Physics – Uspekhi **67** (12) 1257 – 1270 (2024)

METHODOLOGICAL NOTES

PACS numbers: 41.20.Gz, 41.20.Jb, **75.70.–i**

Electromagnetic waves in a tangentially magnetized bi-gyrotropic layer (with an example of analysis of spin wave characteristics in a ferrite plate)

E H Lock, S V Gerus

DOI: https://doi.org/10.3367/UFNe.2024.09.039768

Contents

Abstract. **We discuss difficulties arising from the description of spin waves in the magnetostatic approximation, in which neither the microwave electric field nor the Poynting vector is associated with the wave. To overcome these difficulties, we present for the first time a correct solution to the problem of electromagnetic wave propagation in an arbitrary direction along a tangentially magnetized bi-gyrotropic layer (a special case of this problem is the propagation of spin waves in a ferrite plate). It is shown that the wave distribution over the layer thickness is described by two different wave numbers k_{x21} and k_{x22}, which can take real or imaginary values; in particular, three types of spin wave distributions can occur inside the ferrite plate — surface-surface (when k_{x21} and k_{x22} are real numbers), volume-surface (k_{x21} is imaginary and k_{x22} is real), and volume-volume (k_{x21} and k_{x22} are imaginary numbers), which fundamentally distinguishes the obtained description of spin waves from their description in the magnetostatic approximation.**

Keywords: spin waves, ferrite plates, electromagnetic waves, bi-gyrotropic layers, wave distribution over layer thickness

E H Lock [(*)], **S V Gerus** [(**)]
Kotelnikov Institute of Radio Engineering and Electronics, Russian Academy of Sciences, Fryazino Branch,
pl. Vvedenskogo 1, 141190 Fryazino, Moscow region, Russian Federation
E-mail: [(*)] edwin@fireras.su, [(**)] svg318@ire216.msk.su

Received 30 January 2024, revised 9 September 2024
Uspekhi Fizicheskikh Nauk **194** (12) 1330 – 1344 (2024)
Translated by S D Danilov

1. Introduction

Studies of electromagnetic wave propagation in anisotropic media started long ago as applied to magnetically ordered media, uniaxial crystals, and plasmas (see, e.g., monographs [1–12] and references therein). An analysis of the results of these studies is given in Ref. [13], which describes some regularities (similar to the laws of geometrical optics in isotropic media) that accompany the propagation, reflection, and refraction of waves in anisotropic media in two-dimensional geometries. It turns out that, in these media, the occurrence of certain phenomena (nonreciprocal wave propagation, the appearance of multiple reflected or refracted rays, negative reflection and refraction, the absence of reflection, etc.) is defined by the geometrical and mathematical properties of the isofrequency wave dependence, such as the presence of asymptotes, inflection points, central or axial symmetry, or the uniqueness or multi-valued character of the dependence. In addition, as is well known (see, e.g., [14, 15]), the presence of inflection points in the wave dispersion dependences leads to the appearance of quasi-linear intervals in them, which can be used for a distortion-free transmission of a signal modulating the wave. Later, it was also found that the isofrequency dependences also define wave diffraction properties. In particular, Ref. [16] shows that the angular beam width in anisotropic media depends on the curvature of the isofrequency wave dependence and can not only be larger or smaller than λ_0/D (where λ_0 is the wavelength and D is the transducer size) but, under certain conditions (when the wave vector $\mathbf{k}$ is directed to the inflection point in the wave isofrequency dependence), can be zero, which was confirmed in subsequent experimental studies of spin-wave beams [17, 18].

It is therefore of great significance in the field of wave processes to ascertain the isofrequency and dispersion dependences of waves in anisotropic media. Nevertheless, a precise dispersion wave equation is required for a specific anisotropic medium or structure in order to perform calculations.

That said, even in such a successfully developing branch of science as magnonics (see, e.g., reviews [19–24]), the spin waves (SWs) in ferrite slabs and structures are still described in the magnetostatic approximation. It is well known that SWs are of an electromagnetic nature and represent oscillations of atomic magnetic moments propagating from atom to atom [25]. In Ref. [26], it has been proposed to consider that the wavenumber of the SW $k \gg k_0 \equiv \omega/c$ (where ω is the SW cyclic frequency and c is the speed of light), so that the terms containing ω/c as multipliers in Maxwell's equations can be omitted, resulting in the equations of magnetostatics. Because of its mathematical simplicity, this approximation has been used to date to calculate the properties of SWs for wavenumbers $k < 10^5$ cm^{-1} (when the exchange interaction can be discarded), with the waves themselves being referred to as magnetostatic waves. In the time since the publication of [26], researchers have used the magnetostatic approximation to derive dispersion relations for SWs in various structures based on ferrite slabs in an analytical form, to compute a set of SW characteristics in these structures, and to construct a variety of setups and devices using SWs on this basis [10–12, 19–24].

Over time, the use of the magnetostatic approximation became a kind of scientific tradition to be followed, so that researchers often began to assume that they were dealing with a special 'magnetostatic' wave — in fact, it is not associated with a microwave electric field (which is considered to be negligibly small) nor with the Poynting vector (although there have been attempts to calculate it, without using the electric field, the expressions that followed were incorrect [10, 12, 27]). Moreover, within the framework of the magnetostatic equations, it followed that the characteristics of such a magnetostatic wave are independent of the dielectric permittivities of the ferrite slab and the half-spaces on both its sides.

At the same time, approximately in the 1980s, some researchers working with magnetostatic waves began to face questions that could not be solved in the magnetostatic approximation. As a result, papers began to appear in which SWs were treated without the magnetostatic approximation [1, 10–12, 27–42]. Some of these papers [28–32, 34–37, 43] studied the change in characteristics of SWs in the region of small wavenumbers k; among them are also [31, 32, 35], where the mechanism of radiation accompanying SW propagation in a non-uniformly magnetized structure ferrite– dielectric for $k \to k_0$ has been explored. In Ref. [33], the effect of negative dielectric permittivity of the media attached to the ferrite layer on the dispersion dependences and properties of SWs was studied. In Ref. [36], as a result of simplifying Maxwell's equations for SWs, propagating in the plane of an arbitrarily magnetized ferrite slab, a system of two differential equations was obtained that defines the dependences of the tangent components of the microwave electric field in the SW on the coordinate normal to the slab plane. The same study also considers the characteristics of the volume SW for special cases of this geometry, when the slab is magnetized either longitudinally or tangentially, and its two surfaces are metallized. References [40–42] have calculated the distribution of the vector lines of the microwave fields of an SW propagating in different ferrite structures perpendicularly to the vector of the homogeneous magnetic field $\mathbf{H}_0$. In particular, it was found that the vector lines of microwave magnetic induction in the SW form two rows of oppositely directed vortices localized near the opposite surfaces of the ferrite slab, with the vector lines of adjacent vortices being oppositely directed. The boundary between these rows of vortices is a plane (located inside the ferrite slab) where the microwave electric field in the SW is zero. Thus, such SWs can be considered vortices of magnetic induction propagating in time and space along the ferrite slab.

It should be noted that, in almost all the studies mentioned above, which do not rely on the magnetostatic approximation, the properties of SWs have been studied only for the case when the vectors of SW group and phase velocities are collinear [1] (i.e., when the wave propagates perpendicular to or along the direction of the external magnetic field).

It is obvious that for the further successful development of magnonics it would be valuable to find an analytical solution for the propagation of SWs in any direction, based on Maxwell's equations without using the magnetostatic approximation. The solution to this problem would bring the description of SWs to a qualitatively new level and would allow us, in the end, not only to perform exact calculations of the characteristics of SWs with noncollinear orientation of the wave and group velocity vectors but also to associate the microwave electric field with this wave and, for the first time, to compute the Poynting vector, the direction and density of the energy flux, the polarization, and the structure of the force lines of magnetic and electric microwave fields. We note here that an attempt was made earlier (see, e.g., [27]) to derive a formula for the Poynting vector in the magnetostatic approximation. However, the derived formula [2] $\mathbf{P} = -\omega \operatorname{Re}(\mathrm{i}\Psi^*\mathbf{B})/8\pi$, which also appears in the known monographs (see formulas (5.8) in [10] and (6.10) in [12]), turned out to be incorrect, as proved by calculations in Ref. [39].

In the following, we present an exact analytical solution for the propagation of electromagnetic waves in an arbitrary direction in a tangentially magnetized bi-gyrotropic layer characterized by dielectric permittivity and magnetic permeability described by Hermitian second-rank tensors. Obviously, the propagation of spin waves in a ferrite slab is a special case of this general problem. Using this particular case, we will show below how the obtained theoretical results can be used to study SWs, which (in the absence of the magnetostatic approximation) have six components of the microwave electromagnetic field — three magnetic and three electric.

It should be noted that at present a bi-gyrotropic medium is no longer a hypothetical scientific abstraction but a rather promising and desired medium that can be realized based on rapidly developing techniques aimed at designing new artificial media and metamaterials, in which the propagation of electromagnetic waves is actively studied, motivated by the possibility of developing various functional arrangements for the terahertz frequency range (see, e.g., Refs [44, 45] devoted

[1] The exceptions are only [29, 37], which describe the propagation of SWs in an arbitrary direction; however, as will be shown further, the results obtained in these studies are incorrect.

[2] Here, Ψ is the magnetic potential and $\mathbf{B}$ is the vector of microwave magnetic induction.

to ferro- and antiferromagnetic semiconductors and Ref. [46] devoted to the imitation of left media based on ferromagnetic metamaterials).

It is obvious that the above-mentioned papers show in a transparent way that the theory presented below can be used to study the characteristics of electromagnetic waves in anisotropic antiferromagnetic layers, plasmas, and uniaxial optical crystals, as well as in new artificial media and metamaterials.

2. Problem statement

Consider a bi-gyrotropic layer of thickness s tangentially magnetized by a homogeneous magnetic field $\mathbf{H}_0$ (Fig. 1). Such a layer, as is well known [1], is characterized by a dielectric permittivity and a magnetic permeability which are described by the Hermitian second-rank tensors $\overleftrightarrow{\varepsilon_2}$ and $\overleftrightarrow{\mu_2}$:

$$\overleftrightarrow{\mu_2} = \begin{vmatrix} \mu & \mathrm{i}\nu & 0 \\ -\mathrm{i}\nu & \mu & 0 \\ 0 & 0 & \mu_{zz} \end{vmatrix}, \tag{1}$$

$$\overleftrightarrow{\varepsilon_2} = \begin{vmatrix} \varepsilon & \mathrm{i}g & 0 \\ -\mathrm{i}g & \varepsilon & 0 \\ 0 & 0 & \varepsilon_{zz} \end{vmatrix}. \tag{2}$$

It will be shown below that it is possible to solve analytically the system of Maxwell's equations (without any approximations) and to find the dispersion equation for electromagnetic waves propagating in an arbitrary direction along such a bi-gyrotropic layer.

Since the results to be derived can be used to study electromagnetic waves in various anisotropic media — gyrotropic layers of ferrite, antiferromagnetic, or plasma (which are special cases of bi-gyrotropic media and have either the tensor $\overleftrightarrow{\varepsilon_2}$ or the tensor $\overleftrightarrow{\mu_2}$ described by expressions (1) and (2)) — all mathematical derivations and formulas are given for the general case of electromagnetic wave propagation in a bi-gyrotropic layer[3] and are valid for arbitrary frequency dependences of the components of the tensors $\overleftrightarrow{\varepsilon_2}$ and $\overleftrightarrow{\mu_2}$.

At the same time, in order to make the proposed theory less abstract, we will use the formulas obtained to calculate the characteristics of electromagnetic waves propagating along a ferrite slab,[4] for which the diagonal and off-diagonal components of the tensor $\overleftrightarrow{\mu_2}$ are described, as is well known [11], by the expressions

$$\mu = 1 + \frac{\omega_M \omega_H}{\omega_H^2 - \omega^2}, \qquad \nu = \frac{\omega_M \omega}{\omega_H^2 - \omega^2}, \tag{3}$$

where $\omega_H = \gamma H_0$, $\omega_M = 4\pi\gamma M_0$, $\omega = 2\pi f$, γ is the gyromagnetic constant, $4\pi M_0$ is the saturation magnetization of the ferrite, and f is the frequency of electromagnetic oscillations.

We note that the mathematical description of the problem to be solved is somewhat cumbersome, so that in the following we are forced to omit some intermediate manipulations and to introduce special notations for various intermediate quantities in order to write the obtained results compactly.

[3] Furthermore, the results obtained can be used to describe light propagation in a layer of uniaxial optical crystal with the dielectric permittivity tensor in the diagonal form [7], which will correspond to expression (2) for $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon$ if we set $g = 0$.

