

METHODOLOGICAL NOTES

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On R L Stratonovich's formula for transition from dynamic to probabilistic measurements and its connection with operations on distribution functions of random variables

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Contents

Abstract. The applications of R L Stratonovich's formula [1] for the exact transition from dynamic measurements to probabilistic or thermodynamic measurements are considered. The connection of this formula with known expressions for operations on probability density functions of random variables is given. The application of the formula to the problem of determining the observed physical parameters — fields, potentials, moments — produced by (stochastically moving) charged particles of different multipolar types is demonstrated.

Keywords: delta-function, probability density, microfield distribution

1. Introduction

R L Stratonovich's original formula describes the transition from measurements of the stochastic dynamics of many particles (with the total number N) in the phase space of coordinates and momenta $\{\mathbf{Z}\} = (\mathbf{r}_1, \ldots, \mathbf{r}_N; \mathbf{p}_1, \ldots, \mathbf{p}_N)$ to distributions of some thermodynamic (observable) variables A [1]. It is written as a definition of the transformation from the probability density of the distribution $\rho\{\mathbf{Z}\}$ to the probability density of the distribution $W(A)$,

$$W(A) = \int \delta\big(A - B\{\mathbf{Z}\}\big)\, \rho\{\mathbf{Z}\}\, \mathrm{d}\mathbf{Z}\,, \qquad (1)$$

where $B\{\mathbf{Z}\}$ is some function defining a physical variable.[1] All $B\{\mathbf{Z}\}$ and $\rho\{\mathbf{Z}\}$ that we are interested in should be known to make formula (1) useful. Formula (1) is the generalization of the identity

$$W_A(A) = \int \delta(z - A)\, W_A(z)\, \mathrm{d}z\,, \qquad (1a)$$

or, in the generally accepted formalism with angle brackets used to denote the direct averaging for the random variable A, which depends on the coordinate x,

$$W_A(A, x) = \big\langle \delta\big[A(x) - A\big]\big\rangle_A\,. \qquad (1b)$$

Formula (1b) allows generalizations to K-point distribution functions

$$W_A(A_1, A_2, \ldots, A_K, x_1, x_2, \ldots, x_K) = \left\langle \prod_{i=1}^{K} \delta\big[A_K(x_K) - A\big] \right\rangle_A$$

and, respectively, the distribution functions of different random variables

$$W_A(A_1, A_2, \ldots, A_K) = \left\langle \prod_{i=1}^{K} \delta[A_K - A] \right\rangle_A\,.$$

This formalism extends that of R L Stratonovich's and has been used, e.g., in Ref. [2] to establish connections

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[1] This variable can be, for example, the magnetic dipole moment $\mathbf{M}_H$, $\mathbf{M}_H = (1/2c)\sum_{i=1}^{N}(e_i/m_i)\,[\mathbf{r}_i\mathbf{p}_i]$ for a system of stochastically moving charged particles with coordinates $(r_1, \ldots, r_N)$, momenta $(p_1, \ldots, p_N)$, and charges $(e_1, \ldots, e_N)$.

between the Lagrangian and Eulerian statistical descriptions of hydrodynamic random fields, allowing one to find analytical expressions for probability densities, velocity spectra, etc. in the Eulerian representation.

It is obvious that (1) connects the probability densities of random variables. To see this, it is sufficient to integrate (1) over A in order to obtain the coinciding normalizations for the probability densities $\rho\{\mathbf{Z}\}$ and $W(A)$. Furthermore, formula (1) defines the transition from the instantaneous probability density $\rho\{\mathbf{Z}\}$ in the phase space to the instantaneous probability density for (thermodynamic quantities) $W(A)$. The study of the dynamics of $\rho\{\mathbf{Z}\}$ and $W(A)$ [1] is beyond the scope of this methodological note.

The first part of this note (Sections 2 and 3) will demonstrate how to work with (1) in practice. It will be shown how the known formulas for the probability density of the distributions of sum, difference, product, and quotient of independent random variables can be reduced to R L Stratonovich's formula. For a practically important case of sums and products of functions of an arbitrary number of random variables, the performance of formula (1) will be illustrated as applied to finding probability densities of random variables that are the above-mentioned sums and products of these functions. The results obtained in this way will be exact, in contrast to the asymptotic approximation of the A A Markov method [3, 4].

In the second part (Sections 4 and 5), devoted to applications of (1) for finding observable physical quantities — fields, potentials, moments created by stochastically moving charged particles — we will deal with instantaneous probability density functions and moments of distributions, i.e., instantaneous observations will be described.

We also note that (1) can be considered a theorem — one must prove that $W(A)$ is a sought-after probability density function for the random variable $B\{\mathbf{Z}\}$. The proof is elementary — we multiply both sides of (1) by A^k, where k is an arbitrary natural number, and integrate both sides over $\mathrm{d}A$. By definition, the left side contains the kth moment of A. On the right side, by virtue of Dirac's δ-function, we obtain the kth moment of $B\{\mathbf{Z}\}$. Since k is arbitrary, it follows that all moments of the probability density of the distribution of $B\{\mathbf{Z}\}$ coincide with the corresponding moments of $W(A)$, which is one of the definitions of the equivalence of probability densities [5], i.e., it is proved that (1) defines the probability density of the distribution $B\{\mathbf{Z}\}$.

2. Description of arithmetic operations on random variables which are reducible to R L Stratonovich's formula

In addition to the change from phase variables to thermodynamic variables, it turns out that formula (1) allows convenient calculation of probability densities for random variables (there is a strong motivation to consider them to be thermodynamic) which are the sum, difference, product, and quotient of an (arbitrary) number of independent random variables. This need not be restricted to the random variables in the phase space (i.e., random coordinates and momenta of some particles) and can be of an arbitrary nature. It is sufficient that they be independent.

Furthermore, one can conveniently, in many cases analytically, calculate the probability density of random quantities that are functions of other random variables. This is especially productive in physically important situations where the observable is a sum of similar functions of other random variables (see the example in the footnote in the Introduction).

Let us consider the well-known formula describing the probability density $F(x)$ of a random variable ζ, which is the sum of two independent random variables [5, 6] $\xi + \chi$ with probability density functions $f(x)$ and $g(x)$, respectively,

$$h(x) = \int f(x-y)\, g(y)\, \mathrm{d}y\,, \tag{2}$$

with the integration over the entire definition domain of the functions $f(x)$ and $g(x)$. In the following, we will assume for simplicity that all our distribution functions are defined for variables[2] varying from $-\infty$ to $+\infty$, leaving more complicated cases to the reader. Furthermore, we will avoid introducing new notations for values, variables, and functions that differ from those already in use, assuming that for each new mathematical operation these quantities can be quite different and have different meanings.

