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On effective range expansion in a multichannel system and compositeness of near-threshold resonance

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Contents

Abstract. We discuss effective range expansion in a two-channel system and establishing the nature of a near-threshold hadronic resonance with the help of parameters obtained from a fit to experimental data. In particular, the situation of two strong thresholds located close to each other and a hadronic resonance residing at the lower of them is considered. For the case of a bound state, simple expressions are derived that allow one to estimate the probability of finding the given resonance in each hadronic channel and as a compact quark state if the resonance wave function contains such a component.

Keywords: exotic hadrons, near-threshold resonance, effective range expansion

1. Introduction

In the last two decades, a vast amount of experimental data has appeared on so-called exotic hadrons, that is, strongly interacting states with the quark contents beyond a quark–antiquark pair in a meson or three quarks in a baryon. The possibility of the existence of multiquark states (for example, compact tetraquarks) was predicted already at the early stages of the development of quantum chromodynamics as a theory of strong interactions [1, 2]. Moreover, the non-Abelian nature of strong interactions implies that gluons can be constituents of hadrons, which makes possible the existence of both hadrons composed simultaneously of quarks and gluons (for example, hybrid mesons) and purely gluonic states (glueballs). So far, no reliable experimental evidence has been obtained for the existence of such hadrons in nature, although promising candidates exist. At the same time, the corresponding theoretical predictions are widely discussed in the literature, and the physical program of any experiment in the field of high-energy physics necessarily includes proposals for the search for such states. An intrinsic feature of the above hadrons is that they are formed by the confining forces and, therefore, have a rather small size, that is, they are compact.

Meanwhile, significant recent progress in experimental studies of hadrons containing heavy (charm and bottom) quarks has made it possible to raise the energy above the open-flavor threshold and thus to allow for the decay of a meson consisting of a heavy quark and its antiquark (that is, charmonium or bottomonium) into a pair of heavy-light mesons through the production of a light quark–antiquark pair from the vacuum (for a review of the experimental results on the inclusive cross section of the e^+e^- annihilation into hadrons, see [3]). If a hadronic state resides near such a threshold, the latter has a strong effect on its properties — in the absence of physical reasons preventing the decay of the hadron through the corresponding strong channel, the probability of such a transition cannot be neglected, so the reliability of naive calculations based on the quark model is questionable.

Indeed, the very first exotic state — X(3872), discovered by the Belle Collaboration in the spectrum of the charmonium above the open-charm threshold in 2003 — demonstrated properties at odds with the predictions of the quark model [4]. The extremely close (within 200 keV) location of X(3872) to the two-body threshold $D^0\bar{D}^{*0}$ implies a large probability of detecting this hadronic state in the form of the above mesonic pair, so the molecular model was naturally invoked to describe the properties of X(3872) (see, for example, review [5] and the references therein, as well as the review of the molecular model for exotic hadrons with heavy quarks [6]).

In the almost 20 years that have passed since the first observation of X(3872), the number of discovered exotic hadronic states in the spectrum of the charmonium and bottomonium is already several dozen (see review [7]). The Belle II experiment is actively collecting data [8, 9]. In

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addition, the work of the planned Super-c-τ factories [10, 11] planned in the near future will allow us to collect a large amount of experimental data on the exotic near-threshold states. These all together make the task of constructing a unified theoretical approach to the description of such states extremely relevant and timely. As a primary task, such an approach should allow all the information on the studied resonance encoded in the experimental data to be extracted.

2. Weinberg method and its generalizations

Since the dynamics of the molecule constituents near the two-body threshold is nonrelativistic, one of the most commonly accepted approaches to the description of line shapes of near-threshold resonances is the low-energy expansion of the amplitude in the near-threshold region and extraction of the corresponding expansion parameters (scattering length a and effective radius r) from fits to the data. In particular, in many cases, a successful description of the data is gained using the Flatté distribution [12],

$$F(E) = \frac{(1/2)g}{E - E_f + (\mathrm{i}/2)gk} = -\left(\frac{2(E_f - E)}{g} - \mathrm{i}k\right)^{-1} = -\left(\frac{1}{a} + \frac{1}{2}\,rk^2 - \mathrm{i}k\right)^{-1}, \quad (1)$$

where E is the energy of the system, E_f is a parameter of the distribution with the dimension of energy, and g is the coupling constant with the hadronic channel, in which the constituents have the momentum k and angular momentum $l = 0$ (S-wave threshold). This distribution can be regarded as a generalization of the nonrelativistic Breit–Wigner formula that takes into account an explicit energy dependence of the width arising from the proximity to the threshold of an S-wave channel, with the coupling strength given by the constant g. Notably, account for the explicit energy dependence of the momentum $k \propto \sqrt{E}$ allows one to study various threshold phenomena. Moreover, it is easy to see that it is the unitary term $\mathrm{i}k$ in the denominator that provides the leading energy dependence near the threshold.

A way of deciphering the information encoded in the low-energy parameters of the amplitude was suggested in the pioneering work by S Weinberg [13] — in particular, the extracted values of the scattering length a and effective range r for the deuteron were employed to determine the Z-factor, interpreted as the probability of observing the deuteron (a bound state of a proton and neutron with binding energy E_B) as an elementary particle,

$$a = \frac{2(1-Z)}{2-Z}\,\frac{1}{\varkappa_\mathrm{B}} + \ldots, \quad r = -\frac{Z}{1-Z}\,\frac{1}{\varkappa_\mathrm{B}} + \ldots, \quad (2)$$

where $\varkappa_\mathrm{B} = \sqrt{2\mu E_\mathrm{B}}$ is the binding momentum and μ is the reduced mass of the constituents. The ellipsis stands for the corrections suppressed by the small ratio

$$\frac{\varkappa_\mathrm{B}}{\Lambda} \ll 1, \quad (3)$$

where Λ is the inverse range of the force. The 'compositeness' of the deuteron implies that Z is close to zero, so the scattering length takes a natural value, $a \sim 1/\varkappa_\mathrm{B}$, while the leading contribution to the effective range is small and neither its value not even its sign can be established in a model-independent way, since they are both determined by the corrections in the small ratio (3). This results in a specific pattern for the near-threshold poles, since, in disregarding the small effective range term, the equation defining the position of the zeros in the denominator (see Eqn (1)) becomes linear in the momentum and provides only one near-threshold pole at $k = \mathrm{i}\varkappa_\mathrm{B}$ responsible for the bound state of the proton and neutron.