[4] It is obvious that, according to the definition of an SW [4, 11], SWs with wavenumbers $k < 10^5\ \mathrm{cm}^{-1}$ will correspond to these waves; the exchange interaction can be ignored in their description.

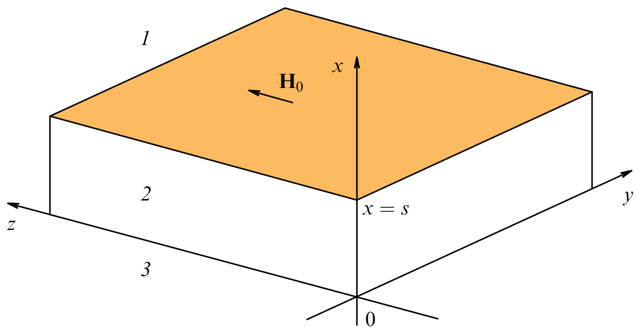

Figure 1. Problem geometry: *1* and *3* are half-spaces of isotropic dielectric (or vacuum), *2* is a bi-gyrotropic layer (a ferrite slab in a particular case) with thickness s.

3. Equations describing propagation of electromagnetic waves in a tangentially magnetized bi-gyrotropic layer

We will describe the electromagnetic fields in the bi-gyrotropic layer *2* and the adjacent dielectric half-spaces *1* and *3* (see Fig. 1) using the indices $j = 1, 2$, and 3. We assume that the half-spaces *1* and *3* have scalar relative dielectric permittivities and magnetic permeabilities ε_1, μ_1 and ε_3, μ_3.

An electromagnetic field of frequency ω and a harmonic time dependence $\sim \exp(\mathrm{i}\omega t)$ propagating in the plane of the bi-gyrotropic layer should satisfy Maxwell's equations for the complex amplitude in each media,

$$\begin{cases} \operatorname{rot} \mathbf{E}_j + \dfrac{\mathrm{i}\omega \mathbf{B}_j}{c} = 0, \\ \operatorname{div} \mathbf{B}_j = 0, \\ \operatorname{rot} \mathbf{H}_j - \dfrac{\mathrm{i}\omega \mathbf{D}_j}{c} = 0, \\ \operatorname{div} \mathbf{D}_j = 0, \end{cases} \tag{4}$$

where $\mathbf{E}_j$, $\mathbf{H}_j$ and $\mathbf{D}_j$, $\mathbf{B}_j$ are the complex amplitudes of the vectors of the microwave electric and magnetic fields and also of the electric and magnetic inductions, which are related as

$$\mathbf{D}_j = \overleftrightarrow{\varepsilon_j}\, \mathbf{E}_j \quad \text{and} \quad \mathbf{B}_j = \overleftrightarrow{\mu_j}\, \mathbf{H}_j. \tag{5}$$

We note here that, in previous studies that tried to solve this problem (see, e.g., [29, 37]), it was proposed from the very beginning to look for the solution to system (4) in the form of a plane wave $\sim \exp(-\mathrm{i}k_x x - \mathrm{i}k_y y - \mathrm{i}k_z z)$. We claim that this approach is incorrect from a mathematical standpoint and does not allow finding a *general solution* to the system of Maxwell's equations (4). In a mathematically correct approach, the dependence of the wave on the coordinate x (normal to the bi-gyrotropic layer) should be found in the process of solving the differential equations that follow as a result of adapting system (4) to the problem geometry. Therefore, we should seek the solution to system (4) in the form of a homogeneous plane wave propagating in the plane yz of the layer and characterized by an arbitrary vector $\mathbf{k}$. In other words, in contrast to [29, 37], we allow, as in the previous study [36], an arbitrary dependence of the fields $\mathbf{E}$ and $\mathbf{H}$ on the coordinate x normal to the layer and assume that these components vary harmonically in the plane of the layer as well as with time according to

$$\mathbf{E}_j = \mathbf{e}_j(x)\exp(-\mathrm{i}\mathbf{kr}) \text{ or } E_{xj,\,yj,\,zj} = e_{xj,\,yj,\,zj}(x)\exp(-\mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{6}$$

$$\mathbf{H}_j = \mathbf{h}_j(x)\exp(-\mathrm{i}\mathbf{kr}) \text{ or } H_{xj,\,yj,\,zj} = h_{xj,\,yj,\,zj}(x)\exp(-\mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{7}$$

where, together with the Cartesian coordinate system $\Sigma_{\mathrm{D}} = \{x; y; z\}$, we have also introduced the related polar (cylindrical) reference frame $\Sigma_{\mathrm{P}} = \{x; r; \varphi\}$, in which angles φ are counted from the y-axis, and the counterclockwise direction is taken as positive. The coordinates of the systems Σ_{P} and Σ_{D} are connected by the relationships $y = r\cos\varphi$, $z = r\sin\varphi$. Obviously, the module of the wave vector k and its components k_y and k_z are also connected by the relationships $k_y = k\cos\varphi$, $k_z = k\sin\varphi$, and $k^2 = k_y^2 + k_z^2$.

Inserting expressions (6) and (7) into (5) and then (5) into (4), and solving system (4), we find for the bi-gyrotropic medium *2*, by analogy with Refs [1, 47], the system of two equations containing only the x-dependent amplitudes e_{z2} and h_{z2} of the components E_{z2} and H_{z2},

$$\begin{cases} \dfrac{1}{k_0^2}\dfrac{\partial^2 e_{z2}}{\partial x^2} - F_v e_{z2} - \mathrm{i}\mu_{zz}F_{vg}h_{z2} = 0\,, \\ \dfrac{1}{k_0^2}\dfrac{\partial^2 h_{z2}}{\partial x^2} - F_g h_{z2} + \mathrm{i}\varepsilon_{zz}F_{vg}e_{z2} = 0\,, \end{cases} \tag{8}$$

where the dimensionless functions F_v, F_g, and F_{vg} have the form

$$F_v = \frac{k_y^2}{k_0^2} + \frac{\varepsilon_{zz}}{\varepsilon}\frac{k_z^2}{k_0^2} - \frac{\varepsilon_{zz}}{\mu}(\mu^2 - v^2) = \frac{k^2}{k_0^2}\left(\cos^2\varphi + \frac{\varepsilon_{zz}}{\varepsilon}\sin^2\varphi\right) - \varepsilon_{zz}\mu_{\perp}\,, \tag{9}$$

$$F_g = \frac{k_y^2}{k_0^2} + \frac{\mu_{zz}}{\mu}\frac{k_z^2}{k_0^2} - \frac{\mu_{zz}}{\varepsilon}(\varepsilon^2 - g^2) = \frac{k^2}{k_0^2}\left(\cos^2\varphi + \frac{\mu_{zz}}{\mu}\sin^2\varphi\right) - \mu_{zz}\varepsilon_{\perp}\,, \tag{10}$$

$$F_{vg} = \frac{k_z}{k_0}\left(\frac{g}{\varepsilon} + \frac{v}{\mu}\right) = \frac{k}{k_0}\sin\varphi\left(\frac{g}{\varepsilon} + \frac{v}{\mu}\right), \tag{11}$$

and also the notation

$$\mu_{\perp} = \frac{\mu^2 - v^2}{\mu}\,, \tag{12}$$

$$\varepsilon_{\perp} = \frac{\varepsilon^2 - g^2}{\varepsilon} \tag{13}$$

is used. Note that both the off-diagonal components v and g of the tensors $\overleftrightarrow{\varepsilon_2}$ and $\overleftrightarrow{\mu_2}$ enter only the function F_{vg}, while the function F_v contains only the component v, and the function F_g contains only the component g (this explains the notation used).

By substituting the quantity h_{z2} from the first equation of system (8) into the second equation, we obtain the following differential equation for the amplitude e_{z2}:

$$\frac{\partial^4 e_{z2}}{\partial x^4} + 2\eta\frac{\partial^2 e_{z2}}{\partial x^2} + \alpha e_{z2} = 0\,, \tag{14}$$

where

$$\eta = -\frac{k_0^2(F_v + F_g)}{2}\,, \tag{15}$$

$$\alpha = k_0^4 F_v F_g - \mu_{zz}\varepsilon_{zz}k_0^4 F_{vg}^2\,. \tag{16}$$

Equation (14) defines the following characteristic equation for the values of the wavenumber k_{x2} inside the bi-gyrotropic layer

$$k_{x2}^4 + 2\eta k_{x2}^2 + \alpha = 0\,. \tag{17}$$

Finding the discriminant of (17), taking into account (15) and (16), it can be easily shown that it is positive,

$$\eta^2 - \alpha = \frac{k_0^4(F_v + F_g)^2}{4} - k_0^4(F_v F_g - \mu_{zz}\varepsilon_{zz}F_{vg}^2) = k_0^4\left[\frac{(F_v - F_g)^2}{4} + \mu_{zz}\varepsilon_{zz}F_{vg}^2\right] > 0\,. \tag{18}$$

The characteristic equation (17) has *four* roots which are given by the relationship

$$k_{x2}^2 = -\eta \pm \sqrt{\eta^2 - \alpha} = \frac{k_0^2}{2}\left(F_v + F_g \pm \sqrt{(F_v - F_g)^2 + 4\mu_{zz}\varepsilon_{zz}F_{vg}^2}\right), \tag{19}$$

and all of them are *simple* (no multiple roots):

$$k_{x21} = \sqrt{-\eta - \sqrt{\eta^2 - \alpha}} = k_0\sqrt{\frac{F_v + F_g}{2} - \frac{1}{2}\sqrt{(F_v - F_g)^2 + 4\mu_{zz}\varepsilon_{zz}F_{vg}^2}}\,, \tag{20}$$

$$k_{x22} = \sqrt{-\eta + \sqrt{\eta^2 - \alpha}} = k_0\sqrt{\frac{F_v + F_g}{2} + \frac{1}{2}\sqrt{(F_v - F_g)^2 + 4\mu_{zz}\varepsilon_{zz}F_{vg}^2}}\,, \tag{21}$$

$$k_{x23} = -k_{x21}\,, \tag{22}$$

$$k_{x24} = -k_{x22}\,. \tag{23}$$

4. Solutions describing electromagnetic waves in a bi-gyrotropic layer

To write down a solution to differential equation (14), we need to find what values the roots $k_{x21} - k_{x24}$ of the characteristic equation can take. We start by noting that the roots $k_{x21} - k_{x24}$ cannot be *complex* (since $\eta^2 - \alpha > 0$ always, according to (18)), but are either *real* or *imaginary*, depending on the signs of the expressions under the roots in (20) and (21). As can be seen from these expressions, if $\alpha < 0$, then $|\eta|$ is always smaller than $\sqrt{\eta^2 - \alpha}$, whose sign defines the sign of the expression under the root; in this case, the root k_{x21} is always imaginary, while the root k_{x22} is real (regardless of the sign of η). If $\alpha > 0$, then, conversely, $|\eta| > \sqrt{\eta^2 - \alpha}$ always, and for $\eta > 0$, the quantities k_{x21} and k_{x22} are imaginary, and for $\eta < 0$, they are real.

Thus, based on the conditions formulated above and using concrete parameters for the bi-gyrotropic layer and expressions (15), (16), and (9)–(11), we can construct in the coordinate space $\{k_y, k_z, f\}$ (or in the coordinate space

$\{k, \varphi, f\}$) *boundary surfaces*

$$\eta = 0 \ \text{ or } \ F_v + F_g = 0\,, \tag{24}$$

$$\alpha = 0 \ \text{ or } \ F_v F_g - \mu_{zz}\varepsilon_{zz} F_{vg}^2 = 0\,. \tag{25}$$

The *boundary surfaces* have the following physical meaning: *when these surfaces cross a certain* dispersion surface $f(k_y, k_z)$ of electromagnetic waves,[5] *they will separate on it the regions with real and imaginary values* of the roots $k_{x21} - k_{x24}$ of characteristic equation (17).

We note here that equality (25) is *identical to the dispersion equation for electromagnetic waves in an unbounded bi-gyrotropic medium* (see expressions (20)–(23) in Ref. [47]) when this equation is reduced to a two-dimensional case by setting to zero a wavenumber corresponding to one of the coordinates normal to $\mathbf{H}_0$ (e.g., k_x).

To get an idea of the shape of the surfaces $\alpha = 0$ and $\eta = 0$, we perform calculations for the case when layer *2* in Fig. 1 is a ferrite slab (which is a special case of a bi-gyrotropic layer) with the saturation magnetization $4\pi M_0 = 1750$ G and the dielectric permittivity $\varepsilon_2 = 15$, magnetized to saturation by a permanent homogeneous magnetic field $H_0 = 300$ Oe.

