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'Quantum' values of the extrema of 'classical' macroscopic quantities

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Contents

Abstract. Fundamental constants play an important role in nature. They determine many high-energy processes. It turns out that these constants also set bounds for the 'ordinary' properties of condensed matter, such as viscosity, thermal conductivity, elastic moduli, and the speed of sound. Kinematic viscosity has a global minimum point on the (P, T) diagram, and the same is true for the thermal diffusivity of substances (except at the critical point). The minimum values of these quantities are determined only by the Planck constant $\hbar$ and the masses of the electron m and the atom or molecule M. A nontrivial conclusion is that the kinematic viscosity values for ordinary fluids and for quark–gluon plasma are close to each other. Similarly, the extrema of the elastic characteristics of substances, the mechanical properties of materials, and the speed of sound are also determined only by the Planck constant, the masses of the electron and ions, and the electron charge. The use of fundamental constants allows proposing reasonable estimates for the speed of sound of substances and the elastic characteristics of low-dimensional systems. We also note a possible connection between extreme values of macroscopic quantities and the anthropic principle.

Keywords: fundamental constants, extreme values, viscosity, thermal conductivity, speed of sound

1. Introduction

The role of fundamental constants in physics is difficult to overestimate [1]. In atomic physics, there is a natural scale of length, given by the Bohr radius $a \sim \hbar^2/me^2$, and energy, given by the Rydberg $E \sim me^4/\hbar^2$, where $\hbar$ is Planck's constant and m and e are the mass and charge of the electron. Sometimes, the unit of energy is chosen to be the Hartree energy, which is twice the Rydberg value. It was D Hartree who first proposed the atomic system of units, where, in addition to mass, charge, angular momentum (action), length, and energy, the 'atomic' quantities of time, electric field strength, speed, force, current, temperature, and pressure are expressed through fundamental constants [2]. This system of units is currently not very popular, although it is still used in quantum electrodynamics, atomic physics, and spectroscopy.

It is usually believed that, if a certain physical quantity is expressed as a combination of fundamental constants, then these are macroscopic manifestations of purely quantum phenomena. Thus, the combination $e^2/\hbar$ is a metal conductivity 'quantum' in two-dimensional systems, which is observed, for example, in the quantum Hall effect and related phenomena; the combination $\hbar/e$ determines the magnetic flux quantum, in particular in vortices in a superconductor; the fine structure constant $\alpha = e^2/\hbar c$ (where c is the speed of light), for example, determines the absorption coefficient of a two-dimensional single-layer graphene film, etc.

At the same time, because interparticle interaction in condensed matter has a quantum electromagnetic nature, the atomic units of length and energy actually determine the interparticle distance and binding energy in such media. Indeed, the interparticle distance in substances ranges from 1 to 5 Å, which is 2 to 10 times greater than the Bohr radius, and the cohesion energy in media with strong (ionic, covalent, and metallic) bonds is 1 to 10 eV per atom, which is only several times less than a Rydberg (13.6 eV). In contrast to these fairly well-known and trivial facts, other atomic units related to condensed matter are used quite rarely. For example, the atomic unit of pressure, which is the atomic energy density, is a Rydberg per Bohr-radius cube, $P \sim E/a^3 \sim m^4e^{10}/\hbar^8$, corresponding to the pressure above which shell effects start manifesting themselves and the

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properties of all substances gradually become universal. The numerical atomic pressure value is 150 Mbar if a Rydberg is used as the unit of energy and 300 Mbar if a Hartree is used. Such pressures are unattainable in static experiments, but are quite accessible in shock wave experiments [3]. We note that the elastic moduli of substances also have the dimension of pressure, and the atomic unit of pressure is at the same time the atomic unit of moduli. The actual moduli of substances are 2 to 3 orders of magnitude smaller: diamond has the maximum modulus among all substances, 4.5 Mbar for the bulk compression modulus and 5.5 Mbar for the shear modulus (of a polycrystal) [4]. Obviously, the estimates of the moduli of substances from fundamental constants are greatly inflated. We note that very good estimates for the elastic moduli can be obtained when the actual values of the cohesion energy and interatomic distances are taken into account [4]. In contrast to the static properties of materials, systematic assessments of the kinetic properties of condensed matter based on atomic units have not previously been made. In what follows, we consider how the viscosity of fluids and heat conduction of condensed media are related to fundamental constants; we show that estimates of the universal minimum values of kinematic viscosity and thermal diffusivity are possible. We then return to the static properties of low-dimensional systems and to the speed of sound in condensed matter in terms of fundamental constants. Finally, we briefly discuss the connection between extreme values of viscosity and thermal conductivity and the anthropic principle. For more details, we refer the reader to the original papers [5–9].

