(\chi, l) = \chi - \left(\zeta \, {\rm E}(\varphi, k) - \frac{2}{\zeta} \, {\rm F}(\varphi, k) - \varphi \right),$$

where

$$\zeta = \sqrt{4 + B^2 (v_0^2 + P_0^2)}, \quad \varphi = \arcsin \frac{P_0}{\sqrt{v_0^2 + P_0^2}},$$
$$k = \sqrt{\frac{B^2 (v_0^2 + P_0^2)}{4 + B^2 (v_0^2 + P_0^2)}}.$$

Nonzero values of the relativistic parameter β for a fixed dimensionless amplitude of the plasma field a (or for a fixed parameter $A \equiv aL^2/\varDelta^2$) correspond to finite values of the dimensionless parameter $B \equiv \omega_0 \varDelta / c$. We therefore associate B with the relativism parameter. The case of the nonrelativistic approximation ($B \ll 1$) was considered in [14]. We note that the condition $k_y \varDelta = B \sin\theta \ll 1$, corresponding to the effect of resonance amplification of the potential component of the plasma field [3, 14], can still be satisfied in our case for laser radiation incident on the plasma layer at sufficiently small angles.

Considering the interaction of laser radiation with plasma at a moderate laser intensity, we limit ourselves to the case where the oscillatory motion of electrons in the laser field is nonrelativistic, $eE_{\rm laser}/m\omega_0 \ll c$, where $E_{\rm laser}$ is the laser field amplitude. Equations (2.6) and (2.7) were obtained under the assumption that relativistic effects are significant only for the longitudinal component of the electron velocity near the plasma resonance, where the effect of resonance amplification of the electric field manifests itself. The most significant nonlinear effect in our theory is the nonlinearity of the longitudinal motion of electrons in the vicinity of the plasma

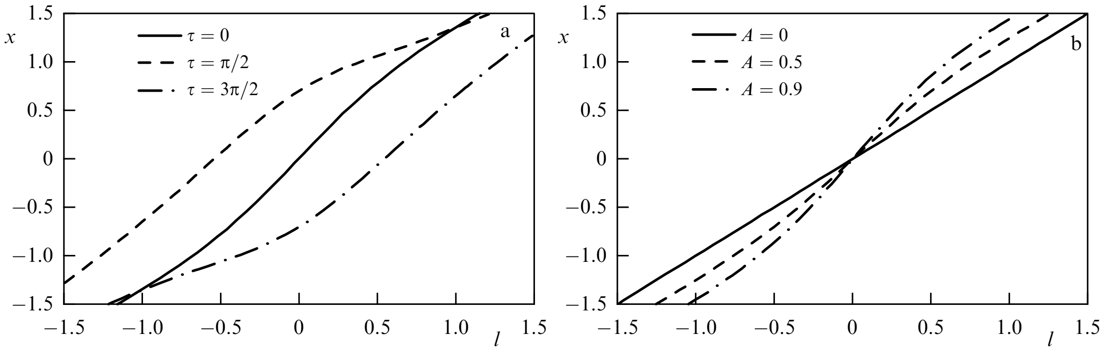

Figure 3. Dependences $x(l)$ (a) at different times τ and (b) for different values of dimensionless amplitude A at time $\tau = 0$. Bisector $x = l$ corresponds to the case $A = 0$.

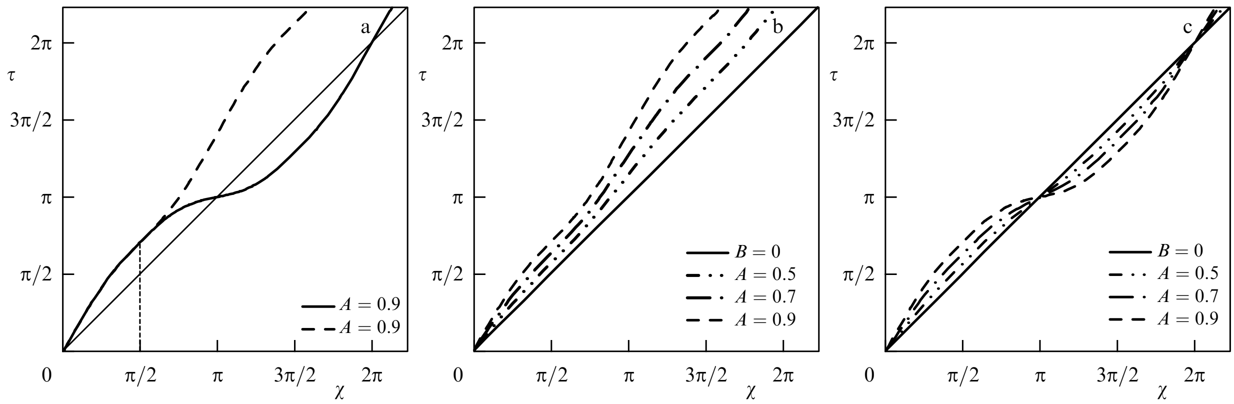

Figure 4. (a) Two branches of the solution corresponding to two ways of continuing $\tau(\chi, l)$ from branch point $\chi = \pi/2$ at $l = 0$. Both curves were plotted at $A = 0.9$ and $B = 1.8$. Bisector corresponds to nonrelativistic limit $B = 0$. (b, c) Plot of expression $\tau(\chi, l)$ for two branches of the solution at point $l = 0$ for different values of parameter A and fixed parameter $B = 1.8$.

resonance, calculated using the group theory approach. For relativistic laser fields, a more complex theory must be developed that goes beyond the hydrodynamic model. At such high laser radiation intensities, the dynamics of the electron component of the plasma must be described in a multi-flow regime, which requires the use of kinetic equations.[4] However, the threshold of the breaking of plasma oscillations can also be studied in the framework of the hydrodynamic approximation, as is demonstrated in what follows.

2.2 Stationary and nonstationary relativistic electron oscillations on a plasma resonance

We study formulas describing the longitudinal and transverse components of the electric field and electron velocity near a plasma resonance. We first discuss the expressions in (2.21) for the electric field P, velocity v_{I}, and coordinates x_{I} (see also formulas (2.15)), which are the result of the first step of the RG transformation and completely coincide with the nonlinear nonrelativistic solution found previously [14]. The transformation $x = \eta - Pa\omega_0^{-2}$ (or, in dimensionless form, $x_0 = l - P_0$) can be viewed as a Lagrangian replacement of the Euler coordinate x if $Pa\omega_0^{-2}$ is understood as the Lagrangian displacement of electrons relative to the coordinates η. The dependences of the dimensionless coordinate $x = x_{\mathrm{I}}/\Delta$ on $l = \eta/\Delta$ at different times and for different values of the parameter A are presented in Fig. 3. When the pump field amplitude is linear, $x = \eta$, the plasma resonance field was monochromatic. It is due to the nonlinear connection between P and x that the electric field P in the nonlinear nonrelativistic case has a spectrum containing higher harmonics with frequencies that are multiples of the fundamental frequency ω_0 of the laser radiation field.

Next, including relativistic nonlinearity and hence expressions for the transformed speed v and time τ, Eqns (2.18), into consideration, we note the branching of the solution due to the existence of two possibilities to continue from the branch point χ_0 of the functions involved in the expression for $\tau = \tau(\chi, l)$, which characterizes the change in the phase of oscillations of relativistic electrons in a resonantly amplified plasma field. The branching of solutions of nonlinear equations of motion is well known, for example, in some problems in mechanics [101]. Taking this property into account, the choice of the method to continue functions from the point χ_0 determines one of the two types of excited plasma oscillations. To illustrate this, we first consider the graphical representation of the expression for $\tau(\chi, l)$. We fix, e.g., $B = 1.8$, thereby fixing the plasma parameters. Taking into account that different values of the parametric variable l correspond to different branch points χ_0, for definiteness, we consider the case $l = 0$, where $\chi_0 = \pi/2$. Figure 4a shows two branches of the solution for the same values of A and B. Disregarding relativistic effects, i.e., setting $B = 0$, we have a linear dependence of $\tau(\chi, l)$ on χ: $\tau = \chi$. In the figure, this case corresponds to the bisector. In the transition to the relativistic regime, this dependence is replaced with a dependence of the

[4] One example of the use of such a transition from single-flow motion to a multi-flow regime is provided by the problem of the dynamics of cold dissipative gas in the expanding Universe [100].

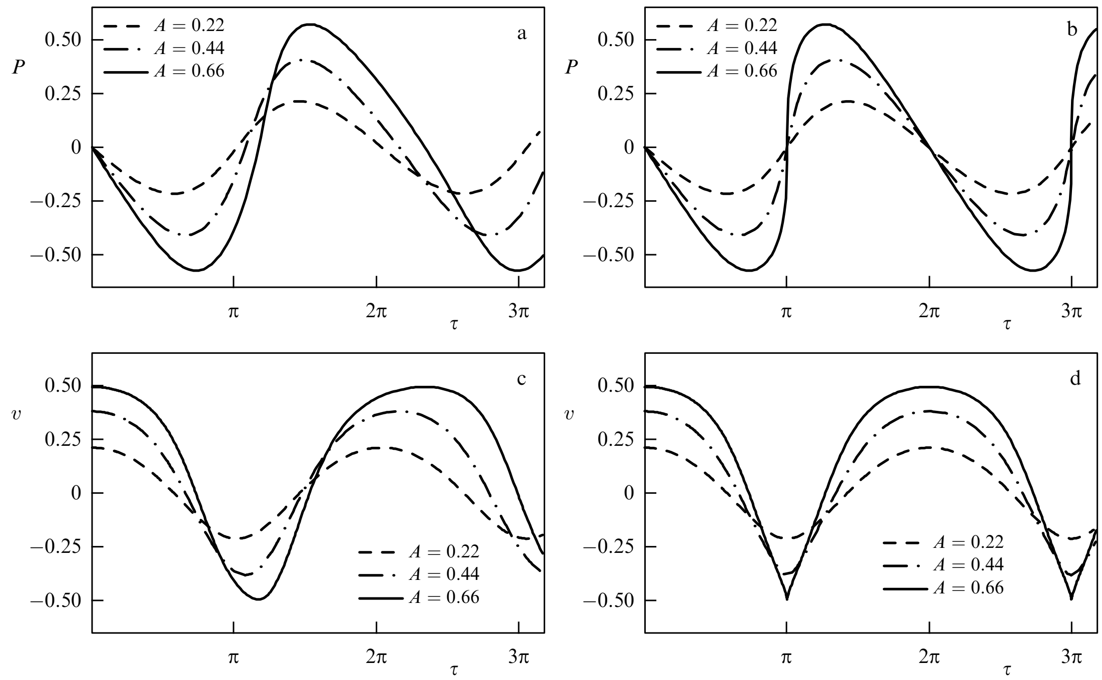

Figure 5. Time dependences of the longitudinal components of electric plasma field P and electron velocity v in (a, c) nonstationary and (b, d) stationary cases at plasma resonance point $x = 0$ for various values of parameter A. Graphs are plotted at $B = 1.5$.

form $\tau = \Psi\chi$, where $\Psi = \Psi(\chi, l) \neq 1$. On one of the solution branches (dashed line), two trends in the manifestation of relativistic effects are discernible. The first is a slow periodic dependence of $\Psi(\chi, l)$ corresponding to a space–time phase modulation, and the second is a change in the angle of inclination $\tau(\chi, l)$, characterized by the derivative $\partial_\chi \tau$, which corresponds to a shift in the electron vibration frequency to the lower-frequency range. On the other branch of the solution (solid curve in Fig. 4a), only phase modulation is observed. The nature of the change in the phase of oscillations with increasing parameter A for two branches of the solution is presented in Figs 4b and 4c: the phase modulation increases with increasing dimensionless amplitude A. In the first case (Fig. 4b), the dependence of τ on χ remains single valued at any A. For the second branch of the solution (Fig. 4c), an increase in modulation above a certain critical value $A_{\rm cr}$ leads to the formation of kinks in the function $\tau(\chi, l = 0)$ at the points $\chi = \pi + 2\pi n$, $n = 0, 1, \ldots$, and hence the 'time' τ is no longer a single-valued function of χ; therefore, the electron velocity v and electric field P functions also lose their single-valuedness. Staying within the hydrodynamic model, we consider only single-valued dependences of functions of physical quantities, which corresponds to the A and B parameter values such that $A^2 B^2 < 2\sqrt{3}$, an inequality that follows from the condition $\partial_\chi \tau(\pi, 0) = 0$ and determines the upper applicability bound of the formula for $\tau(\chi, l)$. The formulas that determine the implicit dependence of the functions P and $v_{\rm I}$ on the coordinate x in (2.15) also have the applicability bound determined by the limit value of the dimensionless parameter $A = A_0 = 1$, which characterizes the contribution of the nonrelativistic electron nonlinearity. In [14], it was found that, for $A \geqslant 1$, the dependence $x(l)$ is not single valued, which means that P and $v_{\rm I}$, as functions of the x coordinate, lose their single-valuedness. However, as we show in what follows, the constraints $A^2 B^2 < 2\sqrt{3}$ and $A < A_0$ are necessary but not sufficient for the single-valuedness of the functions $P(\tau, x)$ and $v(\tau, x)$.

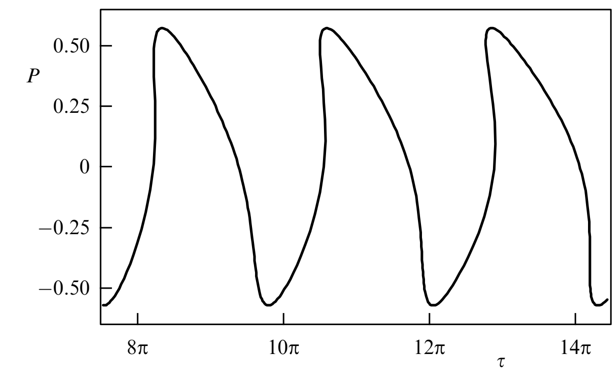

Figure 6. Breaking of profile of nonstationary plasma oscillations at point $x = 0$ at $B = 1.5$ and $A = 0.66$.

Let us discuss the physical consequences of the expression for $\tau(\chi, l)$. Figure 5 shows the time dependences of the longitudinal components of the plasma electric field P and electron velocity v at the point $x = 0$ for different values of the parameter A. It is clear from Figs 5a and 5c that the shift of the oscillation frequency to the low-frequency region, characteristic of the first branch of the solution (Fig. 4b), leads to a loss of the property of stationary oscillations: at different points in space, oscillations occur with different frequencies, and over time this leads to an increase in the phase difference between them. A secularly increasing phase difference, whose existence is associated with the spatial inhomogeneity of the amplitude of plasma oscillations [93, 94], ultimately leads to the intersection of the trajectories of neighboring particles, i.e., to the breaking of the oscillation profile. We emphasize that, in the nonstationary regime, the profile breaking occurs at arbitrarily small amplitudes in a finite time interval, in accordance with the well-known results obtained previously (see, e.g., [92] and review [102]). The breaking in a finite time is demonstrated in Fig. 6: as time progresses, a gradual

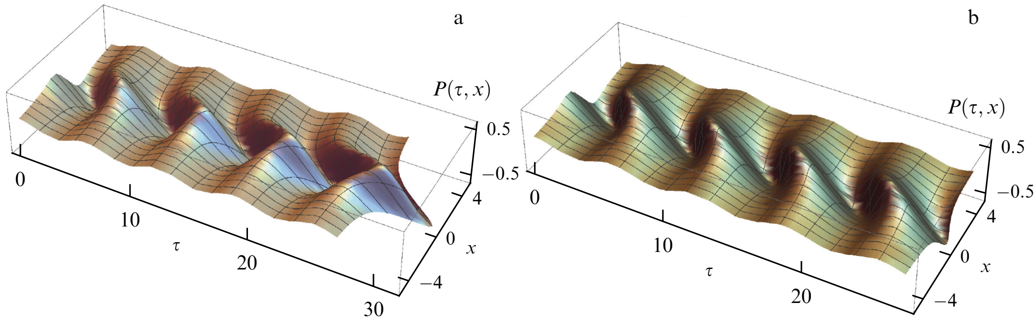

Figure 7. Space–time distributions of resonantly amplified longitudinal electric field P in (a) nonstationary and (b) stationary oscillation regimes. Plots are constructed at $A = 0.66$ and $B = 1.5$.

steepening of the profile of plasma oscillations is observed, ultimately leading to the multivaluedness of the longitudinal electric field $P(\tau)$. The second branch of solution (2.21) determines the stationary regime of plasma oscillations with a modulated phase (Figs 5b, d), in which case breaking occurs only when the field amplitude exceeds a certain threshold value: with increasing parameter A, a steepening of the profile of stationary plasma oscillations is observed up to a limit value of A at which the derivative $\partial_x P$ becomes infinity and the hydrodynamic model becomes inapplicable.

The nonstationary and stationary nature of plasma oscillations is clearly illustrated by the space–time distributions of the resonantly amplified field $P(\tau, x)$, shown in Fig. 7. The space–time structure of the field far from the resonance region corresponds to weakly perturbed stationary plasma oscillations. In the vicinity of the resonance, where nonlinear effects are significant, a considerable distortion of the field structure is observed, which manifests itself in both nonstationary (Fig. 7a) and stationary (Fig. 7b) regimes, depending on the choice of the branch of solution (2.21).

2.3 Breaking and spatial characteristics of a stationary resonantly amplified electric field

Solving the RA and HG problems for a nonlinear plasma resonance naturally implies determining the nature of electron dynamics near the critical plasma density, where nonlinear currents are formed that generate secondary radiation and determine its spectral composition. Classical theories [8, 11, 14] deal with a boundary value setting of the problem, where the dynamics of *stationary* oscillations of the electron component of the plasma are considered with boundary conditions imposed at infinity. In a number of studies of strongly nonlinear (relativistic) plasma oscillations, a choice is made in favor of the problem with an initial time [91–93]. In that approach, a nonstationary regime of nonlinear plasma oscillations arises, not requiring the development of theories of absorption and generation of laser radiation harmonics in the region of relativistic plasma resonance as a natural generalization of the results obtained previously within stationary nonlinear nonrelativistic theories [11, 14] and the linear theory [1, 37]. The need for such a generalization is dictated by the actual slow[5] dynamics of changes in laser intensity corresponding to an electromagnetic radiation pulse incident on the plasma, for which a linear plasma resonance occurs initially [1, 37, 103] and, with an increase in the pulse intensity, is subsequently superseded by a nonlinear nonrelativistic resonance structure [14], with relativistic effects starting to appear as the intensity increases even further. We give the allowed duration of the laser pulse taking the dynamics of plasma resonance formation into account. According to the theory describing how a stationary plasma resonance sets in [103], the time of formation of a stationary structure of a resonantly amplified plasma field can be estimated as $t_{\text{st}} \approx \sqrt{12L\Delta}/V_{\text{T}}$. Given the inhomogeneity scale $L \approx 10\lambda$ and plasma temperature $T \approx 2$ keV, we obtain $t_{\text{st}} \approx 10^{-13}$ s, i.e., the minimum laser pulse duration allowed by our theory is approximately a hundred femtoseconds for the characteristic plasma parameters indicated. The construction of an analytic *stationary* theory of a *relativistic plasma resonance* is also of interest, because the multivaluedness of the known nonlinear solutions [14, 40, 92, 93, 104] that describe the interaction of a powerful electromagnetic field with an inhomogeneous plasma suggests the possibility of realizing unexplored dynamical regimes of plasma oscillations, searching for and studying which are of universal importance both for the theory of laser–plasma interaction and for the theory of nonlinear oscillations in general.

In what follows, based on the above physical considerations, we focus on studying just the stationary regime of relativistic plasma oscillations, which occurs under the condition of an adiabatically slow increase in the laser pulse intensity. We do not discuss issues of stability and implementation of the nonstationary branch of the solution.[6] We emphasize that the breaking of plasma oscillations in the case under discussion results from the superposition of electron nonlinearities of two types, nonrelativistic and relativistic, and the above constraints $A^2B^2 < 2\sqrt{3}$ and $A < 1$ (the first of which follows from the single-valuedness of $\tau(\chi, l)$ as a function of χ, and the second, from the condition of single-valuedness of P and v as functions of the coordinate x), generally speaking, do not define a criterion for the single-valuedness of the functions $P(\tau, x)$ and $v(\tau, x)$. The manifestation of nonlinear effects is most significant at the plasma resonance point $x = 0$ (or at $l = 0$). In this case, the loss of single-valuedness with increasing A, as can be seen, e.g., from Fig. 5b, first occurs at $\tau = \pi$. Therefore, to obtain the complete condition for the breaking of the profile of stationary plasma oscillations, we must calculate the derivative of the electric field with respect to the coordinate, $\partial_x P(\tau, l)$, at $\{\tau = \pi, l = 0\}$

[5] Compared with the dynamics of the electron component of the plasma. This reasoning is valid in the case of not too short laser pulses with a duration of nano-, pico-, and hundreds of femtoseconds.

[6] As was done, e.g., in [21].

