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METHODOLOGICAL NOTES

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On the problem of boundary conditions for mixed type equations arising in the description of astrophysical transonic flows

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Abstract. Using the example of an exactly solvable problem, it is shown that the number of boundary conditions which is necessary to determine the transonic hydrodynamic flow near the so-called 'nonstandard' singular point does not depend on whether the separatrix characteristic passes or does not pass through this point. Thus, a quite popular statement is called into question, according to which the critical surface, on which the regularity condition determines the flow structure, generally coincides with the surface of separatrix characteristics, and not with the sonic surface.

Keywords: hydrodynamics

The rapid development in analytical studies of strongly magnetized flows, carried out from the 1970s to 1990s [1–7] (they were simulated by the need to determine the structure of the flow of matter in the vicinity of compact astrophysical objects), forced us to turn to questions lying at the very foundations of the hydrodynamical approach. In particular, for astrophysical applications, this was linked to the need to study a transonic version of the hydrodynamical equations that inevitably appeared in astrophysical problems.

In fact, even in the simplest spherically symmetric hydrodynamical models (Bondi accretion on a gravitating center [8] and Parker solar wind [9]), it is necessary to consider critical conditions on the sonic surface $v = c_{\mathrm{s}}$. In this case, the problem requires finding three unknowns (any two thermodynamic functions, for example, the density $\rho(r)$ and the sound speed $c_{\mathrm{s}}(r)$, and also the radial component of velocity $v_r(r)$) as functions of radius r. Three scalar equations defining the flow structure (the continuity equation, the radial component of Euler's equation, and the energy equation) can be easily integrated, leading to the conservation of the flux $\Phi = 4\pi\rho v_r r^2$, nonrelativistic Bernoulli integral $E_{\mathrm{n}} = v_r^2/2 + w(\rho, s) + \varphi_{\mathrm{g}}$, and entropy s. Here, $w(\rho, s)$ is the enthalpy and $\varphi_{\mathrm{g}}(r)$ is the gravity potential. The problem, as we see, is three-parametric. However, the relationship

$$\frac{r}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{2v_r^2 + \varphi_{\mathrm{g}}}{v_r^2 - c_{\mathrm{s}}^2} \tag{1}$$

for transonic flows requires an additional condition — that the numerator be zero on the sonic surface $v_r = c_{\mathrm{s}}$. As a result, the number of necessary boundary conditions is one fewer than the number of equations and unknowns. These boundary conditions are usually taken to be the density and sound speed at infinity (for Bondi accretion) or on the surface of a star (for the solar wind). We stress that in both cases the boundary conditions are imposed in the subsonic regime.

As concerns multi-dimensional transonic flows, it is well known that in this case there are no general methods to construct solutions to a direct problem, for example, the problem of finding the flow structure by the known boundary in a transonic flow domain or by the boundary conditions on some surface in a subsonic domain [10–12], also including the case of stationary hydrodynamical equations depending only on two spatial coordinates, to which we will be limited here. Such uncertainty, which inevitably and repeatedly arose in concrete problems, was either explicitly or implicitly present in all studies in this area.

Consider, for example, the textbook case of plane potential flow, characterized by the potential $\phi(x, y)$ which defies the flow velocity as $\mathbf{v} = \nabla\phi$ (see Ref. [12], Ch. XII). In this case, the hydrodynamical equations can be reduced to a single second-order equation

$$\phi_{xx} + \phi_{yy} + \frac{(\phi_y)^2\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (\phi_x)^2\phi_{yy}}{(\nabla\phi)^2 D} = 0\,, \tag{2}$$

where

$$D = -1 + \frac{c_{\mathrm{s}}^2}{v^2}\,, \tag{3}$$

and the subscripts denote partial derivatives. Transforming this equation into the canonical form $\mathcal{A}\phi_{xx} + 2\mathcal{B}\phi_{xy} + \mathcal{C}\phi_{yy} = 0$, we obtain $\mathcal{AC} - \mathcal{B}^2 = D(D+1)$, so that the equation changes from elliptical in the subsonic domain to hyperbolic in the supersonic domain.