2. Dynamical and kinematic viscosity of fluids

Viscosity is an important physical quantity characterizing the fluidity of fluids (liquids and gases). There are dynamical and kinematic viscosities, the latter being equal to the ratio of the dynamical viscosity to density.

Dynamical viscosity characterizes the resistance of a liquid to the motion of objects inside it and varies for different fluids within a record wide range: from 10^{-6} Pa s for the normal component of liquid helium to 10^{12} Pa s for vitrifying liquids near their glass transition point. The viscosity of liquids generally increases exponentially as the temperature decreases or pressure increases. The viscosity of gases increases with increasing temperature and varies slightly with pressure. Of course, there are no universal formulas for calculating the viscosity of fluids for arbitrary T and P parameters. At the same time, it turns out that the extreme (minimum) values of the viscosity of various fluids are quite universal and are determined by fundamental constants.

The rapid decrease in the dynamical viscosity of liquids η with increasing temperature is usually described in terms of the Arrhenius dependence

$$\eta = \eta_0 \exp\left(\frac{E_{\text{act}}}{T}\right), \tag{1}$$

where η_0 is a preexponential factor and E_{act} is the activation barrier for the diffusion jump of an atom (molecule). Energy barriers, of course, are also related to the interparticle interaction potential, but it is extremely difficult to calculate them. The viscosity of fluids can also be expressed in terms of the relaxation time τ (in fact, the particle 'lifetime' in an unchanged environment) and the high-frequency shear

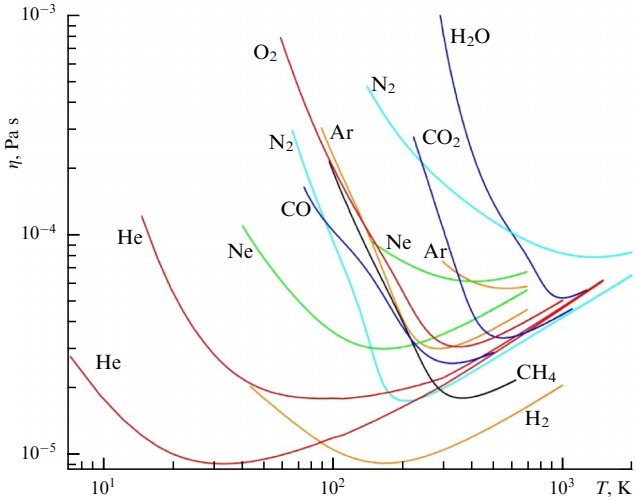

Figure 1. Experimental dynamical viscosity (NIST database https://webbook.nist.gov/chemistry/fluid) exhibiting minima for noble, molecular, and reticulate fluids. Curves for H_2, H_2O, and CH_4 are shown for the respective pressures $P = 50$ MPa, 100 MPa, and 20 MPa. For He, Ne, Ar, and N_2, the data are shown at two pressures each: 20 and 100 MPa for He, 50 and 300 MPa for Ne, 20 and 100 MPa for Ar, and 10 and 500 MPa for N_2. Higher-pressure minimum is located higher than the lower-pressure minimum for each fluid. Details are given in [5].

modulus G [10]:

$$\eta \sim G\tau\,. \tag{2}$$

The effective shear modulus changes weakly, almost linearly, with temperature, while the relaxation time decreases with increasing temperature, approximately following the Arrhenius law

$$\tau = \tau_0 \exp\left(\frac{E_{\text{act}}}{T}\right), \tag{3}$$

where τ_0 is the period of oscillations of a particle in a 'settled' state, and is in fact the inverse of the effective Debye frequency. The shear modulus can be estimated as $G = \rho V_{\text{t}}^2$, where ρ is the density, $V_{\text{t}} \sim a/\tau_0$ is the transverse speed of sound, and

$$G \sim \frac{\rho a^2}{\tau_0^2}\,. \tag{4}$$

In the gaseous phase, viscosity is determined by the particle collision rate

$$\eta \sim \rho V_0 L\,, \tag{5}$$

where ρ is the density, V_0 is the particle velocity (of the order of thermal velocity), and L is the particle free path. For an almost ideal gas, the viscosity is simply proportional to speed, which is in turn proportional to $T^{1/2}$. At pressures above critical, the liquid-to-gas transition is continuous, and the regime of viscosity decrease with temperature (expression (2)) smoothly transforms into the 'gaseous' regime of growth (expression (5)), and hence the viscosity of all fluids passes through a minimum [10] (Fig. 1). It was established previously in [10] that this minimum is located near the crossover of many physical properties of fluids, which we call the Frenkel