Since, on the one hand, the Weinberg method is a very useful approach to establishing the nature of a near-threshold state but, on the other hand, it is not directly applicable to systems with unstable constituents or in the absence of a bound state, there are many attempts at its generalization in the literature. For example, in [14], a so-called spectral density was suggested to observe an elementary state in the continuum spectrum. This spectral density contains the same information about the compositeness of the near-threshold state under study but allows a natural generalization to unstable constituents and the nature of resonance different from a bound state. A method to establish the nature of a resonance based on the number of corresponding near-threshold poles on the complex momentum plane was suggested in paper [15]. Accounting for the finite-range corrections is discussed, for example, in recent studies [16, 17].

In paper [18], an attempt is made to employ relations (2) to extract some information that allows a simple and straightforward generalization from bound states to virtual states and resonances. In particular, the binding momentum $\varkappa_\mathrm{B}$ is excluded from expressions for the scattering length and effective range (2), which allowed the authors to arrive at the compositeness, defined as $X = 1 - Z$, in the form

$$X = \left(1 + \left|\frac{2r}{a}\right|\right)^{-1/2}, \quad (4)$$

and the authors justified the appearance of the absolute value by the necessity of a smooth transition from a bound to virtual state that corresponds to different signs of the scattering length. From formula (4), it can be seen that a small ratio r/a leads to $X \simeq 1$, which can be interpreted as the molecular nature of the studied state.

Thus, extraction of the low-energy parameters of the amplitude from a fit to the data and using them to establish the nature of the studied resonance provide a relatively simple but promising phenomenological approach employed by both theorists and experimental collaborations (see, for example, recent studies [19, 20] on the extraction of the parameters of the tetraquark state $\mathrm{T}_{\mathrm{cc}}^{+}$ from the data obtained recently by the LHCb collaboration in the processes of charm production in proton–proton collisions [21]).

3. Two-channel system: Flatté formula and effective range

The interpretation of near-threshold state X(3872) as a molecule was criticized in a recent paper [22]. The reason for the criticism is the large (in absolute value, about several Fermi) and negative effective range extracted from the LHCb fit to the data on X(3872). Notably, it was argued that, even by taking into account corrections due to the finite range of force (in this case, the role of Λ is played by the pion mass m_π), the value of the effective radius could not be made positive, as required by general quantum mechanical principles for a molecular state [23, 24]. Accordingly, a conclusion was made on the incompatibility of the data with the molecular

nature of X(3872) and the inevitable presence of a compact component in its wave function.[1] The analysis and conclusions of [22] were criticized in paper [25] on a number of points, the main one stressing incorrect extraction of the effective radius from the low-energy expansion of the scattering amplitude in the considered two-channel problem. Indeed, since the neutral and charged $D\bar{D}^*$ channels are split by only $\Delta \approx 8$ MeV [26], any realistic model for X(3872) needs to take into account both channels.

The simplest generalization of the distribution (1) to a two-channel case is

$$F(E) = \frac{(1/2)g}{E - E_f + (\mathrm{i}/2)g(k_1 + k_2)}\,, \tag{5}$$

where the unitary term now contains two contributions, $\mathrm{i}k_1$ and $\mathrm{i}k_2$, where k_1 and k_2 are the momenta in the neutral and charged $D\bar{D}^*$ channels, respectively. For a fixed total energy E of the $D\bar{D}^*$ system, we have

$$E = \frac{k_1^2}{2\mu} = \frac{k_2^2}{2\mu} + \Delta\,, \tag{6}$$

where the mass difference between the charged and neutral D mesons is taken into account in the quantity Δ but disregarded in the corresponding reduced masses μ_1 and μ_2, so, in the last formula (and in what follows), we set $\mu_1 = \mu_2 = \mu$. Then, excluding the energy from the relations contained in formula (6), it is easy to find

$$k_2 = \mathrm{i}\sqrt{2\mu\Delta - k_1^2} \approx \mathrm{i}\sqrt{2\mu\Delta} - \mathrm{i}\,\frac{k_1^2}{2\sqrt{2\mu\Delta}} + \ldots\,, \tag{7}$$

where the momentum k_2 in the second channel was expanded near the threshold of the first (lower) channel in the limit $k_1 \to 0$.

The coefficient for the term k_1^2 in the denominator of the amplitude (5) has the meaning of the effective range (see equation (1)), for which it is easy to derive

$$r = r_0 + \delta r\,, \tag{8}$$

where

$$r_0 = -\frac{2}{\mu g}\,, \quad \delta r = -\frac{1}{\sqrt{2\mu\Delta}}\,, \tag{9}$$

and δr 'blows up' in the limit $\Delta \to 0$. It is easy to see that taking this limit is not legitimate, because it contradicts the expansion made in equation (7) that requires $\Delta \gg k_1^2/(2\mu)$.