It will be important for us here, however, that, in order to close this equation, as is well known, we must additionally use Bernoulli's equation $E_{\mathrm{n}} = (\nabla\phi)^2/2 + w(c_{\mathrm{s}}, s)$ to express the sound speed c_{s}, which enters Eqn (2), in terms of the unknown function ϕ. This requires, as is seen, that the boundary conditions specify two further motion integrals, E_{n} and s. This does not present any difficulties for subsonic flows, since, to fully define the solution, we need to specify two

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thermodynamic functions and two components of the velocity (i.e., the potential ϕ and its derivative). These four boundary conditions correspond to four scalar equations (the continuity equation, two components of Euler's equation, and the energy equation), which fully define the flow structure.

However, if in this problem statement we try to determine the structure of a transonic flow, for which, as we have already seen, we need to impose an additional critical condition (whose localization, by the way, is unknown until the problem is solved!), it suffices to limit ourselves to only three boundary conditions — one fewer than in the subsonic case, as in the simplest cases mentioned above. The fundamental difficulty here is that, in this case, the boundary conditions are insufficient to determine Bernoulli's integral, which depends on all four quantities. As a consequence, Eqn (2), which explicitly contains the quantity $E_{\rm n}$, is indeterminate.

This statement is fully related to axisymmetric flows, which are already described by five algebraic equations (the continuity, three components of Euler's equation, and the energy equation), which are reduced to three integrals of motion (the specific angular momentum $L_{\rm n} = rv_\varphi$ is added) and the second-order equation on the potential $\Phi(r, \theta)$ [13]

$$
\begin{aligned}
&- r^2 \sin^2\theta\, \nabla_k \left(\frac{1}{r^2 \sin^2\theta}\, \nabla^k \Phi \right) - \frac{\nabla^i \Phi\, \nabla^k \Phi\, \nabla_i \nabla_k \Phi}{(\nabla\Phi)^2 D} \\
&+ \frac{\nabla(r^2 \sin^2\theta)\, \nabla\Phi}{2r^2 \sin^2\theta\, D} - 4\pi\rho^2 r^2 \sin^2\theta\, \frac{\nabla\varphi_{\rm g}\, \nabla\Phi}{D(\nabla\Phi)^2} \\
&- 4\pi^2\rho^2\, \frac{D+1}{D}\, L_{\rm n}\, \frac{{\rm d}L_{\rm n}}{{\rm d}\Phi} + 2\pi^2\rho^2\, \frac{\nabla(r^2\sin^2\theta)\, \nabla\Phi}{r^2 \sin^2\theta (\nabla\Phi)^2 D}\, L_{\rm n}^2 \\
&+ 4\pi^2\rho^2 r^2 \sin^2\theta\, \frac{D+1}{D}\, \frac{{\rm d}E_{\rm n}}{{\rm d}\Phi} \\
&- 4\pi^2\rho^2 r^2 \sin^2\theta \left[\frac{D+1}{D}\, \frac{T}{m_{\rm p}} + \frac{1}{D\rho} \left(\frac{\partial P}{\partial s} \right)_\rho \right] \frac{{\rm d}s}{{\rm d}\Phi} = 0\,, \qquad (4)
\end{aligned}
$$

which defines the flux $\rho\mathbf{v}$,

$$
\rho\mathbf{v} = \frac{\nabla\Phi \times \mathbf{e}_\varphi}{2\pi r \sin\theta}\,. \qquad (5)
$$

Equation (4) represents a generalization of the Grad–Shafranov equation [14, 15] to the case of nonzero velocities. Despite the cumbersome form of Eqn (4), the key property $\mathcal{AC} - \mathcal{B}^2$ is $D(D+1)$ as before, so that this equation also changes from elliptic to hyperbolic on the sonic surface.