line. Near the Frenkel line, the gas regime 'merges' with the liquid one, the relaxation time in the liquid levels with τ_0, the speed of transverse high-frequency sound is close to the thermal speed of particles, and the free path in the gas phase levels with the interparticle distance [10]. Expressions (2) and (5) then lead to the same result,

$$\eta_{\min} \sim M\,\frac{\omega_0}{a}\,, \tag{6}$$

where M is the particle mass and ω_0 is the maximum oscillation frequency ($\omega_0 \sim 1/\tau_0$). The Debye energy can be estimated as

$$\hbar\omega_0 \sim \hbar\left(\frac{E}{Ma^2}\right)^{1/2} \sim E\left(\frac{m}{M}\right)^{1/2}, \tag{7}$$

where E is the interparticle binding energy.

If we take atomic units as the interparticle distance and binding energy, then an estimate for the minimum viscosity would be given by a rather cumbersome expression:

$$\eta_{\min} \sim \frac{m^{5/2}M^{1/2}e^6}{\hbar^5}\,. \tag{8}$$

This expression can also be obtained in another way. For many systems, especially with a simple interparticle interaction potential, it is convenient to introduce the characteristic viscosity [11]

$$\eta^* \sim \frac{E^{1/2}M^{1/2}}{a^2}\,, \tag{9}$$

where E and a characterize the energetic and spatial scales of the interaction potential. The characteristic viscosity determines a suitable scale for the actual viscosity. For example, for a fluid with the Lennard-Jones potential, the viscosity of the liquid is approximately three times higher than η^* near the triple point, and approximately four times less than η^* near the critical point [11]. Obviously, the minimum viscosity $\eta_{\min}$ is also given by the characteristic viscosity times a certain numerical coefficient. If we take the atomic units for E and a, we immediately obtain an expression similar to (8),

$$\eta^* \sim \frac{m^{5/2}M^{1/2}e^6}{\hbar^5}\,.$$

The value of this 'atomic' viscosity is approximately 10^{-2} Pa s, which is an order of magnitude greater than the viscosity of liquid metals and 2 to 3 orders of magnitude greater than the viscosity of simple molecular liquids. This is unsurprising, because expression (9) makes it clear that a high value of the atomic energy and a small value of the atomic distance (compared to the actual values of E and a for fluids) result in overestimating the dynamical viscosity.

The situation with the kinematic viscosity $\nu = \eta/\rho$ is much more interesting. It is the kinematic viscosity that enters the Navier–Stokes equation and determines the nature of the fluid flow,

$$\frac{\partial \upsilon}{\partial t} = \frac{\eta}{\rho}\,\nabla^2\upsilon\,.$$

Just like the dynamical viscosity, the kinematic viscosity for all fluids at over-critical pressures passes through a minimum with a characteristic value of 10^{-7} m^2 s^{-1} (Fig. 2). Substituting the density $\rho \sim M/a^3$, we use (8) to immediately obtain the minimum atomic kinematic viscosity

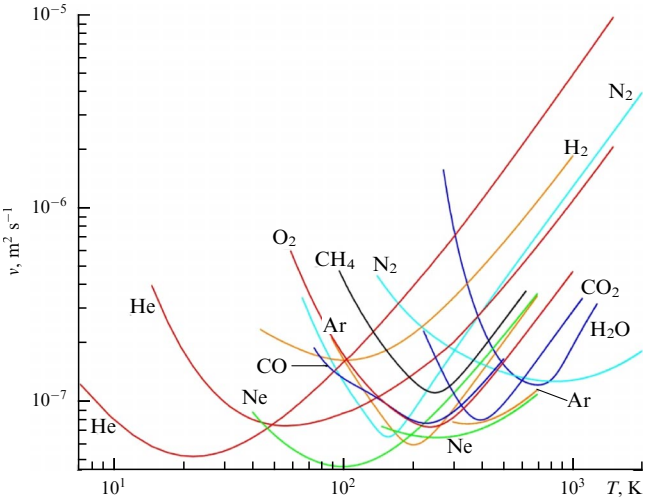

Figure 2. Experimental kinematic viscosity of fluids. Same fluids as in Fig. 1 are shown under the same conditions. Details are given in [5].