Meanwhile, the case $\Delta = 0$ is obviously an adequate limit in many physical applications; it is reached, for example, in the exact isospin or heavy quark spin symmetry limit, so it should not cause trouble. Indeed, setting $\Delta = 0$ under the square root in equation (7) before expanding it, we arrive at the result $k_2 = k_1$, which is quite natural in this limit. Thus the problem is reduced to a single-channel case and the effective range expansion can be performed without concern about its convergence.

There is no formal logical flaw in the effective range expansion in the considered two-channel system for $\Delta \neq 0$, since this expansion is single channel by definition, so the threshold of the second channel only limits its radius of convergence. The absence of any restrictions implies $\Delta \to \infty$, that is, the limit of the large threshold splitting compared with all typical energy scales of the problem. However, most realistic physical systems considered to employ this approach correspond neither to $\Delta = 0$ nor to $\Delta \to \infty$. Thus, for X(3872), the negative contribution of the term δr from equation (9) obtained in [22] is numerically large (comparable in magnitude to r_0). Since the appearance of the δr contribution is no more than a manifestation of the presence of the second channel in the system that has nothing to do with the interaction (this contribution is universal and does not depend on the distribution parameters — the coupling constant and/or binding energy), it was suggested in [25] that the effective radius should be defined without taking this contribution into account, which eventually made it possible to obtain an effective range of X(3872) still negative but compatible in magnitude with the finite range corrections that can eventually allow for its positive value, by analogy with Weinberg's description of the deuteron. This removes the contradiction with the molecular model for X(3872) described in [22].

The same phenomenological approach allowed realistic estimates of the low-energy parameters for the T_{cc}^+ state [20] that resides in the close vicinity of (slightly below) the DD^* threshold and is an analogue of X(3872). At the same time, the LHCb data are also available below the threshold that allows one to build a reliable fit to the line shape and extract the parameters of the amplitude. In addition, the compositeness of T_{cc}^+ obtained from the extracted parameters of the low-energy expansion is about unity [20], thus hinting at its molecular nature. However, the analysis demonstrates the same difficulties as in the case of X(3872), namely, the naive single-channel low-energy expansion of the amplitude is applied to a two-channel problem. This issue generated lively discussion in the scientific community; for its synopsis see, for example, recent paper [27]. Since the situation with a physical state located near two strong thresholds occurs quite often, in this paper we address the low-energy expansion in such a system. In particular, bypassing auxiliary formulas like (4), simple expressions will be derived for the compositeness of the state under study in each hadronic channel evaluated on the basis of the parameters extracted directly from fits to the experimental data.

4. Low-energy expansion in a multichannel system

First, we pose and solve the problem as follows. There are two mesonic channels, labeled as 1 and 2, with thresholds split by Δ (the threshold of channel 2 lies above that of channel 1). The splitting Δ is neglected in the reduced masses in both channels, so, as before, we always set $\mu_1 = \mu_2 = \mu$. It also proves convenient to define the parameter $\lambda = \sqrt{2\mu\Delta}$.

The interaction potential is chosen in a separable form,

$$V_{\alpha\beta}(p, p') = f(p) v_{\alpha\beta} f(p'), \quad f(p) = \frac{\Lambda^2}{\Lambda^2 + p^2}\,, \quad \alpha, \beta = 1, 2\,, \tag{10}$$

where Λ is the inverse range of the force and

$$v_{\alpha\beta} = \begin{pmatrix} v_0 & v_1 \\ v_1 & v_0 \end{pmatrix} \tag{11}$$

[1] This conclusion is not at odds with the X(3872) model often discussed in the literature and described in detail, for example, in review [5].

is a real constant matrix of contact potentials. The employed condition for the diagonal matrix elements $v_{11} = v_{22}$ simplifies the formulas but does oversimplify the problem, since, in many physical applications, such conditions arise naturally due to various internal symmetries of the system under study such as the above-mentioned isospin or heavy quark spin symmetry.

Let the system at hand possess a bound state with the binding energy E_B, associated with a physical state. The bound state momentum $\varkappa_B$ is introduced as given in equation (2). We work in the limit

$$\frac{\varkappa_B}{\Lambda} \ll 1\,, \quad \frac{\lambda}{\Lambda} \ll 1\,, \tag{12}$$

and expansions in both ratios will be performed below. No relation between the parameters $\varkappa_B$ and λ (that is, between the binding energy E_B and the threshold splitting Δ) is assumed; in what follows, different limiting cases will be studied.

The Lippmann–Schwinger equation for potential (10) takes the form

$$T_{\alpha\beta}(p,p',E) = V_{\alpha\beta}(p,p') - \sum_{\gamma=1}^{2}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, V_{\alpha\gamma}(p,q) G_\gamma(q,E) T_{\gamma\beta}(q,p',E)\,, \tag{13}$$

where the two-body propagator in the channel $\alpha = 1, 2$ is

$$G_\alpha(p,E) = \left(M_{\mathrm{th}_\alpha} + \frac{p^2}{2\mu_\alpha} - M - \mathrm{i}0\right)^{-1}, \tag{14}$$

M_{th_α} denotes the threshold in the channel α, and the energy is counted from the lower threshold, $E = M - M_{\mathrm{th}_1}$. The widths of the constituents are disregarded.

A separable form of the potential allows one to find an explicit dependence of the amplitude on the momenta,

$$T_{\alpha\beta}(p,p',E) = f(p)\tau_{\alpha\beta}(E) f(p')\,, \tag{15}$$

where the elements of the matrix τ obey an algebraic system of equations

$$\tau_{\alpha\beta}(E) = v_{\alpha\beta} - \sum_{\gamma=1}^{2} v_{\alpha\gamma}(p,p') J_\gamma(E) \tau_{\gamma\beta}(E)\,. \tag{16}$$

The loop integral can be evaluated explicitly,

$$J_\alpha(E) = \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\, f^2(q) G_\alpha(q,E) = -\frac{\mu}{2\pi} f^2(k_\alpha)\left(\frac{\Lambda^2 - k_\alpha^2}{2\Lambda} + \mathrm{i}k_\alpha\right), \tag{17}$$

where k_α is the momentum in the channel α.