The elementary transformation

$$f(x-y) = \int_{-\infty}^{\infty} \delta\big(z-(x-y)\big)\, f(z)\, \mathrm{d}z\,,$$

where $\delta(x)$ is Dirac's delta, converts (2) into the expression (we rely on the delta function being an even function)

$$h(x) = \iint_{-\infty}^{\infty} \delta(x-z-y)\, f(z)\, g(y)\, \mathrm{d}y\, \mathrm{d}z\,, \tag{3}$$

i.e., into form (1). Here, the role of the 'thermodynamic' variable is played by x, and y and z are phase variables, although it is obvious that they can be arbitrary random variables.

For the sum of three random variables $\zeta = \xi + \chi + \psi$ with probability densities $f(x)$, $g(x)$, and $l(x)$, respectively, the probability density $h(x)$ of the random variable ζ will be

$$\begin{aligned} h(z) &= \int_{-\infty}^{\infty} F(z-t) l(t)\, \mathrm{d}t \\ &= \int_{-\infty}^{\infty} l(t)\, \mathrm{d}t \int_{-\infty}^{\infty} f(z-t-y)\, g(y)\, \mathrm{d}y \\ &= \iint_{-\infty}^{\infty} l(t) g(y) \int_{-\infty}^{\infty} \delta(z-t-y-x)\, f(x)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \\ &= \iiint_{-\infty}^{\infty} f(x) g(y) l(t) \delta(z-x-y-t)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t\,. \end{aligned} \tag{4}$$

Here, $F(x)$ is the probability density of the distribution of the sum of two random variables $\xi + \chi$. Thus, by induction, we arrive at the expression for the probability density $f(x)$ for the random variable ζ, which is an arbitrary sum of independent random variables $\xi_1 + \xi_2 + \ldots + \xi_N$ with probability densities $f_1(x_1), f_2(x_2), \ldots, f_N(x_N)$:

$$f(x) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \delta\left(x - \sum_{i=1}^{N} x_i\right) \times \left[\prod_{i=1}^{N} f_i(x_i)\right] \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_N\,. \tag{5}$$

[2] For the phase space variables, this usually holds for momenta, while the spatial coordinates are usually restricted to a finite volume. This does not affect the derivations further.

The probability density function $h(x)$ for the distribution of the difference of two random variables ξ and χ with probability densities $f(x)$ and $g(x)$ is essentially the same, the only difference being the sign,

$$h(x) = \int f(x+y) g(y)\,\mathrm{d}y\,.$$

Correspondingly, formula (1) for the difference in random variables will be

$$h(x) = \iint_{-\infty}^{\infty} \delta(x - z + y)\, f(z) g(y)\,\mathrm{d}y\,\mathrm{d}z\,.$$

The generalization of (5) to the case when some number of random variables enter the random variable ζ with a plus sign while all others enter with a minus sign is clear: they will appear in the argument of the delta function with opposite signs.

The result given by (5) is quite obvious so far. If we take the Fourier transform of both sides of (5), the characteristic function of the random variable, which is the sum of independent random variables, appears on the left. On the right, taking into account

$$\delta(x - A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\big(\mathrm{i}K(x - A)\big)\,\mathrm{d}K$$

and having computed each integral of the form $\int_{-\infty}^{\infty} f_i(x_i) \exp(-\mathrm{i}Kx_i)\,\mathrm{d}x_i$, which is the characteristic function of the random variable ξ_i, we obtain that the characteristic function of the sum is simply the product of the characteristic functions of these random variables — a classical result [5–8]. R L Stratonovich's formula enables operations not only with the sums of independent random variables but also with functions of them, as well as with quantities which are the result of other mathematical operations.

For the product ζ of independent random variables ξ and χ, the probability density $h(x)$ of the former is expressed via the probability densities $f(x)$ and $g(x)$ of the latter as [5]

$$h(x) = \int_{-\infty}^{\infty} \frac{1}{|y|} f\left(\frac{x}{y}\right) g(y)\,\mathrm{d}y\,. \tag{6}$$

The form of (6) causes great inconvenience in calculating the probability density for a random variable that is a product of a large number of random variables. The same formula (1) makes this task an easy one. With a methodological aim, we will carry out the proof in a 'reciprocal way,' i.e., by transforming an expression like (1) into the form (6). Indeed,

$$\begin{aligned} h(x) &= \iint_{-\infty}^{\infty} \delta(x - yz)\, f(z) g(y)\,\mathrm{d}z\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} g(y)\,\mathrm{d}y \int_{-\infty}^{\infty} \frac{1}{|y|}\, \delta(x - yz)\, f\left(\frac{yz}{y}\right) \mathrm{d}(yz) \\ &= \int_{-\infty}^{\infty} \frac{1}{|y|} f\left(\frac{x}{y}\right) g(y)\,\mathrm{d}y\,. \end{aligned} \tag{7}$$

For the probability density $h(x)$ of the random variable ζ, which is the product of three random variables $\xi\chi\psi$ with probability densities $f(x)$, $g(x)$, and $l(x)$, respectively, we write

$$h(x) = \iiint_{-\infty}^{\infty} \delta(x - yzt)\, f(y) g(z) l(t)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\,. \tag{8}$$

If we denote $s = zt$, then the probability density $h_2(x)$ of the product of two independent random variables χ and ψ is expressed by the first line of expression (7),

$$h_2(s) = \iint_{-\infty}^{\infty} \delta(s - zt) g(z) l(t)\,\mathrm{d}z\,\mathrm{d}t\,,$$

and the probability density of the independent random variables ξ and $\chi\psi$ is

$$h(x) = \iint_{-\infty}^{\infty} \delta(x - ys)\, f(y) h_2(s)\,\mathrm{d}y\,\mathrm{d}s\,.$$

Inserting $h_2(s)$ from the previous expression, we find

$$\begin{aligned} h(x) &= \iint_{-\infty}^{\infty} \delta(x - ys)\, f(y) h_2(s)\,\mathrm{d}y\,\mathrm{d}s \\ &\quad \times \iint_{-\infty}^{\infty} \delta(s - zt) g(z) l(t)\,\mathrm{d}z\,\mathrm{d}t\,. \end{aligned}$$