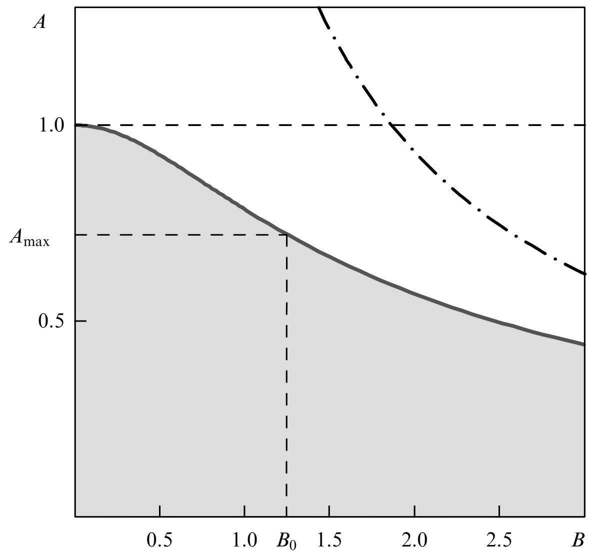

Figure 8. Region of admissible values of parameters A and B. Solid curve corresponds to the breaking threshold in relativistic hydrodynamics, dashed line ($A = 1$) corresponds to breaking in the nonrelativistic theory [14]. Dashed-dotted line corresponds to the condition $A^2B^2 = 2\sqrt{3}$.

or, equivalently, at $\{\chi = \pi, l = 0\}$,

$$\partial_x P\big|_{\chi=\pi,\, l=0} = \frac{1}{D}\left(\partial_\chi \tau\, \partial_l P - \partial_l \tau\, \partial_\chi P\right)\Big|_{\chi=\pi,\, l=0}, \tag{2.22}$$

where $D \equiv \partial_l x\, \partial_\chi \tau - \partial_\chi x\, \partial_l \tau$ is the Jacobian of the transition from the variables τ and x to χ and l. Writing the vanishing condition for the Jacobian $D|_{\chi=\pi,\, l=0}$, which corresponds to the derivative $\partial_x P$ turning to infinity and to the breaking of a nonlinear plasma wave, we obtain the following relation between the parameters A and B:

$$A + \frac{2 + A^2B^2}{\sqrt{4 + A^2B^2}} = 2\,. \tag{2.23}$$

Relation (2.23) complements the inequalities $A < 1$ and $A^2B^2 < 2\sqrt{3}$, thereby defining the domain of A and B values that are admissible as regards the single-valuedness of the functions $P(\tau, x)$ and $v(\tau, x)$ (Fig. 8). The superposition of two nonlinearity types leads to a different form of the breaking threshold compared to the nonrelativistic theory [14]: for a fixed $B_0 > 0$, the value of A cannot exceed the maximum $A_{\max} < A_0 = 1$, which is determined by solving a fourth-order equation following from condition (2.23),

$$(B^2 - B^4)A^4 - 4B^2A^3 + 4A^2 - 16A + 12 = 0\,. \tag{2.24}$$

When $A > A_{\max}$, a multi-flow regime arises and solution (2.21) becomes inapplicable.

It may seem that a comparison of the curve representing the breaking threshold in the relativistic theory with the dashed line corresponding to the breaking in the nonrelativistic theory [14] (see Fig. 8) demonstrates a more stringent constraint imposed in the relativistic theory on the maximum possible amplitude of the A field at which the hydrodynamic model formulas for physical quantities remain single valued. From this, we would be able to conclude that the transition to the relativistic theory does not allow advancing into the region of higher laser intensities I_0 than those considered in [14]. However, this is not so. We show in Section 3.3 below that, due to the effect of saturation of the plasma resonance field amplitude, the breaking threshold in the relativistic theory actually shifts to the region of higher (relativistic) laser radiation intensities. We obtain a nonlinear relation between the magnetic field amplitude at the resonance and laser pumping field amplitude, which allows moving from the dimensionless parameters A and B to the natural physical parameters of the laser–plasma system and exploring the applicability limits of the hydrodynamic approximation in terms of the laser radiation intensity I_0 and the plasma temperature T.

We note that expressions (2.15), (2.18), and (2.20) for the electric field and electron velocity structure contain the linear plasma resonance width Δ determined by dissipative or thermal effects, although it does not appear in the original nonlinear equations (2.6) and (2.7). We explain how the Δ appears in the solution. In the classical linear theory [1, 3], equations that do not contain dissipative terms due to their smallness are also used as the initial ones, and Δ appears when small dissipation effects are taken into account by introducing a finitely small imaginary part into the frequency ω. It would be possible to introduce terms associated with dissipation into the original equations, but the corresponding effects are quite small for the RA process, because corrections of the order of ν_{eff}/ω occur in the solution. And the presence of Δ in (2.15), (2.18), and (2.20) is due to the method of RG symmetries, which is based on the procedure of continuing a linear solution (in the electric field amplitude) into the region of laser and plasma parameters where nonlinear effects are significant. For moderate intensities of laser radiation and relatively long laser pulses (with durations of pico and nanoseconds and hundreds of femtoseconds), a stationary solution for the electric field near the plasma resonance is realized, which corresponds to an adiabatically slow (on the pulse time scale) increase in the laser field compared to the dynamics of electrons in plasma resonance. In this regime, there is a continuous transition from a stationary linear resonance to a stationary nonlinear resonance structure. In other words, the spatial structure of a solution of the form $1/\varepsilon(x) \sim L/(x - \mathrm{i}\Delta)$ is inherited from the solution obtained in the linear plasma resonance theory [1, 3]. The electric field in the linear approximation plays the role of the 'initial' condition and subsequently changes with an adiabatic increase in the laser pulse amplitude. Accordingly, Δ, which corresponds to the width of the linear plasma resonance, loses its meaning in the nonlinear regime and becomes simply a fixed constant defined by the effective collision frequency ν_{eff}. We also note that the laser field, generally speaking, can affect the rate of pair collisions in (2.16), and this can be taken into account by renormalizing ν_{eff} in accordance with [105], for example.

It follows from formulas (2.15), (2.18), and (2.20) that the electrostatic plasma field in the vicinity of the critical density somewhat 'swells' and, due to nonlinear effects, is characterized by the nonlinear resonance width $\delta > \Delta$. To verify this, we consider the spatial and spectral characteristics of the stationary solution of Eqns. (2.6) with the electric field expanded in a series in the incident wave harmonics about the point $x = 0$:

$$P^{(n)} = \frac{1}{2\pi}\int_0^{2\pi} P(\tau, x)\, \exp(-\mathrm{i}n\tau)\, \mathrm{d}\tau\,, \quad n = 0, 1, \ldots\,. \tag{2.25}$$

Figure 9a shows the spatial distributions of the amplitude of the first harmonic $P^{(1)}(x)$ in the nonlinear nonrelativistic case

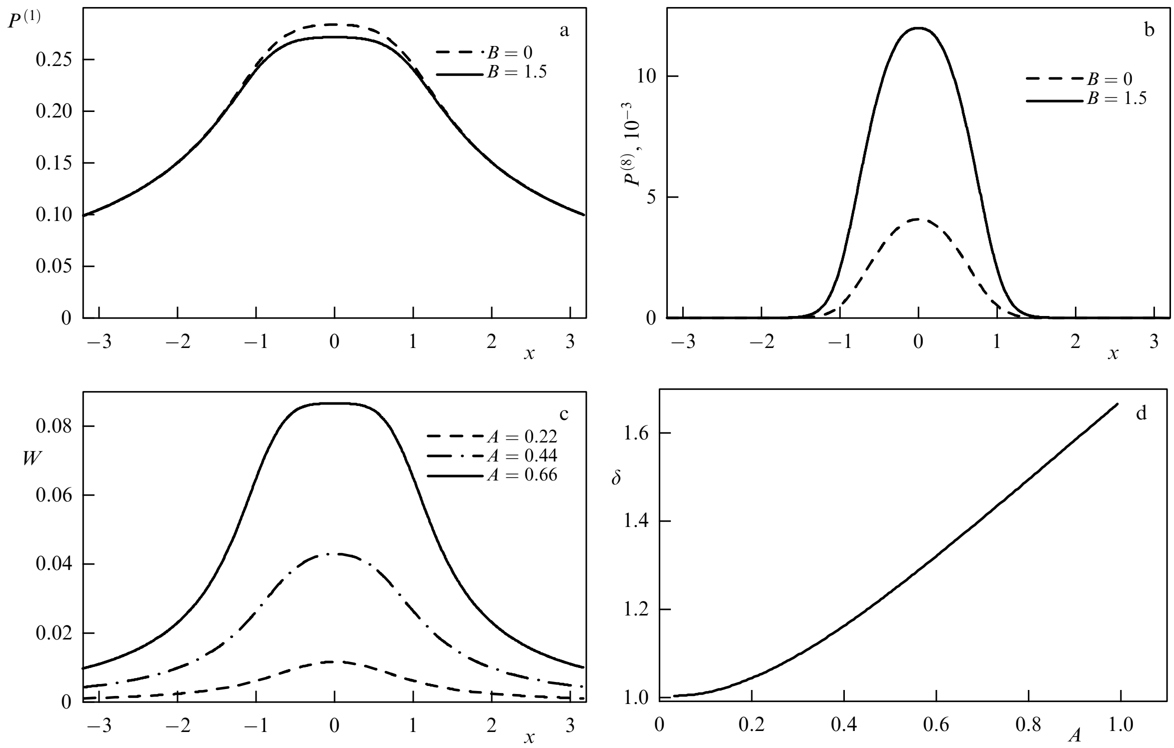

Figure 9. (a) Spatial distributions of first harmonic of longitudinal electric field of plasma resonance without ($B = 0$, dashed line) and with ($B = 1.5$, solid line) the relativistic nature of plasma oscillations taken into account at a fixed field amplitude ($A = 0.66$). (b) Spatial distributions of eighth harmonic of longitudinal electric field of plasma resonance. Dashed line corresponds to nonrelativistic approximation ($B = 0$), solid line corresponds to the result of taking relativistic nonlinearity into account ($B = 1.5$). Plots correspond to fixed parameter $A = 0.66$. (c) Spatial distributions of plasma field energy corresponding to different values of dimensionless amplitude A. (d) Dependence of plasma resonance width (in units of linear-theory width Δ) on A.

($B = 0$) and with the relativistic nonlinearity ($B = 1.5$) taken into account at a fixed value of the parameter A. According to our theory, taking the relativistic nature of plasma oscillations into account does not lead to a shift in the maximum electric field relative to the point $x = 0$, but results in a decrease in the amplitude of the electric field component $P^{(1)}(x, \tau)$ at the fundamental frequency, which is associated with a redistribution of the field energy between harmonics during the transition to the relativistic oscillation regime. Namely, due to the phase modulation effect, the amplitudes of harmonics with numbers $n \geqslant 2$ are amplified, which significantly changes the spectral composition of the field, enriching it with higher harmonics. In Fig. 9b, to illustrate the growth of the amplitudes of higher harmonics, we compare the spatial distributions of the amplitudes of the eighth harmonic of the longitudinal electric field $P^{(8)}(x, \tau)$ in the absence ($B = 0$) and in the presence ($B = 1.5$) of the relativistic nonlinearity.

We now discuss the spatial localization region of the plasma field energy, and take the plasma resonance width δ as a quantitative measure of that region; we define it as HWHM, half width at half maximum. In our case, this region corresponds to the localization of the total energy (intensity) W of all spectral components of the field, $W = \sum_{n=0}^{\infty} |P_n|^2$. It was shown in [43] that keeping the contribution corresponding to relativistic nonlinearity in (2.6) does not lead to a change in the total energy of the longitudinal component of the electric field P and hence to a change in the plasma resonance width δ. This inference is consistent with the above statement: as a result of taking relativistic effects into account, part of the energy is transferred into the energy of higher harmonics, while the total energy of the electric field does not change. However, as can be seen from Figs 9c, d, the width of the nonlinear plasma resonance depends on the dimensionless amplitude A. Plots of the spatial distribution of the field energy W for different values of A up to the breaking point are shown in Fig. 9c, and a slight increase in the resonance width δ with increasing A is illustrated in Fig. 9d. Thus, as the field amplitude increases, a slight 'swelling' of the plasma resonance is observed: near the breaking threshold, for different values of the plasma parameters, the nonlinear plasma resonance width δ is on average approximately 1.5 times greater than the resonance width Δ in the linear theory.

A detailed study of analytic solutions (2.21) of nonlinear differential equations (2.6) and (2.7), which describe relativistic plasma oscillations in the vicinity of the critical density of an inhomogeneous plasma, showed that taking the relativistic nonlinearity of electron motion into account leads to two classes of solutions. The first extends the standard linear [1] and nonlinear nonrelativistic [14] solutions, thereby describing the stationary regime of relativistic plasma oscillations with a modulated phase. This regime corresponds to a physical picture where the field amplitude in a laser pulse changes quite slowly in comparison to the dynamics of plasma electrons. It is then natural to believe that, for a finite-duration laser pulse, as the pump field amplitude increases in time, the solution describing the plasma resonance field passes through three stages: from the linear stage at a low amplitude to the nonlinear nonrelativistic stage and then to the relativistic stage. Solutions of the other type correspond to nonstationary plasma oscillations in the vicinity of the critical density and are similar to solutions obtained in the framework of other models [40, 91, 92, 94]. Here, the breaking occurs at arbitrarily small amplitudes of the pump field. The possibility of implementing such a solution is not yet obvious, but it may well be associated with a violation of the condition of quasistationary transition from a linear to a nonlinear regime, and it requires a separate study.

The results obtained in this section are fundamentally important, because the revealed relativistic-nonlinear stationary structure of the resonantly enhanced electric field and electron velocity underlies the construction of stationary theories of RA and HG, as well as calculations of the static electric field in the vicinity of the critical plasma density. These issues are discussed in subsequent sections.

3. Nonlinear resonance absorption and electrostatic field generation

The analytic theory of linear RA based on the Ginzburg–Denisov linear plasma resonance model [1, 2] was developed by Hirsch, Shmoys, and Piliya [106–108] for cold [106, 107] and hot [108] plasmas. It follows from the results in [106–108] that the RA coefficient G is independent of the plasma temperature, electron collision rate, and pump field intensity, but is characterized by a self-similar dependence on a single variable ρ determined by laser–plasma parameters, $\rho \equiv (\omega_0 L/c)^{2/3} \sin^2\theta$, with the maximum $G_{\mathrm{m}} \approx 0.4$ at the optimum value $\rho = \rho_{\mathrm{opt}} \approx 0.2$. Here, c is the speed of light in a vacuum and θ is the angle of incidence on an inhomogeneous plasma with an inhomogeneity scale L. Due to the approximation of the analytic solutions of the field equations for optimum values of ρ, quantitative estimates of the absorption efficiency given in [106–108] were not completely accurate, as was demonstrated analytically for cold plasma by Omel'chenko and Stepanov in the limit of small angles [109] and by Tang in the limit of large angles [110] of incidence of laser radiation on the plasma, and later by Omel'chenko, Kelly, Forslund, and others using numerical solutions of the wave equation for arbitrary angles [6, 111, 112]. A systematic description of the shape of the resonance curve $G(\rho)$ in [6, 111] allowed clarifying the maximum value of the absorption coefficient and the corresponding optimal ρ: $G_{\mathrm{m}} \approx 0.5$ at $\rho_{\mathrm{opt}} \approx 0.5$ (Fig. 10). In addition, in [6], a negligibly weak dependence of the absorption coefficient on temperature was demonstrated, which confirmed the general conclusion about the decisive role of the variable ρ in describing absorption in the linear regime. We note that the significant absorption of a p-polarized electromagnetic wave in the vicinity of plasma resonance was also noted in Budden's book [113].

The analytic approach developed for cold plasmas by Speziale and Catto [37] has led to significant improvements in solutions [109, 110] in the limit cases of small and large incidence angles; in the case of hot plasmas, Pert [7] obtained a solution in the power series form, generalizing previous results [108] to a wide range of angles. Subsequent analytic studies of linear radiation conversion in an inhomogeneous plasma refined the dependence of the absorption coefficient $G(\rho)$ in the case of a linear plasma density profile [114]; in the limit of small incidence angles, a density profile with a local stepwise jump in density near the plasma resonance was considered in [67], and the shape of the $G(\rho)$ curve for a parabolic profile was studied in [115–117].

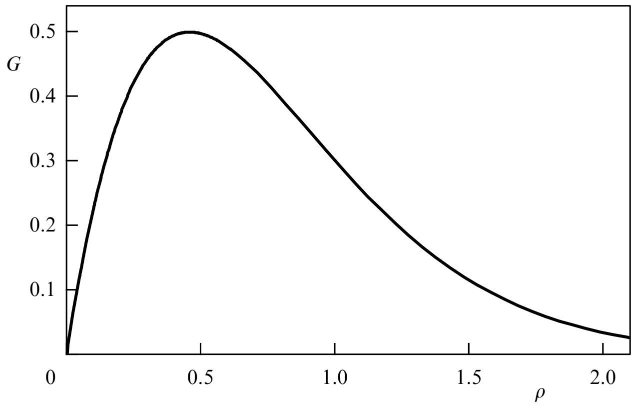

Figure 10. Dependence of linear RA coefficient G on self-similar variable ρ based on the numerical solution of the field equations [6, 67].

In connection with the problem of the efficiency of RA, interest arose quite early [118] in the nonlinear influence of the ponderomotive force of laser radiation on the plasma density. It was shown in a number of studies [119–121] that, under the action of the ponderomotive force of a p-polarized electromagnetic wave incident on a plasma in the vicinity of the plasma resonance, stepwise distributions of the dielectric constant are formed that can contribute to an increase in RA. A distinctive feature of these studies is that, in writing the field equations in plasma, the striction nonlinearity that changes the plasma density is taken into account, but the nonlinearity of electron motion in a strongly nonlinear self-consistent plasma resonance field is ignored. More recent studies on RA were devoted to the effect of nonlinear (including relativistic) electron motion and the deformation of the plasma density profile on the absorption coefficient [38–41]. The authors of [39], working within a one-dimensional PIC model, used estimates based on the 'capacitor' model [18, 21, 22] to demonstrate the influence of relativistic and ponderomotive nonlinearities near the critical density of inhomogeneous plasma on the process of absorption of laser radiation of moderate and subrelativistic intensity. According to their calculations, with an increase in the laser intensity to $\approx 3 \times 10^{17}$ W cm^{-2}, the absorption level in the vicinity of the plasma resonance can decrease from 50% to 30%. This decrease in the absorption coefficient was associated with the effect of an electron acquiring an extra 'weight' when oscillating in a resonantly amplified plasma field. With a further increase in the laser pumping amplitude, due to the ponderomotive nonlinearity, the decrease in the absorption coefficient is superseded by its increase, which results in absorption at the level of 60–70% at a laser radiation intensity of $\approx 10^{18}$ W cm^{-2}.

In another paper of the same year [38], a simplified hydrodynamical model was used together with Maxwell's equations in the approximation where the electric field in the plasma is divided into two parts: an electromagnetic component associated with the incident laser wave and an electrostatic component corresponding to electron plasma oscillations. As a result, it was concluded that the optimal[7] angle of incidence θ_{opt} of laser radiation on the plasma remains constant and the absorption coefficient monotonically increases as the laser pump field intensity increases. The

[7] The optimum angle of incidence is the one for which RA is maximum: θ_{opt} corresponds to ρ_{opt} at a fixed inhomogeneity scale L.

dependence of the optimal angle on the plasma density inhomogeneity scale was revealed. We note, however, that a very wide range of inhomogeneity scales $k_0 L = 10^{-2} - 10$, $k_0 = \omega_0/c$ was considered in [38], but the applicability limits of the model were not indicated and the possibility of its use within such wide limits was not justified. The trends in the dependence of absorption on the laser intensity identified in the previous study [39] were later confirmed and refined using the improved technique of numerical calculation of the two-dimensional PIC model [40]. Then, in [41], an attempt was made to consider the relativistic nonlinear RA effects by a numerical–analytic solution of the wave equation; this partially confirmed the conclusions in [39, 40], but at the same time suggested a shift of the maximum of the nonlinear absorption curve relative to the position of the maximum in the linear theory due to changes in the laser radiation flux.

In contrast to numerical calculations [38–41], recent studies [44–46] were aimed at constructing a systematic analytic theory of nonlinear RA at a relativistic plasma resonance, which is a natural generalization of classical theories [7, 37, 108] and [14] based on the Ginzburg–Denisov model. Using this theory, characteristic dependences of the absorption coefficient on many control parameters of the laser–plasma system, including the laser pumping field intensity, can be identified. The results of analytic studies [44–46] constitute the main part of this section.

3.1 Basic equations and the general solution to the boundary value problem for nonlinear resonance absorption

To describe the process of nonlinear absorption (reflection) of a p-polarized electromagnetic wave (2.1) incident on a weakly inhomogeneous plasma along the x coordinate at an angle θ, we take the initial equations, as in the preceding section, to be Eqns (2.2), which take form (2.3) in component notation. Representing the velocities and fields v, u, P, Q, and R entering (2.3) in the form of series expansions in harmonics of incident wave (2.1), we associate each of these quantities with its Fourier component v_n, u_n, P_n, Q_n, and R_n:

$$\{v, u, P, Q, R\} = \sum_{n=-\infty}^{\infty} \{v, u, P, Q, R\}_n \exp\left[-\mathrm{i}n(\omega_0 t - k_y y)\right] . \tag{3.1}$$

From system of equations (2.3), using expansion (3.1), we obtain the following equation for the nth harmonic of the magnetic field:

$$\begin{aligned}
&\partial_{xx} R_n - \frac{\partial_x \varepsilon_n}{\varepsilon_n}\,\partial_x R_n + \left(\frac{n\omega_0}{c}\right)^2 (\varepsilon_n - \sin^2\theta) R_n \\
&= \frac{\mathrm{i}an\omega_0}{c^2}\left\{v(\partial_x P + \partial_y Q)\right\}_n \sin\theta \\
&+ \frac{\omega_{\mathrm{L}}^2}{c^2}\left\{av\partial_x(\gamma v) + au\partial_y(\gamma v) + \partial_t\big(v(\gamma-1)\big) - \frac{au}{c}\,R\right\}_n \\
&- \frac{a}{c}\left\{\partial_x\big(u(\partial_x P + \partial_y Q)\big)\right\}_n + \frac{a\partial_x\varepsilon_n}{c\varepsilon_n}\left\{u(\partial_x P + \partial_y Q)\right\}_n \\
&+ \frac{\mathrm{i}\omega_{\mathrm{L}}^2}{cn\omega_0}\,\partial_x\left\{av\partial_x(\gamma u) + au\partial_y(\gamma u) + \partial_t\big(u(\gamma-1)\big) + \frac{au}{c}\,R\right\}_n \\
&- \frac{\mathrm{i}n\omega_0\varepsilon_n}{c}\,\partial_x\left(\frac{\varepsilon_n - 1}{\varepsilon_n}\right) \\
&\times\left\{av\partial_x(\gamma u) + au\partial_y(\gamma u) + \partial_t\big(u(\gamma-1)\big) + \frac{av}{c}\,R\right\}_n .
\end{aligned} \tag{3.2}$$

The subscript n in (3.2) indicates taking the nth Fourier component of the corresponding function. Here, $\varepsilon_n = 1 - \omega_{\mathrm{L}}^2/(n^2\omega_0^2)$ is the complex dielectric permittivity of the plasma at the frequency $n\omega_0$. The right-hand side of Eqn (3.2) corresponds to the nonlinear source of HG in the plasma. In the absence of nonlinear effects (at $a \to 0$), this source disappears, and Eqn (3.2) becomes an equation for the free propagation of a p-polarized electromagnetic wave with the frequency $n\omega_0$ in an inhomogeneous plasma.