Then, the solution of matrix equation (16) for the diagonal components takes the form

$$\tau_{11} = \frac{J_2(v_0^2 - v_1^2) + v_0}{J_1 J_2 (v_0^2 - v_1^2) + (J_1 + J_2) v_0 + 1}\,, \quad \tau_{22} = \tau_{11}(J_1 \leftrightarrow J_2)\,. \tag{18}$$

Let us study the low-energy expansion of the amplitudes (18) and to this end consider their common denominator. The bound state present in the system implies that there is a pole on the complex energy plane residing on the first (physical) sheet of the four-sheet Riemann surface at the energy $E = -E_B$ below the lower threshold. The denominator vanishes at the pole,

$$J_1(-E_B) J_2(-E_B)(v_0^2 - v_1^2) + \big[J_1(-E_B) + J_2(-E_B)\big] v_0 + 1 = 0\,. \tag{19}$$

In a two-channel system with the thresholds split by Δ, it proves convenient to use the generalization of the single-channel complex k-plane, which provides a convenient visualization of the four-sheet Riemann surface in the complex energy plane. The idea of [28] is to make a conformal transformation that turns the complex energy plane into the unitary-cut-free complex plane of a new parameter. In particular, for a given energy E, instead of the momenta k_1 and k_2 related with the energy through relation (6), a single complex quantity ω is introduced as

$$k_1 = \frac{\lambda}{2}\left(\omega + \frac{1}{\omega}\right), \quad k_2 = \frac{\lambda}{2}\left(\omega - \frac{1}{\omega}\right), \tag{20}$$

which gives for the energy

$$E = \frac{\Delta}{4}\left(\omega^2 + \frac{1}{\omega^2} + 2\right). \tag{21}$$

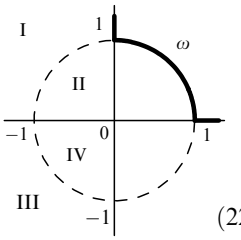
The complex ω-plane is cut-free, and a one-to-one correspondence between the Riemann sheets (labeled as RS-X, where X = I, II, III, IV) of the complex energy plane and different regions in the complex ω-plane reads

RS-I: $\mathrm{Im}\, k_1 > 0\,, \quad \mathrm{Im}\, k_2 > 0\,,$

RS-II: $\mathrm{Im}\, k_1 < 0\,, \quad \mathrm{Im}\, k_2 > 0\,,$

RS-III: $\mathrm{Im}\, k_1 < 0\,, \quad \mathrm{Im}\, k_2 < 0\,,$

RS-IV: $\mathrm{Im}\, k_1 > 0\,, \quad \mathrm{Im}\, k_2 < 0\,.$



(22)

The threshold of channel 1 corresponds to the points $\omega = \pm\mathrm{i}$, the threshold of channel 2, to the points $\omega = \pm 1$, and the bold black line, to real-valued energies E lying on the first sheet of the complex E-plane (the physical region). Moreover, it is easy to see that the bound state with binding energy E_B corresponds to a pole on the complex ω-plane lying on the imaginary axis outside the circle $|\omega| = 1$,

$$\omega_0 = \mathrm{i}(1 + b)\,, \quad b > 0\,, \tag{23}$$

and it is easy to find from equation (21)

$$\omega_0 = \mathrm{i}\,\frac{\sqrt{E_B} + \sqrt{E_B + \Delta}}{\sqrt{\Delta}} = \mathrm{i}\big(\sqrt{\delta} + \sqrt{1+\delta}\big)\,, \tag{24}$$

where we introduced the distances $\Delta_1 = E_B \equiv \Delta\delta$ and $\Delta_2 = E_B + \Delta = \Delta(1 + \delta)$ from the pole position at $E = -E_B$ to the first and second threshold, respectively. Then,

$$b = \sqrt{\delta} + \sqrt{1+\delta} - 1\,. \tag{25}$$

Traveling alone, the segment of the circle $|\omega| = 1$ from the lower threshold ($\omega_1 = \mathrm{i}$) to the upper one ($\omega_2 = 1$) corresponds to ω changing as

$$\omega = \exp\left[\mathrm{i}\left(\frac{\pi}{2} - \xi\right)\right] = \mathrm{i}\exp(-\mathrm{i}\xi)\,, \quad \xi \in \left[0, \frac{\pi}{2}\right]. \tag{26}$$

It is easy then to find the following expressions for the two momenta and the energy:

$$k_1 = \lambda \sin\xi\,, \quad k_2 = -\mathrm{i}\lambda\cos\xi; \quad E = \Delta \sin^2\xi\,. \tag{27}$$

The low-energy expansion of the amplitude near the lower threshold corresponds to the expansion at $\xi \to 0$. Then, it is easy to demonstrate that, up to an irrelevant overall factor, the expansion of the denominator of the amplitudes (18) takes the form

$$f(\xi) = -b - \mathrm{i}\xi - \frac{1}{2}\,\xi^2 + \frac{\lambda}{\Delta}\left(\frac{b^2\big(4 + b(7b+12)\big)}{4(1+b)^2} + \xi^2\right) + \ldots, \tag{28}$$