Exchanging the order of integration and performing one integration over $\mathrm{d}s$, by the relation

$$\delta(x - yzt) = \int_{-\infty}^{\infty} \delta(x - ys)\, \delta(s - zt)\,\mathrm{d}s$$

we obtain formula (8). Using induction, we write the general form (similar to the form of (5)) for the probability density $f(x)$ for the random variable ζ, which is now the product of any number of independent random variables $\xi_1 \xi_2 \ldots \xi_N$ with the probability densities $f_1(x_1), f_2(x_2), \ldots, f_N(x_N)$:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \delta\left(x - \prod_{i=1}^{N} x_i\right) \\ &\quad \times \left[\prod_{i=1}^{N} f_i(x_i)\right] \mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_N\,. \end{aligned} \tag{9}$$

For the quotient ζ of independent random variables ξ and χ, the probability density $h(x)$ of the former is expressed in terms of the respective probability densities $f(x)$ and $g(x)$ as [5, 6]

$$h(x) = \int_{-\infty}^{\infty} |y|\, f(xy)\, g(y)\,\mathrm{d}y\,. \tag{10}$$

On the other hand,

$$\begin{aligned} h(x) &= \iint_{-\infty}^{\infty} \delta\left(x - \frac{y}{z}\right) f(y) g(z)\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_{-\infty}^{\infty} g(z)\,\mathrm{d}z \int_{-\infty}^{\infty} \delta\left(x - \frac{y}{z}\right) f\left(z\,\frac{y}{z}\right) |z|\,\mathrm{d}\left(\frac{y}{z}\right) \\ &= \int_{-\infty}^{\infty} |z|\, f(xz) g(z)\,\mathrm{d}z\,, \end{aligned}$$

which coincides with (10). The form of the general expression for the probability density of a random variable that is a combination of products and quotients of a set of independent random variables is obvious.

Thus, the calculation of probability densities for random variables that arise as a result of arithmetic operations on a set of other independent random variables is much more convenient using R L Stratonovich's formula than using the expressions in classical textbooks [5–8]. We will see further that the utility of this formula is not limited to the calculation of distributions of random variables obtained as a result of arithmetic operations on independent random variables.

3. Operations on functions of random variables using R L Stratonovich's formula

The transition from calculations of probability densities of random variables that are the result of arithmetic operations on independent random variables to those of functions of random variables is based on the relation for the probability density of a function of random variable [5]. Writing (1) in the form

$$W(A) = \int_{-\infty}^{\infty} \delta\big(A - B(z)\big)\, f(z)\, \mathrm{d}z\,,$$

we derive it directly from the definition of the δ-function. The $f(z)$ is now a 'usual' probability density function of the distribution of the random variable ξ; $B(z)$ is the 'usual' function of the argument z, whose values are those of the random variable ξ, i.e., B is also a random variable. Assuming for simplicity that B is a strictly monotone function[3] of z, we introduce the function $b(B) = z$, the inverse of $B(z)$. Then, $\mathrm{d}z = (\mathrm{d}z/\mathrm{d}B)\,\mathrm{d}B = b'(B)\,\mathrm{d}B$ [6]. The probability density $F(B)$ of the random variable B is connected to $f(z)$ as $F(B) = f[z(B)]b'(B)$. Expressing $f(z)$ and $\mathrm{d}z$ by $F(B)$ and $\mathrm{d}B$, we get

$$W(A) = \int_{-\infty}^{\infty} \delta(A - B)F(B)\,\mathrm{d}B = \int_{-\infty}^{\infty} \delta\big(A - B(z)\big)\, f(z)\,\mathrm{d}z\,,$$

which is the original R L Stratonovich's formula with $\rho(z) = f(z)$.

It is obvious that (1) cannot be computed in a general case, so we move on to practically important special cases. The use of (1) becomes especially convenient when $B\{\mathbf{Z}\}$ is the sum of functions $\sum_{i=1}^{N} g_i(x_i) = G(x)$ of independent random variables ξ_i with probability densities $f_i(x)$. We start from relationship (5) assuming that g_i is a random variable with the domain (to be denoted by Q), which need not to be from $-\infty$ to $+\infty$,

$$\begin{aligned} W(G) &= \int_Q \ldots \int_Q \delta\Big(G - \sum_{i=1}^{N} g_i\Big) \times \Big[\prod_{i=1}^{N} f_i(g_i)\Big]\,\mathrm{d}g_1\,\mathrm{d}g_2 \ldots \mathrm{d}g_N \\ &= \int_Q \ldots \int_Q \delta\Big(G - \sum_{i=1}^{N} g_i(x_i)\Big) \times \Big[\prod_{i=1}^{N} f_i\big(g_i(x_i)\big)\,\frac{\mathrm{d}x_i}{\mathrm{d}g_i}\Big]\,\mathrm{d}g_1\,\mathrm{d}g_2 \ldots \mathrm{d}g_N| \\ &= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \delta\Big(G - \sum_{i=1}^{N} g_i(x_i)\Big) \times \Big[\prod_{i=1}^{N} f_i(x_i)\Big]\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_N\,. \end{aligned} \tag{11}$$

[3] Once again, we leave the case of simple, but tedious, manipulations for nonmonotonic $B(z)$ to the reader. In this case, one should use the 'point' properties of the δ-function.

Here, it is convenient to turn to the Fourier transform of the δ-function. As a result, we get

$$\begin{aligned} W(G) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\Big\{\mathrm{i}K\Big[G - \sum_{i=1}^{N} g_i(x_i)\Big]\Big\} \times \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \Big[\prod_{i=1}^{N} f_i(x_i)\Big]\,\mathrm{d}K\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_N \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(\mathrm{i}KG)\,\mathrm{d}K \times \prod_{i=1}^{N}\Big[\int_{-\infty}^{\infty} \exp\big(-\mathrm{i}Kg_i(x_i)\big)\, f_i(x_i)\,\mathrm{d}x_i\Big]\,, \end{aligned} \tag{12}$$

which is another proof of the theorem that the characteristic function of the sum of independent random variables is equal to the product of the characteristic functions of these random variables. When all $f_i(x_i)$ and $g_i(x_i)$ are equal, only the integral $\int_{-\infty}^{\infty} \exp\big(-\mathrm{i}Kg_i(x_i)\big) f_i(x_i)\,\mathrm{d}x_i$ needs to be calculated. Examples of the use of this formula are given below.