We take into account that the dependence of electromagnetic fields and electron velocities on the x coordinate along the density gradient near the plasma resonance is inversely proportional to the width of the plasma resonance $\delta \ll L$ and is much more significant than the dependence on the transverse coordinate y, which is proportional to k_y: $\partial_x(\ldots) \gg \partial_y(\ldots)$. In addition, in the considered case of a weakly inhomogeneous plasma with a characteristic inhomogeneity scale $L \gg 1/k_0$, we can disregard contributions proportional to the inhomogeneity gradient on the right-hand side of (3.2). Collecting the above conditions, we arrive at a set of inequalities that define the domain of applicability of the model:

$$k_0 L \gg 1\,, \quad k_y\delta \ll 1\,, \quad \delta \ll L\,. \tag{3.3}$$

With the main assumptions and model limitations of the theory of plasma resonance, which are specified by inequalities (3.3) and thereby determine the hierarchy of fields and velocities near the critical density (2.5), we use (3.2) to obtain an equation for the magnetic field amplitude reflected from the plasma to the vacuum at the fundamental frequency, with the right-hand side of the equation containing a radiation source in the form of a nonlinear current localized near the critical plasma density. In the case of interest, for the magnetic component of the field at the laser frequency ω_0, we take into account that, for $n = 1$ in (3.2), the largest contribution to the source is made by the resonant terms $\sim 1/\varepsilon_1$, where $\varepsilon_1 = (\mathrm{i}\Delta - x)/L$.

Under the above assumptions, Eqn (3.2) with $n = 1$ becomes

$$\begin{aligned}
&\partial_{xx} R_1 - \frac{\partial_x\varepsilon_1}{\varepsilon_1}\,\partial_x R_1 + \left(\frac{\omega_0}{c}\right)^2 (\varepsilon_1 - \sin^2\theta) R_1 \\
&\quad = -\frac{4\pi}{c}\left\{\frac{a}{4\pi}\ \mathrm{rot}\,\mathbf{J}_1\right\}_z \\
&\quad + \frac{a/c}{x - \mathrm{i}\Delta}\left[u\partial_x P - \mathrm{i}\omega_0 v\,\partial_x(\gamma_0 u) - \frac{\omega_0^2}{a}(\gamma_0 - 1)u\right]_1 ,
\end{aligned} \tag{3.4}$$

where the components of the vector $\mathbf{J}_1$ are given by

$$\begin{aligned}
\mathbf{J}_1 = &\left\{v\,\partial_x P - \mathrm{i}\omega_0 v\,\partial_x(\gamma_0 v) - \frac{\omega_0^2}{a}(\gamma_0 - 1)v\,,\right. \\
&\left. u\partial_x P - \mathrm{i}\omega_0 v\,\partial_x(\gamma_0 u) - \frac{\omega_0^2}{a}(\gamma_0 - 1)u\,,\ 0\right\}_1 .
\end{aligned} \tag{3.5}$$

Subscript 1 in (3.4) and (3.5) indicates taking the first Fourier component of the corresponding function. In (3.4), we assume that $\omega_{\mathrm{L}} = \omega_0$, i.e., we ignore the dependence of the frequency ω_{L} on the x coordinate. This approximation is justified for a weakly inhomogeneous plasma, when the localization region of the plasma resonance field is small compared to the characteristic plasma inhomogeneity scale L. It follows from Eqn (3.4) and relation (3.5) that the first harmonic of the magnetic field in the vacuum is determined by the electric field and the electron velocity in the plasma resonance region that were found in Section 2.

The solution of inhomogeneous equation (3.4) can be written in terms of the fundamental system of solutions Ψ^+ and Ψ^- of the homogeneous equation as

$$\begin{aligned} R_1(x) &= \alpha_1^+ \Psi_1^+(x) + \alpha_1^- \Psi_1^-(x) + \int_{-\infty}^{x} \mathrm{d}\xi\, \mathcal{G}(x,\xi) f_1(\xi)\,, \\ \mathcal{G}(x,\xi) &= -\frac{\Psi_1^+(x)\Psi_1^-(\xi) - \Psi_1^+(\xi)\Psi_1^-(x)}{\Psi_1^+(\xi)\Psi_1^{-\prime}(\xi) - \Psi_1^{+\prime}(\xi)\Psi_1^-(\xi)}\,, \\ f_1(x) &= \frac{a}{c}\left\{ \mathrm{i}k_y \left(vP_x - \mathrm{i}\omega_0 v(\gamma_0 v)_x - \frac{\omega_0^2}{a}(\gamma_0 - 1)v \right) \right. \\ &\quad + \frac{1}{x - \mathrm{i}\Delta}\left[uP_x - \mathrm{i}\omega_0 v(\gamma_0 u)_x - \frac{\omega_0^2}{a}(\gamma_0 - 1)u \right] \\ &\quad \left. - \left[uP_x - \mathrm{i}\omega_0 v(\gamma_0 u)_x - \frac{\omega_0^2}{a}(\gamma_0 - 1)u \right]_x \right\}_1 , \end{aligned} \tag{3.6}$$

where $\mathcal{G}(x,\xi)$ is the Green's function, and the functions Ψ_1^+ and Ψ_1^- satisfy the homogeneous equation

$$\partial_{xx}\Psi_1^{\pm} - \frac{\partial_x \varepsilon_1}{\varepsilon_1}\,\partial_x \Psi_1^{\pm} + \left(\frac{\omega_0}{c}\right)^2 (\varepsilon_1 - \sin^2\theta)\Psi_1^{\pm} = 0\,. \tag{3.7}$$

In formulas (3.6), $\alpha_1^{\pm}$ are constants determined by the boundary conditions for Eqn (3.4) applied to solution (3.6). The boundary conditions, in turn, follow from the form of the magnetic field $R_1(x)$ as $x \to \pm\infty$,

$$\begin{aligned} R_1 &= \widetilde{C}_1^+ \exp\left(\mathrm{i}\,\frac{\omega_0}{c}\,x\cos\theta\right) + \widetilde{C}_1^- \exp\left(-\mathrm{i}\,\frac{\omega_0}{c}\,x\cos\theta\right), \\ &\qquad\qquad x \to -\infty\,, \\ R_1 &= 0\,, \quad x \to +\infty\,, \end{aligned} \tag{3.8}$$

where the complex amplitudes $\widetilde{C}_1^+$ and $\widetilde{C}_1^-$ correspond to the respective incident and reflected waves. We emphasize that, for convenience, we consider the field R (and the corresponding amplitude $\widetilde{C}_1^-$) normalized to the dimensionless amplitude a, but the ultimate goal is to obtain formulas for the magnetic field B_z (and the amplitude C_1^-). The equation for the Fourier component B_1 follows from the equation for R_1 after multiplication by am_e/e, with the relation between the amplitudes $\widetilde{C}_1^-$ and C_1^- being the same as between R_1 and B_1. Therefore, when moving to the formula for the Fourier component C_1^- of the magnetic field B_z at the fundamental frequency, the amplitude $\widetilde{C}_1^-$ must be multiplied by am_e/e.

In constructing nonlinear solution (3.6), we rely on linear solutions of (3.7), which are currently available in analytic form only in two limit cases: for $\rho \ll 1$ and $\rho > 1$ [37, 109]. For a fixed plasma inhomogeneity scale $L \gg 1/k_0$ (more precisely, we consider $L > 10/k_0$) and a given laser frequency ω_0, these two cases correspond to the limits of small ($\rho \ll 1$) and large ($\rho > 1$) angles of incidence θ of laser radiation on the plasma. In this respect, the nonlinear solutions found are asymptotic in nature, allowing absorption to be found to the right and to the left of the maximum of the function $G(\rho)$ (see Fig. 10).

Thus, solving Eqn (3.4) with the Green's function method with boundary conditions (3.8) (see Appendix B and [44–46]), we obtain expressions for the amplitude of the first harmonic of a magnetic field in a vacuum in the case of not small angles of incidence of laser radiation on the plasma ($\rho > 1$),

$$C_1^- = \Omega_1 C_1^+ + \frac{m_e}{e}\,\frac{\omega_0^3 \Delta}{(2\pi)^{3/2}}\left(\frac{L}{c\omega_0|\cos\theta|}\right)^{1/2} \Omega_2 \mathcal{I}\,, \tag{3.9}$$

where

$$\begin{aligned} \Omega_1 &= R_L^{1/2} \exp\left[2\mathrm{i}\,\frac{\omega_0}{c}\,\mathcal{L}_+(-\infty) - \mathrm{i}\,\frac{\pi}{2}\right], \\ \Omega_2 &= \left(\frac{G_L}{2}\right)^{1/2} \exp\left[\mathrm{i}\mathcal{Z} + \mathrm{i}\arg B_1(0) - \mathrm{i}\pi\right], \end{aligned} \tag{3.10}$$

and in the limit of small angles of incidence ($\rho \ll 1$),

$$\begin{aligned} C_1^- &= R_L^{1/2} \exp(2\mathrm{i}\mathcal{Z})C_1^+ - \frac{2|C_1^+|\rho L}{A_L \Delta}\left(\mathrm{Ai}'(0)\right)^2 \\ &\quad \times \left[1 - \frac{1}{2}(1 - R_L^{1/2})(1 + \mathrm{i}\sqrt{3})\right] \exp\left(\mathrm{i}\mathcal{Z} + \mathrm{i}\arg B_1(0)\right)\mathcal{I}\,. \end{aligned} \tag{3.11}$$

In these formulas, $R_L = |C_{1L}^-/C_1^+|^2$ and $G_L = 1 - R_L$ are the reflection and absorption coefficients in the linear theory, and C_{1L}^- is the amplitude of the magnetic field of the reflected wave in the linear approximation; $A_L = a_L L^2/\Delta^2$, where a_L is the dimensionless amplitude of the magnetic field at the plasma resonance point in the linear theory; and $\mathcal{I}$ is given by (B.11). At large angles of incidence ($\rho > 1$),

$$\begin{aligned} R_L &= \left(\frac{1 - (1/2)\exp\left[-(4/3)\rho^{3/2}\right]}{1 + (1/2)\exp\left[-(4/3)\rho^{3/2}\right]}\right)^2, \\ a_L &= \left|\frac{cB_0^2 e^2}{\pi m^2}\,\frac{|\cos\theta|}{\omega_0^5 L^3}(1 - R_L)\right|^{1/2}, \quad B_0 = 2C_1^+. \end{aligned} \tag{3.12}$$

At small angles of incidence ($\rho \ll 1$),

$$\begin{aligned} R_L &= \left[1 - 2\pi^2\left(\mathrm{Ai}'(0)\right)^2 \rho\right]^2, \\ a_L &= \left|\frac{2e\sin\theta}{m\omega_0^2 L}\left(\frac{c}{\omega_0 L}\right)^{1/6}\left(\pi|\cos\theta|\right)^{1/2}\mathrm{Ai}'(0)(1 - R_L^{1/2})C_1^+\right|. \end{aligned} \tag{3.13}$$

The prime after Ai in (3.13) denotes taking the first derivative of the Airy function.

3.2 Removing divergence in the reflection coefficient: taking relativistic effects into account

It follows from (3.9) and (3.11) that the reflected magnetic field amplitude is represented as the sum of a linear term, C_{1L}^-, which is independent of a, and a nonlinear term, C_{1N}^-, viz. $C^- = C_{1L}^- + C_{1N}^-$, where the dependence of C_{1N}^- on a (actually, on the laser light intensity) is given by the integral $\mathcal{I}$ in (B.11). In linear theory, the dimensionless amplitude $a = a_L$ is determined by relations (3.12) and (3.13), connecting the magnetic field amplitude on the plasma resonance with the incident wave amplitude via the reflection coefficient R_L [37, 108]. It is easy to verify that, if $a_L \propto B_0$ (see (3.12) and (3.13)) is used in (3.9) and (3.11) (more precisely, in the integral $\mathcal{I}$) instead of the amplitude a that takes the nonlinearity into account, the C_{1N}^- values are overestimated as the laser radiation intensity increases.[8] We demonstrate this using formula (3.9) as an example.

[8] Naturally, at intensities that are limited by the condition that plasma oscillations do not break at the resonance.

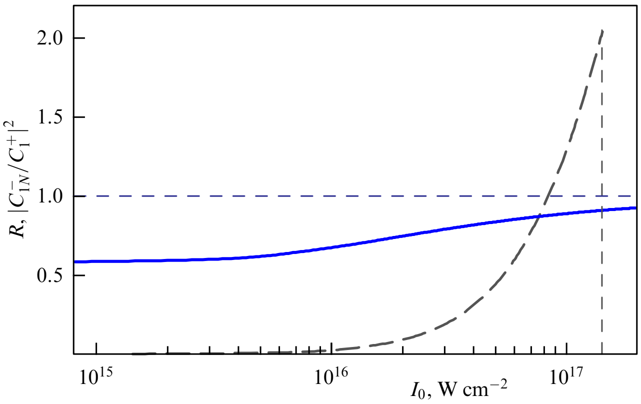

Figure 11. Nonlinear reflection coefficient R at the fundamental frequency after renormalization procedure (blue solid curve) and the value of $|C_{1N}^-/C_1^+|^2$ without taking renormalization into account (dashed curve) depending on intensity of Nd laser field I_0 for temperature $T = 2$ keV and plasma inhomogeneity scale $L = 30\lambda$ at angle of incidence of laser radiation $\theta_{\min} = 10°$. Dashed vertical line corresponds to threshold of the breaking of electron plasma oscillations at the resonance point when using amplitude a_L from the linear theory. Dashed horizontal line corresponds to asymptotic limit of total reflection $R = 1$.

Below, when substituting specific laser and plasma parameters, for definiteness, we speak of the laser intensity I_0 [W cm^{-2}] of an Nd laser with the wavelength $\lambda = 1.064$ μm, inhomogeneity scale $L\,[\lambda]$ expressed in laser radiation wavelengths, plasma temperature T [keV], laser radiation incidence angle θ, and self-similar variable ρ. Because the nonlinear solution of Eqn (3.4) in (3.9) is constructed on the basis of a linear solution of the homogeneous equation, which exists at not too small angles for $\rho > 1$, the value of the parameter $\rho_{\min} \equiv (\omega_0 L/c)^{2/3} \sin^2\theta_{\min} = 1$ can be regarded as the limit for the theory in this case, and $\theta_{\min}$, as the minimum admissible angle of incidence of laser radiation on the plasma for given ω_0 and L.

Figure 11 shows the unbounded increase in $C_{1N}^-(a_L)$, as a result of which, starting with a certain laser intensity level, $C_{1N}^-(a_L)$ exceeds the field amplitude of laser pumping, which is physically impossible. This behavior of the nonlinear contribution is associated with the unjustified use in the source $f_1(x)$ of the amplitude a_L taken in the linear approximation, Eqn (3.12), which does account for the nonlinear relation between the plasma field amplitude at $x = 0$ and the laser radiation field amplitude. This demonstrates the need to construct a self-consistent nonlinear theory of reflection of laser radiation by an inhomogeneous plasma with a magnetic field amplitude different from (3.12) and (3.13) at the resonance point, $a \neq a_L$, which would take the nonlinear dependence of $|B_1(0)|$ on C_1^+ into account and where the source function $f_1(x)$ would be renormalized to the new amplitude a. As we see in what follows, the unbounded increase in the nonlinear part of the reflection coefficient $C_{1N}^-(a)$ is replaced by saturation as a result of such renormalization. The dependence of the total nonlinear reflection coefficient $R = |C_1^-/C_1^+|^2$ at the fundamental frequency on the laser intensity for $\rho > 1$, after renormalizing to the amplitude a with the nonlinearity taken into account, is shown by the blue solid curve in Fig. 11. Finding the analytic relation between $B_1(0)$ and C_1^+ and obtaining a formula corresponding to the saturation of the nonlinear increase in $C_{1N}^-(a)$ observed in Fig. 11 is the subject of this section.

We explicitly outline a sequence of steps to eliminate the divergence in the case $\rho > 1$; for $\rho \ll 1$, we present the final result obtained by following the scheme described below (see [44] for the details). Based on (B.5), we write a formula for the first harmonic of the magnetic field at the resonance point $x = 0$:

$$R_1(0) = \alpha_1^+ \Psi_1^+(0) + \alpha_1^- \Psi_1^-(0) + \frac{\pi c L}{\omega_0} \int_{-\infty}^{0} \mathrm{d}\xi\, \frac{f_1(\xi)}{\mathrm{i}\Delta - \xi} \left[\Psi_1^+(0)\Psi_1^-(\xi) - \Psi_1^+(\xi)\Psi_1^-(0)\right]. \tag{3.14}$$

Substituting expression (B.6) that relates α_1^- to α_1^+ into (3.14), we obtain

$$\begin{aligned} R_1(0) &= \alpha_1^+ \left(\Psi_1^+(0) + \frac{\mathrm{i}}{2} \exp\left[\frac{2\omega_0}{c}\,\mathcal{L}_+(0)\right] \Psi_1^-(0)\right) \\ &\quad - \frac{\mathrm{i}}{2}\left(\frac{2cL}{\omega_0}\right)^{1/2} \Psi_1^-(0) \exp\left[\frac{\omega_0}{c}\,\mathcal{L}_+(0)\right] \\ &\quad \times \int_{-\infty}^{\infty} \mathrm{d}\xi\, f_1(\xi)\, \mathrm{K}_1\left[(\xi - \mathrm{i}\Delta)k_y\right] \\ &\quad + \frac{\pi c L}{\omega_0} \int_{-\infty}^{0} \mathrm{d}\xi\, \frac{f_1(\xi)}{\mathrm{i}\Delta - \xi} \left[\Psi_1^+(0)\Psi_1^-(\xi) - \Psi_1^+(\xi)\Psi_1^-(0)\right] \\ &= \Lambda_1 + \Lambda_2 + \Lambda_3\,. \end{aligned} \tag{3.15}$$

We compare the two integral contributions Λ_2 and Λ_3 to (3.15). To estimate the integrals, we use (B.2) and approximate formulas for the Infeld and Macdonald functions at small values of the argument, $\mathrm{K}_1(x)|_{x\approx 0} \approx 1/x$ and $\mathrm{I}_1(x)|_{x\approx 0} \approx x/2$:

$$\begin{aligned} &\Psi_1^+(0)\Psi_1^-(\xi) - \Psi_1^+(\xi)\Psi_1^-(0) \approx \frac{\omega_0 \xi (2\mathrm{i}\Delta - \xi)}{2\pi L c}\,, \\ &\Lambda_2 \approx \frac{\mathrm{i}}{2\pi(\xi - \mathrm{i}\Delta)k_y^2}\,, \quad \Lambda_3 \approx \frac{\xi(2\mathrm{i}\Delta - \xi)}{2(\mathrm{i}\Delta - \xi)}\,. \end{aligned} \tag{3.16}$$

It follows from the second inequality in the applicability conditions of our model in (3.3) that $\Lambda_2 \gg \Lambda_3$, and therefore the contribution of Λ_3 can be disregarded, given the smallness parameter $k_y^2 \Delta^2 \ll 1$. Thus, we obtain a simplified expression for the magnetic field amplitude at the plasma resonance:

$$R_1(0) = \Lambda_1 + \Lambda_2 = \alpha_1^+ \Psi_1^+(0) + \alpha_1^- \Psi_1^-(0)\,. \tag{3.17}$$

Hence, using the relation of α_1^+ to α_1^- in (B.6) together with formulas (B.8), we find

$$\begin{aligned} R_1(0) &= \left(\frac{2c|\cos\theta|}{\pi\omega_0 L \sin^2\theta}\right)^{1/2} \frac{\exp\left[\mathcal{L}_+(0) + \mathrm{i}\mathcal{Z}\right]}{1 + (1/2)\exp\left[(2\omega_0/c)\mathcal{L}_+(0)\right]}\, \widetilde{C}_1^+ \\ &\quad - \frac{\mathrm{i}/\pi k_y^2}{1 + (1/2)\exp\left[(2\omega_0/c)\mathcal{L}_+(0)\right]} \int_{-\infty}^{\infty} \mathrm{d}\xi\, \frac{f_1(\xi)}{\xi - \mathrm{i}\Delta}\,. \end{aligned} \tag{3.18}$$

Formula (3.18) can be interpreted as a nonlinear equation for the magnetic field amplitude at the plasma resonance, because the right-hand side of this expression contains the integral determined by the source $f_1(\xi)$, where all functions are normalized to the sought amplitude. The solution of (3.18) for this amplitude determines its nonlinear relation to the pump field amplitude.