where the expansion was made for $\xi \ll 1$ and $\lambda/\Delta \ll 1$ simultaneously, and the ellipsis denotes disregarded higher terms in both expansion parameters. No additional assumptions about the quantity b were made. The last term on the right-hand side provides corrections due to the finite range of the force. They are model dependent (depend on the regularization scheme for the loop integral and the form factor shape) and disappear in the limit $\Delta \to \infty$. The necessity of taking such (and higher order) corrections into account depends on the required precision of the calculations performed and the quality of the available experimental data. It is instructive to estimate the value of the expansion parameter λ/Δ for typical near-threshold systems in the spectrum of the charmonium and bottomonium. For example, for X(3872) and $\mathrm{T}_{\mathrm{cc}}^{+}$, we have $\Delta \simeq 1$ GeV and $\mu \approx 1$ GeV, while threshold splitting appears as a result of isospin symmetry breaking, $\Delta = M(\mathrm{D}^{+}\mathrm{D}^{*-}) - M(\mathrm{D}^{0}\bar{\mathrm{D}}^{*0}) \approx 8.2$ MeV for X(3872) and $\Delta = M(\mathrm{D}^{+}\mathrm{D}^{*0}) - M(\mathrm{D}^{0}\mathrm{D}^{*+}) = 1.4$ MeV for $\mathrm{T}_{\mathrm{cc}}^{+}$. Then, it is easy to find that $\lambda/\Delta \lesssim 0.1$. For a bottomonium system, the mass difference between the charged and neutral B mesons is negligibly small, so, to a high accuracy, the corresponding charged and neutral channels can be treated as degenerate in energy. Thus, in the b-sector, the splitting of the thresholds appears due to heavy quark symmetry breaking, $\Delta = m_{\mathrm{B}^*} - m_{\mathrm{B}} \approx 45$ MeV. This gives $\lambda/\Delta \simeq 0.5$. A ratio of the same order of magnitude appears in charmonium systems if one considers the thresholds split by the value $\Delta = m_{\mathrm{D}^*} - m_{\mathrm{D}} \approx 140$ MeV. Therefore, taking into account finite-range corrections definitely lies beyond the reasonable accuracy (at least for the experimental data currently available) for the states X(3872) and $\mathrm{T}_{\mathrm{cc}}^{+}$; however, in studies of, for instance, the bottomonium-like resonances $\mathrm{Z_b}(10610)$ and $\mathrm{Z_b}(10650)$ and their spin partners, as well as those of X(3872) and $\mathrm{T}_{\mathrm{cc}}^{+}$, a more adequate approach is provided by the effective field theory, which allows a systematic and model-independent account for higher orders in the expansion parameter. Further discussions of the corrections in the ratio λ/Δ go beyond the scope of the present work and in what follows they will be omitted, so formula (28) can be presented in a simple form,

$$f(\xi) \approx -b - \mathrm{i}\xi - \frac{1}{2}\,\xi^2\,, \tag{29}$$

that resembles effective range expansion in a single-channel problem performed in terms of the dimensionless quantity ξ. It is seen from equation (27) that, in the limit $\xi \to 0$, $k_1 \approx \lambda\xi$, so expansion (29) can be re-written as the standard effective range expansion in terms of the momentum k_1,

$$-\lambda b - \mathrm{i}k_1 - \frac{k_1^2}{2\lambda}\,, \tag{30}$$

where one can see that the last term causes the aforementioned divergence problem in the limit $\Delta \to 0$. Meanwhile, it is easy to see that the term $\propto \xi^2$ stems from the expansion of $\cos\xi$ contained in k_2 (see the corresponding expression in equation (27)).

We will revisit this discussion later and now turn to the derivation of the expressions for the compositeness of the considered resonance. It is important to mention that, in setting up the task itself, we are dealing with a purely molecular state without any admixture of the compact component, so its total compositeness is $X = 1$. Once there are two channels, one can define the probabilities of finding the physical state either in one of them or in the other — X_1 and X_2, respectively. Then, a self-consistency condition,

$$X_1 + X_2 = 1\,, \tag{31}$$

should hold.

General physical considerations allow establishing some natural properties of the quantities X_1 and X_2. Indeed, the closer the state resides to a threshold, the higher the probability of observing it in this particular channel. Furthermore, it is obvious that in the limit of coinciding thresholds ($\Delta = 0$) the probabilities of observing the resonance in either of the two channels constitutes one half. In other words, the following conditions should hold:

$$\begin{aligned} &X_1(\delta = 0) = 1\,, \quad X_2(\delta = 0) = 0\,, \\ &X_1(\delta \to \infty) = X_2(\delta \to \infty) = \frac{1}{2}\,. \end{aligned} \tag{32}$$

It is well known [29–32] that the compositeness of a resonance with respect to a particular channel α can be evaluated as a product of the derivative from the corresponding loop integral and the residue of the amplitude at the pole,

$$X_\alpha = \lim_{M \to M_{\mathrm{pole}}} \left[(M^2 - M_{\mathrm{pole}}^2)\tau_{\alpha\alpha}(M)\,\frac{\mathrm{d}J_\alpha(M)}{\mathrm{d}M^2} \right], \tag{33}$$

where no summation over the repeated index α is implied. Then, by an explicit calculation with the help of the formulas from equation (18), we can find

$$X_1 = \frac{(\ln J_1)'(1 + v_0 J_2)}{(\ln J_1)'(1 + v_0 J_2) + (\ln J_2)'(1 + v_0 J_1)}\,, \tag{34}$$

$$X_2 = \frac{(\ln J_2)'(1 + v_0 J_1)}{(\ln J_1)'(1 + v_0 J_2) + (\ln J_2)'(1 + v_0 J_1)}\,, \tag{35}$$

where the prime denotes a derivative with respect to the energy E (or the dimensionless quantity ω). All loop functions and their derivatives are taken at the pole (the argument $-E_{\mathrm{B}}$ is omitted for simplicity). In the above derivation, the following trivially verified relations valid at the pole were used:

$$\begin{aligned} &J_1(-E_{\mathrm{B}})(v_0^2 - v_1^2) + v_0 = -\frac{1}{J_2(-E_{\mathrm{B}})}\big(1 + v_0 J_1(-E_{\mathrm{B}})\big)\,, \\ &J_2(-E_{\mathrm{B}})(v_0^2 - v_1^2) + v_0 = -\frac{1}{J_1(-E_{\mathrm{B}})}\big(1 + v_0 J_2(-E_{\mathrm{B}})\big)\,. \end{aligned} \tag{36}$$

It is easy to see that expressions (34) and (35) obey the condition $X_1 + X_2 = 1$ discussed above.