Among other arithmetic operations on functions of random variables, one can choose only to calculate the probability density of a function of random variables that is a product of independent random variables ξ_i with probability density functions $f_i(x)$. In this case,

$$W(G) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \delta\Big(G - \prod_{i=1}^{N} g_i(x_i)\Big) \times \Big[\prod_{i=1}^{N} f_i(x_i)\Big]\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_N\,.$$

One cannot proceed further with the Fourier transform of the δ-function, since the integrand cannot be factorized as in expression (12). However, another approach can be used.

If ζ is a product of independent random quantities $\xi_1\xi_2\ldots\xi_N$, we have $\ln\zeta = \ln\xi_1 + \ln\xi_2 + \ldots + \ln\xi_N$. Then,

$$W(\ln G) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \delta\Big(\ln G - \sum_{i=1}^{N} \ln g_i(x_i)\Big) \times \Big[\prod_{i=1}^{N} f_i(x_i)\Big]\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_N\,,$$

where the probability density functions $f_i(x_i)$ are related to the random variables ξ_i (11). In this case, the probability density $W(G)$ is easily determined from [6],

$$W(G) = \frac{W(\ln G)}{G} = \frac{1}{2\pi G}\int_{-\infty}^{\infty} \exp(\mathrm{i}K\ln G)\,\mathrm{d}K \times \prod_{i=1}^{N}\Big[\int_{-\infty}^{\infty} \exp\big(-\mathrm{i}K\ln g_i(x_i)\big) f_i(x_i)\,\mathrm{d}x_i\Big]\,. \tag{13}$$

For mixed operations with functions of probability density of a random variable resulting from combinations with arithmetic operations on independent random variables (subtraction or division), such simple formulas cannot be obtained, but in most cases they are not needed.

4. Application of R L Stratonovich's formula to calculations of random fields and potentials

Historically, the first work on calculations of (electric) microfields of various multipole multiplicity was that by

J Holtsmark [9]. At the time it was written, only A A Markov's method [3] for calculating random walks in the description of Brownian motion (see also [4]) was known from the apparatus of probability theory, and the characteristic function had not yet been introduced by P Lévy. As a result, calculations of the probability density for the electric field distribution were accompanied by significant difficulties — it was later found to hold for microfields of an ideal plasma [10]. Calculations of the probability density distribution for a (gravitational) microfield [4] in practice already used an analog of (1). The simplest method of applying (1) was done in [11], and we describe it here. Briefly, the steps of computing the final probability density of a concrete physical quantity are as follows: the general form of R L Stratonovich's formula is written, then, in the integrand, one moves to the Fourier representation of the δ-function. The subsequent change in the integration order leads to the resultant form of the probability density as the Fourier transform of the characteristic function, where, in an overwhelming number of cases, one can exploit the equality of the characteristic function for the sum of independent random variables and the product of individual characteristic functions of random variables. We first compute the electric microfield in an ideal plasma: in this case, the kinetic energy of charged particle motion is much larger than that of Coulomb interaction between the particles.

The charges in an ideal plasma are located independently from each other, their total number being N. We will measure the electric field at the coordinate center, $\mathbf{E} = \sum_{i=1}^{N} e_i \mathbf{r}_i / r_i^3$. In turn, the probability density of finding any of the charges e_i is simply $1/V$, where V is the volume of the (ideal) plasma. From (11), the probability density of the distribution of a (random) variable $\mathbf{E}$ is

$$W(\mathbf{E}) = \frac{1}{V^N} \int_V \ldots \int_V \delta\left(\mathbf{E} - \sum_{i=1}^{N} \frac{e_i \mathbf{r}_i}{r_i^3}\right) \prod_{i=1}^{N} \mathrm{d}\mathbf{r}_i$$
$$= \frac{1}{8\pi^3 V^N} \iiint_{-\infty}^{\infty} \mathrm{d}\mathbf{K} \exp(\mathrm{i}\mathbf{K}\mathbf{E}) \times \left[\prod_{i=1}^{N} \left(\int_V \exp\left(-\mathrm{i}\mathbf{K}\mathbf{r}_i \frac{e_i}{r_i^3}\right) \mathrm{d}\mathbf{r}_i\right)\right]. \qquad (14)$$

For example, we determine the electron microfield in an ideal plasma where all the charges are the same and equal to e. To compute the only integral

$$\int_V \exp\left(-\mathrm{i}\mathbf{K}\mathbf{r} \frac{e}{r^3}\right) \mathrm{d}\mathbf{r},$$

we use the following approach. Since $\int_V \mathrm{d}\mathbf{r} = V$, we add this integral and subtract it in the integral in the product to obtain

$$W(\mathbf{E}) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \mathrm{d}\mathbf{K} \exp(\mathrm{i}\mathbf{K}\mathbf{E}) \times \left\{1 - \frac{1}{V}\int_V \left[1 - \exp\left(-\mathrm{i}\mathbf{K}\mathbf{r} \frac{e}{r^3}\right)\right] \mathrm{d}\mathbf{r}\right\}^N$$

or, moving to the limit $N \to \infty$,

$$W(\mathbf{E}) \cong \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \mathrm{d}\mathbf{K} \exp\left\{\mathrm{i}\mathbf{K}\mathbf{E} - n\int_V \left[1 - \exp\left(-\mathrm{i}\mathbf{K}\mathbf{r} \frac{e}{r^3}\right)\right] \mathrm{d}\mathbf{r}\right\}.$$

We assume here that both the volume V and the number of particles N can go to infinity, but their ratio n, the density, is finite. In the integral over volume, one changes to polar coordinates with the axis directed along the vector $\mathbf{K}$. After the standard procedure of angular integration, the integration over the radius from 0 to infinity is performed twice by parts [4, 11], meaning the old condition $V \to \infty$. As a result, we get

$$W(\mathbf{E}) = \frac{1}{8\pi^3} \iiint_{-\infty}^{\infty} \mathrm{d}\mathbf{K} \exp\left(\mathrm{i}\mathbf{K}\mathbf{E} - K^{3/2} E_0^{3/2}\right),$$

where $E_0 = 2\pi(4/15)^{2/3} e n^{2/3}$. For the density of the distribution of the absolute value of the random field at zero, using the obvious field isotropy and the relation $W(E) = 4\pi E^2 W(\mathbf{E})$, and also assuming $E/E_0 = \beta$, we get the Holtsmark distribution

$$W(\beta) = \frac{2\beta}{\pi} \int_0^{\infty} x \sin(\beta x) \exp(-x^{3/2}) \, \mathrm{d}x. \qquad (15)$$

This elementary derivation (15) can be compared to the volumetric calculations in [9].[4]

A very similar result follows for the distribution of the magnetic microfield in an ideal isotropic plasma, in this case $\beta = H/H_0$, $H_0 = \pi^{5/3} q v_T n^{2/3} / 2^{4/3} c$ [15, 16]. In addition to the electric field, an obvious relativistic factor $\sim v_T/c$ appears here (v_T is the mean thermal velocity of charged particles and c is the speed of light in a vacuum), because in probability density calculations the phase coordinates are now not only space coordinates but also momenta (velocities) of particles.