Passing from $\widetilde{C}_1^+$ to C_1^+ by multiplying $\widetilde{C}_1^+$ by (am_e/e) and using normalization (2.4), albeit now outside the linear

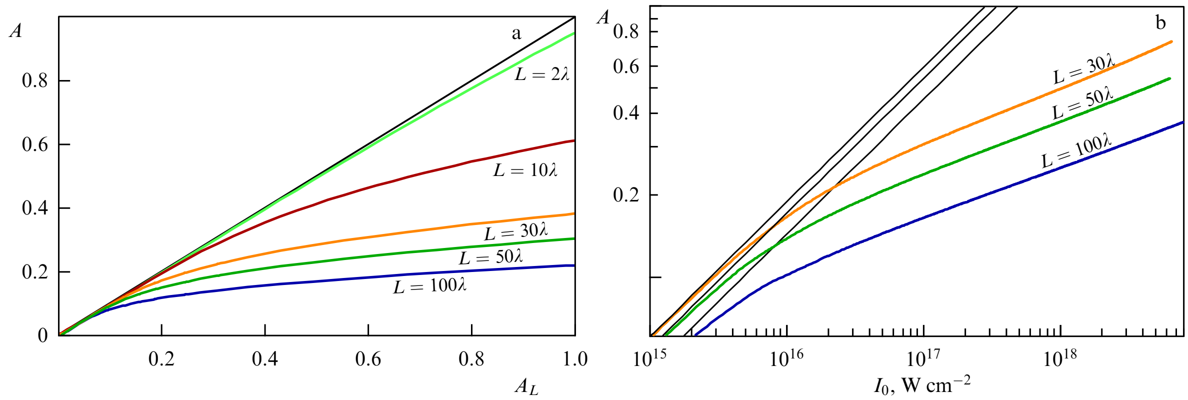

Figure 12. Dependences of amplitude A in the nonlinear theory (a) on amplitude A_L defined by the linear approximation, and (b) on energy flux density I_0 of Nd laser radiation, calculated for plasma electron temperature $T = 2$ keV, inhomogeneity scales $L = 2\lambda$, $L = 10\lambda$, $L = 30\lambda$, $L = 50\lambda$, and $L = 100\lambda$, at corresponding minimum angles of incidence of laser radiation on the plasma $\theta = \theta_{\min}$. Black bisector in (a) and black lines in (b) correspond to linear-theory limit.

approximation (3.12), we rewrite Eqn (3.18) in the form

$$\frac{\omega_0^2}{2}\left(aL - \frac{\Delta\mathcal{I}}{\pi^2\left(1+(1/2)\exp\left[(2\omega_0/c)\mathcal{L}_+(0)\right]\right)}\right)\exp\left[\mathrm{i}\arg B_1(0)\right]$$
$$= \frac{|e|}{m}\,|C_1^+|\left(\frac{2c|\cos\theta|}{\pi\omega_0 L}\right)^{1/2}\frac{\exp\left[\mathcal{L}_+(0)+\mathrm{i}\mathcal{Z}+\mathrm{i}\arg C_1^+\right]}{1+(1/2)\exp\left[(2\omega_0/c)\mathcal{L}_+(0)\right]} \tag{3.19}$$

or, equivalently, in terms of dimensionless variables (amplitudes) $A = aL^2/\Delta^2$ and $A_L = a_L L^2/\Delta^2$, where a_L is determined from (3.12), in the form

$$\left(A - \frac{L/\Delta}{2\pi^2}(1+R_L^{1/2})\mathcal{I}\right)\exp\left[\mathrm{i}\arg B_1(0)\right]$$
$$= A_L\exp\left[\mathrm{i}\mathcal{Z}+\mathrm{i}\arg C_1^+\right]. \tag{3.20}$$

Because the integral $\mathcal{I}$ in (B.11) is a complex function of the amplitude a, and hence of the dimensionless amplitude A, it follows that expression (3.20) is a nonlinear transcendental complex equation for the dimensionless magnetic field amplitude at the plasma resonance as a function of the linear-theory amplitude $A = A(A_L)$, and therefore as a function of the incident wave amplitude $A = A(C_1^+)$.

Calculating A in (3.20) solves the problem of finding the nonlinear relation between the magnetic field amplitude at $x = 0$ and the pump field amplitude; it allows renormalizing the functions in (3.9) to a new amplitude by using the established connection. Complex equation (3.20) is equivalent to a pair of equations for the moduli and arguments of complex functions on the right- and left-hand sides of (3.20),

$$\left|F(A)\right| = A_L\,,\quad \phi+\phi_1 = \mathcal{Z}+\phi_0\,, \tag{3.21}$$

where

$$F(A) = A - \frac{L/\Delta}{2\pi^2}(1+R_L^{1/2})\mathcal{I}\,,$$
$$\phi = \arg F(A)\,,\quad \phi_0 = \arg C_1^+\,,\quad \phi_1 = \arg B_1(0)\,. \tag{3.22}$$

To calculate the integral in (B.11) and solve the moduli equation in (3.21), we used the Wolfram Mathematica computer algebra package [122]. Figure 12a shows the dependences of the amplitude A on the linear-theory amplitude A_L for five values of the inhomogeneity scale L. The dependences of A on the laser radiation energy flux density I_0 for three values of the inhomogeneity scale L are shown in Fig. 12b. The plasma temperature T is fixed, and the angles of incidence are equal to the minimum ones for the corresponding L, $\theta = \theta_{\min}$. The nonlinearity in the resonance region leads to a drop in the field amplitude at $x = 0$ compared with the linear theory result, and a saturation effect is observed: deceleration of the growth of the resonance field amplitude as the pump field amplitude increases. In the case of a weakly inhomogeneous plasma, $L \simeq (10-100)\lambda$, the difference between A and A_L increases quite significantly as the laser intensity increases, whereas, for relatively sharper inhomogeneity gradients $L \simeq \lambda$, the difference between A and the amplitude A_L is weakly noticeable.

We similarly obtain equations for the resonant amplitude and phase in the limit of small angles of incidence ($\rho \ll 1$),

$$\left|F(A)\right| = A_L\,,\quad \phi+\phi_1 = \mathcal{Z}+\phi_0\,, \tag{3.23}$$

where

$$F(A) = A + \frac{L/\Delta}{\pi^2}(1-R_L^{1/2})\exp\left(\frac{\mathrm{i}\pi}{3}\right)\mathcal{I}\,,$$
$$\phi = \arg F(A)\,,\quad \phi_0 = \arg C_1^+\,,\quad \phi_1 = \arg B_1(0)\,, \tag{3.24}$$

and the dimensionless amplitude a_L and the linear reflection coefficient R_L at the fundamental frequency are given by formulas (3.13).

3.3 Applicability limits of the hydrodynamic model in terms of physical parameters

Renormalizing the source function to the plasma field amplitude at the resonance and taking the nonlinearity into account allows the range of applicability of the hydrodynamical model (see Fig. 8) to be studied in terms of the physical parameters of the laser and plasma. The applicability limit is determined by the breaking of the plasma oscillation profile at the resonance point [14], which occurs when the laser energy flux density reaches a threshold. Therefore, the drop in the resonant amplitude identified above in the relativistic regime shifts this threshold value to the region of higher laser intensities compared with the values following from the nonrelativistic theory [14].

Figure 13 shows the breaking thresholds on the $\{T, I_0\}$ plane for different values of the plasma inhomogeneity scale. The figure confirms the conclusion that the applic-

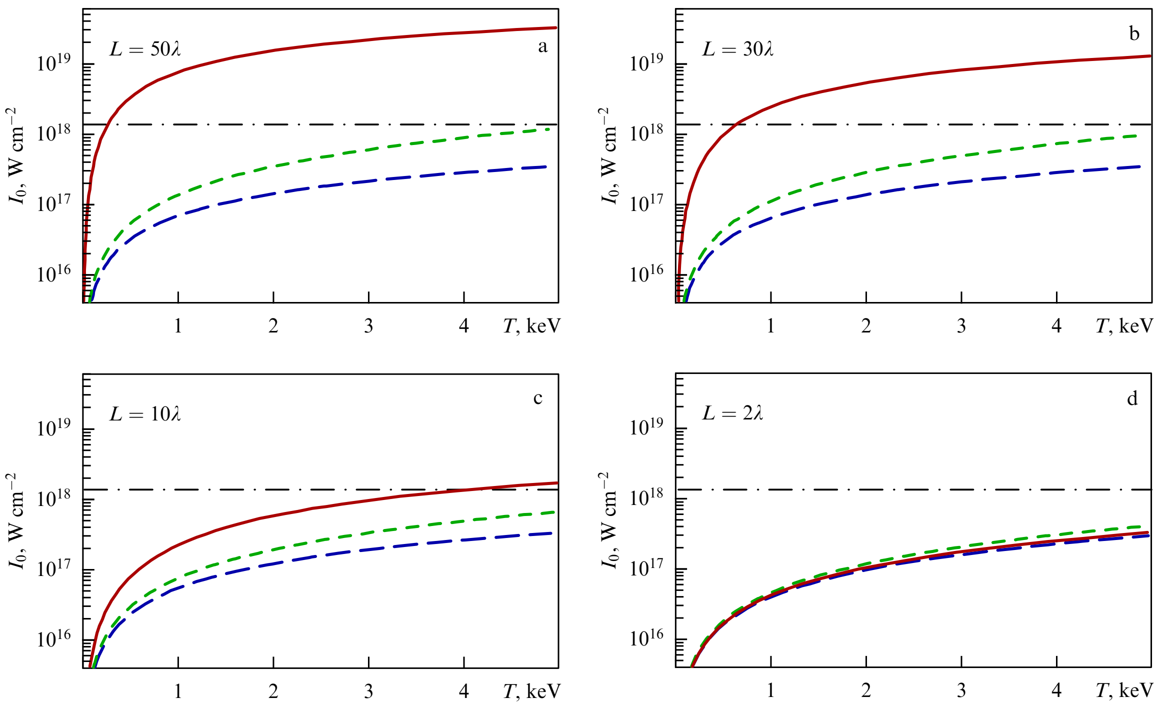

Figure 13. Domains of applicability of the theory (hydrodynamic model) on the parameter plane $\{T, I_0\}$ for an Nd laser and various plasma density inhomogeneity scales (a) $L = 50\lambda$, (b) $L = 30\lambda$, (c) $L = 10\lambda$, and (d) $L = 2\lambda$. Solid red curve shows the breaking threshold in relativistic hydrodynamics, with nonlinear renormalization to the amplitude A at the resonance. Blue dashed line corresponds to the relativistic breaking threshold, but without taking nonlinear renormalization into account. Green dashed curve corresponds to the breaking threshold in the nonrelativistic theory [14]. Angles of incidence are everywhere equal to the minimum, $\theta = \theta_{\min}$, for corresponding L. Horizontal dashed-dotted line delineates the relativistic intensities of the laser field.

ability limits of the plasma resonance theory are expanded to the range of the maximum possible laser field intensity, where the formulas of the model that we use for collisionless hydrodynamics of cold electron plasma remain single valued. Because a significant difference between the amplitudes in the nonlinear theory and those in the linear approximation is observed for smooth plasma density gradients $L \simeq (10-100)\lambda$ (see Fig. 12), it follows that the expansion of the applicability domain of the hydrodynamical model is most conspicuous in this case: the breaking threshold shifts to the range of relativistic laser intensities, far beyond the limits predicted by the nonlinear nonrelativistic theory [14] (green dashed curve). With a sharper inhomogeneity gradient (Fig. 13d), the positions of the breaking thresholds differ insignificantly. Figure 13 also shows the breaking thresholds calculated with the relativistic effects of the motion of plasma electrons partly taken into account,[9] but with the use of linear relation (3.12) between the amplitude of the magnetic component of the electromagnetic field at $x = 0$ and the pump field amplitude. In this case, on the contrary, a comparison with the nonrelativistic theory shows a shift of the breaking threshold to the range of lower laser intensities. We emphasize once again that relativistic effects are taken into account in our constructions only in a narrow region of the plasma resonance, where an electron moves in a self-consistent resonantly amplified plasma field, while nonrelativistic laser field intensities are understood. Therefore, in subsequent calculations of the absorption and HG efficiencies, we do not consider laser radiation intensities exceeding the relativistic threshold,[10] but limit ourself from above to the intensity at which the dimensionless amplitude of the laser field $a_0 = 0.85 \times (I_0\,[10^{18}\ \text{W cm}^{-2}]\,\lambda\,[\mu\text{m}]^2)^{1/2}$ is equal to unity, $a_0 = 1$. In this regard, the breaking thresholds shown in Fig. 13 that lie in the relativistic region, where $a_0 > 1$, should be considered from a formal standpoint. We indicate this by dashed-dotted horizontal lines in Fig. 13, which cut off the regions of relativistic laser field intensities.

We note that the breaking condition obtained in [14] (the green dashed curve in Fig. 13) coincides with the well-known condition $k_p e E_p / m\omega_p^2 = 1$, where $k_p = (\lambda_D^2 L)^{-1/3}$, $\omega_p \equiv \omega_L$, and E_p is the plasma electric field (see, e.g., [88]). The breaking of stationary plasma oscillations discussed in this study has a threshold nature and occurs only when the oscillation amplitude exceeds a threshold value. This regime does not correspond to that of plasma oscillations [18, 22] used in discussing the nonrelativistic dynamics of plasma electrons in the vicinity of the critical density in the framework of the capacitor model and in considering forced oscillations of an inhomogeneous plasma layer in an external uniform high-frequency electric field. Due to the plasma inhomogeneity, such oscillations are nonstationary and the breaking occurs in a finite period of time.

3.4 Relativistic nonlinear resonance absorption coefficient

With the relativistic nonlinear relation between the amplitude of the plasma resonance field and the pump field amplitude

[9] In the language of RG transformations, this means that only the velocity amplitude transformation is taken into account, and the time transformation, which is responsible for phase modulation, is disregarded.

[10] A laser field is called relativistic if an electron moving over the distance of one wavelength in this field gains kinetic energy equal to the rest energy.

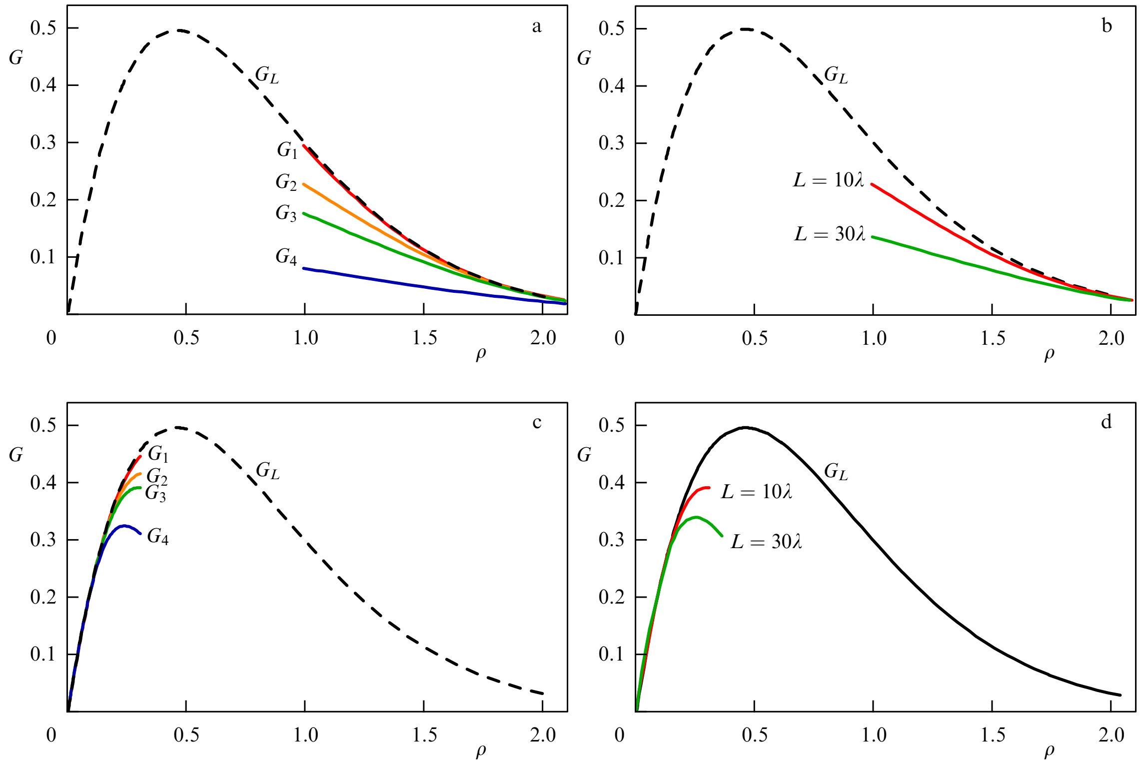

Figure 14. Dependences of the RA coefficient G on the self-similar variable ρ at (a, b) large and (c, d) small angles of incidence. (a, c) Graphs are plotted at fixed values of plasma temperature $T = 2$ keV and plasma inhomogeneity scale $L = 10\lambda$ for different intensities of an Nd laser: $I_0 = 10^{16}$ W cm^{-2} (G_1), $I_0 = 5 \times 10^{16}$ W cm^{-2} (G_2), $I_0 = 10^{17}$ W cm^{-2} (G_3), and $I_0 = 5 \times 10^{17}$ W cm^{-2} (G_4). (b, d) Laser intensity is fixed at $I_0 = 5 \times 10^{16}$ W cm^{-2}, and solid colored curves correspond to plasma inhomogeneity scales $L = 10\lambda$ and $L = 30\lambda$. Dashed black curve G_L corresponds to the linear-theory absorption coefficient.

found in Section 3.2, we now obtain the RA coefficient. We consider the case $\rho > 1$. Substituting $\mathcal{I}$ found from the first equation in (3.21) into C_1^- given by (3.9) and taking the equality of phases into account, we use the second equation in (3.21) to find the amplitude of the magnetic field emitted from the plasma at the fundamental frequency:

$$C_1^- = -\mathrm{i} C_1^+ \exp\left(2\mathrm{i}\,\frac{\omega_0}{c}\,\mathcal{L}(-\infty)\right)\left\{R_L^{1/2} + (1 - R_L^{1/2}) \times \left[1 - \frac{A}{A_L}\exp(-\mathrm{i}\phi)\right]\right\}. \tag{3.25}$$

Formula (3.25) takes the nonlinearity of the plasma resonance field amplitude into account, which is expressed by moving from the amplitude A_L in the linear theory of reflection to the amplitude A calculated from (3.21). The reflection coefficient can then be written in the form

$$R = \left|\frac{C_1^-}{C_1^+}\right|^2 = \left|R_L^{1/2} + (1 - R_L^{1/2})\left[1 - \frac{A}{A_L}\exp(-\mathrm{i}\phi)\right]\right|^2. \tag{3.26}$$

Numerical and analytic (see Appendix B) studies of the integral $\mathcal{I}$ in (B.11) show that it is a purely imaginary quantity. Therefore, from (3.21) and (3.22), we then have the equalities

$$\phi = \arg F(A) = \arccos\frac{A}{|F(A)|} = \arccos\frac{A}{A_L}\,, \quad \cos\phi = \frac{A}{A_L}\,, \tag{3.27}$$

which allow writing the reflection coefficient in a more compact form. A chain of identical transformations gives

$$\left|R_L^{1/2} + (1 - R_L^{1/2})\left[1 - \frac{A}{A_L}(\cos\phi - \mathrm{i}\sin\phi)\right]\right|^2 = 1 - \left(\frac{A}{A_L}\right)^2 (1 - R_L)\,. \tag{3.28}$$

Finally, we obtain the nonlinear reflection coefficients R and the absorption coefficient G for $\rho > 1$:

$$R = 1 - \left(\frac{A}{A_L}\right)^2 (1 - R_L)\,, \quad G = \left(\frac{A}{A_L}\right)^2 G_L\,. \tag{3.29}$$

Formulas (3.29) clearly demonstrate the limit transition to the linear theory as $A \to A_L$, i.e., the upper bound for the reflection coefficient R and the absorption coefficient G by their limit values R_L and G_L. The decrease in the RA coefficient with increasing laser energy flux density can be seen from Fig. 14a, which shows the dependences of G on the self-similar variable ρ at a fixed plasma temperature and inhomogeneity scale ($\rho > 1$). Figure 14b demonstrates the decrease in G under the transition to a weakly inhomogeneous plasma at a fixed laser intensity.

In a similar way (see details in [44]), we find the reflection coefficient $R = |C_1^-/C_1^+|^2$ and the absorption coefficient G in the limit $\rho \ll 1$:

$$R = \left|R_L^{1/2} + \left[\exp\left(\frac{\mathrm{i}\pi}{3}\right) - R_L^{1/2}\right]\left[1 - \frac{A}{A_L}\exp(-\mathrm{i}\phi)\right]\right|^2, \quad G = 1 - R\,. \tag{3.30}$$

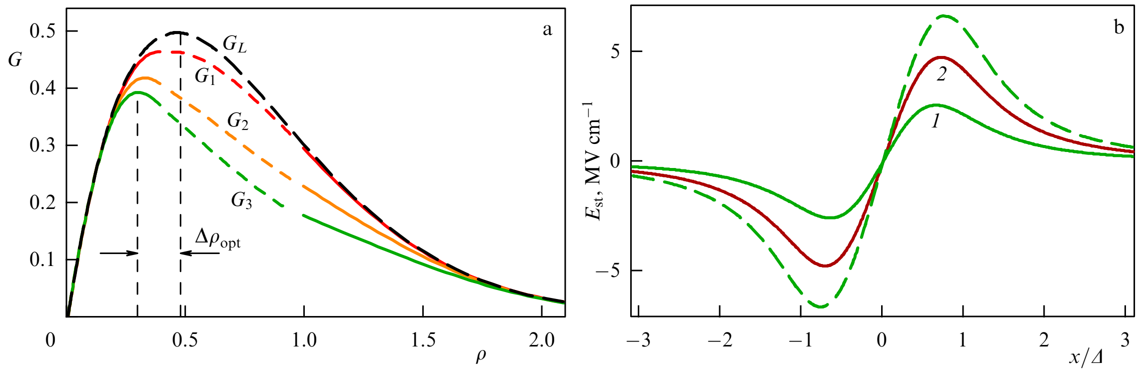

Figure 15. (a) Asymptotic forms of nonlinear absorption coefficient $G(\rho)$ reconstructed in intermediate range $\rho \approx 0.5$ for different intensities of an Nd laser $I_0 = 10^{16}$, 5×10^{16}, and 10^{17} W cm^{-2} (G_1, G_2, and G_3, respectively). Plots correspond to fixed temperature $T = 2$ keV and plasma inhomogeneity scale $L = 10\lambda$. Black dashed curve G_L corresponds to absorption in the linear theory [6, 67]. (b) Spatial distributions of quasistatic electric field of plasma resonance calculated at different values of flux density I_0 of Nd laser radiation for plasma temperature $T = 2$ keV, inhomogeneity scale $L = 30\lambda$, and angle of incidence $\theta_{min} = 10°$. Curves *1* and *2* correspond to respective intensities $I_0 = 10^{17}$ W cm^{-2} and $I_0 = 10^{18}$ W cm^{-2}. Dashed curve describes spatial profile of the field in the nonlinear nonrelativistic theory [14] at intensity $I_0 = 10^{17}$ W cm^{-2}.