It is instructive to study decoupled channels first, that is, take the limit $v_1 = 0$. Then, the system of the Lippmann–Schwinger equations splits into two decoupled equations for the components of the amplitude τ_{11} and τ_{22}, with the solutions trivially found in the form

$$\tau_{11} = \frac{1}{v_0^{-1} + J_1(E)}\,, \quad \tau_{22} = \frac{1}{v_0^{-1} + J_2(E)}\,. \tag{37}$$

Assuming a pole in the first (second) channel and setting $v_0 J_1(-E_B) + 1 = 0$ ($v_0 J_2(-E_B) + 1 = 0$), it is easy to verify that formulas (34) and (35) give $X_1 = 1$, $X_2 = 0$ ($X_1 = 0$, $X_2 = 1$), in agreement with natural expectations.

For a nonvanishing off-diagonal potential $v_1 \neq 0$ and, therefore, for the two channels coupled to each other, neither $v_0 J_1 + 1$ nor $v_0 J_2 + 1$ vanishes at the pole individually, allowing us to simplify expressions (34) and (35) employing the fact that the leading term in both loop integrals is given by the large inverse range of the force Λ, so we can set $J_1 \approx J_2$ everywhere but in the terms with derivatives where the leading nonvanishing contribution stems from the subleading term in the integral (see the explicit expression for the loop integral (17)). Then, it is easy to find

$$\begin{aligned} X_1 &= \frac{J_1'}{J_1' + J_2'} = \frac{1}{2}\left(1 - \frac{1}{\omega_0^2}\right), \\ X_2 &= \frac{J_2'}{J_1' + J_2'} = \frac{1}{2}\left(1 + \frac{1}{\omega_0^2}\right), \end{aligned} \tag{38}$$

where the explicit expression for the pole position ω_0 from formula (24) should be used. It is then straightforward to arrive at the following final result:

$$X_1 = \frac{\sqrt{1+\delta}}{\sqrt{\delta} + \sqrt{1+\delta}}, \quad X_2 = \frac{\sqrt{\delta}}{\sqrt{\delta} + \sqrt{1+\delta}}\,, \tag{39}$$

which can be verified to satisfy all the properties of the quantities X established above and enumerated in equation (32). It should also be noted that the obtained expressions for the compositeness are free of any troubles in the limit $\Delta \to 0$. Also, it is worth mentioning that no effective range expansion was employed in the derivation.

Meanwhile, the formulation of the problem just solved differs from a typical situation encountered in the analysis of the experimental data for near-threshold resonances. Indeed, a pure molecule was studied above, so the absence of a compact component in the wave function of the resonance was assumed from the beginning. On the contrary, a typical task in studies of a near-threshold resonance is to extract the information on its nature — the position of the pole on the complex plane and the contents of the wave function — directly from a fit to the data.

In [33], the interplay of the hadronic and quark dynamics in a near-threshold resonance was studied for the case of several hadronic channels present in the system. In particular, in the two-channel case, the wave function of the resonance was taken in the form

$$|\psi\rangle = \begin{pmatrix} \sqrt{Z}\,|\psi_0\rangle \\ \chi_1(\mathbf{p})|M_{11}M_{12}\rangle \\ \chi_2(\mathbf{p})|M_{21}M_{22}\rangle \end{pmatrix}, \tag{40}$$

where $|\psi_0\rangle$ is a compact state, $|M_{\alpha 1}M_{\alpha 2}\rangle$ ($\alpha = 1, 2$) are hadronic channels, and $\chi_\alpha(\mathbf{p})$ is the momentum–space wave function in the channel α. In the simplest case of no near-threshold poles produced by the hadronic dynamics alone, the expression for the amplitude is given by distribution (5). Then, if the system possesses a bound state with the binding energy E_B, the Z-factor describing the probability of observing the resonance as an elementary particle takes the form [33][2]

$$Z = \left(1 + \frac{\mu g(\kappa_1 + \kappa_2)}{2\kappa_1\kappa_2}\right)^{-1}, \tag{41}$$

where $\kappa_\alpha = \mathrm{Im}\, k_\alpha(E = -E_B)$, that is,

$$\kappa_1 = \sqrt{2\mu E_B} = \lambda\sqrt{\delta}, \quad \kappa_2 = \sqrt{2\mu(E_B + \Delta)} = \lambda\sqrt{1+\delta}\,. \tag{42}$$

It is easily seen from expression (41) that the limit $\Delta \to 0$ is trouble free and amounts to setting $\kappa_2 = \kappa_1$, which gives

$$Z(\Delta = 0) = \left(1 + \frac{\mu g}{\sqrt{2\mu E_B}}\right)^{-1}. \tag{43}$$

In the opposite limit of $\Delta \gg E_B$, the quantity Z also takes a simple form,

$$Z(\Delta \to \infty) = \left(1 + \frac{\mu g}{2\sqrt{2\mu E_B}}\right)^{-1}, \tag{44}$$

where the splitting Δ dropped out, as expected.