First, it should be mentioned that, for the ferrite slab, the quantity α additionally changes its sign when the frequency is varied from the values $f < f_\perp$ to the values $f > f_\perp$ (since, according to (3) and (12), for $f_\perp = \omega_\perp/2\pi = (\omega_H^2 + \omega_H\omega_M)^{1/2}/2\pi$, we have $\mu = 0$ and $\mu_\perp \to \infty$).

In order to identify the values that the roots k_{x21} and k_{x22} of the characteristic equation take in different regions of the space $\{k_y, k_z, f\}$, we determine and construct the surfaces $\alpha = 0$, $\eta = 0$ and the plane $f = f_\perp$ in this space. The regions of space bounded by these surfaces and the sections of these surfaces by the planes $k_y = 0$, $k_z = 0$, and $f = 7000$ MHz are shown in Fig. 2.

The schematics in Fig. 2 offer the capability to learn what the distribution of the wave amplitude is within the tangentially magnetized ferrite layer (of arbitrary thickness) when the wave dispersion surface is in one region or another of the space $\{k_y, k_z, f\}$. In reality, we already know this relying only on the properties of differential equation (14), even though we have not yet obtained the wave dispersion equation!

By analyzing Fig. 2, we can see that the largest regions are those labeled SS and colored yellow, where $\alpha > 0$ and $\eta < 0$. The part of the dispersion surface located in region SS will correspond to solutions with *real* values of the roots k_{x21} and k_{x22} and the *general* solution of equation (14) in the form

$$e_{z2} = A\exp(k_{x21}x) + B\exp(-k_{x21}x) + C\exp(k_{x22}x) + D\exp(-k_{x22}x)\,. \tag{26}$$

The wave distribution inside the ferrite slab is described only by exponential functions, and such a wave can be provisionally called *surface–surface*, or an SS-wave.

The smallest domain in Fig. 2 is occupied by the region VV colored orange, where $\alpha > 0$ and $\eta > 0$. The part of the wave dispersion surface located in region VV will describe the solution with *imaginary* roots k_{x21} and k_{x22}, which correspond to the *general* solution of equation (22) in the form

$$e_{z2} = A\cos(|k_{x21}|x) + B\sin(-|k_{x21}|x) + C\cos(|k_{x22}|x) + D\sin(-|k_{x22}|x)\,. \tag{27}$$

[5] For example, the dispersion surface for any type of spin wave propagating in a ferrite slab.

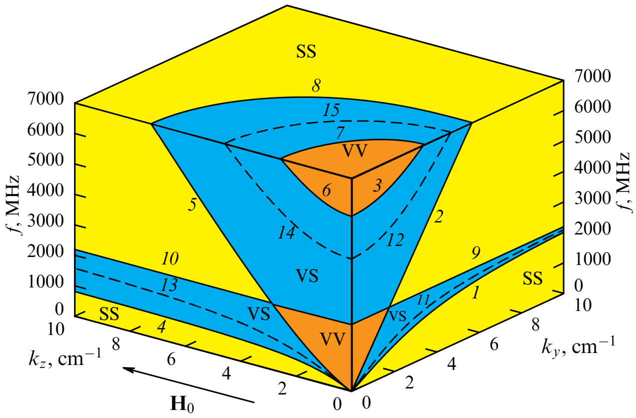

Figure 2. Space domains SS, VS, and VV, which define the character of wave distribution in ferrite slab section. Boundaries of domains SS, VS, and VV are formed by composite surface $\alpha = 0$ and plane $f = f_\perp$. Curves *1–3*, *4–6*, and *7*, *8* describe sections of surface $\alpha = 0$ by planes $k_y = 0$, $k_z = 0$, and $f = 7000$ MHz, respectively, straight lines *9* and *10* correspond to sections of plane $f = f_\perp$ by planes $k_y = 0$ and $k_z = 0$, respectively, dashed curves *11*, *12*, *13*, *14*, and *15* correspond to section of the surface $\eta = 0$ by planes $k_y = 0$, $k_z = 0$, and $f = 7000$ MHz, respectively.

The wave distribution within the slab is described only by trigonometric functions, and such a wave can be provisionally called *volume–volume*, or a VV-wave.

The regions VS colored blue in Fig. 2 are characterized by $\alpha < 0$, while the surfaces $\eta = 0$ (their sections *11–15* are shown by the dashed lines) are *always* located inside these regions. As mentioned above, in this case, regardless of the sign of η, the root k_{x21} takes *imaginary* values and the root k_{x22} takes *real* values. In other words, the part of the dispersion surface located in region VS will describe the waves corresponding to the *general* solution of (22) in the form

$$e_{z2} = A\cos(|k_{x21}|x) + B\sin(-|k_{x21}|x) + C\exp(k_{x22}x) + D\exp(-k_{x22}x)\,. \tag{28}$$

Thus, the wave distribution within the ferrite slab for this part of the dispersion surface will be described by both trigonometric and exponential functions, and this wave can be provisionally called *volume–surface*, or a VS-wave.

It should be noted that the remaining case, where the *general* solution of differential equation (22) is of the form

$$e_{z2} = A\exp(k_{x21}x) + B\exp(-k_{x21}x) + C\cos(|k_{x22}|x) + D\sin(-|k_{x22}|x)\,, \tag{29}$$

which corresponds to an SV-wave with the *real* root k_{x21} and *imaginary* root k_{x22}, is never realized in ferrite slabs.[6]

Thus, the wave distribution inside a ferrite slab can vary depending on the wave parameters, and on one part of the dispersion or isofrequency dependence[7] this distribution can correspond to, e.g., an SS-wave (described by expression (26)), and on the other part, to a VS-wave (described by expression (28)). This is the *fundamental difference between the exact description* of spin waves and their description in the

[6] However, it cannot be ruled out that an SV-wave can occur in layers of other anisotropic media, which are special cases of the bi-gyrotropic layer.

[7] As is well known, the dispersion and isofrequency dependences are the sections of the dispersion surface, so everything discussed here regarding the intersection of this surface with boundary surfaces also applies to these dependences.

magnetostatic approximation [26] in which each dispersion surface of spin waves is characterized by a certain distribution type (surface or volume).

It is also obvious that the one-to-one correspondence we found between the region in space $\{k_y, k_z, f\}$ and the type of wave distribution in the ferrite layer does not depend on the boundary conditions to be further used and is valid for all structures based on the ferrite slab, be it a ferrite slab proper surrounded by vacuum half-spaces, a ferrite slab metallized on one side, a structure metal–dielectric–ferrite–dielectric–metal, or another.

Furthermore, based on equations (14)–(25), a general conclusion can be drawn: *The wave distribution in the bi-gyrotropic layer* (*including a layer of ferrite, antiferromagnetic, plasma, or uniaxial optical crystal*) *and in structures based on such a layer is always described in terms of two*[8] *wavenumbers* k_{x21} *and* k_{x22}. Depending on the type of anisotropic medium, its parameters, and the values of k_{x21} and k_{x22}, this distribution is given by one of the expressions (26)–(29).

In the following, we will derive the dispersion relation and the expressions for the microwave components of an SS-wave propagating along a bi-gyrotropic layer.

5. Expressions for components of an electromagnetic field in a bi-gyrotropic layer

To write down expressions for all the microwave components of an electromagnetic wave inside a bi-gyrotropic layer, we must first obtain expressions for their amplitudes e_{x2}, e_{y2}, h_{x2}, h_{y2}, and h_{z2}.

Substituting (26) into the first equation of system (8), we obtain the expression for the amplitude h_{z2}:

$$h_{z2} = \mathrm{i}\beta_1\big(A\exp(k_{x21}x) + B\exp(-k_{x21}x)\big) + \mathrm{i}\beta_2\big(C\exp(k_{x22}x) + D\exp(-k_{x22}x)\big)\,, \tag{30}$$

where

$$\beta_1 = \frac{1}{\mu_{zz}F_{vg}}\left(F_v - \frac{k_{x21}^2}{k_0^2}\right), \tag{31}$$

$$\beta_2 = \frac{1}{\mu_{zz}F_{vg}}\left(F_v - \frac{k_{x22}^2}{k_0^2}\right). \tag{32}$$

Inserting expressions (6) and (7) into (5) and then (5) into system of equations (4), we find

$$k_y e_{z2} - k_z e_{y2} - k_0\mu h_{x2} - \mathrm{i}k_0 v h_{y2} = 0\,, \tag{33}$$

$$-\,\mathrm{i}k_z e_{x2} - \frac{\partial e_{z2}}{\partial x} + k_0 v h_{x2} + \mathrm{i}k_0\mu h_{y2} = 0\,, \tag{34}$$

$$\frac{\partial e_{y2}}{\partial x} + \mathrm{i}k_y e_{x2} + \mathrm{i}k_0\mu_{zz}h_{z2} = 0\,, \tag{35}$$

$$\mu\left(\frac{\partial h_{x2}}{\partial x} - \mathrm{i}k_y h_{y2}\right) + \mathrm{i}v\left(\frac{\partial h_{y2}}{\partial x} + \mathrm{i}k_y h_{x2}\right) - \mathrm{i}\mu_{zz}k_z h_{z2} = 0\,, \tag{36}$$

$$k_y h_{z2} - k_z h_{y2} + k_0\varepsilon e_{x2} + \mathrm{i}k_0 g e_{y2} = 0\,, \tag{37}$$

$$\mathrm{i}k_z h_{x2} + \frac{\partial h_{z2}}{\partial x} + k_0 g e_{x2} + \mathrm{i}k_0\varepsilon e_{y2} = 0\,, \tag{38}$$

$$\frac{\partial h_{y2}}{\partial x} + \mathrm{i}k_y h_{x2} - \mathrm{i}k_0\varepsilon_{zz}e_{z2} = 0\,, \tag{39}$$

$$\varepsilon\left(\frac{\partial e_{x2}}{\partial x} - \mathrm{i}k_y e_{y2}\right) + \mathrm{i}g\left(\frac{\partial e_{y2}}{\partial x} + \mathrm{i}k_y e_{x2}\right) - \mathrm{i}k_z\varepsilon_{zz}e_{z2} = 0\,. \tag{40}$$

From equations (33) and (37), one can obtain the equations

$$h_{x2} = \frac{k_y}{\mu k_0}\,e_{z2} - \frac{k_z}{\mu k_0}\,e_{y2} - \mathrm{i}\,\frac{v}{\mu}\,h_{y2}\,, \tag{41}$$

$$e_{x2} = -\frac{k_y}{\varepsilon k_0}\,h_{z2} + \frac{k_z}{\varepsilon k_0}\,h_{y2} - \mathrm{i}\,\frac{g}{\varepsilon}\,e_{y2}\,. \tag{42}$$

Inserting expressions (41), (42) into equations (34) and (38), we find the following relations:

$$\mathrm{i}\,\frac{k_y k_z}{\varepsilon k_0^2}\,h_{z2} - \mathrm{i}F_{v2}h_{y2} - F_{vg}e_{y2} + \frac{vk_y}{\mu k_0}\,e_{z2} - \frac{1}{k_0}\frac{\partial e_{z2}}{\partial x} = 0\,, \tag{43}$$

$$-\,\mathrm{i}\,\frac{k_y k_z}{\mu k_0^2}\,e_{z2} + \mathrm{i}F_{g2}e_{y2} - F_{vg}h_{y2} + \frac{gk_y}{\varepsilon k_0}\,h_{z2} - \frac{1}{k_0}\frac{\partial h_{z2}}{\partial x} = 0\,, \tag{44}$$

where the dimensionless functions F_{v2} and F_{g2} take the form

$$F_{v2} = \frac{k_z^2}{\varepsilon k_0^2} - \mu_\perp\,, \tag{45}$$

$$F_{g2} = \frac{k_z^2}{\mu k_0^2} - \varepsilon_\perp\,. \tag{46}$$

Multiplying (43) by $\mathrm{i}F_{vg}$ and (44) by F_{v2} and summing the results, we find the expression for the amplitude e_{y2}:

$$e_{y2} = \frac{1}{F_2}\left[a_0 e_{z2} - \mathrm{i}a_2 h_{z2} + \frac{F_{vg}}{k_0}\frac{\partial e_{z2}}{\partial x} - \mathrm{i}\,\frac{F_{v2}}{k_0}\frac{\partial h_{z2}}{\partial x}\right], \tag{47}$$

and multiplying (43) by $\mathrm{i}F_{g2}$ and (44) by F_{vg} and summing the results, we find the expression for the amplitude h_{y2}:

$$h_{y2} = \frac{1}{F_2}\left[\mathrm{i}b_0 e_{z2} + b_2 h_{z2} + \frac{F_{vg}}{k_0}\frac{\partial h_{z2}}{\partial x} + \mathrm{i}\,\frac{F_{g2}}{k_0}\frac{\partial e_{z2}}{\partial x}\right]. \tag{48}$$

The following notations are used in expressions (47) and (48):

$$F_2 = F_{v2}F_{g2} - F_{vg}^2\,, \tag{49}$$

$$a_0 = \frac{k_y k_z}{\mu k_0^2}\,F_{v2} - \frac{vk_y}{\mu k_0}\,F_{vg}\,, \tag{50}$$

$$a_2 = \frac{k_y k_z}{\varepsilon k_0^2}\,F_{vg} - \frac{gk_y}{\varepsilon k_0}\,F_{v2}\,, \tag{51}$$

$$b_0 = \frac{k_y k_z}{\mu k_0^2}\,F_{vg} - \frac{vk_y}{\mu k_0}\,F_{g2}\,, \tag{52}$$

$$b_2 = \frac{k_y k_z}{\varepsilon k_0^2}\,F_{g2} - \frac{gk_y}{\varepsilon k_0}\,F_{vg}\,. \tag{53}$$

As we can see, the amplitudes e_{y2} and h_{y2} in (47) and (48) are expressed only through the amplitudes e_{z2} and h_{z2} and their derivatives over the coordinate x. Inserting (47) and (48) into (41) and (42), we write analogous expressions for the amplitudes e_{x2} and h_{x2}. To shorten the expressions for all amplitudes on the coordinate x, we introduce the *dimensionless* functions Σ_0, Σ_1, Σ_2, and Σ_3 of the coordinate x that satisfy the relations

$$e_{z2} = \Sigma_0(x)\,,\quad \frac{\partial e_{z2}}{\partial x} = k_0\Sigma_1(x)\,,$$
$$h_{z2} = \mathrm{i}\Sigma_2(x)\,,\quad \frac{\partial h_{z2}}{\partial x} = \mathrm{i}k_0\Sigma_3(x)\,. \tag{54}$$

[8] In Section 10, we will clarify in which special cases the wave distribution in a bi-gyrotropic layer is described by a single wavenumber.