We will make further use of another important property of transonic flows, which has been studied in detail for a plane potential flow, i.e., for Eqn (2); this property naturally also applies to an axisymmetric flow, i.e., to Eqn (4). It concerns the so-called singular point where the streamlines are perpendicular to the sonic surface. An analysis based on an exactly solvable Tricomi problem, to which Eqns (2) and (4) can be reduced near this point, showed that the solution is analytic near this point [16]. Furthermore, the problem is exactly solvable if its boundary condition is not specified in the subsonic domain but on the streamline passing through this singular point,

$$
v_x(x) = c_\star + k(x - x_\star)\,. \qquad (6)
$$

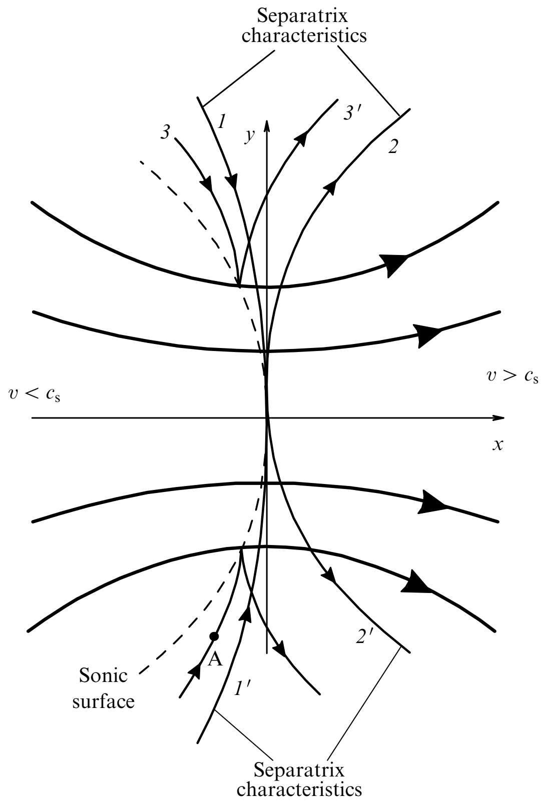

Figure 1. Classical singular point of a transonic flow which corresponds to converging streamlines (bold solid lines) on a sonic surface (dashed line). Shown are the characteristics in a hyperbolic flow domain, including two separatrix characteristics which touch the sonic surface at the singular point. Hyperbolic domain between the sonic surface and separatrix characteristics (for example, point A) affects subsonic domain along characteristics that terminate on the sonic surface.

For a polytropic equation of state $P \propto \rho^\Gamma$ ($\Gamma = \text{const}$), the solution of Eqn (2) in the vicinity of the singular point takes the form [12]

$$
\begin{aligned}
\phi(x, y) = {} & c_\star(x - x_\star) + \frac{k(x - x_\star)^2}{2} \\
& + \frac{k^2(\Gamma + 1)}{2c_\star}(x - x_\star)\, y^2 + \frac{k^3(\Gamma + 1)^2}{24 c_\star^2}\, y^4\,. \qquad (7)
\end{aligned}
$$

Here, the sound speed on the sonic surface $c_\star$ is expressed through Bernoulli's integral as $E_{\rm n} = c_\star^2/2 + w(c_\star, s)$.

The analysis of this solution showed that, in the majority of points on the sonic surface where the streamlines and this surface are not perpendicular, two branches of characteristics are directed in opposite directions (lines *3* and *3′* in Fig. 1). At the singular point, however, there are already two characteristics that touch the sonic surface and continue in the same direction (characteristic *1* merges into characteristic *2′*, and characteristic *1′* merges into characteristic *2*).