Formula (1) allows calculations not only for equilibrium ideal plasma but also for nonequilibrium plasma [16]. Instead of coordinates of particles in plasma, one can also deal with elementary microcurrents in corresponding media (e.g., in plasma screens or microchips) [17]. In many such media, the microcurrents can be considered to be noninteracting. For the probability densities, the result depends on the geometry of the system generating the magnetic field — in three dimensional geometry, one gets the distribution

$$W(\mathbf{B}) = \int_V \ldots \int_V \iiint_{-\infty}^{\infty} \ldots \iiint_{-\infty}^{\infty} \frac{1}{(\sqrt{\pi} v_T^3 V)^N} \times \delta\left(\mathbf{B} - \sum_{i=1}^{N} \frac{\mu\mu_0 e_i [\mathbf{v}_i \mathbf{r}_i]}{c r_i^3}\right) \prod_{i=1}^{N} \mathrm{d}\mathbf{r}_i \, \mathrm{d}\mathbf{v}_i$$
$$= \frac{1}{8\pi^{3(1+N/2)} v_T^{3N} V^N} \iiint_{-\infty}^{\infty} \mathrm{d}\mathbf{K} \exp(\mathrm{i}\mathbf{K}\mathbf{B}) \times \left\{\prod_{i=1}^{N} \left[\int_V \iiint_{-\infty}^{\infty} \exp\left(-\mathrm{i}\mathbf{K}[\mathbf{v}_i \mathbf{r}_i] \frac{\mu\mu_0 e_i}{c r_i^3}\right) \mathrm{d}\mathbf{r}_i \, \mathrm{d}\mathbf{v}_i\right]\right\}$$

where $\beta = B/B_0$, the measured quantity is now the magnetic field B, $B_0 = 2^{5/3} \pi^{4/3} [\Gamma(7/4)]^{2/3} \mu\mu_0 \langle l^{3/2} \rangle^{2/3} / 15^{2/3} c$, Γ is the Euler gamma function, μ is the medium magnetic permeability, μ_0 is the vacuum magnetic permeability, and $\langle l^{3/2} \rangle$ is the mean of the half cube of the current element. In the two-dimensional geometry of a microchip layer, the result is different (see [17]).

R L Stratonovich's method allows one to calculate the microfields not only of charged particles and microcurrents but also, for example, of two stochastically and independently

[4] Note that these calculations for the probability density of the electric field are valid if there are no charges at the origin of the coordinates (see discussions in Refs [12–14]). The same is also true for the calculations of magnetic fields below and for potentials.

located parallel point dipoles [18, 19].[5] The component of the field of random dipoles along one of the coordinate axes (e.g., z) ε has the probability distribution

$$f(\varepsilon)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(\mathrm{i}k\varepsilon)\times\left\{\iiint_{-\infty}^{\infty}\exp\left[-\mathrm{i}k\left(\frac{d}{r^3}-3\,\frac{d(\mathbf{e}_z\mathbf{r})^2}{r^5}\right)\right]\frac{\mathrm{d}^3\mathbf{r}}{V}\right\}^N\mathrm{d}k\,.$$

Here, N is the total number of dipoles, V is the volume where they are located, as before $N\to\infty$, $V\to\infty$, but the density $n=N/V$ is finite, $\mathbf{e}_z$ is the unit vector along the z-axis, and d is the magnitude of the dipole moment of one point particle. The calculations follow the method that is analogous to the calculations of the microfield (14), (15) and give

$$f(\varepsilon)=\frac{1}{\pi}\,\frac{G}{(\varepsilon-G_c)^2+G^2}\,,\qquad(16)$$

where $G=4\sqrt{2}\pi nd/9\sqrt{3}$,

$$G_c=4\left(3+\sqrt{3}\,\frac{\ln\sqrt{3}-1}{\ln\sqrt{3}+1}\right)\frac{nd}{27}\,.$$

Thus, using R L Stratonovich's formula, we obtain the Lorentz-shifted distribution (see also the result (17) for $D/a=1$).

Another 'application point' of (1) in the form of (12) is the computation of the fluctuations of the potential generated by particles at a neutral point in space. Here, one can turn to the idea of Holtsmark [9] to compute fluctuations of (a field) of different multipolar types. Applied to the potential, the problem is posed in the simplest case as follows [20]: suppose there are N point particles which generate potentials of the form q/r_i^a, which are independently located in the domain with volume V. We assume, as before, that $N\to\infty$, $V\to\infty$, but the density of the particles $n=N/V$ remains finite.

The problem can be generalized to a space of arbitrary dimension [20]. Particles (without loss of generality, they can be taken to be the same) generate at the coordinate origin the potential

$$\varphi(0)=\sum_{i=1}^{N}\frac{q}{r_i^a}+\text{const}\,.$$

Calculation of the potential distribution density by formula (12) (we assume const $=0$) gives

$$W(\beta)=\frac{2}{\pi}\int_0^{\infty}\cos(\beta x)\exp(-x^{D/a})\,\mathrm{d}x\,.\qquad(17)$$

Here, $\beta=\varphi/\varphi_0$,

$$\varphi_0=q\left(\frac{\pi d}{2D\sin(\pi D/2a)\Gamma(D/a)}\right)^{a/D},$$

D is the dimension of the space, and d is the total solid angle in the space of the respective dimension ($d=1$, 2π, 4π, $4\pi^2$ for one-, two-, three- or four-dimensional space and so on).

Distribution (17) is the standard Lévy function [21] and exists only for $0<D/a\leqslant 2$. This means, for example, that in three-dimensional space it is impossible to calculate the probability density distribution of Coulomb (and gravitational) potential of randomly placed noninteracting particles. If interactions between particles are taken into account, this turns out to be possible in a certain approximation of the collective potential [22]. In this case, the Lévy distribution is transformed into a truncated Lévy distribution (see, e.g., [23, 24]).