Substituting formulas (26) and (30) into relations (54), we obtain the following expressions for the functions Σ_0, Σ_1, Σ_2, and Σ_3:

$$\Sigma_0(x) = A\exp(k_{x21}x) + B\exp(-k_{x21}x) + C\exp(k_{x22}x) + D\exp(-k_{x22}x)\,, \tag{55}$$

$$\Sigma_1(x) = \frac{k_{x21}}{k_0}\big(A\exp(k_{x21}x) - B\exp(-k_{x21}x)\big) + \frac{k_{x22}}{k_0}\big(C\exp(k_{x22}x) - D\exp(-k_{x22}x)\big)\,, \tag{56}$$

$$\Sigma_2(x) = \beta_1\big(A\exp(k_{x21}x) + B\exp(-k_{x21}x)\big) + \beta_2\big(C\exp(k_{x22}x) + D\exp(-k_{x22}x)\big)\,, \tag{57}$$

$$\Sigma_3(x) = \frac{k_{x21}}{k_0}\,\beta_1\big(A\exp(k_{x21}x) - B\exp(-k_{x21}x)\big) + \frac{k_{x22}}{k_0}\,\beta_2\big(C\exp(k_{x22}x) - D\exp(-k_{x22}x)\big)\,. \tag{58}$$

To clarify our notation, we note that the numerical index in the functions $\Sigma_0 - \Sigma_3$ corresponds to the maximum power of the wavenumbers k_{x21} and k_{x22} in the pre-exponent factors in expressions (55)–(58) (taking into account their power in expressions (31) and (32) for the quantities β_1 and β_2).

Taking into account expressions (54)–(58) in the expressions for the amplitudes (26), (30), (41), (42), (47), and (48) and inserting these six expressions into formulas (6) and (7), we find expressions for all the electromagnetic field components in the bi-gyrotropic layer:

$$E_{x2} = \frac{\mathrm{i}}{\varepsilon F_2}\left[\frac{k_z}{k_0}(b_0\Sigma_0 + F_{g2}\Sigma_1 + b_2\Sigma_2 + F_{vg}\Sigma_3) - \frac{k_y}{k_0}\,F_2\Sigma_2 - g(a_0\Sigma_0 + F_{vg}\Sigma_1 + a_2\Sigma_2 + F_{v2}\Sigma_3)\right]\exp(-\mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{59}$$

$$H_{x2} = \frac{1}{\mu F_2}\left[\frac{k_y}{k_0}\,F_2\Sigma_0 - \frac{k_z}{k_0}(a_0\Sigma_0 + F_{vg}\Sigma_1 + a_2\Sigma_2 + F_{v2}\Sigma_3) + v(b_0\Sigma_0 + F_{g2}\Sigma_1 + b_2\Sigma_2 + F_{vg}\Sigma_3)\right]\exp(-\mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{60}$$

$$E_{y2} = \frac{1}{F_2}\left[a_0\Sigma_0 + F_{vg}\Sigma_1 + a_2\Sigma_2 + F_{v2}\Sigma_3\right]\exp(-\mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{61}$$

$$H_{y2} = \frac{\mathrm{i}}{F_2}\left[b_0\Sigma_0 + F_{g2}\Sigma_1 + b_2\Sigma_2 + F_{vg}\Sigma_3\right]\exp(-\mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{62}$$

$$E_{z2} = \Sigma_0(x)\exp(-\mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{63}$$

$$H_{z2} = \mathrm{i}\Sigma_2(x)\exp(-\mathrm{i}k_y y - \mathrm{i}k_z z)\,. \tag{64}$$

6. Expressions for electric field components outside bi-gyrotropic layer

Now consider the microwave fields outside the bi-gyrotropic layer in media *1* and *3* characterized by scalar dielectric permittivities and magnetic permeabilities ε_1, μ_1 and ε_3, μ_3. Substituting solutions in the form of (6) and (7) into Maxwell's equations (4), we obtain two independent differential equations on the amplitudes $e_{z1,3}$ and $h_{z1,3}$ instead of system (8):

$$\frac{\partial^2 e_{z1,3}}{\partial x^2} - (k_z^2 + k_y^2 - k_0^2\varepsilon_{1,3}\mu_{1,3})e_{z1,3} = 0\,, \tag{65}$$

$$\frac{\partial^2 h_{z1,3}}{\partial x^2} - (k_z^2 + k_y^2 - k_0^2\varepsilon_{1,3}\mu_{1,3})h_{z1,3} = 0\,. \tag{66}$$

Solutions of equations (65) and (66) are defined by the characteristic equation

$$k_{x1,3}^2 = k_z^2 + k_y^2 - k_0^2\varepsilon_{1,3}\mu_{1,3}\,. \tag{67}$$

Since microwave fields should decay exponentially with distance from the layer, solutions of equations (65) and (66) in medium *1* are sought in the form

$$e_{z1} = N\exp(-k_{x1}x)\,, \tag{68}$$

$$h_{z1} = \mathrm{i}G\exp(-k_{x1}x)\,, \tag{69}$$

and in medium *3*, in the form

$$e_{z3} = K\exp(k_{x3}x)\,, \tag{70}$$

$$h_{z3} = \mathrm{i}L\exp(k_{x3}x)\,, \tag{71}$$

where N, G, L, and K are independent coefficients.

Transforming the system of Maxwell's equations (4), we express the quantities $e_{y1,3}$, $e_{x1,3}$, $h_{y1,3}$, and $h_{x1,3}$ in terms of the quantities $e_{z1,3}$ and $h_{z1,3}$, which are described by expressions (68)–(71), and then, inserting the expressions obtained into relationships (6), (7), we obtain the expressions for the microwave field components in half-spaces *1* and *3*:

$$E_{x1} = \frac{\mathrm{i}}{q_1^2}(Gk_yk_0\mu_1 - Nk_zk_{x1})\exp(-k_{x1}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{72}$$

$$H_{x1} = \frac{1}{q_1^2}(Gk_zk_{x1} - Nk_yk_0\varepsilon_1)\exp(-k_{x1}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{73}$$

$$E_{y1} = \frac{1}{q_1^2}(Nk_yk_z - Gk_{x1}k_0\mu_1)\exp(-k_{x1}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{74}$$

$$H_{y1} = \frac{\mathrm{i}}{q_1^2}(Gk_yk_z - Nk_{x1}k_0\varepsilon_1)\exp(-k_{x1}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{75}$$

$$E_{z1} = N\exp(-k_{x1}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{76}$$

$$H_{z1} = \mathrm{i}G\exp(-k_{x1}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{77}$$

$$E_{x3} = \frac{\mathrm{i}}{q_3^2}(Lk_yk_0\mu_3 + Kk_zk_{x3})\exp(k_{x3}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{78}$$

$$H_{x3} = -\frac{1}{q_3^2}(Lk_zk_{x3} + Kk_yk_0\varepsilon_3)\exp(k_{x3}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{79}$$

$$E_{y3} = \frac{1}{q_3^2}(Kk_yk_z + Lk_{x3}k_0\mu_3)\exp(k_{x3}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{80}$$

$$H_{y3} = \frac{\mathrm{i}}{q_3^2}(Lk_yk_z + Kk_{x3}k_0\varepsilon_3)\exp(k_{x3}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{81}$$

$$E_{z3} = K\exp(k_{x3}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{82}$$

$$H_{z3} = \mathrm{i}L\exp(k_{x3}x - \mathrm{i}k_y y - \mathrm{i}k_z z)\,, \tag{83}$$

where the quantities q_1 and q_3 are described by the following expression:

$$q_{1,3}^2 = k_z^2 - k_0^2\varepsilon_{1,3}\mu_{1,3}\,. \tag{84}$$

7. Dispersion relation for electromagnetic waves in a bi-gyrotropic layer

We now turn to the derivation of the dispersion equation describing the propagation of electromagnetic waves in a bi-gyrotropic layer. Satisfying the boundary condition of continuity of the tangential components E_y, E_z, H_y, and H_z at $x = 0$ and $x = s$, we obtain the following system of eight equations on the constant coefficients A, B, C, D, G, N, K, L:

$$
\begin{aligned}
& N\exp(-k_{x1}s) = \Sigma_0(s)\,,\\
& F_2\,\frac{Nk_yk_z - G\mu_1 k_{x1}k_0}{q_1^2\exp(k_{x1}s)} = a_0\Sigma_0(s) + F_{vg}\Sigma_1(s)\\
& \qquad + a_2\Sigma_2(s) + F_{v2}\Sigma_3(s)\,,\\
& G\exp(-k_{x1}s) = \Sigma_2(s)\,,\\
& F_2\,\frac{Gk_yk_z - N\varepsilon_1 k_{x1}k_0}{q_1^2\exp(k_{x1}s)} = b_0\Sigma_0(s) + F_{g2}\Sigma_1(s)\\
& \qquad + b_2\Sigma_2(s) + F_{vg}\Sigma_3(s)\,,\\
& K = \Sigma_0(0)\,,\\
& F_2\,\frac{Kk_yk_z + L\mu_3 k_{x3}k_0}{q_3^2} = a_0\Sigma_0(0) + F_{vg}\Sigma_1(0)\\
& \qquad + a_2\Sigma_2(0) + F_{v2}\Sigma_3(0)\,,\\
& L = \Sigma_2(0)\,,\\
& F_2\,\frac{Lk_yk_z + K\varepsilon_3 k_{x3}k_0}{q_3^2} = b_0\Sigma_0(0) + F_{g2}\Sigma_1(0)\\
& \qquad + b_2\Sigma_2(0) + F_{vg}\Sigma_3(0)\,.
\end{aligned}
\tag{85}
$$

Inserting the constants N, G, K, and L from the first, third, fifth, and seventh equations into the second, fourth, sixth, and eighth equations of system (85), we reduce it to a system of four equations for the coefficients A, B, C, and D:

$$
\begin{aligned}
& \left(a_0 - \frac{k_yk_z}{q_1^2}\,F_2\right)\Sigma_0(s) + F_{vg}\Sigma_1(s)\\
& \qquad + \left(a_2 + \frac{\mu_1 k_{x1}k_0}{q_1^2}\,F_2\right)\Sigma_2(s) + F_{v2}\Sigma_3(s) = 0\,,\\
& \left(b_0 + \frac{\varepsilon_1 k_{x1}k_0}{q_1^2}\,F_2\right)\Sigma_0(s) + F_{g2}\Sigma_1(s)\\
& \qquad + \left(b_2 - \frac{k_yk_z}{q_1^2}\,F_2\right)\Sigma_2(s) + F_{vg}\Sigma_3(s) = 0\,,\\
& \left(a_0 - \frac{k_yk_z}{q_3^2}\,F_2\right)\Sigma_0(0) + F_{vg}\Sigma_1(0)\\
& \qquad + \left(a_2 - \frac{\mu_3 k_{x3}k_0}{q_3^2}\,F_2\right)\Sigma_2(0) + F_{v2}\Sigma_3(0) = 0\,,\\
& \left(b_0 - \frac{\varepsilon_3 k_{x3}k_0}{q_3^2}\,F_2\right)\Sigma_0(0) + F_{g2}\Sigma_1(0)\\
& \qquad + \left(b_2 - \frac{k_yk_z}{q_3^2}\,F_2\right)\Sigma_2(0) + F_{vg}\Sigma_3(0) = 0\,.
\end{aligned}
\tag{86}
$$

Inserting into system (86) expressions (55)–(58), describing the quantities Σ_0, Σ_1, Σ_2, and Σ_3, and collecting the terms by the coefficients A, B, C, and D, we obtain the system of equations

$$
\begin{cases}
d_{11}A + d_{12}B + d_{13}C + d_{14}D = 0\,,\\
d_{21}A + d_{22}B + d_{23}C + d_{24}D = 0\,,\\
d_{31}A + d_{32}B + d_{33}C + d_{34}D = 0\,,\\
d_{41}A + d_{42}B + d_{43}C + d_{44}D = 0\,.
\end{cases}
\tag{87}
$$

The entries $d_{11} - d_{44}$ of the matrix defined by (87) are given in the Appendix.