It should be noted that, in the initial phase, studies devoted to astrophysical questions did not pay proper attention to the question of the number of boundary conditions. The point is that there was a certain gap between these studies and the intensive research in this field carried out from the beginning of the 20th century [16, 17]. In particular, the generalization of the Grad–Shafranov equation to the case of nonzero plasma velocity, first obtained by L S Solov'ev in 1963 [18], was in fact repeated once again in [2, 3, 6] in the

1970s. Similarly, the magnetorotational instability, discovered by E P Velikhov [19] and S Chandrasekhar [20] in the 1950s, was only rediscovered for astrophysical applications by S Balbus and J Hawley in 1991 [21]. It is therefore not surprising that in those cases where questions about the correct formulation of the problem came to the fore, the astrophysical literature resorted to basically simplified approaches, based to a considerable extent on heuristic arguments.

Another bad joke was played by the circumstance that quite early a large family of self-similar substitutions of the form $\Phi(r,\theta) = r^{\alpha}\Theta(\theta)$ was found for equation (4) [4, 22–29], leading to the ordinary differential equation $K\,\mathrm{d}^2\Theta/\mathrm{d}\theta^2 + \ldots = 0$, many key properties of which differed substantially from those of two-dimensional equations. In particular, the singularity (the condition $K = 0$ for the coefficient with the highest derivative) did not occur on the sonic surface (in magnetohydrodynamics — on fast and slow magnetoacoustic surfaces), but precisely on the separatrix characteristic. Based on this important property (which, by the way, was well known in classical hydrodynamics [11]), a statement was formulated that is now accepted by the majority of researchers, namely that the separatrix characteristic, and not the sonic surface, is such a special surface on which critical conditions define the unique solution [27, 30].

Several clarifications are certainly in order here. First of all, we do not question the number of boundary conditions needed to determine the structure of the transonic flow. In a general case, the number of boundary conditions b can be written in the form [13]

$$b = 2 + i - s' , \tag{8}$$

since, for the second-order equation on the potential $\phi(x,y)$ or the potential $\Phi(r,\theta)$, one needs to fix i motion integrals, and s' critical conditions (the absence of singularity) will play the role of additional constraints that fix the flow structure. In the case of ideal hydrodynamics, when there is only one critical surface ($s' = 1$), we obtain $b = 4$, given the required three integrals of motion (Bernoulli integral E_{n}, the specific angular momentum L_{n}, and entropy s). Note that formula (8) is not applicable for a spherically symmetric case, since there is no need to solve the second-order equation. The spherically symmetric flow is also degenerate, because its sonic surface coincides with the separatrix characteristic.

Furthermore, there is no doubt that it is the surface of separatrix characteristics and not the sonic surface that defines the boundary of influence on the solution in the subsonic domain. As shown in Fig. 1, the hyperbolic domain between the sonic surface and separatrix characteristics affects the subsonic domain along the characteristics ending at the sonic surface. This property is essentially connected with the argument in favor of the singularity on the separatrix characteristic.

Indeed, let us assume that two conditions on some boundary in the subsonic domain together with the regularization condition on the sonic surface completely determine the solution upstream from the critical surface. But, according to, for example, Ref. [30], this leads to a contradiction, since signals that can propagate from perturbations between the critical and separatrix surfaces will affect the solution in the domain bounded by the critical surface, and hence the solution obtained will not be unique. In simpler terms, assuming that a unique solution is constructed for a hydrodynamic flow, a perturbation is introduced in the domain between the sonic surface and separatrix characteristics (an 'obstacle' at point A, the modification of the flow boundary in the supersonic domain), which will inevitably modify the flow in its subsonic domain.

As we see, the main argument in favor of singularity not on the sonic surface but on the separatrix characteristic is not the existence of the singularity itself but reasoning related to causality. This is also an interesting moment. On the separatrix characteristics, no critical condition can be formulated, whereas on the sonic surface the derivative of density ρ over coordinates,

$$\nabla_i\,\rho = \frac{N_i}{D} , \tag{9}$$

should satisfy the conditions $N_i = 0$. And in all cases when exact solutions to concrete problems were constructed [1, 31–36] (they were all obtained as small perturbations of known spherically symmetric cases, when the position of the sonic surface was known), namely the condition on the sonic surface $N_r = 0$ determined the flow structure. Granted, in all these cases, owing to the smallness of the perturbation to the spherical flow, the separatrix surface coincided with the sonic one in the zeroth approximation. This is why this consideration could not serve as an argument against the statement that the separatrix characteristic is a singular surface.