$$\nu_{\min} \sim \frac{\hbar}{(Mm)^{1/2}}\,. \tag{10}$$

The proportionality coefficient in (10) is 0.1–0.2 [5]. Expression (10) is very elegant and gives the exact order of magnitude for the minimum value of the kinematic viscosity of all liquids, even for simple molecular fluids (see Fig. 2). This is easy to explain. From expression (9) for the characteristic kinematic viscosity, we have

$$\nu^* \sim \frac{E^{1/2}a}{M^{1/2}}\,. \tag{11}$$

The interaction in molecular and inert-gas fluids is of a dipole–dipole nature and is two orders of magnitude weaker than the strong purely Coulomb interaction, and the interparticle distances are 2 to 4 times greater than in strongly interacting systems. As a result, for molecular fluids and for strongly interacting liquids, the combination $E^{1/2}a$ differs by only a few times [5]. Expression (10) also gives an estimate for the minimum kinematic viscosity for metallic and covalent fluids in the still unexplored region of high pressures and temperatures near the critical point. In addition, this expression predicts the dependence of the minimum viscosity on the mass of an atom (molecule) in the form $\nu_{\min} \sim M^{-1/2}$. There are virtually no experimental data on the viscosity of atomic fluids in the supercritical region, but data on the viscosity of liquid metals at high temperatures are quite consistent with this dependence. For example, melts of light metals, Mg ($M = 24$) and Al ($M = 27$), at a temperature of 1300 K (which is significantly higher than the melting point) have the respective kinematic viscosities of 5.5×10^{-7} m^2 s^{-1} and 4.5×10^{-7} m^2 s^{-1}. At the same time, melts of heavy metals, Pb ($M = 207$) and Bi ($M = 209$), also at 1300 K, have respective viscosity values 3 to 5 times lower: $\sim 1.5 \times 10^{-7}$ m^2 s^{-1} and 0.9×10^{-7} m^2 s^{-1} [12]. We add that all the above estimates of dynamical and kinematic viscosity refer to moderate pressure values, when the characteristic interaction energy significantly exceeds the PV contribution to the chemical potential.

The simplicity of expression (10) is very attractive. The minimum possible value of kinematic viscosity (or, equivalently, maximum fluidity) of any fluid is determined only by Planck's constant and the masses of the electron and atom (molecule). We note that, in early versions of Eyring's formula for viscosity, Eqn (1), the expression for η_0 also included Planck's constant [13]. Eyring and Frenkel estimated the vibration frequency of a molecule in a potential well as $\hbar\omega \sim k_B T$, which is not an entirely correct expression, whence $\omega \sim k_B T/\hbar$, where k_B is Boltzmann's constant. Subsequently, η_0 was considered an empirical fitting parameter, and the 'quantum' nature of 'classical' viscosity remained obscured. In 1977, Nobel laureate Purcell published a note [14] where he expressed puzzlement at the fact that the viscosity of all liquids decreases with increasing temperature to almost the same value. Purcell noted that there must be some fundamental reason for this universality in the values of η_0 and ν_0. It has now become clear that the lower limit of viscosity is simply given by Planck's constant [6].

Sometimes, to describe the flow of fluids, the concept of elementary kinematic viscosity is introduced, which is simply equal to the viscosity times the mass of the molecule [5],

$$\iota = \nu M .$$

Elementary viscosity can also be defined as the product of dynamical viscosity and volume per molecule. The meaning of elementary viscosity lies in estimating the angular momentum per particle during fluid motion. We note that elementary viscosity has the same dimension as Planck's constant. From (10), we then have

$$\iota_{\min} \sim \hbar \left(\frac{M}{m}\right)^{1/2} . \tag{12}$$

The minimum possible value of elementary viscosity of all substances is, naturally, that of atomic hydrogen, $\iota_{\min} \sim \hbar (M_p/m)^{1/2} \approx 40\hbar$, where M_p is the proton mass. Figure 3 shows data on the elementary kinematic viscosity for molecular hydrogen.

We recall that, along with the fine structure constant, the dimensionless fundamental constant given by the ratio of the proton and electron masses is one of the most important fundamental constants that determine many processes in the Universe.