Thus, the dispersion equation for electromagnetic SS-waves propagating along a bi-gyrotropic layer is the fourth-order determinant of the system of homogeneous equations (87).

The dispersion relations and expressions for the microwave field components of VV-, VS-, and SV-waves, which can appear in various special cases of a bi-gyrotropic layer, can be obtained similarly.

8. Proof of continuity of normal components of electric and magnetic induction at boundaries of bi-gyrotropic layer

In the previous Section 7, we derived the dispersion equation for electromagnetic waves in a bi-gyrotropic layer. To derive this equation, we used the boundary conditions that the tangential components E_y, E_z, H_y, and H_z are continuous at the layer surfaces (at $x = 0$ and $x = s$).

It is quite possible that researchers describing the SW in the magnetostatic approximation would doubt the applicability of our solution for describing an SW, since, in the magnetostatic approximation, no microwave electric field is associated with this wave (it is assumed to be small and thus can be neglected), and, instead of the condition that the tangential components of the electric field be continuous across the boundaries, it is required that the normal component of the magnetic induction be continuous [26].

In the following, we will prove that our solution is unique and thus can be used to describe all electromagnetic waves (including the SW) in anisotropic media described by the Hermitian second-rank tensors $\overleftrightarrow{\varepsilon_2}$ and $\overleftrightarrow{\mu_2}$ given by expressions (1) and (2).

As is well known, according to the theorem on the uniqueness of solutions to Maxwell's equations, if the condition that the tangential components of the electric and magnetic fields **E** and **H** be continuous at the interface between some media is satisfied, this simultaneously ensures the continuity of the normal components of the electric and magnetic induction **D** and **B** at the same interface (see § 9.2 in Ref. [48]). However, this theorem was proved only for electromagnetic waves in isotropic media: electromagnetic waves in gyrotropic media were excluded from consideration [48].

Since in the derivation of the dispersion relation we used the condition that the tangential components E_y, E_z, H_y, and H_z be continuous across boundaries, we must additionally prove the continuity of the normal components of the **D** and **B** vectors across these boundaries.

Comparing expressions (59)–(64), and also from expressions (41) and (42), it can be seen that, inside the bi-gyrotropic layer, the field components E_{x2} and H_{x2} can be written as follows:

$$
E_{x2} = -\frac{k_y}{\varepsilon k_0}\,H_{z2} + \frac{k_z}{\varepsilon k_0}\,H_{y2} - \mathrm{i}\,\frac{g}{\varepsilon}\,E_{y2}\,,
\tag{88}
$$

$$
H_{x2} = \frac{k_y}{\mu k_0}\,E_{z2} - \frac{k_z}{\mu k_0}\,E_{y2} - \mathrm{i}\,\frac{\nu}{\mu}\,H_{y2}\,.
\tag{89}
$$

Departing from expressions (5) and using expressions (88) and (89), the normal components of electric and magnetic induction D_{x2} and B_{x2} can be written as follows:

$$D_{x2} = \varepsilon E_{x2} + \mathrm{i}gE_{y2} = -\frac{k_y}{k_0}\,H_{z2} + \frac{k_z}{k_0}\,H_{y2}\,, \tag{90}$$

$$B_{x2} = \mu H_{x2} + \mathrm{i}\nu H_{y2} = \frac{k_y}{k_0}\,E_{z2} - \frac{k_z}{k_0}\,E_{y2}\,. \tag{91}$$

In isotropic half-spaces *1* and *3*, the normal components of electric and magnetic induction D_{x1}, B_{x1}, and D_{x3}, B_{x3}, equal to $\varepsilon_1 E_{x1}$, $\mu_1 H_{x1}$, and $\varepsilon_3 E_{x3}$, $\mu_3 H_{x3}$, respectively, can be written analogously as[9]

$$D_{x1} = -\frac{k_y}{k_0}\,H_{z1} + \frac{k_z}{k_0}\,H_{y1}\,, \tag{92}$$

$$B_{x1} = \frac{k_y}{k_0}\,E_{z1} - \frac{k_z}{k_0}\,E_{y1}\,, \tag{93}$$

$$D_{x3} = -\frac{k_y}{k_0}\,H_{z3} + \frac{k_z}{k_0}\,H_{y3}\,, \tag{94}$$

$$B_{x3} = \frac{k_y}{k_0}\,E_{z3} - \frac{k_z}{k_0}\,E_{y3}\,. \tag{95}$$

As can be seen from expressions (90)–(95), in a bi-gyrotropic layer, as well as in isotropic half-spaces, *the normal component of the electric induction D_x is defined by the sum of the tangential components of the magnetic field, and the normal component of the magnetic induction B_x is defined by the sum of the tangential components of the electric field*, and related terms appear with equal coefficients.

It should be recalled now that the following boundary conditions on the tangential components E_y, E_z, H_y, and H_z were used in the derivation of (85):

$$\begin{aligned}
&E_{z1}(x=s) = E_{z2}(x=s)\,,\\
&E_{y1}(x=s) = E_{y2}(x=s)\,,\\
&H_{z1}(x=s) = H_{z2}(x=s)\,,\\
&H_{y1}(x=s) = H_{y2}(x=s)\,,\\
&E_{z3}(x=0) = E_{z2}(x=0)\,,\\
&E_{y3}(x=0) = E_{y2}(x=0)\,,\\
&H_{z3}(x=0) = H_{z2}(x=0)\,,\\
&H_{y3}(x=0) = H_{y2}(x=0)\,.
\end{aligned} \tag{96}$$

From equations (90)–(95), it follows that the fulfillment of the continuity conditions for the components E_y, E_z, H_y, and H_z at the boundaries between the media also ensures the fulfillment of other conditions: the equality (and thus continuity) of the normal components of the vectors **D** and **B** at these boundaries,

$$\begin{aligned}
&D_{x1}(x=s) = D_{x2}(x=s)\,,\\
&B_{x1}(x=s) = B_{x2}(x=s)\,,\\
&D_{x3}(x=0) = D_{x2}(x=0)\,,\\
&B_{x3}(x=0) = B_{x2}(x=0)\,.
\end{aligned} \tag{97}$$

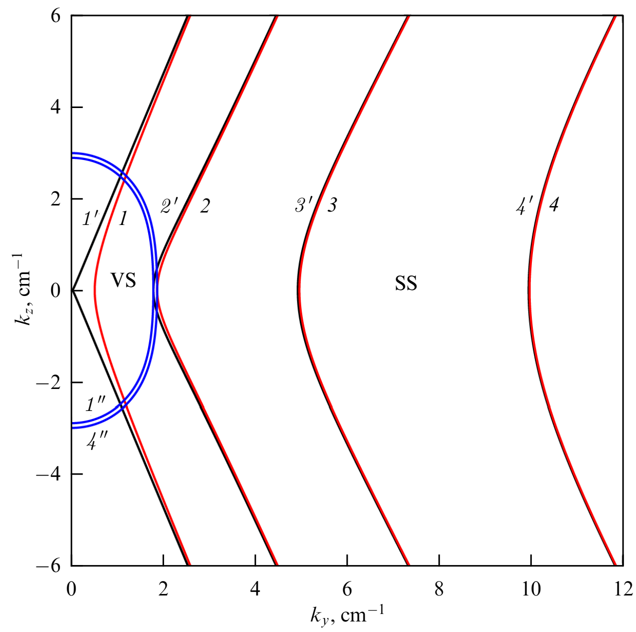

Figure 3. Isofrequency dependences for spin waves in a tangentially magnetized ferrite slab for frequencies 2198 (*1* and *1′*), 2216.3 (*2* and *2′*), 2250 (*3* and *3′*), and 2300 MHz (*4* and *4′*) (half-plane $k_y > 0$ is shown). Curves *1*–*4* are calculated without magnetostatic approximation, curves *1′*–*4′* are calculated in this approximation. Also shown are curves *1″* and *4″* which are intersection of surface $\alpha = 0$ and, respectively, surfaces $f = 2198$ and $f = 2300$ MHz (curve *1″* separates on curve *1* regions that correspond to SS-wave and region that corresponds to VS-wave).

Thus, whatever boundary conditions are used in the derivation of the dispersion relation, we arrive at one and the same *unique* dispersion relation, describing the propagation of electromagnetic waves in a bi-gyrotropic layer.

9. Calculations of spin wave characteristics in a ferrite slab

As an example of using the dispersion equation (87), we calculate some characteristics of the SW in a ferrite slab surrounded by vacuum half-spaces.

As already mentioned, the fundamental difference in the description of SWs here from the previous ones obtained in the magnetostatic approximation [26] and without it [29, 37] lies in the fact that *the wave distribution inside the ferrite slab is described by two wavenumbers*[10] — k_{x21} and k_{x22}. We will show how this is reflected in the characteristics of SWs with the spectrum located above the frequency $f_\perp$. As is well known, such a wave is called a surface magnetostatic wave [26], since, in the magnetostatic approximation, its distribution in the slab is of a purely surface type and is characterized by a single wavenumber k_{x2ms}.

Isofrequency dependences for this SW at different frequencies are shown in Fig. 3, where curves *1*–*4* are calculated using the description of SWs given above, while curves *1′*–*4′* are calculated in the magnetostatic approximation according to Ref. [26] (for brevity, they will be further referred to as 'magnetostatic' dependences). The calculations were performed for the following parameters: $H_0 = 300$ Oe, $4\pi M_0 = 1750$ G, $s = 40$ μm, $\varepsilon_2 = 15$.

[9] These relations can be easily obtained from equations (33) and (37) by expressing the x-components of the field by the y- and z-components, taking $\nu = 0$ and $g = 0$ there, and changing indices 2 to 1 and 3.

[10] Which confirms the earlier results for the reciprocal SW propagating along the direction of the vector $\mathbf{H}_0$ tangent to a magnetized ferrite slab [38].

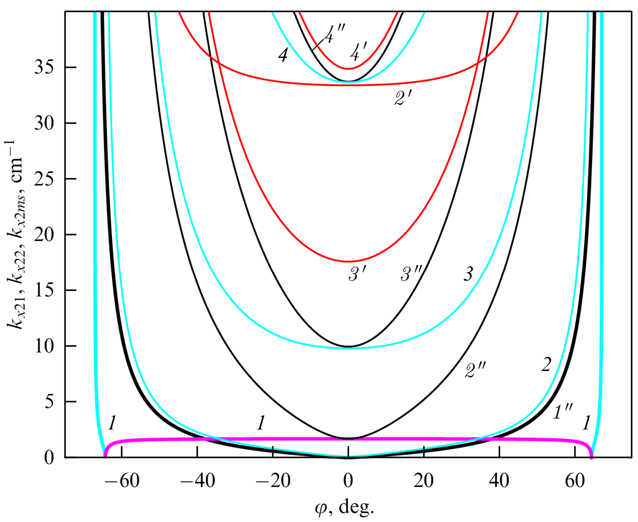

Figure 4. Dependences of transverse wavenumbers k_{x21} (curves *1–4*), k_{x22} (curves *1′–4′*), and k_{x2ms} (curves *1″–4″*) on angle φ which sets the orientation of wavevector for frequencies of 2198, 2216.3, 2300, and 2500 MHz, respectively. Curve *1′* is not shown in this figure because it is located above value of 250 cm^{-1}. On part of curve *1* plotted in violet, k_{x21} takes imaginary values which correspond a VS-wave (in this case, the absolute value $|k_{x21}|$ is plotted along the ordinate).