We emphasize the fundamental importance of taking relativistic effects on a plasma resonance into account when finding the nonlinear RA coefficient. An analytic estimate of the integral $\mathcal{I}$ in (B.11) shows that it is nonzero only when relativistic nonlinearity is taken into account (see Appendix B). Namely, for finite values of A at $c = \infty$ and $B = 0$, integral (B.11) strictly vanishes:

$$\mathcal{I}|_{c=\infty} = \mathcal{I}_0 = 0\,. \tag{3.31}$$

Moreover, it follows from (3.21), (3.23) and (3.29), (3.30) that $A = A_L$ and $R = R_L$, i.e., there is no nonlinear contribution to the reflection coefficient in the nonrelativistic limit, even when nonrelativistic nonlinearity is taken into account, and linear relations (3.12) and (3.13) remain valid. Thus, the difference between R and R_L arises only in the relativistic theory of nonlinear plasma resonance.

Figure 14c shows the curves of the nonlinear RA coefficient G calculated in the limit $\rho \ll 1$ using formula (3.30) for various radiation flux densities of an Nd laser at a fixed plasma temperature and a fixed inhomogeneity scale. As in the case $\rho > 1$, the RA coefficient decreases as the laser radiation intensity increases. Figure 14d corresponds to a fixed laser intensity and demonstrates an increase in the absorption suppression effect under the transition to a weakly inhomogeneous plasma, which is consistent with the tendency for the resonant amplitude of the plasma field to saturate.

We show that an analysis of the RA coefficient $G(\rho)$ for $\rho > 1$ and $\rho \ll 1$ not only leads to a general conclusion about the suppression of absorption but also allows formulating a hypothesis on a change in the optimal angle of incidence of laser radiation on the plasma, i.e., on the shift of the maximum of the $G(\rho)$ curve relative to the maximum predicted in the linear theory. Indeed, comparing the $G(\rho)$ curves for $\rho > 1$ and $\rho \ll 1$ (see Fig. 14) at a fixed intensity I_0, we see an asymmetry of suppression to the left and to the right of the maximum of $G(\rho)$. Namely, in the region of large angles of incidence ($\rho > 1$), absorption is suppressed more significantly than at small angles. This asymmetry may mean that an increase in the intensity of the pump field leads not only to a decrease in the absorption coefficient in a wide range of ρ but also to a shift of the absorption maximum to the region of smaller ρ or, equivalently, to the range of smaller incidence angles θ of laser radiation onto the plasma. Although, as already mentioned, expressions (3.30) and (3.29) are asymptotic in nature and reflect the behavior of the absorption coefficient in two limit cases, and hence, strictly speaking, do not allow the coefficient to be calculated in the entire range of ρ, we can nevertheless use general similarity considerations to interpolate the function $G(\rho)$ in the intermediate region $\rho \approx 0.5$, combining the asymptotics for $\rho > 1$ and $\rho \ll 1$ at a fixed laser radiation intensity (Fig. 15a). As an example, we take $I_0 = 10^{16}$, 5×10^{16}, and 10^{17} W cm^{-2} and connect the asymptotic forms corresponding to these intensities (dashed colored curves in Fig. 15a). In these cases, the absorption maximum decreases to the respective values $G_m \approx 0.47$, $G_m \approx 0.43$, and $G_m \approx 0.4$, which is in qualitative agreement with the results in [39, 40], and the optimum value ρ_{opt} corresponding to such maxima shifts from the point $\rho_{opt} \approx 0.5$ towards lower ρ and reaches $\rho_{opt} \approx 0.3$ at $I_0 = 10^{17}$ W cm^{-2}.

Let us emphasize the following circumstance. We compare the absorption coefficients at different laser intensities I_0 keeping the scale of plasma density inhomogeneity L fixed, although it is known that L can change as a result of plasma expansion and the action of the ponderomotive force of the laser field [6, 64, 118–121, 123–129], which we do not take into account, but which ultimately influence the absorption value. For example, according to [39, 40], when the laser radiation intensity surpasses $I_0 \approx 3 \times 10^{17}$ W cm^{-2}, the decrease in the absorption coefficient is replaced by its increase due to the nonlinear effect of steepening of the plasma density profile by the ponderomotive effect of incident electromagnetic radiation. On the other hand, we note that the jump in the density profile in the resonance region tends to smear over time due to the hydrodynamical spreading of plasma over times of the order of a nanosecond and fractions of a nanosecond (see, e.g., [130]). We believe that the change in the density profile can be taken into account in our theory in the first approximation in terms of the parameter L by assuming that the condition of weak plasma inhomogeneity is satisfied. Although L depends on I_0 in general, L and I_0 are external, formally independent control parameters in our case, and the relation between them can be determined, e.g., by methods of numerical modeling of plasma hydrodynamics. The theory 'demonstrates how the absorption value is influenced just by the relativistic nonlinearity of electron motion in a resonantly amplified plasma field for given I_0, T, and L and can be regarded as part of larger-scale computation schemes, including the use of numerical methods to study laser–plasma interaction. Systematic analytic consideration of the self-consistent deformation of the density profile of an inhomogeneous plasma under space–time modulation of the phase of relativistic electron oscillations is a nontrivial problem in need of a separate detailed consideration in the further development of the theory.

3.5 Generation of an electrostatic field near the critical density

When studying electromagnetic fields excited near the critical density, *quasistatic fields*, in addition to fundamental and higher harmonics, are also of interest. Their generation mechanism is based on the nonlinear interaction of high-frequency oscillations such that the frequency of the resulting mode formed when adding them is zero. The effect of a quasistatic field generation is trivial when considered for weak nonlinearity, when the ratio of the amplitude of electron oscillations in a resonantly amplified potential electric field to the width of the plasma resonance is much smaller than unity; such a mechanism corresponds to the beating of two main harmonics of the plasma field and gives a well-known expression for the force of high-frequency radiation pressure (the Miller force). For strong nonlinearity, when that ratio is comparable to unity, the spectrum of the nonlinear plasma field in the vicinity of the resonance is enriched with higher harmonics of finite intensity. In this case [14], the contribution to the generation of a quasistatic field comes from the entire variety of interactions of harmonics of the fundamental frequency $\{1+1, 2+2, \ldots, n+n, \ldots\}$, whose calculation in the presence of various types of nonlinearity is a nontrivial problem, important in estimating the field amplitude. In this section, we show that assessing the amplitude of a quasistatic field in the framework of a weakly nonlinear or strongly nonlinear nonrelativistic theory gives an overestimated result, which is corrected when taking relativistic nonlinearity into account [45].

A weakly nonlinear theory of generation of a quasistatic electric field by a p-polarized electromagnetic wave in inhomogeneous plasma, with the dissipation of radiation at the plasma resonance taken into account, was constructed by Bychenkov, Abdullaev, Aliev, and Frolov [17]. The authors found a stationary spatial distribution of the quasistatic electric field in the vicinity of the plasma resonance and showed that it has a bipolar shape. Later, Kovalev and Pustovalov [14] found the structure of quasistatic fields in the regime of strong nonrelativistic nonlinearity and came to a conclusion similar to the one in [17] regarding the bipolarity of the spatial distribution of the electric field, indicating that the quasistatic electric field component changes sign when passing through the plasma resonance. The theory presented here allows estimating not only the nonlinear absorption and reflection coefficients at the fundamental frequency but also the efficiency of generating higher harmonics of laser radiation, as well as the quasistatic electric field in the vicinity of plasma resonance. The next section is devoted to finding plasma emission spectra and their detailed analysis. Here, we illustrate the progress achieved in the theory of plasma resonance using the example of a quasistatic electric field E_{st} localized in the critical density region. This field is the time average of the total longitudinal electric field of the plasma resonance with all spectral components included,

$$E_{\mathrm{st}} = \frac{1}{2\pi} \int_0^{2\pi} E_{p_x}(\tau, x)\, \mathrm{d}\tau \,, \tag{3.32}$$

where E_{p_x} is determined implicitly by the functions $P_0(\chi, l)$, $v_0(\chi, l)$, $x_0(\chi, l)$, and $\tau(\chi, l)$ in (2.21) in terms of the parametric variables χ and l.

Figure 15b shows the spatial distribution of the quasistatic electric field in the vicinity of the plasma resonance point $x = 0$ at fixed plasma temperature and inhomogeneity scale for various laser energy flux densities. It can be seen that electric field (3.32), as in the weakly nonlinear [17] and strongly nonlinear nonrelativistic [14] theories, has a universal bipolar form and changes sign at the plasma resonance point $x = 0$. The nonlinear nonrelativistic theory [14] gives overestimated values of the field amplitude (dashed line in Fig. 15b). Thus, for fixed values of T and L, the maximum amplitude of the static field $E_{\mathrm{st}} \approx 7$ MV cm^{-1} in the nonrelativistic theory is attained already at $I_0 = 10^{17}$ W cm^{-2}, while, in the relativistic theory, we have $E_{\mathrm{st}} \approx 2.5$ MV cm^{-1} at the same laser intensity.

We estimate the characteristic energy $W \sim eE_{\mathrm{st}}\Delta$ acquired by an electron accelerated by a quasistatic electric field over a distance of the order of the plasma resonance width. For the parameters $L = 30\lambda$, $T = 2$ keV, $I_0 \approx 10^{18}$ W cm^{-2}, and $\theta_{\min} = 10°$, the energy of accelerated electrons is $W \simeq 30$ keV. The possible consequences of such a mechanism for the generation of fast electrons have yet to be understood, although it can already be concluded that, due to the bipolarity of the electric field, fast electrons experience the action of a multidirectional static electric force at the plasma resonance and therefore do not leave its narrow spatial region. This allows us to conclude that such electrons do not exert an additional parasitic effect of preheating the ICF target, suppressing which is one of the most important tasks in identifying the optimal conditions for the implementation of ICF in the direct heating scheme [18–21, 24–26, 33].

4. Generation of higher harmonics

Processes of HG in laser plasma formed when laser radiation interacts with solid targets have been the subject of both experimental and theoretical studies for half a century [15]. HG was first observed in 1970 in [131], where the second harmonic of laser radiation was obtained, and was subsequently also demonstrated in [132, 133]. It soon became clear that a detailed study of the characteristics of secondary radiation could serve as a source of rich information about the state of laser plasma. What is meant here is a plasma diagnostics technique based on the study of the second harmonic, which allows obtaining information about the evolution of laser plasma parameters with high temporal and spatial resolution. The first proposals for using measurements of the intensity and spectrum of the second harmonic to determine various laser plasma parameters in the vicinity of the critical density were put forward by Pustovalov, Vinogradov, and Silin [134, 135]. Estimates of the local temperature and the characteristic density inhomogeneity scale of laser plasma obtained later by various groups [136–139] demonstrated the feasibility of developing methods for diagnosing parameters of the plasma from its radiation in the frequency range near the double frequency of the pump field. In addition, it was noted in [140] that, in the framework of diagnostics with the use of the second harmonic, information can be gleaned not only about the macroscopic parameters of the plasma but also about the level and spectrum of parametric turbulence developing in it.

The experiments in [131–133] were soon followed by others to detect the third harmonic [141] and higher harmonics up to the 11th and $n = 2-5$ for a CO_2 [54] and neodymium laser [142], respectively. Higher-order harmonics were registered in 1981 during experiments in [55, 143] on the interaction of a CO_2 laser at an intensity of 10^{14}–10^{16} W cm^{-2} with solid-state targets. Laser frequency harmonics up to the 29th [143] and even 46th [55] were observed. Moreover, the spectra of secondary radiation were obtained for the first time, characterized by a slow (nonexponential) decrease in the intensities of harmonics with increasing number, as were 'pale fencing' spectra, with the intensities of neighboring harmonics equalized.

Many theoretical studies of HG in inhomogeneous plasma rely on a mathematical model based on the plasma resonance effect [1, 3]. By the time of the experimental discovery of the second harmonic [131], theoretical ideas were already developed [8] indicating a possible connection of this effect with the excitation of strong Langmuir longitudinal electron oscillations in laser plasma due to a linear conversion of laser radiation. In the range where the frequency of plasma oscillations is close to that of radiation incident on the plasma further interaction of longitudinal plasma oscillations with each other or with laser pump waves leads to the generation of secondary radiation at twice the laser light frequency. Based on these ideas, Erokhin, Zakharov, Moiseev, and Mukhin constructed a weakly nonlinear theory of second harmonic generation [8, 10]. By generalizing the approach proposed in [8], Silin, Vladimirskii, and Trotsenko solved the problem of generating harmonics of arbitrary multiplicity in cold [11] and hot [13] plasmas. They showed that strong plasma oscillations resulting from resonance generate nonlinear electron currents that serve as the source of secondary radiation enriched in higher harmonics. The same mechanism was considered in [144] in connection with the explanation of a number of experiments [55, 143]. The assumption of weak nonlinearity in the vicinity of the resonance allowed the authors of [8, 11, 144] to use the perturbation theory in the pump field amplitude. This assumption allows neglecting the influence of higher harmonics on lower ones, including the fundamental harmonic, whose amplitude is considered fixed. It follows from the perturbation theory that the flux density of secondary radiation at the frequency $n\omega_0$ is proportional to the nth power of the flux density of laser radiation incident on the plasma. This dependence leads to an exponentially fast decrease in harmonic intensities as the number n increases. However, as the pump field amplitude increases, the assumption of weak nonlinearity is violated, and the effects caused by strong nonlinearity must be taken into account. The presence of strong nonlinearity qualitatively changes the process of generating integer harmonics; the higher harmonics, which, according to the standard (weakly nonlinear) perturbation theory [8, 11], were exponentially small, are then significantly amplified, which leads to a smoother decay or even equalization of the intensities of neighboring harmonics [55].

Indications of the possibility of the existence of more gently decaying spectra of radiation from plasma were given by Isichenko and Yankov [145, 146] in a qualitative discussion of the generation of laser radiation harmonics due to the breaking of nonrelativistic electron flows, but without a rigorous theoretical justification. In a series of theoretical studies in the late 1980s [14, 147–153], an analytic theory of the generation of higher harmonics was constructed under conditions of strong nonrelativistic nonlinearity in the vicinity of the critical density of an inhomogeneous plasma. It was shown that, even at laser intensities of $\sim 10^{16}$ W cm^{-2}, which are low by modern standards, it is possible to generate electromagnetic radiation with a spectrum with a much slower decay of harmonic amplitudes with an increase in their number compared with the standard perturbation theory [11]. This was confirmed experimentally in [55, 143]. The basic formulas for the coefficient of laser radiation conversion into harmonics for cold and hot plasmas were then obtained, and their temperature dependences were analyzed [148, 150–152]. This work was continued in recent studies [56, 57], where the issue of the influence of the relativistic effects of the dynamics of the electron component of a plasma near the critical density and under the condition of effective RA on the generation of harmonics was discussed. In what follows, we present the results of these latest theoretical studies, which allow describing the spectral composition of harmonics emitted from the plasma resonance region.

4.1 Solution to the problem of generating higher harmonics on a relativistic plasma resonance

To describe the HG process, we proceed from the previously obtained formula (3.2). With the inequalities (3.3) for $n \geqslant 2$, Eqn (3.2) takes the form

$$\partial_{xx} R_n - \frac{\partial_x \varepsilon_n}{\varepsilon_n}\,\partial_x R_n + \left(\frac{n\omega_0}{c}\right)^2 (\varepsilon_n - \sin^2\theta) R_n = \frac{a}{c}\left\{ \mathrm{i}nk_y v\,\partial_x P + k_y\omega_0 v\partial_x(\gamma_0 v) + \frac{k_y\omega_0}{a}\,\partial_t\big(v(\gamma_0 - 1)\big) - \partial_x(u\partial_x P) + \frac{\mathrm{i}\omega_0}{n}\,\partial_x\left[v\partial_x(\gamma_0 u) + \frac{1}{a}\,\partial_t\big(u(\gamma_0-1)\big)\right]\right\}_n . \tag{4.1}$$

In deriving (4.1), as in previous sections, we neglected the dependence of the frequency on the x coordinate, $\omega_{\mathrm{L}} = \omega_0$, which is justified for weakly inhomogeneous plasma, where the width of the plasma resonance is small compared with the characteristic density inhomogeneity scale L. The hierarchy of electromagnetic field components near the critical density (2.5) allows us to keep only terms proportional to k_y on the right-hand side of (4.1). As a result, we obtain a wave equation with a source in the standard form:

$$\partial_{xx} R_n - \frac{\partial_x \varepsilon_n}{\varepsilon_n}\,\partial_x R_n + \left(\frac{n\omega_0}{c}\right)^2 (\varepsilon_n - \sin^2\theta) R_n = -\frac{4\pi}{c}\left\{\frac{a}{4\pi}\,\mathrm{rot}\,\mathbf{J}_n\right\}_z , \quad n \geqslant 2 , \tag{4.2}$$

where the nonlinear current components are given by

$$\mathbf{J}_n = \left\{ v\,\partial_x P - \frac{\mathrm{i}\omega_0}{n}\,v\,\partial_x(\gamma_0 v) - \frac{\omega_0^2}{a}(\gamma_0 - 1)v ,\; u\partial_x P - \frac{\mathrm{i}\omega_0}{n}\,v\partial_x(\gamma_0 u) - \frac{\omega_0^2}{a}(\gamma_0-1)u ,\; 0 \right\}_n . \tag{4.3}$$

At electron speeds much lower than the speed of light ($\gamma_0 \to 1$), expression (4.3) turns into a formula for the current obtained in constructing the nonlinear nonrelativistic theory [14]. As in the case $n = 1$, it follows from Eqn (4.2) and relation (4.3) that calculating the nth harmonic of the magnetic field in a vacuum requires knowing the nonlinear current determined by the structure of the electric field and the electron velocity in the plasma resonance region described in (2.21).

Similarly to the case of the fundamental harmonic, we write the solution of the inhomogeneous equation (4.2) in terms of the fundamental system Ψ^+ and Ψ^- of solutions of the homogeneous equation as

$$R_n(x) = \alpha_n^+ \Psi_n^+(x) + \alpha_n^- \Psi_n^-(x) + \int_{-\infty}^{x} \mathrm{d}\xi\, \mathcal{G}(x,\xi) f_n(\xi)\,,$$
$$\mathcal{G}(x,\xi) = -\frac{\Psi_n^+(x)\Psi_n^-(\xi) - \Psi_n^+(\xi)\Psi_n^-(x)}{\Psi_n^+(\xi)\Psi_n^{-\prime}(\xi) - \Psi_n^{+\prime}(\xi)\Psi_n^-(\xi)}\,, \tag{4.4}$$
$$f_n(x) = \frac{a}{c}\left\{ \mathrm{i}nk_y\left(vP_x - \frac{\mathrm{i}\omega_0}{n} v(\gamma_0 v)_x - \frac{\omega_0^2}{a}(\gamma_0 - 1)v\right) + \left[uP_x - \frac{\mathrm{i}\omega_0}{n}\, v(\gamma_0 u)_x - \frac{\omega_0^2}{a}(\gamma_0 - 1)u\right]_x \right\}_n .$$

Here, $\alpha_n^\pm$ are constants determined by the boundary conditions for Eqn (4.2) when applied to solution (4.4). The boundary conditions in turn follow from the form of the magnetic field $R_n(x)$ as $x \to \pm\infty$,

$$R_n = \widetilde{C}_n^- \exp\left(-\mathrm{i}\,\frac{\omega_0}{c}\, x\cos\theta\right), \quad x \to -\infty\,,$$
$$R_n = 0\,, \quad x \to +\infty\,, \tag{4.5}$$

where the complex amplitude $\widetilde{C}_n^-$ corresponds to the wave reflected from the plasma layer.

We find the conversion coefficient to harmonics at the angles θ that are not too small, such that $\rho > 1$. The solution of a homogeneous linear equation in various regions relative to the turning point x_{n0} for the harmonic with a frequency $n\omega_0$ has the form (with σ_1 and σ_2 being some constants)

$$R_n = -\mathrm{i}\mathcal{E}_n\left(\frac{3}{2}\,\mathcal{L}_n^-(x)\right)^{1/6}\left\{\sigma_1 \,\mathrm{Ai}\left[-\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^-(x)\right)^{2/3}\right] + \sigma_2\,\mathrm{Bi}\left[-\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^-(x)\right)^{2/3}\right]\right\}, \quad x < x_{n0}\,,$$
$$R_n = -\mathrm{i}\mathcal{E}_n\left(\frac{3}{2}\,\mathcal{L}_n^+(x)\right)^{1/6}\left\{\sigma_1 \,\mathrm{Ai}\left[\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^+(x)\right)^{2/3}\right] + \sigma_2\,\mathrm{Bi}\left[\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^+(x)\right)^{2/3}\right]\right\}, \quad x > x_{n0}\,,$$

where

$$\mathcal{E}_n = \frac{\sqrt{\varepsilon_n}}{\sqrt[4]{\varepsilon_n - \sin^2\theta}}\,,$$
$$\mathcal{L}_n^-(x) = \int_x^{x_{n0}} \mathrm{d}\tau\,\sqrt{\varepsilon_n - \sin^2\theta}\,,$$
$$\mathcal{L}_n^+(x) = \int_{x_{n0}}^{x} \mathrm{d}\tau\,\sqrt{\sin^2\theta - \varepsilon_n}\,.$$

It follows that a uniformly applicable representation for the functions Ψ^+ and Ψ^- in the entire range of the x coordinate, except a small neighborhood of the higher-order resonance (the region where $\varepsilon(n\omega_0, x) \approx 0$), is given by the functions

$$\Psi_n^+(x) = \mathcal{E}_n\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^-(x)\right)^{1/6}\mathrm{Ai}\left[-\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^-(x)\right)^{2/3}\right], \quad x < x_{n0}\,,$$
$$\Psi_n^+(x) = \mathcal{E}_n\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^+(x)\right)^{1/6}\mathrm{Ai}\left[\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^+(x)\right)^{2/3}\right], \quad x > x_{n0}\,,$$
$$\Psi_n^-(x) = \mathcal{E}_n\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^-(x)\right)^{1/6}\mathrm{Bi}\left[-\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^-(x)\right)^{2/3}\right], \quad x < x_{n0}\,,$$
$$\Psi_n^-(x) = \mathcal{E}_n\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^+(x)\right)^{1/6}\mathrm{Bi}\left[\left(\frac{3n\omega_0}{2c}\,\mathcal{L}_n^+(x)\right)^{2/3}\right], \quad x > x_{n0}\,. \tag{4.6}$$

The turning point x_{n0} of the harmonic with the frequency $n\omega_0$, determined by the condition $\varepsilon(n\omega_0, x_{n0}) = \sin^2\theta$, in the considered case of not too large angles θ such that $\sin^2\theta < 3/4$, lies to the right of the plasma resonance point for all higher harmonics, i.e., $x_{n0} > 0$.