However, the reasoning above contains one implicit assumption which in our opinion violates its strictness. This is the assumption that the position of the sonic surface does not change on adding a perturbation. The solution in the subsonic domain is considered to be given. However, even in the textbook case of plain potential flow, given two known integrals E_{n} and s, there is freedom in the choice of two constants, $x_\star$ and k, in the expression for the velocity along the x-axis (6) which fully specifies a smooth transonic flow, as shown above, i.e., in reality, an infinite set of flows exists with given integrals that do not have a singularity on the critical surface, whatever it is.

Indeed, let us consider flow (7) that corresponds to boundary condition (6) and place its boundary such that it coincides with some streamline with the shape of parabola

$$y_{\mathrm{b}} = y_0 + A(x - x_0)^2 \tag{10}$$

with given values of x_0, y_0, and A. In this case, with quadratic accuracy, we have a solution to the direct problem (i.e., a solution of Eqn (7) with the given flow boundary) with the values

$$x_\star = x_0 + \frac{k(\Gamma+1)}{6c_\star}\,y_0^2 , \tag{11}$$

$$k = \left(\frac{2Ac_\star^2}{(\Gamma+1)y_0}\right)^{1/2} . \tag{12}$$

If we now 'perturb' the flow boundary, also in the region between the sonic and separatrix surfaces (but still within the framework of the quadratic approximation!), we simply move to other values of constants $x_\star$ and k. This is understandable, because the position of the sonic surface is not determined by the condition $D = 0$.

Reference [37] shows that the direct problem can also be solved for the next approximation for the flow boundary $y_{\mathrm{b}} = y_0 + A(x - x_0)^2 + B(x - x_0)^3$, if the quadratic term in

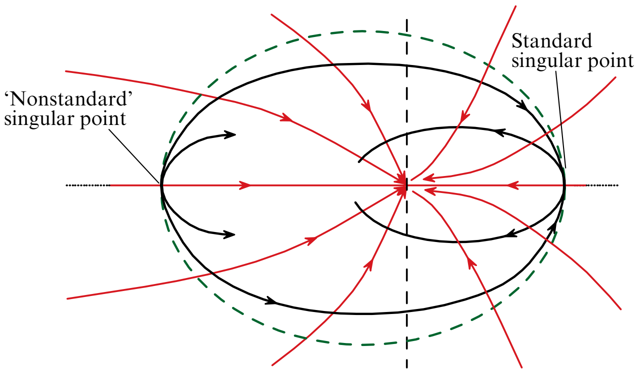

Figure 2. Behavior of characteristics in the Bondi–Hoyle accretion case. A separatrix characteristic exits from the 'nonstandard' singular point on the sonic surface (dashed line), which corresponds to diverging (with respect to the sonic surface) streamlines.

expansion (6) is taken into account. It is true that in this case one has to solve a nonlinear system of algebraic equations to determine the coefficients defining the flow. However, this does not change the essence. Changing the boundary in the supersonic domain and even placing an 'obstacle' in this domain (when it is necessary to consider an infinite series in (6)) will only change the position of the critical surface.

As a further argument in favor of our viewpoint, we analyze below the construction of the solution for an axisymmetric steady flow in the vicinity of a 'nonstandard' singular point. A short historic reminder is in order here: the research with an astrophysical focus which was mentioned above has nevertheless contributed substantially to the theory of transonic flows. It became clear that, in the absence of flow boundaries (accretion on a compact object, stellar wind), there must be another singular point on the sonic surface, where a streamline is perpendicular to the sonic surface. However, unlike the standard singular point shown in Fig. 1, it corresponds to streamlines that diverge with respect to the sonic surface. Such a singular point is impossible for plane potential flows and has therefore not been discussed in classical studies on transonic flows [1] [10–12, 16].