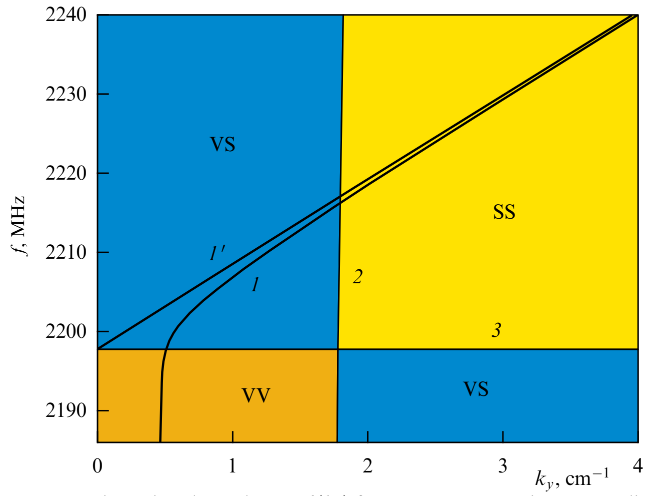

Figure 5. Dispersion dependences $f(k_y)$ for an SW propagating perpendicularly to vector $\mathbf{H}_0$ in a tangentially magnetized ferrite slab: *1* is exact calculation, *1′* is result in magnetostatic approximation, *2* is curve where $\alpha = 0$ (it corresponds to curve *2* in Fig. 2), and *3* is line $f = f_\perp$ (which corresponds to line **9** in Fig. 2). Parts of curve *1* in different regions of the plane (f, k_y) correspond to VV-, VS-, and SS-waves.

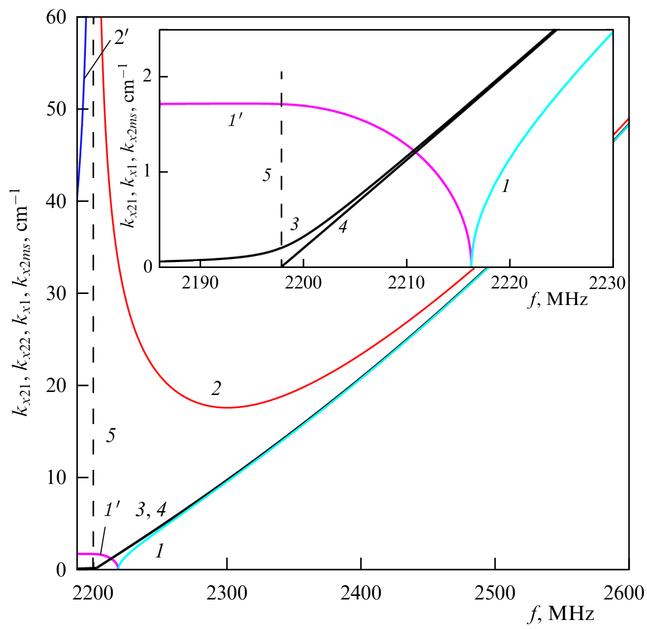

Figure 6. Dependences of transverse wavenumbers k_{x21} (curves *1* and *1′*), k_{x22} (curves *2* and *2′*), k_{x1} (curve *3*), and k_{x2ms} (curve *4* calculated in magnetostatic approximation) on frequency f for an SW propagating perpendicularly to vector $\mathbf{H}_0$ in a tangentially magnetized ferrite slab: curves *1* and *2* correspond to those parts of dependences $k_{x21}(f)$ and $k_{x22}(f)$ where k_{x21} and k_{x22} are real, and curves *1′* and *2′* correspond to the parts of $k_{x21}(f)$ and $k_{x22}(f)$ where k_{x21} and k_{x22} are imaginary (in this case, values $|k_{x21}|$ and $|k_{x22}|$ are shown along the ordinate); *5* is line $f = f_\perp$.

As one would expect from the plots in Fig. 3, in the exact description of this SW, its isofrequency dependences may contain intervals describing VS-waves in addition to the intervals describing SS-waves. For example, SWs with frequencies $f < 2015$ MHz have regions corresponding to a VS-wave (for example, the boundary curve *1″* bounds such a region on isofrequency curve *1* in Fig. 3), whereas SWs with frequencies $f > 2015$ MHz are always SS-waves (for example, boundary curve *4″* does not intersect isofrequency curve *4* in Fig. 3).

Although the exact and magnetostatic isofrequency dependences in Fig. 3 do not differ substantially, the change in the quantities k_{x21} and k_{x22} (which characterize the SW distribution inside the slab) along the isofrequency dependences differs significantly from the analogous change in the related magnetostatic quantity k_{x2ms} (Fig. 4).

As can be seen from Fig. 4, at $\varphi = 0$, the magnetostatic dependences $k_{x2ms}(\varphi)$ pass near the curves $k_{x21}(\varphi)$, and as $|\varphi|$ increases the dependences $k_{x2ms}(\varphi)$ gradually approach the curves $k_{x22}(\varphi)$, and the difference between the values $k_{x22}(\varphi = 0)$ and $k_{x21}(\varphi = 0)$ essentially depends on the frequency, changing from 255 cm^{-1} for $f = 2198$ MHz to 2 cm^{-1} for $f = 2500$ MHz.

10. Comparison of spin wave propagation along *y*-axis with earlier results

In testing any new theory, it is important to compare the results obtained on the basis of that theory with the available data for various limit and special cases.

One such case is the exact description of the SW propagating tangentially to a magnetized ferrite slab and perpendicular to the vector $\mathbf{H}_0$, which was obtained earlier [30, 31]. Figure 5 shows the dispersion dependences of SW $f(k_y)$ for this case for the parameters mentioned above: the magnetostatic dependence is described by curve *1′*, and the exact dependence, calculated on the basis of the theory presented above and on the basis of Ref. [31], is described by one and the same curve *1*, since the results of these calculations are identical.

Figure 5 also shows boundary curves *2* and *3*, which separate the intervals with SS-, VV-, or VS-waves on curve *1*, which are characterized by the distributions of the SW in the ferrite slab given by expressions (26)–(28).

Figure 6 shows the variation in the wavenumbers k_{x21}, k_{x22}, k_{x1}, and k_{x2ms} along the respective dispersion dependences shown in Fig. 5. The dependences $k_{x21}(f)$ and $k_{x22}(f)$ are calculated by (20) and (21), the dependence $k_{x1}(f)$ relies on expression (67), and the dependence $k_{x2ms}(f)$ uses the theory [26].

Before starting a discussion of the results shown in Figs 5 and 6, it should be noted that, in the earlier description of an SW (without the magnetostatic approximation) propagating perpendicular to the vector $\mathbf{H}_0$ [30, 31], the distribution of the SV in the ferrite slab is characterized by a single wavenumber instead of two (as follows from the theory presented here). Thus, at first glance there is an apparent contradiction between the proposed description and the previous one. In reality, this contradiction is only apparent, as will be shown below.

Obviously, the case where the SW propagates perpendicular to the vector $\mathbf{H}_0$ can be obtained from the description we are considering if we set $k_z = 0$ (or $\varphi = 0$). For $k_z = 0$, the system of Maxwell's equations (4) is decomposed into two independent subsystems, one containing only the field components E_{z2}, H_{x2}, and H_{y2}, and the other one containing H_{z2}, E_{x2}, and E_{y2}. Since for $k_z = 0$ we get $F_{vg} = 0$ according to (11), the resulting system (8) for the amplitudes e_{z2} and h_{z2} reduces to two independent equations: one for the amplitude e_{z2} and the other one for the amplitude h_{z2}. The first equation describes the well-known surface SW (an H-wave with the components E_{z2}, H_{x2}, and H_{y2}), for which, using $F_{vg} = 0$ in expression (21), we find

$$k_{x22} = k_0 \sqrt{F_v} = \sqrt{k_y^2 - k_0^2 \varepsilon \mu_\perp}\,, \tag{98}$$

while the second equation describes a surface electromagnetic wave (an E-wave with the components H_{z2}, E_{x2}, and E_{y2}, which is common for a layer of usual dielectric), for which, inserting $F_{vg} = 0$ into expression (20), we find

$$k_{x21} = k_0 \sqrt{F_g} = \sqrt{k_y^2 - k_0^2 \varepsilon \mu_{zz}}\,. \tag{99}$$

These values of k_{x22} and k_{x21} correspond to the results obtained earlier (see, e.g., [1, 11, 30]).

Nevertheless, the reader may still remark that it is clear from Fig. 4 that, in the theory presented here, an SW is not characterized by a single wavenumber at $\varphi = 0$, but by two wavenumbers k_{x21} and k_{x22}.

To answer this remark, we calculate below the change in the coefficients A, B, and C, normalized with the coefficient D, as a function of the wavenumber k_z (Fig. 7). Recall that the coefficients A, B, C, and D define the amplitudes of the exponential functions in expression (26), and now, as can be seen from Fig. 7, the coefficients A and B, appearing with the exponents $\exp(k_{x21}x)$ and $\exp(-k_{x21}x)$, become zero at $\varphi = 0$ (or $k_z = 0$), while the coefficients C and D appearing with the exponents $\exp(k_{x22}x)$ and $\exp(-k_{x22}x)$ remain finite! Obviously, for $\varphi = 0$, the distribution of an SW inside the ferrite slab is described by a single wavenumber — k_{x22}, which has already been found in Refs [30, 31].

As follows from Fig. 7, in practice, the wavenumber k_{x21} does not affect the distribution of the microwave field of the SW inside the ferrite slab for a narrow angular range $|\varphi| \lesssim 2°$, where the coefficients A and B are *close to zero* for all frequencies, and the *main contribution* in this distribution, in accordance with formulas (55)–(64), is introduced by the coefficients C and D and the corresponding wavenumber k_{x22}. Therefore, the dependences shown in Figs 5 and 6 and the change in the distribution of SWs for $\varphi = 0$ described

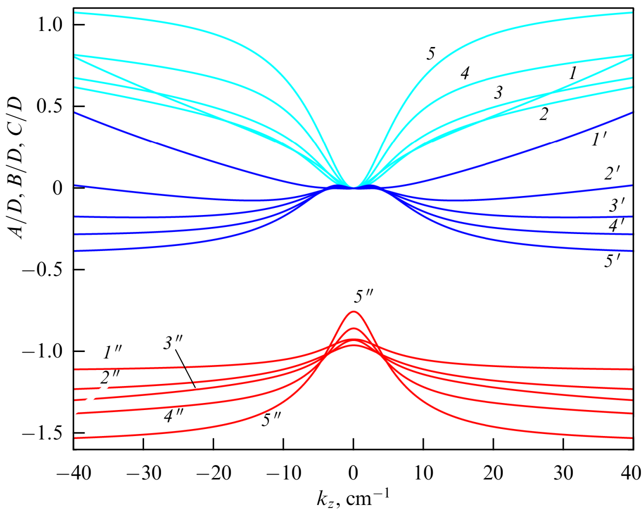

Figure 7. Dependences of coefficient ratios A/D (curves *1–5*), B/D (curves *1′–5′*), and C/D (curves *1″–5″*) on wavenumber k_z for frequencies of 2217, 2300, 2500, 2800, and 3200 MHz, respectively.

below will also be practically preserved for small angles $|\varphi| \lesssim 2°$.

As can be seen from Fig. 6, as $f \to f_\perp$, we have $k_{x22} \to \infty$, while for the analogous magnetostatic dependence we find $k_{x2ms} \to 0$! It is obvious that such a large difference in the values of k_{x22} and k_{x2ms} will lead to significant differences in the description of the SW distribution in the magnetostatic approximation and without it: in fact, the wavenumbers k_{x22} and k_{x2ms} define the penetration depth of the microwave field of the SW into the ferrite slab, and, as can be seen from Fig. 6, the dependence $k_{x2ms}(f)$ is a monotonic one, while the dependence $k_{x22}(f)$ has a minimum about 100 MHz from the frequency $f_\perp$. As a consequence, the distribution of the microwave electric field of the SW (which is defined by the dependence $E_z(x)$ [39]) varies in the following way.

At frequencies close to $f_\perp$ (where $k_y \sim k_0$, k_{x22} are large and k_{x1} is very small), practically all the energy of the SW turns out to be localized in one of the half-spaces (see curve *1* in Fig. 8), and the energy of an SW traveling in the positive direction of the y-axis turns out to be localized in half-space *1* (as in Fig. 8), and the energy of an SW traveling in the negative direction of the y-axis becomes localized in half-space *3*. Thus, despite the large wavelength λ of the SW at frequencies which are close to $f_\perp$, the ferrite slab is in practice an insurmountable obstacle: almost all the SW energy is confined to one of the half-spaces, and this energy extends over tens of centimeters [11] from the slab surface!

With increasing frequency f, the electric field of SW E_z penetrates more strongly through the ferrite slab into half-space *3*, and this penetration reaches its maximum at the frequency $f = 2300$ MHz (see curve *2* in Fig. 8), at which $k_{x22}(f)$ has a minimum (see curve *2* in Fig. 6). A further increase in the frequency reduces the penetration of the SW energy into half-space *3*, and, at frequencies close to the upper spectral limit, the energy of the SW turns out to be localized near the surface $x = s = 40$ μm.