Following the general scheme outlined for $n = 1$ in Appendix B (see the details of calculations for $n \geqslant 2$ in [44, 56, 57]), we obtain the following expression for the amplitude of the magnetic field harmonic B_z with the number $n \geqslant 2$ in a vacuum:

$$C_n^- = \frac{\mathrm{i}m_e\omega_0^3\Delta^2 \exp\left(\mathrm{i}n\arg R_1(0) - \mathrm{i}n\pi + \mathrm{i}\int_{-\infty}^0 k_n(x)\,\mathrm{d}x\right)}{4\pi ec(\cos^2\theta - 1/n^2)^{1/2}(1 - 1/n^2)^{1/4}|\cos\theta|^{1/2}} \times \left[\exp\left\{\frac{\mathrm{i}4n^3L\omega_0}{3c}\left(\cos^2\theta - \frac{1}{n^2}\right)^{3/2}\right\}I_n^- + \mathrm{i}I_n^+\right], \tag{4.7}$$

$$I_n^\pm = \int_{-\infty}^{\infty}\mathrm{d}l\int_0^{2\pi}\mathrm{d}\chi\,\exp\left(\mathrm{i}n\tau \pm \mathrm{i}nBx_0\sqrt{\cos^2\theta - \frac{1}{n^2}}\right)$$
$$\times\left\{\left[\partial_\chi\left(P_0 - \frac{\mathrm{i}}{n}\,\gamma_0 v_1\right)\partial_l\tau - \partial_l\left(P_0 - \frac{\mathrm{i}}{n}\,\gamma_0 v_1\right)\partial_\chi\tau - (\partial_l\tau\,\partial_\chi x_0 - \partial_l x_0\,\partial_\chi\tau)(\gamma_0 - 1)\right]v_1\sin\theta \pm \sqrt{\cos^2\theta - \frac{1}{n^2}}\right.$$
$$\left.\times\left[\left(u_0\partial_\chi P_0 - \frac{\mathrm{i}}{n}\,v_1\partial_\chi(\gamma_0 u_0)\right)\partial_l\tau - \left(u_0\partial_l P_0 - \frac{\mathrm{i}}{n}\,v_1\partial_l(\gamma_0 u_0)\right)\partial_\chi\tau - (\partial_l\tau\,\partial_\chi x_0 - \partial_l x_0\,\partial_\chi\tau)(\gamma_0 - 1)u_0\right]\right\}. \tag{4.8}$$

The presence of two terms in square brackets in the formula for C_n^- corresponds to two different physical mechanisms of HG. The term proportional to I_n^+ describes the harmonic wave directly emitted into the vacuum from the plasma resonance region, and the term proportional to I_n^- describes the harmonic wave emitted from the plasma resonance region deep into the plasma and escaping to the vacuum after reflection from a region of denser plasma. Integrals (4.8) in (4.7) are further calculated using the computer algebra system Wolfram Mathematica [122].

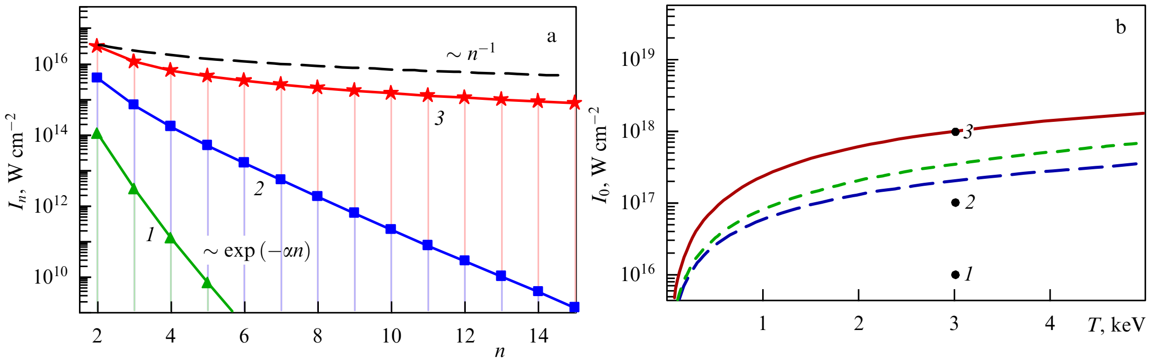

Figure 16. (a) Spectra of radiation emitted from plasma, calculated for flux densities of Nd laser light $I_0 = 10^{16}$ W cm^{-2} (curve *1*), 10^{17} W cm^{-2} (curve *2*), and 10^{18} W cm^{-2} (curve *3*) at fixed plasma temperature $T = 3$ keV, inhomogeneity scale $L = 10\lambda$, and angle of incidence of laser radiation $\theta_{\min} = 14.5°$. (b) Applicability limits of hydrodynamical model on the parameter plane $\{T, I_0\}$ for inhomogeneity scale $L = 10\lambda$. Solid red curve denotes breaking threshold in relativistic hydrodynamics, with nonlinear amplitude renormalization at resonance. Blue dashed line corresponds to relativistic breaking threshold, but without taking nonlinear renormalization into account. Green dashed curve corresponds to breaking threshold in the nonrelativistic theory [14]. Points *1*, *2*, and *3* on the parameter plane $\{T, I_0\}$ correspond to curves *1*, *2*, and *3* in Fig. a.

4.2 Spectral composition of radiation and comparison with the strongly nonlinear nonrelativistic theory

To study the spectral composition of radiation, we take the magnetic field of the first harmonic as $x \to -\infty$ to be of the form

$$B_1 = B_0 \cos(k_x x + k_y y - \omega_0 t + \varphi_0) + B_s \cos\left(-(k_x x - k_y y) - \omega_0 t + \varphi_s\right), \qquad (4.9)$$

where the subscripts 0 and s denote the respective incident and reflected waves. From (4.7), the intensity I_n of the harmonic emitted from the plasma and the reflection coefficient $\mathcal{R}_n$ at the frequency $n\omega_0$ can then be written in the form

$$\mathcal{R}_n = \left|\frac{C_n^-}{C_1^+}\right|^2 = \frac{k_0 L G_L |\Omega_n|^2}{4\pi^3 A_L^2 (1 - 1/n^2)(\cos^2\theta - 1/n^2)^{1/2}},$$

$$\Omega_n = \exp\left\{\frac{4\mathrm{i}n^3 L\omega_0}{3c}\left(\cos^2\theta - \frac{1}{n^2}\right)^{3/2}\right\} I_n^- + \mathrm{i}I_n^+, \qquad (4.10)$$

$$I_n = \mathcal{R}_n I_0, \quad C_1^+ = \frac{B_0}{2},$$

where I_0 is the flux density of radiation incident on the plasma and the $I_n^\pm$ are defined in (4.8). To evaluate the results of the theory at its applicability limit, we consider the characteristics of radiation emitted by plasma at the angle $\theta_{\min}$ for an Nd laser with the wavelength $\lambda = 1.064$ µm. Figure 16a shows the dependences of the intensities I_n of the reflected magnetic field harmonics on the number n at fixed values of the inhomogeneity scale L and plasma temperature T for different laser radiation flux densities I_0 corresponding to points *1*, *2*, and *3* on the parametric plane $\{T, I_0\}$ (Fig. 16b). It follows from a comparison of the $I_n(n)$ spectra that, as the pump field amplitude increases, the slope of the spectral curve changes: it becomes more gentle, and this is most pronounced near the breaking point of resonant plasma oscillations, where the emission spectra are formed such that I_n decreases with increasing n in accordance with a power law faster than $1/n$. In Fig. 17a, we present spectral curves obtained with and without taking relativistic effects into account [14] for fixed parameters I_0, θ, T, and L; the figure shows that, due to saturation of the plasma resonance field amplitude and the shift of the breaking threshold towards higher laser intensities, the radiation spectrum in the relativistic theory demonstrates a faster decay of harmonics with increasing n. In Fig. 17b, the point corresponding to the set of parameters in Fig. 17a is marked; clearly, it is close to the breaking threshold in the nonrelativistic theory [14] and is far from the breaking threshold in the relativistic theory presented here.

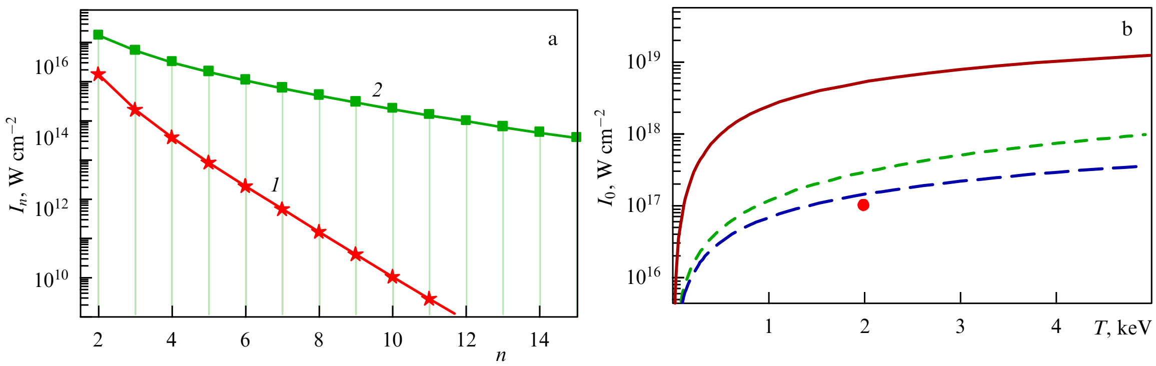

Figure 17. (a) Magnetic field spectra in relativistic (curve *1*) and nonrelativistic [14] (curve *2*) theories at $I_0 = 10^{17}$ W cm^{-2} (Nd laser), $T = 2$ keV, and $\theta = \theta_{\min}$ for $L = 30\lambda$. (b) Applicability limits of the hydrodynamic model on the parameter plane $\{T, I_0\}$ for inhomogeneity scale $L = 30\lambda$. Solid red curve denotes breaking threshold in relativistic hydrodynamics, with nonlinear amplitude renormalization at resonance. Blue dashed line corresponds to relativistic breaking threshold, but without taking nonlinear renormalization into account. Green dashed curve corresponds to breaking threshold in the nonrelativistic theory [14]. Red dot corresponds to the set of parameters $I_0 = 10^{17}$ W cm^{-2} (Nd laser), $T = 2$ keV, $\theta = \theta_{\min}$, and $L = 30\lambda$.

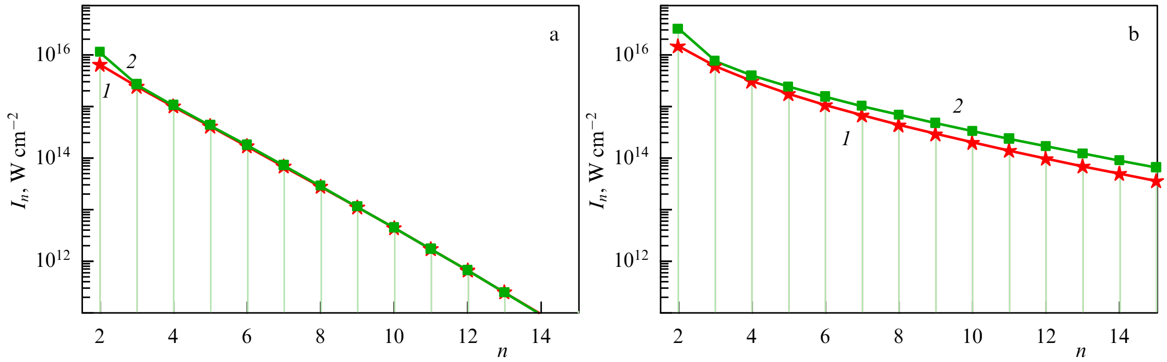

Figure 18. Magnetic field spectra obtained in the nonrelativistic theory [14] (curve *1*) and in the relativistic theory, but without taking phase modulation into account (curve *2*) at $I_0 = 1.3 \times 10^{17}$ W cm^{-2} (Nd laser), $T = 2$ keV, and $\theta = \theta_{\min}$ for (a) $L = 100\lambda$ and (b) $L = 30\lambda$.

Table 1. Ratios of intensities of harmonics with numbers $n = 2, 5, 7, 9, 11, 13, 15$, calculated without taking phase modulation into account ($\tilde{I}_n$) and in the nonlinear nonrelativistic theory ($I_n^{\rm NR}$) [14] at $I_0 = 10^{17}$ W cm^{-2} (Nd laser) and $T = 2$ keV.

L	$\tilde{I}_2/I_2^{\rm NR}$	$\tilde{I}_5/I_5^{\rm NR}$	$\tilde{I}_7/I_7^{\rm NR}$	$\tilde{I}_9/I_9^{\rm NR}$	$\tilde{I}_{11}/I_{11}^{\rm NR}$	$\tilde{I}_{13}/I_{13}^{\rm NR}$	$\tilde{I}_{15}/I_{15}^{\rm NR}$
30λ	2.2	1.4	1.5	1.6	1.7	1.8	1.8
100λ	1.7	1.0	1.0	1.0	1.0	0.9	0.9

We emphasize that the significant differences between the spectral curves in the relativistic and nonrelativistic [14] theories are a direct consequence of the phase modulation of relativistic plasma oscillations in the vicinity of the critical plasma density, which was studied in detail in the previous sections. Taking the relativistic nature of plasma oscillations into account leads to the occurrence in formulas (4.7) and (4.8) of both the Lorentz factor γ_0 and the parametric dependence of τ on x and P, which is responsible for the phase modulation of oscillations. To illustrate the decisive influence of phase modulation on the rearrangement of spectra in moving to the relativistic theory, we compare the spectra of radiation from plasma in the nonrelativistic theory [14] with the spectra calculated using formulas (4.7) and (4.8), but without taking phase modulation into account, i.e., when $\tau(\chi, l) = \chi$ and the $I_n^{\pm}$ become

$$I_n^{\pm} = \int_{-\infty}^{\infty} \mathrm{d}l \int_0^{2\pi} \mathrm{d}\chi \, \exp\left(\mathrm{i}n\chi \pm \mathrm{i}nBx_0\sqrt{\cos^2\theta - \frac{1}{n^2}}\right) \times \left\{\left[(\gamma_0 - 1)\partial_l x_0 - \partial_l\left(P_0 - \frac{\mathrm{i}}{n}\,\gamma_0 v_1\right)\right] v_1 \sin\theta \pm \sqrt{\cos^2\theta - \frac{1}{n^2}} \times \left[(\gamma_0 - 1)u_0\partial_l x_0 - \left(u_0\partial_l P_0 - \frac{\mathrm{i}}{n}\,v_1\,\partial_l(\gamma_0 u_0)\right)\right]\right\}.$$

The comparison (see Fig. 18) shows the absence of significant differences in the intensities of harmonics in the case of less extended ($L = 30\lambda$) and more extended ($L = 100\lambda$) gradients of plasma inhomogeneity (see also Table 1). Thus, the change in the structure of the spectrum in moving to the relativistic theory is mainly determined by the phase modulation of resonantly amplified plasma oscillations.

The change in the $I_n(n)$ dependence with a decrease in the characteristic plasma density inhomogeneity scale L is shown in Fig. 19. For relatively large L (Fig. 19a), with an increase in the inhomogeneity gradient, a transition to a smoother dependence of I_n on n is observed, but a further increase in the inhomogeneity gradient (Fig. 19b) leads to a decrease in the efficiency of HG and gives rise to spectra with an exponential decay law, because the condition $k_0 L \gg 1$ necessary for the plasma resonance mechanism [3] is satisfied

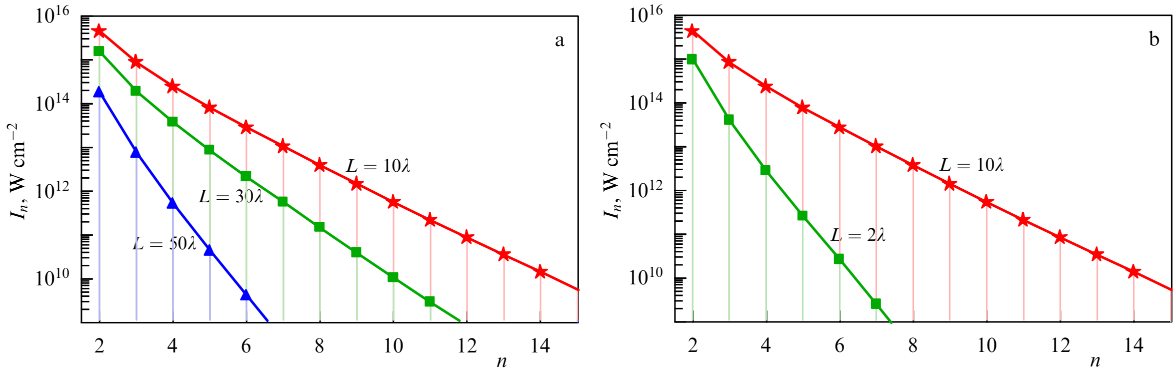

Figure 19. Spectral distributions of the magnetic field in a vacuum at a fixed Nd laser radiation field intensity $I_0 = 10^{17}$ W cm^{-2}, angle of incidence $\theta = \theta_{\min}$, and plasma temperature $T = 2$ keV for different values of the inhomogeneity scale (a) $L = 50\lambda$, $L = 30\lambda$, and $L = 10\lambda$ and (b) $L = 2\lambda$ and $L = 10\lambda$.

less and less accurately. This picture differs from the results in [16, 154, 155], where the relative inhomogeneity scales L/λ in the range of 0.1–1.0 were considered in studying the efficiency of laser radiation absorption and second harmonic generation. In particular, the value $L/\lambda \sim 0.23$ was indicated in [16] as the optimal one for the second harmonic generation by the plasma resonance mechanism. Our theory shows that such inhomogeneity scales are associated with manifestations of the plasma resonance mechanism, although the model used is applicable only for extended gradients $L/\lambda \gg 1/2\pi$. Formally, we are prevented from making a comparison with [16, 154, 155] for $L/\lambda \sim 0.23$ by the applicability conditions of our theory. However, it can be argued that the plasma resonance effect as a mechanism for generating harmonics stops working effectively for such sharp plasma density gradients $L \sim \lambda$.

The results of our theory are in good agreement with the slowly decreasing harmonic spectra of a CO_2 laser observed in experiments more than forty years ago [54, 55] and unexplained until recently. Indeed, at a CO_2 laser intensity $I_0 \simeq 10^{15}$ W cm^{-2} and electron temperature $T \simeq 1$ keV, nonlinear effects in the vicinity of the critical density are so strong that they can lead to the breaking of plasma oscillations. Near this intensity, the theory predicts harmonic spectra with a weak dependence on n. For example, for the incident radiation intensities $I_0 \simeq 10^{15}$ W cm^{-2} at $L = 10\lambda$ and $T \simeq 1$ keV, we obtain $I_2/I_3 \approx 2$, $I_3/I_4 \approx 1.7$, and $I_5/I_6 \approx 1.4$, which corresponds to the experimental results [55]. Where $I_0 \simeq 10^{14} - 10^{15}$ W cm^{-2}, the theory gives $I_2/I_3 \approx 9$, $I_3/I_4 \approx 6.5$, and $I_5/I_6 \approx 5$, which is in good agreement with [54], where the decrease in the energy of harmonics with increasing n was estimated as $I_n/I_{n+1} \approx 6$.

Because experimental data [54, 55] were obtained for sufficiently long laser pulses, we discuss the role of ion motion and give estimates that support our proposed mechanism of the formation of secondary radiation spectra. Along with the fast motion of electrons, which makes a leading contribution to the structure of the spectra, there is a slow hydrodynamic motion of ions, typically with velocities of the order of the speed of ion sound $v_s \approx 10^7$ cm s^{-1}. This manifests itself in the form of motion of the critical density surface itself, as well as in the form of plasma flow through the critical density region [12]. The motion of the critical surface has the effect that the spectral peak of a harmonic can be Doppler-shifted either to the blue or to the red spectral range. The flow of plasma, given the finite time of recording its spectrum, leads to Doppler broadening of the harmonic line, because the flow velocity changes with time. Due to the smallness of the characteristic velocity of ions compared with electron velocities (thermal and oscillatory), taking the motion of ions into account does not lead to significant changes in the global line structure of the harmonic spectrum. Small shifts and broadenings of spectral lines can be neglected, because their characteristic scale is insignificant compared with the separation between harmonics, i.e., in the case of the inequality $\omega_0 \gg v_s/\delta$, which is easy to satisfy because it becomes approximately 10^{15} s$^{-1} \gg 10^{11}$ s^{-1}.

4.3 Relation to the perturbation theory

We discuss the relation between the theory constructed here and the weakly nonlinear theory of HG. The authors of [10] noted the weak dependence of the efficiency of the second harmonic generation on the plasma resonance width (effective collision rate) when nonlinear effects are small. We show that this is indeed the case. In Table 2, we compare the intensities of the second harmonic in the perturbation theory [8, 11] with the intensities obtained in the relativistic theory. It can be seen that, at moderate laser radiation flux densities I_0, the nonlinearity has little effect on the second harmonic generation, but as I_0 increases, i.e., as we approach the breaking threshold, the difference becomes noticeable: the perturbation theory significantly overestimates the intensity of the second harmonic. On the other hand, it was shown in [10] that the applicability limit of the weakly nonlinear theory lies significantly below the breaking threshold. The authors of [10] formulated a criterion for the smallness of nonlinearity, which formally defines the applicability limits of the perturbation theory for estimating the efficiency of second harmonic generation:

Table 2. Second harmonic intensities calculated by the perturbation theory, I_2^{PT}, and in the relativistic theory, I_2, for various flux densities I_0 of Nd laser light at $L = 10\lambda$, $T = 3$ keV, and $\theta = \theta_{\min}$.