As it turns out, the flow in the vicinity of 'nonstandard' singular point has surprising properties. First of all, we note that the separatrix characteristics cannot originate from the hyperbolic domain, but must start at some other point on the sonic surface (Fig. 2). It also turns out that, in the vicinity of 'nonstandard' singular point, characteristic surfaces have a very different structure, as we will show below.

Consider an axisymmetric stationary flow similar to the Bondi–Hoyle accretion, shown in Fig. 2. An important simplifying assumption, which will not affect our arguments, is the absence of axial rotation ($L_{\rm n} = 0$), and also the constancy of Bernoulli's function ($E_{\rm n} = {\rm const}$) and entropy ($s = {\rm const}$) in the entire domain. In this case, the Grad–Shafranov equation (4) for the potential Φ, which enters into the definition (5), in a compact form looks like [13]

$$r^2 \sin^2\theta \, \nabla_k \left(\frac{1}{\rho r^2 \sin^2\theta} \nabla^k \Phi \right) = 0 \,. \qquad (13)$$

[1] More precisely, the authors are unaware of studies on these topics, with the exception of those devoted to astrophysical flows.

It will be recalled that this equation should be complemented by the Bernoulli equation

$$\frac{(\nabla\Phi)^2}{8\pi^2\rho^2 r^2 \sin^2\theta} + \frac{c_{\rm s}^2}{\Gamma - 1} + \varphi_{\rm g}(r) = \frac{(\Gamma+1)c_\star^2}{2(\Gamma-1)} + \varphi_{\rm g}(r_\star) \,, \qquad (14)$$

where

$$\varphi_{\rm g}(r) = -\frac{GM}{r} \,. \qquad (15)$$

It implicitly gives the dependence $\rho = \rho(\Phi)$, accounting for which leads to additional second derivatives of potential Φ.

Writing now Eqn (13) in the standard form $\mathcal{A}\Psi_{rr} + 2\mathcal{B}\Psi_{r\theta} + \mathcal{C}\Psi_{\theta\theta} + \ldots = 0$, the equation on characteristics ${\rm d}r/{\rm d}\theta = [\mathcal{B} \pm (\mathcal{B}^2 - \mathcal{AC})^{1/2}]/\mathcal{C}$ can be rewritten as

$$\frac{{\rm d}R}{{\rm d}\vartheta} = a\vartheta \pm \sqrt{R} \,. \qquad (16)$$

Here, the angle $\vartheta = \theta - \theta_*$,

$$R = \frac{r_*(\vartheta) - r}{r_* D_1} \,, \qquad (17)$$

$D_1 = r_* (\partial D/\partial r)_*$ at $r = r_*$, and r_* and θ_* are the singular point coordinates. Finally [38],

$$a = -\frac{(\partial^2 D/\partial\theta^2)_*}{D_1^2} - \frac{r_*(\partial^2\Phi/\partial r\partial\theta)_*}{(\partial\Phi/\partial\theta)_* D_1} \,. \qquad (18)$$

The positive values of a correspond to the standard singular point (converging streamlines), whereas negative values of a correspond to a 'nonstandard' singular point (diverging streamlines).

It can be readily verified that Eqn (16) has an exact solution in the form of two parabolas,

$$R(\vartheta) = w_{1,2}^2 \vartheta^2 \,, \qquad (19)$$

where

$$w_{1,2} = \frac{1 \pm \sqrt{1 + 8a}}{4} \,. \qquad (20)$$

While for $a > 0$ the smaller root corresponds to two separatrix characteristics entering the standard singular point and the larger root corresponds to two characteristics leaving it (Fig. 3), for $a < 0$, i.e., for the 'nonstandard' singular point, the situation is fundamentally different.