Therefore, by extracting two parameters (the binding energy E_B and the coupling constant g) from a fit to the experimental data, one can evaluate the probability of the studied resonance being an elementary state. For example, for the central values of the parameters used in [22],

$$\Delta = 8.2\ \text{MeV}\,, \quad E_B = 18\ \text{keV}\,, \quad g = g_{\text{LHCb}} = 0.108\,, \tag{45}$$

formula (41) gives $Z \approx 0.1$, which coincides with the estimates found in the cited work. In other words, the probability of X(3872) being a molecule is about 90%.

Meanwhile, such estimates need to be treated with care. On the one hand, final conclusions on the nature of a near-threshold state can only be made after performing a combined analysis of the entire experimental information collected for it, including data on its production in different reactions and its decays into various final states. On the other hand, the parameters used in the estimate may contain uncertainties caused by particular procedures employed to extract them. For example, if the Flatté formula is used in the analysis of the data collected in a narrow near-threshold region, the term $\propto E$ in the denominator is very small; however, if it is neglected, the distribution demonstrates a scaling property [34] and its parameters can be re-scaled with an arbitrary factor without changing the line shape. Then, the fit convergence is spoiled and it may fall into an unphysical local minimum with parameters very different from the correct physical values. For instance, the plot of the likelihood as a function of the parameter E_f given in [35] has a flat section (nearly a plateau) where, however, the coupling constant g significantly changes.

[2] The coupling constant g introduced here is related to the constant g_f from [33] as $g = g_f/2$.

It is instructive to substitute expression (41) into formula (2) for the effective range, which gives

$$r = -\frac{2}{\mu g}\,\frac{\sqrt{E_B + \Delta}}{\sqrt{E_B} + \sqrt{E_B + \Delta}}\,. \tag{46}$$

It is easy to see that this expression does not contain terms singular in the limit $\Delta \to 0$. At the same time, in the case of $\Delta \gg E_B$ (see, for example, the parameters contained in equation (45)), one finds

$$r \approx -\frac{2}{\mu g} = r_0\,, \tag{47}$$

which coincides with the first term in formula (8). In the limit $\Delta \to 0$, the effective radius acquires the factor $\sqrt{E_B + \Delta}/(\sqrt{E_B} + \sqrt{E_B + \Delta}) \approx 1/2$ canceled by the additional factor 2 introduced to take into account the presence of two identical channels in the system. Thus, we revisit the conclusion that $r = r_0$. This result complies well with the suggestion contained in [25] to remove the second term $\delta r \propto 1/\sqrt{\Delta}$ in formula (8) when evaluating the effective range in a two-channel system.

The expressions for the compositeness (39) derived above can be generalized to the case when the wave function contains a compact component. To this end, it is sufficient to modify the normalization condition and require that $X_1 + X_2 = 1 - Z$ while preserving the ratio $X_1/X_2 = \sqrt{1+\delta}/\sqrt{\delta}$ (see formula (39)). Then, it is easy to arrive at the following final expressions:

$$\begin{aligned}
Z &= |r_0|\left(|r_0| + \frac{1}{\sqrt{2\mu E_B}} + \frac{1}{\sqrt{2\mu(E_B+\Delta)}}\right)^{-1},\\
X_1 &= \frac{1}{\sqrt{2\mu E_B}}\left(|r_0| + \frac{1}{\sqrt{2\mu E_B}} + \frac{1}{\sqrt{2\mu(E_B+\Delta)}}\right)^{-1},\\
X_2 &= \frac{1}{\sqrt{2\mu(E_B+\Delta)}}\left(|r_0| + \frac{1}{\sqrt{2\mu E_B}} + \frac{1}{\sqrt{2\mu(E_B+\Delta)}}\right)^{-1},
\end{aligned} \tag{48}$$

where the quantity r_0 defined in formula (9) can be interpreted as the 'effective range' in the considered two-channel system. In the limit of a purely molecular state $g \to \infty$, so the previous expressions turn into $Z = 0$ and the compositenesses quoted in formula (31). On the contrary, in the opposite limit of $g \to 0$, we have $Z = 1$ and $X_1 = X_2 = 0$, in agreement with natural expectations for a purely compact (not molecular) state.

The structure of the derived expressions for X_1 and X_2 can be easily understood. As one can see from definition (33), the additional contribution $|r_0|$ in the denominator stems from the term $\propto E$ in the denominator of the amplitude (see, for example, Flatté formula (5)), which, in turn, arises from the Green's function of the compact state $1/(E - E_0)$ (here, E_0 is the bare energy of the compact state absorbed by the 'renormalized' parameter of the distribution E_f (see the derivation of the corresponding equations, for example, in [33])).

Formulas (48) allow a straightforward generalization to an arbitrary number of thresholds N and take a simple and unified form,

$$X_i = \frac{R_i}{\sum_{j=0}^{N} R_j}\,, \tag{49}$$

where $i = 0$ corresponds to the compact state with $R_0 = |r_0|$, and $R_i = 1/\sqrt{2\mu\Delta_i}$ for all $i > 0$ (see the definition of the quantities Δ_i given after equation (24)). In the formal limit of $\Delta_i \to \infty$, the corresponding channel decouples from the system and does not affect its dynamics. In this limit, $R_i \to 0$ and $X_i \to 0$, in agreement with natural expectations, and formula (49) reduces to the case of $N - 1$ channels. It is worth mentioning, however, that this is a purely academic limit, since large splitting between thresholds implies a large difference in the masses of the constituents in different channels, so the approximation of equal reduced masses employed in this work fails. Notice also that the 'radius' R_i for the ith channel is defined by the quantity Δ_i that has the meaning of the 'binding energy' with respect to the corresponding threshold — a typical long-range behavior of the bound state wave function. It is also instructive to emphasize that the 'radius' R_0 that corresponds to the compact state is given by the quantity $|r_0|$ independent of the threshold splittings.[3]