[11] This phenomenon allows us to explain the efficient transformation of the SW energy into an electromagnetic wave emitted into the surrounding space, which was previously experimentally demonstrated in Ref. [35].

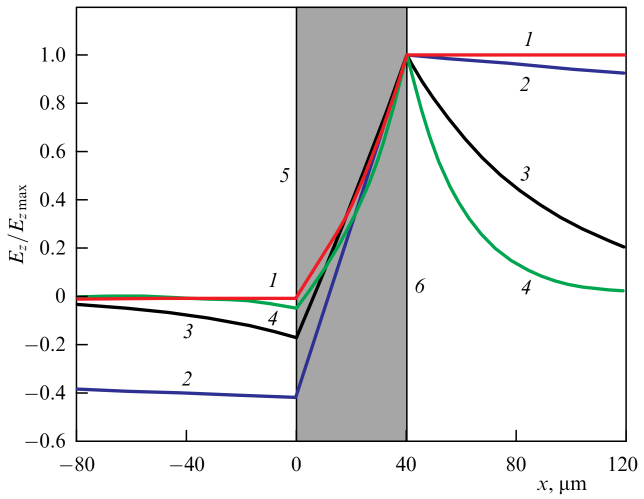

Figure 8. Normalized distribution of microwave electric field $E_z(x)$ in an SW with the following values of frequency f and wavenumber k_y: *1* — $f = 2197.85$ MHz, $k_y = 0.503$ cm^{-1}; *2* — $f = 2300.3$ MHz, $k_y = 10$ cm^{-1}; *3* — $f = 3103$ MHz, $k_y = 200$ cm^{-1}; *4* — $f = 3276$ MHz, $k_y = 500$ cm^{-1}; vertical lines *5* and *6* mark ferrite slab boundaries $x = 0$ and $x = s = 40$ μm.

The picture is very different if the SW is described in the magnetostatic approximation, when, as is well known [26], for $k_z = 0$ it is valid that $k_{x2ms} = k_{x1ms} = k_{x3ms} = |k_{yms}|$, and the distribution of magnetic potential Ψ inside and outside the ferrite slab is described as

$$\Psi_2 = A\exp(k_{x2ms}x) + B\exp(-k_{x2ms}x)\,,$$
$$\Psi_1 = C\exp(-k_{x1ms}x) \quad \text{and} \quad \Psi_3 = D\exp(-k_{x3ms}x)\,.$$

In this case, at frequencies that are close to $f_\perp$ (when $k_{x2ms} = k_{x1ms} = k_{x3ms} \sim 0$, as is seen in Fig. 6), the normalized distribution[12] $\Psi(x)$ will resemble a straight line $\Psi = 1$ (or $\Psi = -1$). As the frequency f is further increased, the quantities $k_{x2ms}, k_{x1ms}, k_{x3ms}$ increase monotonically, defining the growth in the exponential decay of the distribution $\Psi(x)$ with the distance to the slab surface where the SW is localized (see, e.g., curve *2* in Fig. 5 in [39]).

We see that the magnetostatic description of SWs leads to an incorrect representation of SW properties in the initial spectral domain when $f \sim f_\perp$. The reason for this is hidden in the magnetostatic approximation itself, the use of which assumes that, in all equations and formulas, starting from Maxwell's equations, one can consider that $k \gg k_0$. A broad use of this assumption sometimes leads to errors. For example, in the magnetostatic approximation, from the exact formula (98), neglecting the combination $k_0^2\varepsilon\mu_\perp$ as compared to k_y^2, a formula $k_{x2ms} = |k_{yms}|$ can be obtained. However, it is clear that the inequality $k_y^2 \gg k_0^2$ is valid for one interval of k_y values, and the inequality $k_y^2 \gg k_0^2\varepsilon\mu_\perp$ is valid for a quite different interval of k_y, since the product $k_0^2\varepsilon\mu_\perp$ can be several orders of magnitude larger than k_0^2 at frequencies where $\mu_\perp$ is large.

Note that we are not suggesting that all SW researchers should completely abandon the use of the magnetostatic approximation to describe SWs, as many useful and helpful results have been obtained on its basis (e.g., the formulas for SW cutoff angles obtained in the magnetostatic approximation for $k \to \infty$ are certainly also valid in the exact description of SWs). We only want to draw attention to the fact that the magnetostatic approximation provides a simplified and sometimes even incorrect description of SW properties.

Unfortunately, the framework of a single paper does not allow us to discuss all the advantages that the exact description of electromagnetic waves in a bi-gyrotropic layer offers, even for the special case of the SW, in the same manner that the publication of [26] was followed by many publications exploring the properties of SWs. However, we would like to mention here some of the advantages. First of all, the theory developed here would allow the description of SWs in the framework of classical electrodynamics, thus removing an essential limitation of the SW description imposed by the magnetostatic approximation. In particular, with the help of this theory, in addition to the exact distribution of SWs, one can calculate the Poynting vector, the direction and density of the energy flux, as well as the polarization and vector lines of SWs, and in the future even more complex problems can be solved on the basis of modern methods of electrodynamics, which will undoubtedly promote the successful development of magnonics and the perfection of devices using SWs.

11. Conclusions

This paper discusses problems and errors caused by the description of spin waves in the magnetostatic approximation widely used by specialists for more than 60 years. In particular, when describing a spin wave based on the magnetostatic approximation and calling it a magnetostatic wave, researchers associate neither the microwave electric field (which is considered to be negligibly small) nor the Poynting vector (which cannot be found without the electric field) with it.

To alleviate these problems, we present for the first time a correct analytical treatment of the propagation of electromagnetic waves in an arbitrary direction along a tangentially magnetized bi-gyrotropic layer with dielectric permittivity and magnetic permeability described by Hermitian second-rank tensors. The propagation of spin waves in a ferrite slab is a special case of this general treatment.

It is shown that by representing a solution of Maxwell's equations as a wave of the form $\exp(-\mathrm{i}k_y y - \mathrm{i}k_z z)$ propagating in the plane of the layer, with the dependence on the coordinate x normal to the layer left unspecified, it is possible to reduce Maxwell's equations to a system of two differential equations of second order containing only the x-dependent amplitudes of the microwave electric and magnetic fields parallel to the vector of the constant uniform magnetic field $\mathbf{H}_0$. This system is further reduced to a fourth-order differential equation, which defines a bi-quadratic characteristic equation specifying wavenumbers of the electromagnetic wave distribution in the cross section of the bi-gyrotropic layer. It is shown that this equation has four simple (not multiple) roots, k_{x21}, k_{x22}, $k_{x23} = -k_{x21}$, and $k_{x24} = -k_{x22}$, which cannot be *complex*-valued and can take either *real* or *imaginary* values.

We have considered the propagation of electromagnetic waves with real k_{x21} and k_{x22} in an arbitrary direction along a bi-gyrotropic layer surrounded by dielectric half-spaces. For these waves, a dispersion equation has been derived, which is

[12] The distribution $\Psi(x)$ is often associated with the SW energy distribution, which is not correct, as mentioned in Ref. [39]. We analyze here the distribution $\Psi(x)$, since an SW lacks a dependence analogous to $E_z(x)$ in the magnetostatic approximation.

a fourth-order determinant of the system of linear homogeneous equations, and it is also shown that these waves have six components of the microwave electromagnetic field — three electric and three magnetic. It is proved that this dispersion equation is obtained both when the boundary conditions are the continuity of the tangential components of the electric and magnetic fields **E** and **H** and when they are the continuity of the normal component of the vectors of electric and magnetic induction **D** and **B**.

It is shown that *the distribution of the amplitudes of microwave fields inside the bi-gyrotropic layer* (*and its special cases — the layers of ferrite, antiferromagnetic, plasma, or uniaxial optical crystal*) *and in structures based on such a layer, in the general case, is always described by two wavenumbers* — k_{x21} and k_{x22}. Depending on the anisotropic medium, its parameters, and the values of k_{x21} and k_{x22}, propagating waves are characterized by one of four possible distributions, described by trigonometric and exponential functions (see expressions (26)–(29)) and are one of the following wave types: SS-waves, or surface–surface waves (k_{x21} and k_{x22} are real numbers); VS-waves, or volume–surface waves (k_{x21} is imaginary and k_{x22} is real); VV-waves, or volume–volume waves (k_{x21} and k_{x22} are both imaginary); and SV-waves, or surface–volume waves (k_{x21} is real and k_{x22} is imaginary).

On the basis of the developed theory, we have studied the characteristics of SWs in a ferrite slab, which is a special case of a bi-gyrotropic layer. In the coordinate space $\{k_y, k_z, f\}$, *boundary surfaces* were constructed which separate regions with different wave distributions on the dispersion surfaces $f(k_y, k_z)$ for different SW types. It is shown that, on the dispersion surfaces of SWs (and also in their sections — dispersion and isofrequency dependences), there can be regions describing SS-, VS-, and VV-waves. It is found that the boundary surfaces are described by an equation which is identical to the dispersion equation for electromagnetic waves in an unbounded ferrite (bi-gyrotropic) medium (when this equation is reduced to a two-dimensional case).

We have studied the characteristics of SWs at frequencies above the ferromagnetic resonance frequency $f_\perp$ for the ferrite slab. In particular, the dependences of the SW transverse wavenumbers k_{x21} and k_{x22} on the wave vector orientation φ were calculated for different values of the frequency f. It is found that the dependences $k_{x21}(\varphi)$ and $k_{x22}(\varphi)$ differ substantially from each other as well as from the analogous magnetostatic dependences $k_{x2ms}(\varphi)$. The difference in the values of $k_{x22}(\varphi \sim 0)$ and $k_{x21}(\varphi \sim 0)$ depends essentially on the frequency, changing from $\sim 255\ \mathrm{cm}^{-1}$ for $f \sim f_\perp$ to $2\ \mathrm{cm}^{-1}$ for $f \sim f_\perp + 300$ MHz. For the angles $\varphi \sim 0$, the dependences $k_{x2ms}(\varphi)$ pass near the curves $k_{x21}(\varphi)$, and for the angles φ near the wave vector cutoff angles, they pass near the curves $k_{x22}(\varphi)$.

Furthermore, it is established that, if $\varphi = 0$ (or $k_z = 0$), the wave distribution in the ferrite slab is described by one wavenumber k_{x22}, because the coefficients A and B determining the contribution of the second wavenumber k_{x21} and acting as factors with the exponents $\exp(k_{x21}x)$ and $\exp(-k_{x21}x)$ are equal to zero.

It is shown that the microwave electric field of SWs plays an important role in the wave description, governing the energy distribution inside and near the ferrite slab. In particular, calculations of the distribution of the microwave electric wave field $E_z(x)$ for a set of frequencies f showed that, at frequencies close to $f_\perp$, the ferrite slab is almost an impenetrable obstacle for the wave despite its large wavelength: practically all the wave energy is concentrated in one half-space where it penetrates over tens of centimeters from the slab surface (whereas analogous calculations in the magnetostatic approximation give erroneous results according to which the energy is practically equally distributed between the half-spaces)! It is also found that the penetration of the SW energy through the ferrite slab reaches a maximum at a frequency at which the dependence $k_{x22}(f)$ has a minimum.

We assume that the solution found here for the propagation of electromagnetic waves in a tangentially magnetized bi-gyrotropic layer will facilitate accurate calculations of electromagnetic waves not only in layers of ferrite, antiferromagnetic, plasma, and uniaxial crystals, but also in layers of various metamaterials.

This study was carried out in the framework of state assignment to the Kotelnikov Institute of Radio Engineering and Electronics RAS.