First, real roots $w_{1,2}$ exist only for $-1/8 < a < 0$. In this case, both parabolas correspond to the characteristics leaving the singular point. As shown in Fig. 3a, there are infinitely many exiting characteristics in the vicinity of the parabola corresponding to the smaller root w that touch the parabola at the singular point. One of these is the separatrix characteristic, but it cannot be found in the vicinity of the 'nonstandard' singular point, since the trajectory of the characteristics up to the standard singular point should be determined for this. For us, it will only be important that for $-1/8 < a < 0$ the separatrix characteristic passes through the 'nonstandard' singular point.

Second, for $a < -1/8$, the expressions for w (20) become complex. This means that for a slow change in parameters the net of characteristics undergoes an abrupt structural change. This is the remarkable property of the bifurcation of characteristics [27, 30, 38]. As shown in Fig. 3b, the hyperbolic domain in the vicinity of the singular point starts

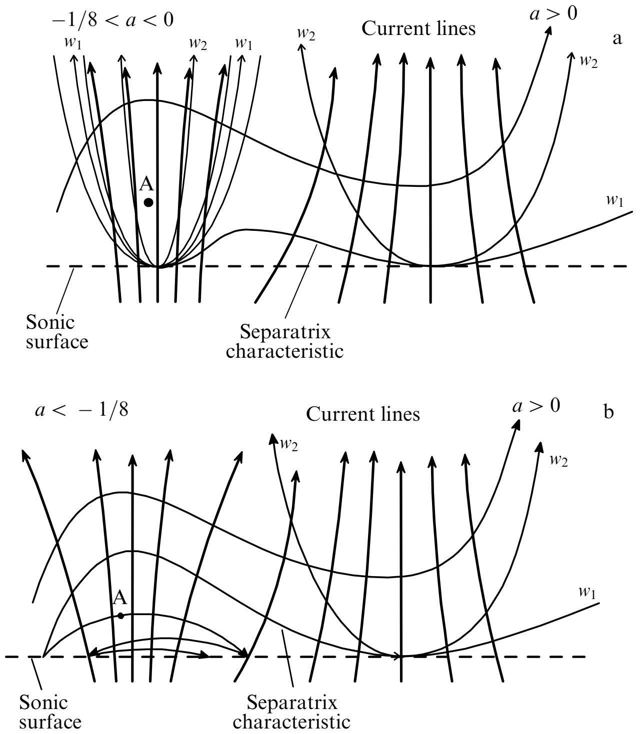

Figure 3. (a) Behavior of characteristic surfaces when parameter a for the 'nonstandard' singular point (left) satisfies the condition $-1/8 < a < 0$. Separatrix characteristics and solutions that correspond to $w = w_1$ and $w = w_2$ are depicted by bolder lines. Standard singular point $a > 0$ is shown on the right. Point A has no influence on the subsonic domain. (b) Same as in (a), but for $a < -1/8$. A perturbation from point A reaches the sonic surface along the characteristic.

to affect the subsonic domain, since perturbations from point A reach the sonic surface along the characteristic. An important point for us here is that for $a < -1/8$ no characteristic passes through the 'nonstandard' singular point, including the separatrix characteristic, which is now located further upstream with respect to the 'nonstandard' singular point.

So, we are sure that for $a > -1/8$ the separatrix characteristic passes through the 'nonstandard' singular point, and for $a < -1/8$ it no longer does. On the other hand, if the separatrix characteristic was indeed a critical surface, i.e., it defined the necessary number of boundary conditions, then the construction of a solution in the vicinity of the 'nonstandard' singular point would require a different number of free functions defining a unique solution when passing through $a = -1/8$. It is easy to show that this is not the case.

Indeed, the analyticity of the singular point allows us to seek a solution to Eqn (13) as an expansion in integer powers of distances from the singular point. In this case, as in the standard approach, we will assume that the longitudinal velocity v_r along the flow axis is given:[2]

$$v_r(r) = -c_\star(1 - kh)\,. \tag{21}$$

Here, by definition,

$$h = \frac{r - r_\star}{r_\star}\,. \tag{22}$$

[2] A different approach has been considered in Ref. [38].