5. Discussion

In this paper, we considered a generalization of effective range expansion to a two-channel system and its application to extracting the nature of near-threshold resonances from the data. Interest in this problem is related to the appearance of experimental data on exotic hadrons and the necessity of studying their properties on the basis of the available experimental information. Fitting the data in a near-threshold region with the help of simple analytical formulas is the only approach available to experimental collaborations. Since the dynamics of the system in the near-threshold region is essentially nonrelativistic, then low-energy expansion of the amplitude is a powerful and convenient method of investigation. Meanwhile, a naive application of the single-channel formulas to a system for which it is necessary to take into account two hadronic thresholds residing close to each other leads to a number of difficulties that are a direct consequence of using the given approach beyond the scope of its applicability. It is demonstrated how these difficulties can be overcome by explicitly accounting for both hadronic channel and the possible presence of a compact component in the resonance wave function. The resulting expressions from equation (48) make it possible to estimate the probability of observing the studied resonance in each hadronic channel separately, solely on the basis of the masses of the constituents (that determine the splitting Δ between the thresholds and the value of the reduced mass μ) and the two parameters (the binding energy E_B and the coupling constant with the hadronic channels g) extracted directly from a fit to the experimental data.

The formulas in this paper were derived in certain approximations. To begin with, the problem of instability of the molecule constituents deserves a comment. Indeed, in actuality, the hadrons forming a molecular state are unstable and have finite widths. Then, the pole identified with a bound state and lying below the lower of the two thresholds considered above does not reside on the real axis but gets shifted to the complex plane. It does not lead to a contradiction of the basic principles of quantum mechanics, since

[3] Strictly speaking, such a dependence arises if the approximation of the equal reduced masses in all channels is relaxed; however, it is very weak and does not lead to a singularity in the limit $\Delta_i \to 0$.

the Riemann surface on the complex energy plane becomes multi-sheeted, with the branch points at all thresholds, including those lying lower in energy and representing the decay channels due to constituent decays. However, direct use of the Weinberg formulas provides an unsatisfactory result in this case — the quantity Z takes complex values and its straightforward interpretation as a probability fails. It is then necessary to employ generalizations of the Weinberg method and consider, for example, the spectral density [14]. The results of the application of a generalized approach to data analysis for the scalar mesons $a_0(980)$ and $f_0(980)$ can be found in [36]. Shortcomings of this approach are a lack of simple and transparent formulas and the theoretical uncertainty related to a particular choice of the interval of integration for the spectral density to estimate the probability of observing the resonance as a compact state — such an integral needs to be taken in the near-threshold region, but the exact limits of integration can not be determined as a matter of principle. Strictly speaking, to tackle this problem, one needs to extend the set of coupled channels to incorporate all final states that provide a nonvanishing width of the studied resonance. For example, if weak decays of the D meson are neglected, then for the states $X(3872)$ and T_{cc}^{+} it is necessary to additionally consider the three-body channels $D\bar{D}\pi/D\bar{D}\gamma$ and $DD\pi/DD\gamma$, respectively. In [37], the line shape of a near-threshold resonance with the three-body dynamics included was studied in such an extended formulation. The formulas derived allowed the authors to arrive at a realistic description of the line shape of the resonance in a near-threshold region and naturally reproduce such effects as a nonvanishing signal below the formal two-body threshold and interference of different contributions to the amplitude coming from nonequivalent ways of the decay reaction. Unfortunately, these formulas are not simple algebraic expressions and can hardly be used in the experimental data analysis.

In [38], and then much later in paper [39], simple analytic formulas for the momentum in the two-body channel and its 'analytic continuation' below the threshold were suggested. As demonstrated in the study [37], the suggested formulas provide a satisfactory agreement with the exact result near the two-body threshold but demonstrate a wrong behavior near the three-body threshold. Employing the formulas with a finite D^* width included as suggested in paper [39] for the $X(3872)$ allows one to have a nonvanishing signal below the nominal two-body $D\bar{D}^*$ threshold and, if there is a bound state in the $D\bar{D}^*$ system, describe the below-threshold peak that appears as a result of the decay of the $\bar{D}^*$ meson as a bound state constituent. However, since the experimental data are missing in the below-threshold region, the influence of this effect on the line shape above the threshold is minimal [40]. Thus, a conclusion on whether or not various extensions of the simple two-channel approach need to be employed in a particular analysis should be based on the quality of the experimental data. Since, for example, the LHCb data for the T_{cc}^{+} [21] are much more precise than the Belle data for $X(3872)$ [4], to analyze the former, it is necessary to resort to approaches that explicitly include multibody effects (see, for instance, [20]).

Another phenomenon that may potentially play an essential role in the description of the exotic near-threshold resonances is the interaction in the final state. In particular, nontrivial effects due to a direct interaction between molecule constituents were studied in [33, 39]. Moreover, for a specific choice of the constituents' masses and the mass of the exchanged particle, the amplitude may possess a logarithmic singularity known as triangle singularity; a related discussion for the $X(3872)$ state can be found, for example, in [41 – 43] and for T_{cc}^{+}, in papers [44, 45].

6. Conclusions

The aim of the present work is to extend the Weinberg analysis to a two-channel system. The derived formulas contain only two parameters: the binding energy and the coupling constant with the hadronic channels that are extracted from the data and absorb all the information about the studied resonance. If the amplitude meets all required physical constraints, such as unitarity (including the multiparticle one) and exact and approximate symmetries present in the system, and takes into account all additional effects like the aforementioned triangle singularity and so on, then the corresponding information will be contained in the extracted parameters, too. If the imaginary part of the binding energy is not large, it can be disregarded for estimates based on the formulas derived in this study. Violation of unitarity due to this disregard can be treated as a model uncertainty; it may not exceed the uncertainties stemming from other effects — for example, experimental errors.

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