I_0, W cm^{-2}	I_2^{PT}, W cm^{-2}	I_2, W cm^{-2}
10^{16}	10^{14}	10^{14}
10^{17}	4×10^{16}	4×10^{15}
10^{18}	10^{18}	3×10^{16}

$$\left| \frac{eH_0 \sin\theta}{2mc\omega_0 \tilde{\beta}} \right| < \left(\frac{\tilde{\beta}}{\rho_0} \right)^{2/3}. \tag{4.11}$$

Here, H_0 is the amplitude of the magnetic field at the plasma resonance point, $\tilde{\beta} = \sqrt{T_e/mc^2}$, and $\rho_0 = \omega_0 L/c$. In terms of the laser radiation intensity I_0 for the set of parameters given in Table 2, this condition is equivalent to the inequality $I_0 < 5 \times 10^{15}$ W cm^{-2}. However, as Table 2 shows, at $I_0 = 10^{16}$ W cm^{-2}, the perturbation theory still gives the second harmonic intensities I_2^{PT} coincident with those of the relativistic theory. Thus, the weakly nonlinear theory of second harmonic generation gives a good estimate of the generation efficiency even beyond the formal applicability limits of the theory.

We compare the generation efficiencies of higher harmonics at different laser radiation flux densities (Fig. 20). At low and moderate laser field intensities I_0, the results of the weakly nonlinear theory [11] coincide with the results of the relativistic theory up to the harmonic numbers $n \approx 10$ (Fig. 20a). Then, with an increase in the laser pump field intensity, the Vladimirskii–Silin theory starts significantly overestimating the harmonic amplitudes (Fig. 20b). Finally, in the range of parameters of the laser–plasma system where relativistic effects are significant in the vicinity of the plasma resonance, the perturbative series diverges (Fig. 20c), and the perturbation theory has no chance of yielding a satisfactory estimate of the harmonic intensities I_n.

To conclude this section, we note that slowly decaying spectra of radiation from plasma have also been obtained in the cases of shorter, more powerful (relativistic) laser pulses of femtosecond duration and sharper gradients of plasma inhomogeneity; there, other harmonic generation mechanisms, different from plasma resonance, are realized [15, 102]. In the limit case of a semiconfined plasma, the HG process can be represented as the reflection of laser radiation from an oscillating electron (plasma) mirror [156–158]. The oscillating mirror model leads to power-law harmonic spectra of the form $\propto \omega^{-8/3}$ in the strongly relativistic limit [159], when the speed of electrons oscillating in the laser field is close to the

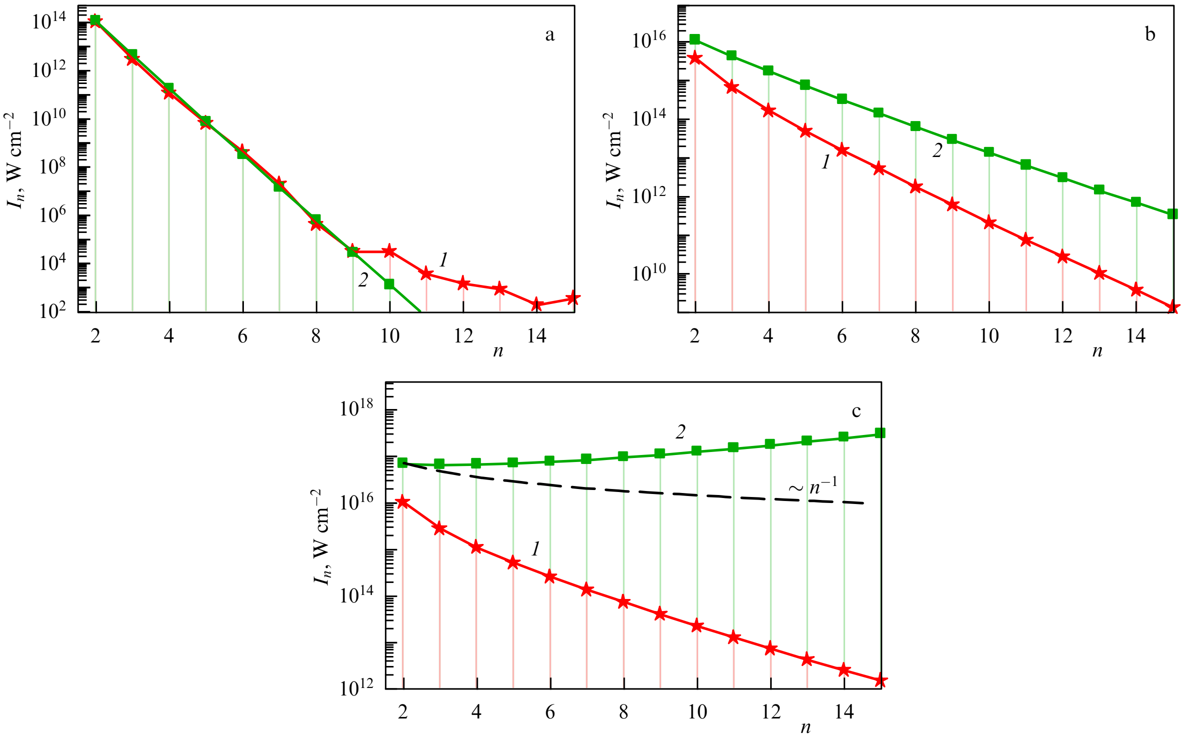

Figure 20. Spectra of the magnetic field in a vacuum obtained in the relativistic theory (*1*) and in the perturbation theory [11] (*2*) for various intensities of Nd laser radiation: (a) $I_0 = 10^{16}$ W cm^{-2}, (b) $I_0 = 10^{17}$ W cm^{-2}, and (c) $I_0 = 2.5 \times 10^{17}$ W cm^{-2}. Plots correspond to fixed values $T = 3$ keV, $L = 10\lambda$, and $\rho = 1$.

speed of light. This HG model [156, 159] was confirmed by numerical simulation using the PIC method. Smooth secondary radiation spectra under conditions of a sharp (stepped) plasma–vacuum interface, which demonstrate a power-law decay of the intensity of the harmonics, were also obtained in [160–163] using numerical modeling, but still have not received a detailed theoretical justification. The methods used in our research allow us to look forward to the construction of a reasonable physical theory in this field, because these methods can be extended in the future to the above regimes of HG at high laser intensities, when the plasma dynamics should be described in terms of kinetic equations.

5. Conclusions

It is hardly disputable that the presence of a consistent analytic theory of any phenomenon, even within certain limitations, is strongly indicative of the level of understanding of its physical foundations and the possibilities of its description in a wide range of control parameters. Of special value are theories applicable in the case of strong nonlinearity, where perturbative methods cannot be used and new approaches are often needed. Since the late 1980s, the search for such approaches has led to the introduction of unique mathematical tools of RG symmetries into the physics of strong laser–plasma interaction [68]. This has allowed describing plasma resonance and the generation of higher harmonics in the case of strong nonrelativistic electron nonlinearity in the region of the critical density of inhomogeneous plasma [14, 147–153], new self-focusing regimes of light [164–168], nonlocal heat transfer [169], acceleration of ions at the expansion of plasma into a vacuum [170–175], and ponderomotive cumulation of ions in a laser–plasma channel [176]. This review provides another illustration of the successful use of RG symmetries in the physics of laser–plasma interaction.

Thirty years after the appearance of [14], it has become possible to develop the RG ideas contained in that paper and construct a theory of nonlinear relativistic plasma resonance proceeding from the Ginzburg–Denisov electromagnetic structure [1, 2]. As a distinctive feature of the method used in finding solutions of nonlinear first-order partial differential equations for the electric field and the electron velocity near the critical plasma density, a two-parameter algebra of RG generators was used, which allowed obtaining solutions that are valid in a wide range of parameters pertaining to different nonlinearity types, and separating the contributions of relativistic and nonrelativistic (convective) nonlinearities (see Section 2.1). This possibility is one of the reasons for using multidimensional RGs for systems with several parameters [177]. The separation of nonlinearities, in turn, led to a systematic two-step transition between solutions. The first step corresponds to the transition from a linear [2] to a strongly nonlinear nonrelativistic solution [14] via an RG transformation with respect to the nonrelativistic nonlinearity parameter. At the second step, the transition to a nonlinear relativistic solution is carried out by the RG transformation with respect to the relativistic parameter. The applicability limits of the hydrodynamic model that we use are naturally determined by the breaking condition of resonantly amplified plasma oscillations derived by taking both convective and relativistic nonlinearity of electron motion into account (see Sections 2.3 and 3.3).

Knowing the electromagnetic field in the plasma resonance region has allowed constructing a theory of relativistic nonlinear RA of electromagnetic radiation in inhomogeneous plasma and thereby significantly pushing the envelope of a theoretical description of RA by laser plasma in the range of high laser intensities: the theory allows the nonlinear RA coefficient G to be calculated in the range of laser intensities up to $I_0\lambda^2 = 10^{18}$ W cm^{-2} μm^2. The coefficient G obtained in Section 3.4 depends on four laser–plasma control parameters: the laser intensity, the plasma inhomogeneity scale, the plasma temperature, and the angle of incidence of laser radiation on the plasma. We note that a number of studies [39–41] have demonstrated the dependence of the absorption coefficient on the laser radiation intensity when taking the relativistic effects of electron motion into account near the critical density of inhomogeneous plasma, although the linear theory does not of course provide such a dependence [6]. At the same time, the obvious successes in the description of nonlinear RA notwithstanding, we see inconsistencies in the currently known results obtained by different scientific groups. For example, some authors [38] assert the universality of the optimal angle of incidence of laser radiation on plasma and the monotonic increase in the absorption coefficient as the laser pump field intensity increases. Others, on the contrary, note the effect of suppression of RA as the laser intensity increases to $I_0 \approx 3 \times 10^{17}$ W cm^{-2} (Nd laser) [39, 40] or indicate a shift of the maximum of the angular function of the nonlinear absorption coefficient away from the position of the maximum in the linear theory [41]. Moreover, no clear-cut applicability limits of the proposed semianalytic models are established in [38, 41], but the range of plasma inhomogeneity gradients considered is so wide that it must, generally speaking, involve transitions between different absorption models, depending on the characteristic density inhomogeneity scales. The approach considered in this review allows us to articulate the applicability limits of the nonlinear plasma resonance model that are associated both with the limit value of the laser radiation flux for the hydrodynamical model and with the maximum admissible value of the plasma density gradient.

The nonlinear effect of suppression of the plasma resonance field amplitude and nonlinear 'termination' of RA with increasing laser intensity, demonstrated in Sections 3.2 and 3.4, are qualitatively consistent with the conclusions of studies based on numerical simulations [39, 40]. So far, only a qualitative comparison of the theory with these simulations has been possible, because studies on PIC modeling give the behavior of the absorption maximum depending on the magnitude of the laser flux at the optimal incidence angle in accordance with the linear theory, whereas, in our theory, the absorption value at that point can be estimated by extrapolating the asymptotic solutions for large and small incidence angles (see Section 3.4). However, such extrapolation also agrees well with numerical simulations. For example, in [39], at $L = 4\lambda$ and $I_0\lambda^2 = 10^{17}$ W cm^{-2} μm^2, the amount of absorbed energy decreases from 50% to 45%, while the theory estimate gives $G \approx 0.4$. In addition, analysis of the asymptotic behavior of the absorption coefficient allowed us to conclude that the effect of relativistic nonlinearity on the angular function of the RA coefficient is asymmetric: at large incidence angles, absorption is suppressed more significantly than at small ones. This asymmetry may mean that an increase in the intensity of the pump field leads not only to a decrease in relativistic-nonlinear RA in a wide range of incidence angles but also to a shift of the absorption maximum towards smaller angles.

The study of the properties of the quasistatic electric field E_{st} of the plasma resonance in Section 3.5 shows that the shape of the spatial distribution of the field in the relativistic theory remains the same as predicted by weakly nonrelativistic [17] and strongly nonlinear nonrelativistic [14] theories: the field is bipolar and changes sign at the plasma resonance point. At the same time, the relativistic nonlinearity noticeably affects the amplitude of E_{st}, lowering the maximum field value compared with the nonrelativistic theory results [14]. For laser fluxes up to $I_0\lambda^2 = 10^{18}$ W cm^{-2} μm^2, the field E_{st} generated in the vicinity of the critical density of inhomogeneous plasma does not exceed a value of several MV cm^{-1}, which limits the maximum energy of electrons accelerated in it to a level of several ten keV. Due to the bipolarity of E_{st}, these electrons do not leave the spatial region of the plasma resonance and do not contribute to the effect of target preheating, which is parasitic for ICF experiments. We note that a full-fledged quantitative study of the dynamical and energy properties of hot electrons generated in the vicinity of the critical density requires a systematic description, not only in the framework of the acceleration mechanism by an electrostatic field in the cold plasma model but also in the regime of plasma wave escape taking the thermal motion of plasma electrons into account, which is beyond the scope of the presented studies and requires a separate analysis.

The theory of HG of laser radiation in an inhomogeneous plasma constructed in Section 4, taking relativistic-nonlinear effects into account in the vicinity of the critical density, allowed studying the spectral characteristics of radiation emitted from plasma resonance in the range of laser intensities up to $I_0\lambda^2 = 10^{18}$ W cm^{-2} μm^2. The formation of energy spectra of radiation from plasma that decay in accordance with a power law near the breaking threshold of resonantly amplified plasma oscillations has been demonstrated. The generation of higher harmonics is most effective at gradient plasma scales of the order of 10λ and in the range of laser radiation intensities $I_0\lambda^2 = 5 \times 10^{17}$–$10^{18}$ W cm^{-2} μm^2. The results in [56, 57] are in good agreement with the results of experiments [54, 55], where slowly decreasing spectra of harmonics emitted by plasma were discovered at low intensities $I_0 \simeq 10^{15}$ W cm^{-2} of CO_2 laser radiation (see Section 4.2). We emphasize that, in the context of ICF, of interest for the developed theory are experiments in the shock ignition regime [33, 178], because they involve precisely those laser radiation intensities at which relativistic effects can appear in the vicinity of the critical plasma density. Unfortunately, there are no dedicated experiments on recording higher harmonics in such a high-intensity ICF regime. Our research presented in this paper could stimulate relevant experiments.

Significant progress in laser technologies over the past 30 years, largely due to the method of amplifying chirped laser pulses [179], has led to a shift in research interest towards the processes of generation of higher harmonics in plasma by relativistically intense, short (femtosecond) laser pulses. When such pulses interact with solid targets, plasma with steep density gradients is formed, where local inhomogeneities in the electron density distribution appear in the form of thin electron bunches [15]. As a result, HG regimes different from the effective plasma resonance regime are realized in the plasma. For example, the relativistic oscillating mirror model has become widespread [102, 156]; in it, the interaction of electromagnetic radiation with local plasma inhomogeneities is viewed as a reflection from a harmonically oscillating electron mirror with a relativistic change in the frequency and amplitude of the reflected wave. Based on this and similar mechanisms, the generation of secondary electromagnetic radiation from plasma in the far ultraviolet and X-ray ranges [15] and the generation of attosecond pulses [160, 180, 181] are studied, and methods are proposed for increasing the frequency and intensity of radiation [182–188]; however, systematic analytic theories are still unavailable. In this regard, the research presented here, from the theoretical and methodological standpoint, suggests a possible strategy to construct a consistent theory of HG of a relativistically strong laser pulse, inasmuch as the approaches of a rather universal applicability that we use can serve as the basis for its development.

To summarize, the theory presented in this review allows quantitatively and qualitatively estimating the absorption and the efficiency of HG of laser radiation at a nonlinear plasma resonance in inhomogeneous plasma, depending on the intensity of laser light, its angle of incidence on the plasma, the characteristic density inhomogeneity scale, and the plasma temperature. The possible improvement of the theory, noted in Section 3.4, by taking the self-consistent deformation of the plasma density profile into account or by considering the obtained analytic expressions in conjunction with codes describing the hydrodynamics of the plasma under laser pumping conditions, will make it possible to refine the conclusions presented here about the behavior of the absorption coefficient with increasing laser energy flux. The results of the theory can find application in problems concerning diagnostics of laser plasma and in planning or interpreting ICF experiments, and can also play an important role in supporting and directing full-scale numerical modeling of the interaction of laser pulses with solid targets.

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6. Appendices

A. Calculating the admissible group

Using the example of equations for the longitudinal components of the electric field and the electron velocity, Eqns (2.6), we outline the main steps in finding the admitted group of transformations. We use the procedure of classical Lie group analysis of differential equations [96], with a distinction that our list of independent variables, along with t, x, v, and P, also includes the parameters a and β. This trick of increasing the number of variables to be used in group transformations is characteristic of the RG method. The coordinates ξ^i, $i = 1, \ldots, 4$ and η^j, $j = 1, 2$, of operator (2.8) are found from the system of defining equations that express the invariance of system (2.6) under the sought group:

$$\begin{aligned} &\underset{1}{X}\left[\partial_t v + av\,\partial_x v - \gamma_0^{-3} P\right]\Big|_{(2.6)} = 0\,, \\ &\underset{1}{X}\left[\partial_t P + av\,\partial_x P + \omega_0^2 v\right]\Big|_{(2.6)} = 0\,. \end{aligned} \tag{A.1}$$

Here, $\underset{1}{X}$ is the first prolongation of the infinitesimal operator X of the sought transformation group (2.8). The symbol $|_{(2.6)}$ means that the action of the prolonged operator $\underset{1}{X}$ is considered on the manifold defined by Eqns (2.6) and all their differential corollaries. The operator $\underset{1}{X}$ is related to X by the prolongation formula

$$\underset{1}{X} = X + \zeta_1^1 \partial_{\partial_t v} + \zeta_2^1 \partial_{\partial_x v} + \zeta_1^2 \partial_{\partial_t P} + \zeta_2^2 \partial_{\partial_x P}\,. \tag{A.2}$$

As above, we use the notation $\partial_{x^i} \equiv \partial/\partial x^i$ for the partial derivative with respect to x^i, and hence $\partial_{\partial_k u}$ denotes the derivative with respect to the derivative $\partial u/\partial x^k$. Extra coordinates ζ_k^j are expressed in terms of ξ^i and η^j by using the total derivative operators D_k:

$$\begin{aligned} &\zeta_k^j = D_k \eta^j - u_i^j D_k \xi^i\,, \quad D_k = \partial_{x^k} + u_k^j \partial_{u^j}\,, \\ &u_i^j \equiv \partial_{x^i} u^j\,, \quad \{u^j\} = \{v, P\}\,, \quad \{x^i\} = \{t, x, a, \beta\}\,. \end{aligned} \tag{A.3}$$

The indices in (A.3) range the values $i = 1, \ldots, 4$, $j = 1, 2$, and $k = 1, 2$, and summation over repeated indices is performed. With (2.6), (2.8) and (A.2), (A.3), the system of defining equations takes the form

$$\begin{aligned} &a\eta^1 + avY\xi^1 - Y\xi^2 + v\xi^3 = 0\,, \\ &Y\eta^1 - z^{3/2}(PY\xi^1 + \eta^2) + \frac{3}{2}(1 - z)Pz^{1/2}\left(\frac{\xi^4}{\beta} + 2\,\frac{\eta^1}{v}\right) = 0\,, \\ &Y(a\eta^2 + \xi^2\omega_0^2 + \xi^3 P) = 0\,, \quad Y\xi^3 = 0\,, \quad Y\xi^4 = 0\,, \end{aligned} \tag{A.4}$$

where

$$Y \equiv \partial_t + av\partial_x + Pz^{3/2}\partial_v - \omega_0^2 v\,\partial_P\,, \quad z \equiv 1 - \beta v^2\,. \tag{A.5}$$

Skipping the details of solving Eqns (A.4), we present the found coordinates of the operator X:

$$\begin{aligned} \xi^1 &= \xi_1(I_1, I_2, I_3) - \frac{\eta^2}{\omega_0^2 v} \\ &\quad + \frac{\xi^4}{4\sqrt{2}\omega_0\beta}\int \mathrm{d}z\,\frac{2 - 3/\sqrt{z} + 1/z^{3/2}}{(1 - z)^{3/2}\sqrt{\beta I_1/2 - 1/\sqrt{z} + 1}}\,, \\ \xi^2 &= \xi_2(I_1, I_2, I_3) - \frac{a\eta^2}{\omega_0^2} - \frac{\xi^3 P}{\omega_0^2}\,, \\ \eta^1 &= \frac{z^{3/2}}{v}\left[\eta_1(I_1, I_2, I_3) - \frac{P}{\omega_0^2}\,\eta^2 - \frac{\xi^4}{2\beta^2}\left(2 - \frac{3}{\sqrt{z}} + \frac{1}{z^{3/2}}\right)\right], \\ \eta^2 &= \eta^2(t, x, v, P)\,, \quad \xi^{3,4} = \xi^{3,4}(a, \beta, I_1, I_2, I_3)\,. \end{aligned} \tag{A.6}$$

Here, $\xi_{1,2}$, $\xi^{3,4}$, and η^2 are arbitrary functions of their arguments, and I_1, I_2, and I_3 are invariants of the operator Y, defined as follows:

$$\begin{aligned} I_1 &= \frac{2}{\beta}\left(\frac{1}{\sqrt{z}} - 1\right) + \frac{P^2}{\omega_0^2}\,, \\ I_2 &= x + \frac{Pa}{\omega_0^2}\,, \\ I_3 &= t - \frac{1}{2\sqrt{\beta}}\int \frac{\mathrm{d}z}{z^{3/2}\sqrt{1 - z}\,\sqrt{I_1 - (2\omega_0^2/\beta)(1/\sqrt{z} - 1)}}\,. \end{aligned} \tag{A.7}$$

Operator (2.8) with the found coordinates (A.6) defines an infinite group of continuous point transformations in the space of six variables, t, x, v, P, a, β, which is the sought widest group allowed by Eqns (2.6), which is needed in constructing the RG. Specifying the coordinates ξ^i, $i = 1, \ldots, 4$, and η^j, $j = 1, 2$, using the restriction [68] of group (2.8), (A.6) to a particular solution defines RG transformations that allow using the results of linear theory when constructing a solution of Eqns (2.6) in a wide range of values of the nonlinearity parameters a and β. Finite transformations with respect to the group parameters a and β are found in the usual way, by solving the corresponding Lie equations.