This selection of signs corresponds to the case $k > 0$ (the velocity increases on approaching the gravitational center). Further, one can readily check that, owing to definition (5), Eqn (13) is nothing but the condition $\mathrm{rot}\,\mathbf{v} = 0$, which, in our case, should hold identically. We then can define the components of velocity $\mathbf{v}$ in the form

$$v_r(r,\theta) = -c_\star\left(1 - kh - \frac{1}{2}\,\varkappa\theta^2\right), \tag{23}$$

$$v_\theta(r,\theta) = c_\star\varkappa\theta\,. \tag{24}$$

Accordingly, the flux function $\Phi(r,\theta)$ should be defined, which we write as

$$\Phi(r,\theta) = -\Phi_\star\left(\frac{\theta^2}{2} + bh\theta^2 - \frac{q}{24}\,\theta^4 + \ldots\right). \tag{25}$$

Here, $\Phi_\star = 2\pi r_\star^2 n_\star c_\star$. In this case, for a monopole solution $\Phi(r,\theta) = \Phi_\star(1 - \cos\theta)$, we have $q = 1$ and $b = 0$. If we also write the density $\rho(r,\theta)$ as

$$\rho(r,\theta) = \rho_\star\left(1 + \eta_1 h + \frac{1}{2}\,\eta_2\theta^2\right), \tag{26}$$

all other coefficients of expansion can be found from the Bernoulli equation (14) and definition (5).

Indeed, expanding the radial component of relationship (5) up to the first powers of h and θ^2 and also using the leading ($\propto \theta$) θ-component of this relationship, we find

$$\eta_1 = k + 2b - 2\,, \tag{27}$$

$$\eta_2 = \varkappa + \frac{1}{3} - \frac{q}{3}\,, \tag{28}$$

$$\varkappa = b\,. \tag{29}$$

Expanding the Bernoulli equation (14) up to the first powers of h and θ^2, we find, taking into account relationships (28) and (29),

$$b = 1 - \frac{GM}{2r_\star c_\star^2}\,, \tag{30}$$

$$\eta_2 = b - b^2\,. \tag{31}$$

As a consequence, denominator D (3) will be written as

$$D = \big[(\Gamma + 1)k - 2(\Gamma - 1)(1 - b)\big]h + b(\Gamma + 1)(1 - b)\,\frac{\theta^2}{2}\,. \tag{32}$$

Finally, we obtain

$$a = -b\,\frac{2(\Gamma + 1)k + (1 - b)(5 - 3\Gamma)}{D_1^2}\,, \tag{33}$$

where

$$D_1 = (\Gamma + 1)k - 2(\Gamma - 1)(1 - b)\,. \tag{34}$$

As one might expect, all the coefficients are expressed in terms of $c_\star$, which is defined by Bernoulli's integral E_{n} and the position of the singular point $r_\star$, which determines the values of b and k. For a spherically symmetric solution ($b = 0$), we have $a = 0$. On the other hand, for a singular point further away from the gravitating center compared to the spherically symmetric case ($b > 0$), we get $a < 0$, i.e., what is required for the 'nonstandard' singular point (see Fig. 2). Correspondingly, for the standard singular point, we have $a > 0$.

However, the most important point for us is certainly that the solution procedure is in no way related to the value of a. For the 'nonstandard' singular point, the solution is constructed in exactly the same way for both $a > -1/8$ and $a < -1/8$, and, to find it in both cases (in addition to two integrals of motion), one needs to specify only one function: in our case — the radial velocity $v_r(r)$. For zero angular momentum $L_{\rm n}$, this number of boundary conditions corresponds exactly to relation (8). Thus, at least for the exactly solvable problem considered here, it is clear that the separatrix characteristic is not a critical surface that imposes additional constraints on the parameters of the supersonic flow.

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