Thus, R L Stratonovich's formula is a powerful tool in the calculation of random microfields and micropotentials of stochastically and independently located (and moving) particles of various multipole types. In a number of cases it is possible to perform such calculations not only for independent but also for interacting particles. The most interesting result here is a principal difference in the derived probability density distributions for microfields and micropotentials from the Gaussian distribution, despite the fact that the number of formally considered particles forming the random field (potential) tends to infinity. All these distributions of microfields and micropotentials are related to a wider class of infinitely divisible distributions [5, 21].

5. Application of R L Stratonovich's formula to calculations of random moments of bounded bodies

Consider a body filled with a moving substance — be it continuous or discrete — held inside the body in one way or another [25]. It is also assumed that the motion of this substance is stochastic. We further assume that point charges can be present inside this body. It is obvious that the calculation of the mean value of any projection of the angular momentum $\mathbf{M}$ in the absence of macroscopic motion will yield exactly zero. According to the relationship for the probability density function for the absolute value of the random angular momentum $W(M)=4\pi M^2W(\mathbf{M})$, the expectation of this absolute value differs from zero. Similarly, if there are point charges inside the body, they are entrained by the substance inside it in a stochastic motion and ensure the existence of a random magnetic moment.[6] Formula (1) allows us to calculate these distributions and their expectations.

We divide the volume of the body into physically small parts (particles). In principle, it is not necessary that the parts be identical, but we can further assume that their masses are equal. Let there be N such parts with masses m_i, with the total mass of the body m_N, $m_N=\sum_{i=1}^{N}m_i$. If we further find that the model of a body composed of smaller parts is preferable to the model of a continuous medium, N will have the sense of the number of these smaller particles, but, in any case, the number will be (physically) large. It can also be shown that a small deviation of the body shape from the spherical one (within 10–15% in radial size) [26]) does not change the results qualitatively. The random angular momentum of such a body is $\mathbf{M}=\sum_{i=1}^{N}m_i[\mathbf{r}_i\mathbf{v}_i]$. It is distributed with the probability density

$$W(\mathbf{M})=\int_V\ldots\int_V\iiint_{-\infty}^{\infty}\ldots\iiint_{-\infty}^{\infty}\delta\left(\mathbf{M}-\sum_{i=1}^{N}m_i[\mathbf{r}_i\mathbf{v}_i]\right)\times P_N(\mathbf{r}_1,\ldots,\mathbf{r}_N,\mathbf{v}_1,\ldots,\mathbf{v}_N)\prod_{i=1}^{N}\mathrm{d}\mathbf{r}_i\,\mathrm{d}\mathbf{v}_i\,.\qquad(18)$$

[5] Note that these papers use the formalism of averaging with the help of angular brackets. This is followed by lengthy calculations of the arising integrals.

[6] To simplify the calculations, we assume that the velocities of the masses (but not of the point charges) inside the body are nonrelativistic.

Here, $P_N(\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N)$ is the probability density of finding the first mass part (particle) at the point with radius vector $\mathbf{r}_1$ moving with velocity $\mathbf{v}_1$, the second one at the point $\mathbf{r}_2$ with the velocity $\mathbf{v}_2$, and so on. It is obvious that the distribution over velocities should be the equilibrium Maxwellian one, which is further expressed through the total energy U of (thermal) stochastic particle motion. To proceed with the calculations of $W(\mathbf{M})$, a Fourier transform is performed on the δ function in formula (18).

The integrals over the velocities are easy to calculate [25]. We concretize the form of $P_N(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ by accounting for the incompressibility of the substance inside the body. This does not impose any restrictions on the substance to be considered later — be it continuous or 'molecular,' similar to a fluid, i.e., consisting of the small particles mentioned above (see above). In classical statistical mechanics, the latter is associated with the model of 'hard spheres' in the simplest case. In the model considered here, this means that positions occupied by some part of body substance (or 'hard sphere') cannot be occupied by other parts of the body (or remaining 'hard spheres'). Hence, $P_N(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{i=1}^N \delta(\mathbf{r}_i - \mathbf{r}_i')$, where the prime indicates the above condition that all the parts of the substance (or small particles — 'hard spheres') occupy different positions within the body. Integrating over the Fourier vector components and taking into account the expression for the moment of the inertia of (solid body) I, we obtain the probability density of the distribution of the absolute value of the random angular momentum of the body containing stochastically moving parts (particles) inside in the form [25]

$$W(\beta) = \frac{4\beta^2}{\sqrt{\pi}} \exp(-\beta^2), \tag{19}$$

i.e., the Maxwell distribution, $\beta = M/M_0$, $M_0 = \sqrt{2UI}$. The first two moments of this distribution are

$$\langle M \rangle = \frac{2}{\sqrt{\pi}} M_0 = 2\sqrt{\frac{2UI}{\pi}}, \qquad \langle M^2 \rangle = \frac{3M_0^2}{2} = 3UI. \tag{20}$$

We will also need the expression for the mean velocity of thermal motion of particles inside the body $v_T = (2U/m_N)^{1/2}$. We now turn to the magnetic moments.

In our model, the magnetic moment of the body is also a random quantity. Physically, this moment is due to stochastic displacements of charges. The charge carriers are considered to be point ones. The relativistic motion of one or more such particles is a physical interest.

The magnetic moment of a system of N point charges e_i moving with velocities v_i in the relativistic case [27] is

$$\mathbf{M}_H = \frac{1}{2c} \sum_{i=1}^N \frac{e_i}{m_i} [\mathbf{r}_i \mathbf{p}_i].$$

Here, $\mathbf{r}_i$ is the radius vector of the point where the ith charge is located, and m_i are the masses of the particles carrying the charges (they are assumed to be small compared to the body, becoming point-like in the limit case). The probability density function of the magnetic dipole moment generated by such a system of moving point charges can be written in a form analogous to (18),

$$W(\mathbf{M}_H) = \int_V \ldots \int_V \iiint_{-\infty}^{\infty} \ldots \iiint_{-\infty}^{\infty} \delta\left(\mathbf{M}_H - \frac{1}{2c} \sum_{i=1}^N \frac{e_i}{m_i} [\mathbf{r}_i \, \mathbf{p}_i]\right) \times P_N(\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N) \prod_{i=1}^N \mathrm{d}\mathbf{r}_i \, \mathrm{d}\mathbf{p}_i. \tag{21}$$