B. Magnetic field amplitude at the fundamental frequency in a vacuum

For weak plasma inhomogeneity, $k_0 L \gg 1$, an approximate solution of Eqn (3.7) can be written in terms of the Airy functions,

$$\Psi_1^+(x) = \mathcal{E}_1 \left(\frac{3\omega_0}{2c}\,\mathcal{L}_-(x)\right)^{1/6} \mathrm{Ai}\left[-\left(\frac{3\omega_0}{2c}\,\mathcal{L}_-(x)\right)^{2/3}\right], \quad x < x_0\,,$$

$$\Psi_1^+(x) = \mathcal{E}_1 \left(-\frac{3\omega_0}{2c}\,\mathcal{L}_+(x)\right)^{1/6} \mathrm{Ai}\left[\left(\frac{3\omega_0}{2c}\,\mathcal{L}_+(x)\right)^{2/3}\right], \quad x > x_0\,,$$

$$\Psi_1^-(x) = \mathcal{E}_1 \left(\frac{3\omega_0}{2c}\,\mathcal{L}_-(x)\right)^{1/6} \mathrm{Bi}\left[-\left(\frac{3\omega_0}{2c}\,\mathcal{L}_-(x)\right)^{2/3}\right], \quad x < x_0\,,$$

$$\Psi_1^-(x) = \mathcal{E}_1 \left(-\frac{3\omega_0}{2c}\,\mathcal{L}_+(x)\right)^{1/6} \mathrm{Bi}\left[\left(\frac{3\omega_0}{2c}\,\mathcal{L}_+(x)\right)^{2/3}\right], \quad x > x_0\,, \tag{B.1}$$

where

$$\mathcal{E}_1 = \frac{\sqrt{\varepsilon_1}}{\sqrt[4]{\varepsilon_1 - \sin^2\theta}}\,, \quad \mathcal{L}_-(x) = \int_x^{x_0} \mathrm{d}\tau\, \sqrt{\varepsilon_1 - \sin^2\theta}\,,$$

$$\mathcal{L}_+(x) = \int_{x_0}^{x} \mathrm{d}\tau\, \sqrt{\sin^2\theta - \varepsilon_1}\,.$$

Here, x_0 is a turning point that satisfies the condition $\varepsilon_1(x_0) = c^2 k_y^2/\omega_0^2$. Formulas (B.1), which define the solution of homogeneous equation (3.7) to the right and to the left of the turning point, are inapplicable in the vicinity of the plasma resonance $x = 0$, where $\varepsilon_1 \to 0$.

Because the source $f_1(\xi)$ is localized in the vicinity of the plasma resonance, it follows that, when calculating the Green's function $\mathcal{G}(x, \xi)$, we must use the formulas for Ψ_+ and Ψ_- at $x \approx 0$. The solution of a homogeneous linear equation near the plasma resonance point $x = 0$ at not too small angles θ such that $\rho > 1$ is expressed in terms of the Infeld and Macdonald functions I_1 and K_1, which leads to the following formulas for the functions $\Psi_1^{\pm}$ to the right and to the left of $x = 0$:

$$\Psi_1^+(x) = \frac{\mathrm{i}\varDelta - x}{\sqrt{2}L}\left(\frac{\omega_0 L}{c}\right)^{1/2} \exp\left[\frac{\omega_0}{c}\,\mathcal{L}_+(0)\right] \times \mathrm{I}_1\big((\mathrm{i}\varDelta - x)k_y\big), \quad x \lesssim 0\,;$$

$$\Psi_1^+(x) = \frac{x - \mathrm{i}\varDelta}{\sqrt{2}L}\left(\frac{\omega_0 L}{c}\right)^{1/2} \exp\left[\frac{\omega_0}{c}\,\mathcal{L}_+(0)\right] \times \mathrm{I}_1\big((x - \mathrm{i}\varDelta)k_y\big)\,, \quad x \gtrsim 0\,;$$

$$\Psi_1^-(x) = \frac{\sqrt{2}(\mathrm{i}\varDelta - x)}{\pi L}\left(\frac{\omega_0 L}{c}\right)^{1/2} \exp\left[-\frac{\omega_0}{c}\,\mathcal{L}_+(0)\right] \times \mathrm{K}_1\big((\mathrm{i}\varDelta - x)k_y\big)\,, \quad x \lesssim 0\,;$$

$$\Psi_1^-(x) = \frac{\sqrt{2}(x - \mathrm{i}\varDelta)}{\pi L}\left(\frac{\omega_0 L}{c}\right)^{1/2} \exp\left[-\frac{\omega_0}{c}\,\mathcal{L}_+(0)\right] \times \left[\mathrm{K}_1\big((x - \mathrm{i}\varDelta)k_y\big) + \mathrm{i}\pi \mathrm{I}_1\big((x - \mathrm{i}\varDelta)k_y\big)\right], \quad x \gtrsim 0\,. \tag{B.2}$$

In the limit of small incidence angles ($\rho \ll 1$), we have

$$\Psi_1^+(x) = q^{-1/6}\left\{\frac{1}{2}\,\mathrm{Bi}(0) q^{4/3}\eta^2 + \mathrm{Bi}'(0)\left[1 + \frac{\rho}{2}\,q^{4/3}\eta^2\left(\ln\,(q^{2/3}\eta) - \frac{1}{2}\right) - \frac{1}{3}\,q^2\eta^3\right]\right\}, \quad x \lesssim 0\,;$$

$$\Psi_1^-(x) = -q^{-1/6}\left\{\frac{1}{2}\,\mathrm{Ai}(0) q^{4/3}\eta^2 + \mathrm{Bi}'(0)\left[1 + \frac{\rho}{2}\,q^{4/3}\eta^2\left(\ln\,(q^{2/3}\eta) - \frac{1}{2}\right) - \frac{1}{3}\,q^2\eta^3\right]\right\}, \quad x \lesssim 0\,;$$

$$\Psi_1^+(x) = q^{1/6}\eta^{1/2}\Big\{\pi^2\rho\big(\mathrm{Bi}'(0)\big)^2 \mathrm{Ai}\big[q^{2/3}(\sin^2\theta - \eta)\big] + \mathrm{i}\big[1 - \mathrm{i}\pi^2\rho\,\mathrm{Ai}'(0)\,\mathrm{Bi}'(0)\big]\,\mathrm{Bi}\big[q^{2/3}(\sin^2\theta - \eta)\big]\Big\}, \quad x \gtrsim 0\,;$$

$$\Psi_1^-(x) = q^{1/6}\eta^{1/2}\Big\{-\pi^2\rho\big(\mathrm{Ai}'(0)\big)^2 \mathrm{Bi}\big[q^{2/3}(\sin^2\theta - \eta)\big] + \mathrm{i}\big[1 + \mathrm{i}\pi^2\rho\,\mathrm{Ai}'(0)\,\mathrm{Bi}'(0)\big]\,\mathrm{Ai}\big[q^{2/3}(\sin^2\theta - \eta)\big]\Big\}, \quad x \gtrsim 0\,. \tag{B.3}$$

Here, the prime means taking the first derivative and the following notation is used: $q \equiv \omega_0 L/c$, $\eta \equiv (\mathrm{i}\varDelta - x)/L$. Simple calculations give

$$\Psi_1^+(x)\Psi_1^{-\prime}(x) - \Psi_1^{+\prime}(x)\Psi_1^-(x)\Big|_{x\simeq 0} = \begin{cases} -\dfrac{\omega_0}{\pi c}\,\dfrac{\mathrm{i}\varDelta - x}{L}\,, & \rho > 1\,, \\[2mm] \dfrac{\omega_0}{\pi c}\,\dfrac{\mathrm{i}\varDelta - x}{L}\,, & \rho \ll 1\,. \end{cases} \tag{B.4}$$

Substituting (B.4) in (3.6), we obtain

$$R_1(x) = \alpha_1^+\Psi_1^+(x) + \alpha_1^-\Psi_1^-(x) \pm \frac{\pi c L}{\omega_0}\int_{-\infty}^{x} \mathrm{d}\xi\,\frac{f_1(\xi)}{\mathrm{i}\varDelta - \xi}\left[\Psi_1^+(x)\Psi_1^-(\xi) - \Psi_1^+(\xi)\Psi_1^-(x)\right], \tag{B.5}$$

where the $+$ sign corresponds to the case $\rho > 1$, and the $-$ sign, to $\rho \ll 1$. Using the condition that the magnetic field $R_1(x)$ vanishes as $x \to \infty$, the exponential decay of the functions $\mathrm{Ai}(x)$ and $\mathrm{K}_1(x)$, and the exponential growth of $\mathrm{Bi}(x)$, we find the relation between the constants α_1^+ and α_1^- for $\rho > 1$ in the form

$$\alpha_1^- = \frac{\mathrm{i}}{2}\,\alpha_1^+ \exp\left[\frac{2\omega_0}{c}\,\mathcal{L}_+(0)\right] - \frac{\mathrm{i}}{2}\left(\frac{2cL}{\omega_0}\right)^{1/2} \exp\left[\frac{\omega_0}{c}\,\mathcal{L}_+(0)\right] \times \int_{-\infty}^{\infty} \mathrm{d}\xi\, f_1(\xi)\,\mathrm{K}_1\left[(\xi - \mathrm{i}\varDelta)k_y\right], \tag{B.6}$$

and for $\rho \ll 1$, in the form

$$\alpha_1^+ = \frac{\mathcal{D} - i\pi^2\rho\big(\mathrm{Ai}'(0)\big)^2\alpha_1^-}{1 - i\pi^2\rho\,\mathrm{Ai}'(0)\,\mathrm{Bi}'(0)}\,, \quad \mathcal{D} \equiv \frac{\pi c L}{\omega_0}\int_{-\infty}^{\infty}\frac{f_1(\xi)}{i\Delta - \xi}\,\mathcal{M}(\xi)\,d\xi\,,$$

$$\mathcal{M}(\xi) \equiv \big[1 - i\pi^2\rho\,\mathrm{Ai}'(0)\,\mathrm{Bi}'(0)\big]\Psi_1^-(\xi) - i\pi^2\rho\big(\mathrm{Ai}'(0)\big)^2\Psi_1^+(\xi)\,. \tag{B.7}$$

In the region where the plasma density vanishes, i.e., at $x \to -\infty$, the magnetic field is represented as a linear combination of the incident and reflected plane waves with the coefficients $\widetilde{C}_1^+$ and $\widetilde{C}_1^-$, Eqn (3.8). These coefficients can be expressed in terms of α_1^+ and α_1^- if we use formulas (B.2) in (B.5) and take the limit $x \to -\infty$ using the asymptotic expressions for the Airy functions. As a result, we have

$$\widetilde{C}_1^+ = (i\alpha_1^+ + \alpha_1^-)\,\frac{\exp(-i\mathcal{Z})}{2\sqrt{\pi|\cos\theta|}}\,,$$

$$\widetilde{C}_1^- = (\alpha_1^- - i\alpha_1^+)\,\frac{\exp(i\mathcal{Z})}{2\sqrt{\pi|\cos\theta|}}\,, \tag{B.8}$$

$$\mathcal{Z} = \frac{\pi}{4} + \frac{\omega_0}{c}\,\mathcal{L}_+(-\infty)\,.$$

To obtain the relation of the reflected amplitude $\widetilde{C}_1^-$ to the incident one $\widetilde{C}_1^+$, we eliminate the coefficients α_1^+ and α_1^- from (B.8) using (B.6) and (B.7); because the source is localized in the region $(\xi - i\Delta)k_y \ll 1$, we substitute the functions $\mathrm{K}_1(\xi)$ and $\mathcal{M}(\xi)$ in the integrand at small values of the argument. This gives

$$\begin{aligned}\widetilde{C}_1^- &= -i\exp\left[2i\,\frac{\omega_0}{c}\,\mathcal{L}_+(-\infty)\right]\\ &\times \frac{1 - (1/2)\exp\big[(2\omega_0/c)\mathcal{L}_+(0)\big]}{1 + (1/2)\exp\big[(2\omega_0/c)\mathcal{L}_+(0)\big]}\,\widetilde{C}_1^+\\ &- \frac{i\exp\big[\mathcal{L}_+(0) + i\mathcal{Z}\big]}{1 + (1/2)\exp\big[(2\omega_0/c)\mathcal{L}_+(0)\big]}\\ &\times \left(\frac{cL}{2\pi\omega_0|\cos\theta|k_y^2}\right)^2\int_{-\infty}^{\infty}d\xi\,\frac{f_1(\xi)}{\xi - i\Delta}\,, \quad \rho > 1\,,\end{aligned} \tag{B.9}$$

$$\widetilde{C}_1^- = \frac{1 - \pi^2\rho\,\mathrm{Ai}'(0)\big[\mathrm{Ai}'(0) + i\,\mathrm{Bi}'(0)\big]}{1 + \pi^2\rho\,\mathrm{Ai}'(0)\big[\mathrm{Ai}'(0) - i\,\mathrm{Bi}'(0)\big]}\exp(2i\mathcal{Z})\,\widetilde{C}_1^+ - \frac{i\exp(i\mathcal{Z})\mathcal{D}}{\big(\pi|\cos\theta|\big)^{1/2}\big(1 + \pi^2\rho\,\mathrm{Ai}'(0)\big[\mathrm{Ai}'(0) - i\,\mathrm{Bi}'(0)\big]\big)}\,, \quad \rho \ll 1\,.$$

Next, because $P(x,t)$, $Q(x,t)$, and the velocities $v(x,t)$ and $u(x,t)$ vanish as $x \to \infty$, we simplify the integral of the source $f_1(x)$ in (B.9) by integrating by parts, thus eliminating the functions of the transverse components of the electric field u and the electron velocity Q from the integrand,

$$\int_{-\infty}^{\infty}dx\,\frac{f_1(x)}{x - i\Delta} = \frac{i\Delta\omega_0^3 k_y}{2\pi c a}\exp\big[-i\pi + i\arg B_1(0)\big]\mathcal{I}\,, \tag{B.10}$$

where

$$\begin{aligned}\mathcal{I} &\equiv \int_{-\infty}^{\infty}dl\int_0^{2\pi}d\chi\,\frac{v_1\exp\big[i\tau(\chi,l)\big]}{x_0 - i}\big[\partial_\chi\tau\,\partial_l(P_0 - i\gamma_0 v_1)\\ &- \partial_l\tau\,\partial_\chi(P_0 - i\gamma_0 v_1) - (\partial_\chi\tau\,\partial_l x_0 - \partial_l\tau\,\partial_\chi x_0)(\gamma_0 - 1)\big]\,,\end{aligned}$$

$$P_0 = -\frac{A}{1 + l^2}(l\cos\chi + \sin\chi)\,, \quad v_0 = -\frac{A}{1 + l^2}(l\sin\chi - \cos\chi)\,,$$

$$\gamma_0 = 1 + \frac{B^2 v_0^2}{2}\,, \quad x_0 = l - P_0\,, \quad v_1 = v_0\,\frac{\big(1 + (1/4)B^2 v_0^2\big)^{1/2}}{1 + (1/2)B^2 v_0^2}\,, \tag{B.11}$$

$$\tau(\chi,l) = \chi - \left(\zeta\,\mathrm{E}(\varphi,k) - \frac{2}{\zeta}\,\mathrm{F}(\varphi,k) - \varphi\right),$$

$$\zeta = \sqrt{4 + B^2(v_0^2 + P_0^2)}\,,$$

$$\varphi = \arcsin\frac{P_0}{\sqrt{v_0^2 + P_0^2}}\,, \quad k = \sqrt{\frac{B^2(v_0^2 + P_0^2)}{4 + B^2(v_0^2 + P_0^2)}}\,.$$

Using (B.10) and moving from the amplitudes characterizing the normalized functions R to the amplitudes that correspond to the true magnetic field B_z, we find expressions for the amplitudes of the first harmonic of the reflected magnetic field in the vacuum, Eqns (3.9) and (3.11).

C. Estimating integral $\mathcal{I}$ analytically

We find approximately the integral in (B.11) for the values of the dimensionless amplitude A such that $A^2 \ll 1$. In the relativistic regime, the parameter B responsible for the relativistic nonlinearity takes the values $B \simeq 1$ at temperatures $T \simeq 1$ keV and the dimensionless amplitude $A \simeq 0.5$, and we can therefore expand the integrands in (B.11) in a series in the parameter $(AB)^2$. The difference between τ and χ can be disregarded. Hence,

$$\mathcal{I} \approx \int_{-\infty}^{\infty}dl\int_0^{2\pi}d\chi\,\frac{v_1\exp(i\chi)}{x_0 - i}\big[\partial_l(P_0 - i\gamma_0 v_0) - \partial_l x_0(\gamma_0 - 1)\big]\,. \tag{C.1}$$

Expressing the speed v_1 in terms of v_0, and x_0 in terms of P_0, and expanding the function v_1 in a series in the parameter $(AB)^2$ through the first order, we obtain

$$\begin{aligned}\mathcal{I} &\approx \int_{-\infty}^{\infty}dl\int_0^{2\pi}d\chi\,\frac{v_0\exp(i\chi)}{x_0 - i}\\ &\times\left[\partial_l(P_0 - iv_0) - \frac{B^2 v_0^2}{2}\left(1 - \frac{1}{4}\,\partial_l P_0\right)\right] = \mathcal{I}_0 + \mathcal{I}_1 + \mathcal{I}_2\,.\end{aligned} \tag{C.2}$$

Let us discuss the first term

$$\mathcal{I}_0 = \int_{-\infty}^{\infty}dl\int_0^{2\pi}d\chi\,\frac{v_0\exp(i\chi)}{x_0 - i}\,\partial_l(P_0 - iv_0)\,, \tag{C.3}$$

which is independent of B and corresponds to taking the nonrelativistic nonlinearity into account, i.e., $\mathcal{I}_0 = \mathcal{I}\big|_{c=\infty}$. Noting that the integrand has the singularity $(x_0 - i)^{-1}$, we calculate $\mathcal{I}_0$ using the theory of residues, replacing integration over χ with integration over a unit-radius circle. Passing to the variable $z = \exp(i\chi)$ gives

$$v_0 = \frac{A(1 - \alpha z^2)}{2z(1 + il)}\,, \quad P_0 = -\frac{A(1 + \alpha z^2)}{2z(l - i)}\,,$$

$$x_0 - i = \frac{A(z - z_1)(z - z_2)}{2z(l + i)}\,, \tag{C.4}$$

where

$$z_{1,2} = -\frac{1+l^2}{A}\left(1 \pm \sqrt{1-\frac{A^2}{\alpha(1+l^2)^2}}\right), \quad \alpha = \frac{1}{z_1 z_2} = \frac{l-\mathrm{i}}{l+\mathrm{i}}, \quad |z_1| > 1, \quad |z_2| < 1. \tag{C.5}$$

The integral of the time χ then reduces to the integral along the circle $|z| = 1$, which is given by the residues at $z = z_2$ and $z = 0$ and turns out to vanish:

$$\int_{|z|=1} \mathrm{d}z \, \frac{1-\alpha z^2}{z(z-z_1)(z-z_2)} = 2\pi\mathrm{i} \sum_{z=z_2,0} \mathrm{res} \, \frac{1-\alpha z^2}{z(z-z_1)(z-z_2)} = 0. \tag{C.6}$$

Thus, the nonrelativistic term $\mathcal{I}_0$ does not contribute to $\mathcal{I}$. The term $\mathcal{I}_1$, after the change of the variable $z = \exp(\mathrm{i}\chi)$, becomes

$$\mathcal{I}_1 = -\frac{B^2}{2}\int_{-\infty}^{\infty} \mathrm{d}l \int_0^{2\pi} \mathrm{d}\chi \, \frac{v_0^3 \exp(\mathrm{i}\chi)}{x_0 - \mathrm{i}} = -\frac{B^2 A^2}{8\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}l \, \frac{l+\mathrm{i}}{(1+\mathrm{i}l)^3} \int_{|z|=1} \mathrm{d}z \, \frac{1-3\alpha z^2 + 3\alpha^2 z^4 - \alpha^3 z^6}{z^2(z-z_1)(z-z_2)}. \tag{C.7}$$

The integral along the circle is given by the sum of residues of the integrand $f(z)$ at the points $z = z_2$ and $z = 0$:

$$\int_{|z|=1} \mathrm{d}z \, f(z) = 2\pi\mathrm{i} \sum_{z=z_2,0} \mathrm{res} \, \frac{1-3\alpha z^2 + 3\alpha^2 z^4 - \alpha^3 z^6}{z^2(z-z_1)(z-z_2)} = 2\pi\mathrm{i} \, \frac{3z_1 - z_2}{z_1^3 z_2} \approx \frac{6\mathrm{i}\pi}{z_1^2 z_2}. \tag{C.8}$$

The last approximate equality is valid because $|z_1| \gg |z_2|$. Switching in Eqn (B.5) for $z_{1,2}$ to the limit $A^2 \ll 1$ and calculating the integral over space again using the theory of residues, we obtain

$$z_2 \approx -\frac{A}{2\alpha(1+l^2)}, \quad \mathcal{I}_1 \approx \frac{3\mathrm{i}\pi A^3 B^2}{8} \int_{-\infty}^{\infty} \mathrm{d}l \, \frac{l^2 + 2\mathrm{i}l - 1}{(1+l^2)^3} = -\frac{3\mathrm{i}\pi^2}{32} A^3 B^2. \tag{C.9}$$

The $\mathcal{I}_2$ contribution can be found similarly and is given by

$$\mathcal{I}_2 = \frac{B^2}{8}\int_{-\infty}^{\infty} \mathrm{d}l \int_0^{2\pi} \mathrm{d}\chi \, \frac{\exp(\mathrm{i}\chi) v_0^3 \partial_l P_0}{x_0 - \mathrm{i}} \approx -\frac{15\mathrm{i}\pi^2 B^2 A^5}{4096}. \tag{C.10}$$

Because $\mathcal{I}_2 \ll \mathcal{I}_1$, we can write the final approximate value of the integral $\mathcal{I}$:

$$\mathcal{I} \approx -\frac{3\mathrm{i}\pi^2}{32} A^3 B^2. \tag{C.11}$$

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