12. Appendix

The elements of matrix (87) are given by the following expressions:

$$d_{11} = \left[a_0 - \frac{k_y k_z}{q_1^2} F_2 + \frac{k_{x21}}{k_0} F_{vg} + \beta_1 \left(a_2 + \frac{\mu_1 k_{x1} k_0}{q_1^2} F_2 + \frac{k_{x21}}{k_0} F_{v2}\right)\right] \exp(k_{x21}s), \quad \text{(A.1)}$$

$$d_{12} = \left[a_0 - \frac{k_y k_z}{q_1^2} F_2 - \frac{k_{x21}}{k_0} F_{vg} + \beta_1 \left(a_2 + \frac{\mu_1 k_{x1} k_0}{q_1^2} F_2 - \frac{k_{x21}}{k_0} F_{v2}\right)\right] \exp(-k_{x21}s), \quad \text{(A.2)}$$

$$d_{13} = \left[a_0 - \frac{k_y k_z}{q_1^2} F_2 + \frac{k_{x22}}{k_0} F_{vg} + \beta_2 \left(a_2 + \frac{\mu_1 k_{x1} k_0}{q_1^2} F_2 + \frac{k_{x22}}{k_0} F_{v2}\right)\right] \exp(k_{x22}s), \quad \text{(A.3)}$$

$$d_{14} = \left[a_0 - \frac{k_y k_z}{q_1^2} F_2 - \frac{k_{x22}}{k_0} F_{vg} + \beta_2 \left(a_2 + \frac{\mu_1 k_{x1} k_0}{q_1^2} F_2 - \frac{k_{x22}}{k_0} F_{v2}\right)\right] \exp(-k_{x22}s), \quad \text{(A.4)}$$

$$d_{21} = \left[b_0 + \frac{\varepsilon_1 k_{x1} k_0}{q_1^2} F_2 + \frac{k_{x21}}{k_0} F_{g2} + \beta_1 \left(b_2 - \frac{k_y k_z}{q_1^2} F_2 + \frac{k_{x21}}{k_0} F_{vg}\right)\right] \exp(k_{x21}s), \quad \text{(A.5)}$$

$$d_{22} = \left[b_0 + \frac{\varepsilon_1 k_{x1} k_0}{q_1^2} F_2 - \frac{k_{x21}}{k_0} F_{g2} + \beta_1 \left(b_2 - \frac{k_y k_z}{q_1^2} F_2 - \frac{k_{x21}}{k_0} F_{vg}\right)\right] \exp(-k_{x21}s), \quad \text{(A.6)}$$

$$d_{23} = \left[b_0 + \frac{\varepsilon_1 k_{x1} k_0}{q_1^2} F_2 + \frac{k_{x22}}{k_0} F_{g2} + \beta_2 \left(b_2 - \frac{k_y k_z}{q_1^2} F_2 + \frac{k_{x22}}{k_0} F_{vg}\right)\right] \exp(k_{x22}s), \quad \text{(A.7)}$$

$$d_{24} = \left[b_0 + \frac{\varepsilon_1 k_{x1} k_0}{q_1^2} F_2 - \frac{k_{x22}}{k_0} F_{g2} + \beta_2 \left(b_2 - \frac{k_y k_z}{q_1^2} F_2 - \frac{k_{x22}}{k_0} F_{vg} \right) \right] \exp(-k_{x22} s) , \quad (A.8)$$

$$d_{31} = a_0 - \frac{k_y k_z}{q_3^2} F_2 + \frac{k_{x21}}{k_0} F_{vg} + \beta_1 \left(a_2 - \frac{\mu_3 k_{x3} k_0}{q_3^2} F_2 + \frac{k_{x21}}{k_0} F_{v2} \right) , \quad (A.9)$$

$$d_{32} = a_0 - \frac{k_y k_z}{q_3^2} F_2 - \frac{k_{x21}}{k_0} F_{vg} + \beta_1 \left(a_2 - \frac{\mu_3 k_{x3} k_0}{q_3^2} F_2 - \frac{k_{x21}}{k_0} F_{v2} \right) , \quad (A.10)$$

$$d_{33} = a_0 - \frac{k_y k_z}{q_3^2} F_2 + \frac{k_{x22}}{k_0} F_{vg} + \beta_2 \left(a_2 - \frac{\mu_3 k_{x3} k_0}{q_3^2} F_2 + \frac{k_{x22}}{k_0} F_{v2} \right) , \quad (A.11)$$

$$d_{34} = a_0 - \frac{k_y k_z}{q_3^2} F_2 - \frac{k_{x22}}{k_0} F_{vg} + \beta_2 \left(a_2 - \frac{\mu_3 k_{x3} k_0}{q_3^2} F_2 - \frac{k_{x22}}{k_0} F_{v2} \right) , \quad (A.12)$$

$$d_{41} = b_0 - \frac{\varepsilon_3 k_{x3} k_0}{q_3^2} F_2 + \frac{k_{x21}}{k_0} F_{g2} + \beta_1 \left(b_2 - \frac{k_y k_z}{q_3^2} F_2 + \frac{k_{x21}}{k_0} F_{vg} \right) , \quad (A.13)$$

$$d_{42} = b_0 - \frac{\varepsilon_3 k_{x3} k_0}{q_3^2} F_2 - \frac{k_{x21}}{k_0} F_{g2} + \beta_1 \left(b_2 - \frac{k_y k_z}{q_3^2} F_2 - \frac{k_{x21}}{k_0} F_{vg} \right) , \quad (A.14)$$

$$d_{43} = b_0 - \frac{\varepsilon_3 k_{x3} k_0}{q_3^2} F_2 + \frac{k_{x22}}{k_0} F_{g2} + \beta_2 \left(b_2 - \frac{k_y k_z}{q_3^2} F_2 + \frac{k_{x22}}{k_0} F_{vg} \right) , \quad (A.15)$$

$$d_{44} = b_0 - \frac{\varepsilon_3 k_{x3} k_0}{q_3^2} F_2 - \frac{k_{x22}}{k_0} F_{g2} + \beta_2 \left(b_2 - \frac{k_y k_z}{q_3^2} F_2 - \frac{k_{x22}}{k_0} F_{vg} \right) . \quad (A.16)$$

References

1. Gurevich A G *Ferrity na Sverkhvysokikh Chastotakh* (Ferrites at Ultrahigh Frequencies) (Moscow: Fizmatgiz, 1960)
2. Lax B, Button K J *Microwave Ferrites and Ferrimagnetics* (New York: McGraw-Hill, 1962)
3. Agranovich V M, Ginzburg V L *Crystal Optics with Spatial Dispersion, and Excitons* (Berlin: Springer-Verlag, 1984); Translated from Russian: *Kristallooptika s Uchetom Prostranstvennoi Dispersii i Teoriya Eksitonov* (Moscow: Nauka, 1965)
4. Akhiezer A I, Bar'yakhtar V G, Peletminskii S V *Spin Waves* (Amsterdam: North-Holland Publ. Co., 1968); Translated from Russian: *Spinovye Volny* (Moscow: Nauka, 1967)
5. Born M, Wolf E *Principles of Optics* (Oxford: Pergamon Press, 1969); Translated into Russian: *Osnovy Optiki* (Moscow: Mir, 1970)
6. Ginzburg V L, Rukhadze A A *Volny v Magnitoaktivnoi Plazme* (Wave in Magnetoactive Plasma) 2nd ed. (Moscow: Nauka, 1975)
7. Landsberg G S *Optika* (Optics) (Moscow: Nauka, 1976)
8. Felsen L B, Marcuvitz N *Radiation and Scattering of Waves* (Englewood Cliffs, NJ: Prentice-Hall, 1973); Translated into Russian: *Izluchenie i Rasseyanie Voln* Vol. 2 (Moscow: Mir, 1978)
9. Vinogradova M B, Rudenko O V, Sukhorukov A P *Teoriya Voln* (Wave Theory) (Moscow: Nauka, 1990)
10. Vashkovskii A V, Stal'makhov V S, Sharaevskii Yu P *Magnitostaticheskie Volny v Elektronike Sverkhvysokikh Chastot* (Magnetostatic Waves in Electronics of Ultrahigh Frequencies) (Saratov: Izd. Saratov. Univ., 1993)
11. Gurevich A G, Melkov G A *Magnetization Oscillations and Waves* (Boca Raton, FL: CRC Press, 1996); Translated from Russian: *Magnitnye Kolebaniya i Volny* (Moscow: Nauka, 1994)
12. Stancil D D, Prabhakar A *Spin Waves. Theory and Applications* (New York: Springer, 2009)
13. Lock E H *Phys. Usp.* **51** 375 (2008); *Usp. Fiz. Nauk* **178** 397 (2008)
14. Mandelstam L I *Lektsii po Optike, Teorii Otnositel'nosti i Kvantovoi Mekhanike* (Lectures on Optics, Relativity Theory and Quantum Mechanics) (Moscow: Nauka, 1972)
15. Lock E H, Gerus S V, Annenkov A Yu *J. Commun. Technol. Electron.* **67** 649 (2022); *Radiotekh. Elektron.* **67** 567 (2022)
16. Lock E H *Phys. Usp.* **55** 1239 (2012); *Usp. Fiz. Nauk* **182** 1327 (2012)
17. Annenkov A Yu, Gerus S V, Lock E H *Europhys. Lett.* **123** 44003 (2018)
18. Annenkov A Yu, Gerus S V, Lock E H *EPJ Web Conf.* **185** 02006 (2018)
19. Nikitov S A et al. *Phys. Usp.* **58** 1002 (2015); *Usp. Fiz. Nauk* **185** 1099 (2015)
20. Chumak A V et al. *Nature Phys.* **11** 453 (2015)
21. Wang X S, Zhang H W, Wang X R *Phys. Rev. Applied* **9** 024029 (2018)
22. Nikitov S A et al. *Phys. Usp.* **63** 945 (2020); *Usp. Fiz. Nauk* **190** 1009 (2020)
23. Pirro P et al. *Nat. Rev. Mater.* **6** 1114 (2021)
24. Chumak A V et al. *IEEE Trans. Magn.* **58** 0800172 (2022)
25. Akhiezer A I, Bar'yakhtar V G, Peletminskii S V *Spin Waves* (Amsterdam: North-Holland Publ. Co., 1968); Translated from Russian: *Spinovye Volny* (Moscow: Nauka, 1967)
26. Damon R W Eshbach J R *J. Phys. Chem. Solids* **19** 308 (1961)
27. Gupta S S, Srivastava N C *J. Appl. Phys.* **50** 6697 (1979)
28. Vendik O G, Kalinikos B A, Miteva S I *Izv. Vyssh. Uchebn. Zaved. SSSR. Radioelektron.* **24** (9) 52 (1981)
29. Ruppin R *J. Appl. Phys.* **62** 11 (1987)
30. Danilov V V, Zavislyak I V, Balinskii M G *Spinvolnovaya Elektrodinamika* (Spin Wave Electrodynamics) (Kiev: Lybid', 1991)
31. Vashkovskii A V, Lokk E G *J. Commun. Technol. Electron.* **46** 674 (2001); *Radiotekh. Elektron.* **46** 729 (2001)
32. Vashkovskii A V, Lokk E G *J. Commun. Technol. Electron.* **46** 1163 (2001); *Radiotekh. Elektron.* **46** 1257 (2001)
33. Vashkovskii A V, Lokk E G *J. Commun. Technol. Electron.* **47** 87 (2002); *Radiotekh. Elektron.* **47** 97 (2002)
34. Lokk E G *J. Commun. Technol. Electron.* **48** 1369 (2003); *Radiotekh. Elektron.* **48** 1484 (2003)
35. Vashkovskii A V, Lokk E G *J. Commun. Technol. Electron.* **49** 904 (2004); *Radiotekh. Elektron.* **49** 966 (2004)
36. Grishin S V, PhD Thesis (Phys.-Math. Sci.) (Saratov: Saratov State Univ., 2006)
37. Vyatkina S A, Babichev R K, Ivanov V N *Elektromagn. Volny Elektromagn. Sistemy* (10) 64 (2011)
38. Vashkovskii A V, Lokk E G *J. Commun. Technol. Electron.* **57** 490 (2012); *Radiotekh. Elektron.* **57** 541 (2012)
39. Vashkovsky A V, Lock E H *Phys. Usp.* **54** 281 (2011); *Usp. Fiz. Nauk* **181** 293 (2011)
40. Lock E H *Solid State Phenomena* **233–234** 476 (2015)
41. Lokk E G *J. Commun. Technol. Electron.* **61** 33 (2016); *Radiotekh. Elektron.* **61** 35 (2016)
42. Lokk E G, Vashkovskii A V *J. Commun. Technol. Electron.* **61** 877 (2016); *Radiotekh. Elektron.* **61** 746 (2016)
43. Lock E H, Gerus S V *J. Commun. Technol. Electron.* **68** 971 (2023); *Radiotekh. Elektron.* **68** 884 (2023)
44. Grishin S V, Bogomolova A V, Nikitov S A *Tech. Phys. Lett.* **48** (3) 37 (2022); *Pis'ma Zh. Tekh. Fiz.* **48** (5) 39 (2022)
45. Bogomolova A V et al. *J. Magn. Magn. Mater.* **587** 171278 (2023)
46. Amel'chenko M D et al. *Phys. Rev. B* **108** 224401 (2023)
47. Lock E H, Lugovskoi A V, Gerus S V *J. Commun. Technol. Electron.* **66** 834 (2021); *Radiotekh. Elektron.* **66** 662 (2021)
48. Stratton J A *Electromagnetic Theory* (Intern. Ser. in Physics) (New York: McGraw-Hill Book Co., 1941); Translated into Russian: *Teoriya Elektromagnetizma* (Moscow–Leningrad: Gostekhizdat, 1948)