Here, the function P_N has the same meaning as the function encountered in angular momentum calculations: the quantity $P_N(\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N)$ is the probability of finding the first particle in the vicinity of point $\mathbf{r}_1$ with momentum $\mathbf{p}_1$, the second particle in the vicinity of $\mathbf{r}_2$ and with momentum $\mathbf{p}_2$, and so on. Note that this function P_N for the magnetic moment is different from the one introduced in (18) for the angular momentum. Using the Fourier transform of the δ-function, the distribution function can be written as

$$W(\mathbf{M}_H) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} \mathrm{d}\mathbf{K} \exp(\mathrm{i}\mathbf{K}\mathbf{M}_H) \times \int_V \ldots \int_V \iiint_{-\infty}^{\infty} \ldots \iiint_{-\infty}^{\infty} \prod_{i=1}^N \exp\left(-\frac{\mathrm{i}}{2c} \frac{e_i}{m_i} \mathbf{K}[\mathbf{r}_i \, \mathbf{p}_i]\right) \times P_N(\mathbf{r}_1, \ldots, \mathbf{r}_N, \mathbf{p}_1, \ldots, \mathbf{p}_N) \prod_{i=1}^N \mathrm{d}\mathbf{r}_i \, \mathrm{d}\mathbf{p}_i. \tag{22}$$

We calculate this function assuming that the body contains only one charged particle with the rest mass m_1 and the charge e_1.

Assume that the probability density of the distribution of this particle is uniform over the volume. The distribution over velocities of this charge should be Gaussian for an equilibrium system. A good approximation for the function P_N in this case is therefore the function P_1 which, as mentioned above, is written from the very beginning for relativistic motion

$$P_1(\mathbf{r}_1, \mathbf{p}_1) = \frac{c}{4\pi m_1 T^2 V K_2(m_1 c^2/T)} \exp\left(-\frac{c}{T}\sqrt{p^2 + m_1^2 c^2}\right). \tag{23}$$

Here, T is the proper relativistic temperature of the stochastic motion of the charge carrier, and K_2 is the Bessel function of the second kind [28]. Again, without loss of generality, we assume that the Fourier vector $\mathbf{k}$ is directed along the z-axis. The dependence on angles disappears due to isotropy, leaving only the dependence on the absolute value k, and the probability density function for the distribution of the absolute value of the magnetic moment M_{H1} of the body containing a single stochastically moving point charge after all possible integrations can be conveniently written as

$$W_1(M_{H1}) = \frac{M_{H1}^2 m_1 c^2}{2VTK_2(m_1 c^2/T)} \times \int_V \mathrm{d}\mathbf{r}_1 \frac{\exp\left(-\dfrac{2m_1 c^2}{Te_1 r_1 \sin\theta} \sqrt{M_{H1}^2 + \dfrac{e_1^2 r_1^2 \sin^2\theta_1}{4}}\right)}{(e_1 r_1 \sin\theta_1/2)^3}. \tag{24}$$

Here, e_1 is the charge of the single particle and θ_1 is the polar angle of the vector $\mathbf{r}_1$. It is easy to see that this distribution is normalized to 1, $\int_0^\infty W_1(M_{H1}) \, \mathrm{d}M_{H1} = 1$.

The expectation of this distribution

$$\langle M_{H1}\rangle = \int_0^\infty M_{H1} W_1(M_{H1})\,\mathrm{d}M_{H1}$$

is, provided the body is spherical with the radius R,

$$\langle M_{H1}\rangle = \frac{3\pi e_1 RT\exp\left(-m_1c^2/T\right)}{8m_1c^2K_2(m_1c^2/T)}\left(1+\frac{3T}{m_1c^2}+\frac{3T^2}{m_1^4c^4}\right). \quad (25)$$

Let us consider now the stochastic magnetic moment of a (spherical) body with two opposite charges of equal absolute value e_1. Introducing the coordinates and the velocity of the center of mass for the system positive–negative charges, the magnetic moment of such a system is written as

$$\mathbf{M}_{H2} = \frac{e}{2c}\left[\mathbf{r}_{\rm cm}\mathbf{v}_{\rm rel}\right] + \frac{e}{2c}\left[\mathbf{r}_{\rm rel}\mathbf{v}_{\rm cm}\right]. \quad (26)$$

Here, $\mathbf{r}_{\rm rel}$ is the vector connecting two charges, $\mathbf{v}_{\rm cm}$ is the velocity of the center of mass of charged particles, $\mathbf{r}_{\rm cm}$ is the coordinate of the center of mass, and $\mathbf{v}_{\rm rel}$ is the velocity of relative motion. Since the particles inside the spherical body are electrically connected and their velocities are actually less than the speed of light, it is easy to see that the second term on the right-hand side of (26) is much smaller than the first one, as $|\mathbf{v}_{\rm cm}| \ll |\mathbf{v}_{\rm rel}|$. Since in the first approximation the random particle velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are independent (they are governed by interactions with the material filling the spherical body rather than by mutual interaction), the distribution of the velocity difference $\mathbf{v}_1-\mathbf{v}_2$ is the same as the distribution of individual velocities, i.e., it is Gaussian. Thus, the magnetic moment of the system is defined by an effective particle. Using the properties of the Gaussian distribution, it is easy to show that the effective temperature of such a particle is $2T$. Comparing expression (26) with the formula for the angular momentum of a system of two particles of equal mass (as in the case considered), it can be seen that the mass of the effective particle is $2m_1$. This means that the distribution over energy of the effective particle is exactly the same as for a single charged particle inside the spherical body. Along with this, the distribution of the random variable $\mathbf{r}_{\rm cm}$ even if the probability densities for the positions of each of the charged particles inside the body are uniform, will not be uniform.

After some volumetric calculations it can be shown that the probability density for the distribution of the projection $x = (x_1+x_2)/2$ of the vector $\mathbf{r}_{\rm cm} = (\mathbf{r}_1+\mathbf{r}_2)/2$ is

$$f_{\rm p}(x) = \frac{6}{5R^5}\left[R^5 \mp x^5 - 5R^2(R^3 \mp x^3) + 5R^3(R^2-x^2)\right],$$

using the upper sign for $x>0$ and the lower sign for $x<0$. Calculating the probability densities for the distributions of the modules $|\mathbf{r}|$, $\mathbf{r}^2$, and also $|\mathbf{r}^3|$, we find that the probability density for the distribution of the vector $\mathbf{r}_{\rm cm}$ can only be the polynomial Ar^3+Br+c. It is easy to see that the probability density normalized over the volume $\int_V f_{\rm p}(\mathbf{r})\,\mathrm{d}\mathbf{r} = V$ (omitting the index in $\mathbf{r}_{\rm cm}$) is

$$f_{\rm p}(\mathbf{r}) = 4\left(\frac{r^3}{R^3} - 3\,\frac{r}{R} + 2\right).$$

Now, we can write the probability density for the distribution of the relativistic magnetic moment of a spherical body containing two charges of opposite sign:

$$W_2(M_{H2}) = \frac{2M_{H2}^2 m_1c^2}{VTK_2(m_1c^2/T)} \times \int_V \mathrm{d}\mathbf{r}\left(\frac{r^3}{R^3} - 3\,\frac{r}{R} + 2\right)\frac{\exp\left(-\dfrac{2m_1c^2}{Te_1 r\sin\theta}\sqrt{M_{H2}^2+\dfrac{e_1^2r^2\sin^2\theta}{4}}\right)}{\left(e_1r\sin\theta/2\right)^3} \quad (27)$$

(θ is the polar angle of the vector $\mathbf{r}_{\rm cm}$), which differs from (24) only by the presence of the additional multiplier $f_{\rm p}(\mathbf{r})$ in the integrand.

The expectation of the magnetic moment of a spherical body with two charges is calculated in the same way as in the case of one charge, first by integration over M_{H2}, and then over $\mathrm{d}\mathbf{r}$. We obtain

$$\langle M_{H2}\rangle = \frac{24}{35}\,\langle M_{H1}\rangle \cong 0.6857\langle M_{H1}\rangle. \quad (28)$$

Thus, the difference in the mean magnetic moment of a spherical body with two charges of opposite signs moving stochastically differs from the magnetic moment of the same spherical body with one charge by $0.6857 = 24/35 = (4\times6)/(5\times7)$ times.[7] Let us recall that, in the model considered here, magnetic moments in both cases are random quantities, with direction distributed isotropically in space and absolute values distributed according to formulas (24) and (27). In experiments, the mean distributions of the absolute values (25) and (28) are measured. The quantum fluctuations of the magnetic moments [29] coincide with the classical ones, calculated in the framework of the given model. The difference between (28) and (25) is physically determined only by the fact that the effective particle, which generates the magnetic moment in a body with two charges, is distributed inhomogeneously in space inside the body. This distribution is maximal at the center and equals zero at the periphery, resulting in a smaller 'mean radius' and a smaller magnetic moment than in a body with one charge.

Note that an attempt to calculate in a similar way the distribution of the electric dipole moment of a spherical body with one or two charges in the framework of the present model does not lead to success — the resulting probability density function turns out to be of changing sign,[8] i.e., a definite electric dipole moment is missing. We see that R L Stratonovich's formula also gives a correct negative result for the existence of certain parameters in physical systems.

[7] If we consider bodies with stochastically moving matter and one or two charges inside as a very rough model of the proton and neutron, this number describes well the ratio of the experimentally measured magnetic moment of the neutron, which is smaller by a factor of 0.68 than the magnetic moment of the proton. In this case, the condition that the mean of the angular momentum (20) is equal to half of the Planck constant $\hbar/2$ and the mean of the magnetic moment (28) is equal to the experimental value for the magnetic moment of the proton $2.793e\hbar/2m_{\rm p}c$ ($m_{\rm p}$ is the proton mass) is fulfilled for the unique condition $T = 60.5$ keV. For classical calculations (20), (24), and (28) to be applicable 'inside the nucleon' [25], it is enough to assume that the matter of the nucleon has a dielectric permittivity of $\sim 10^{-4}-10^{-5}$.

[8] It is known, for example, that nucleons do not have a definite electric dipole moment. If the nucleons are described by the above rough model, this means that the distribution function of the electric dipole moment has no meaning because of the mentioned changing sign, and there is no definite electric dipole moment.

The problem of finding the electromagnetic moments of bodies containing a large number of stochastically moving particles is solved in an asymptotic approximation by A A Markov's method [3, 4], and in this case the electric dipole moment can be found [26].

6. Conclusions

Formula (1) turns out to be a powerful tool for calculating probability densities of distributions and moments of thermodynamic quantities as proposed by R L Stratonovich [1] (see Sections 4 and 5), as well as for operations on distributions of independent random variables, regardless of their dynamical or physical character (Section 3). This point turns out to be particularly important for finding distributions of random variables that are the sum or product of a large number of independent random variables, which is difficult to do with classical methods [5–8]. A significant result is that the obtained probability densities are fundamentally different from the Gaussian ones, and the fact that they belong to a larger class of unlimitedly divisible distributions [5, 21].

Formula (1) also allows one to easily compute the probability densities of random variables that are functions of other random variables. This is done in the simplest way for the sums of the same functions (Section 4), which is already widely used in calculations of plasma microfields and microfields of other objects [11–19]. Formula (1) also defines the probability density of thermodynamic quantities in a general form, but its calculations are possible only in concrete cases [1].

Ruslan Leont'evich Stratonovich himself used his formula exclusively as a working tool, e.g., in calculations of conditional entropy or in the analysis of Markov processes from the point of view of microscopic dynamics [1].

Note that the first analogs of formula (1) appeared already in R L Stratonovich's earlier work [30, 31][9] in the analysis of the distribution of random variables in the representation space [30] (see also the introduction of virtual spaces in Refs [21, 23]) and in the statistical treatment of the quantum theory [31].

Formula (1) was mentioned for the first time as a formula named after Stratonovich apparently in Ref. [12]. At the beginning of the 21st century, formula (1) found wide use in statistical physics, plasma physics, and other branches of physics, chemistry, or related sciences, thus writing the name of its author — Ruslan Leont'evich Stratonovich — forever into the history of modern science and technology.[10]

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[9] R L Stratonovich's paper "On distributions in representation space" (see [30]) was received by the editorial office of *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki* (*ZhETF*) (*Sov. Phys. JETP*) on September 29, 1955, and the paper "On the statistical interpretation of quantum theory" was received by the same office on October 12, 1956 (see [31]).

[10] A short paper on the scientific legacy of Ruslan Leont'vich Stratonovich was published by *Uspekhi Fizicheskikh Nauk* (*Physics–Uspekhi*) (see [32]).