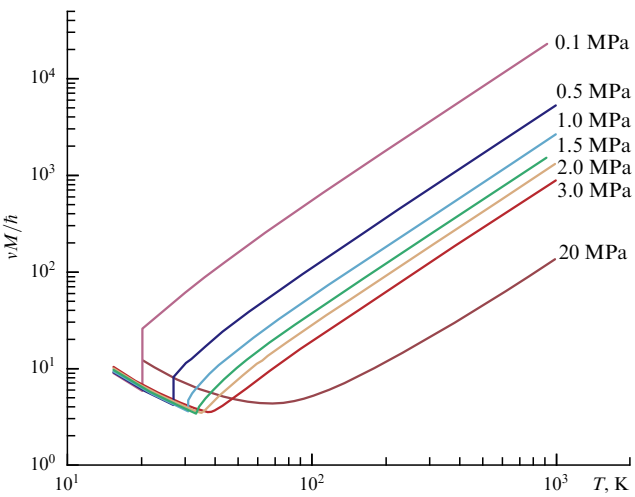

Figure 3. 'Elementary' kinematic viscosity for H_2 depending on temperature at pressures both below and above critical.

Previously, when analyzing the viscosity of a dense relativistic quark–gluon plasma in the strong-coupling limit in the superstring theory framework, an expression for the minimum dynamical viscosity divided by the bulk entropy density S was obtained in the form [15]

$$\frac{\eta}{S} = \frac{\hbar}{4\pi k_B} . \tag{13}$$

This expression bears a remarkable similarity with the expression for atomic kinematic viscosity (10) and elementary viscosity (12). In fact, for the elementary viscosity of the quark–gluon plasma, we can obtain $\iota_{\min} \sim \hbar$, the same as expression (12) but without atom-to-electron mass ratio. This is quite natural, because electromagnetic interaction in ordinary 'classical' fluids is effected by electrons, while the process of viscous flow involves atoms. In the case of quark–gluon plasma, the mass ratio drops out. We note that the dynamical viscosity of the quark–gluon plasma is huge in absolute value: 10^{12} Pa s (like that of glass!) [7]. At the same time, the kinematic viscosity, 10^{-7} m^2 s^{-1}, is practically the same as the minimum values for ordinary fluids [5, 7]; it follows that hadronic matter flows in the same way as low-viscosity 'classical' fluids (!) (see a discussion in [7]).

It was noted in a number of papers, including the first one [15], that the minimum value of the viscosity of hadronic matter can be estimated (of course, up to a numerical coefficient) just from the uncertainty relation. The same can be done in our case of 'classical' fluids. Indeed, the minimum elementary viscosity follows from (5) in the form $\iota_{\min} \sim mVa = pa$, where p is the particle momentum. From the uncertainty relation $\Delta p \Delta x > \hbar$, we then have $pa > \hbar$, $\iota_{\min} > \hbar$. To obtain the correct expression (12), we should take into account that the uncertainty in the particle energy is of the order of the energy of one phonon, $\Delta E = (m/M)^{1/2} E$ (see Eqn (7)). Similarly, the uncertainty in the particle momentum is $(M/m)^{1/2}$ times less than the momentum itself. In general, Planck's constant corresponds to the minimum possible quantum of action (angular momentum). Angular momentum is a quantity that most fully characterizes the dynamics. Therefore, the presence of $\hbar$ in formulas for the minimum possible transport coefficients for molecules and atoms is certainly not accidental. The transition to the classical limit corresponds to the Planck constant going to zero. In quantum mechanics, Planck's constant not only bounds the uncertainties of the characteristics of microobjects from below but also bounds the positive scalar quantities themselves. Thus, if the average value of the particle momentum (a vector) can be equal to zero, the kinetic energy associated with momentum squared cannot be less than a certain value (the Fermi energy for an electron in a metal). We note that, as the particle mass M increases to macroscopic values (macromolecules, colloids), the minimum kinematic viscosity formally tends to zero according to expression (10). But, due to the weak and short-range interparticle interaction in such systems, expression (10) is unlikely to be applicable to them.

3. Thermal conductivity and thermal diffusivity of condensed media

We now consider another physical characteristic, thermal conductivity. If viscosity actually characterizes the transfer of

momentum in a medium, heat conduction is about the transfer of energy. This implies a significant difference between these quantities. Fluidity requires the presence of motion of particles themselves, and therefore the effective viscosities of solids have astronomically enormous values. At the same time, heat transfer can occur both due to the motion of particles and due to the exchange of energy during vibrations, which can be conveniently described in the formalism of quasiparticles, i.e., phonons. The thermal conductivity of all media can then be described by the same expression, similar to the 'gaseous' expression for viscosity,

$$\lambda \sim CVL\,, \tag{14}$$

where λ is the thermal conductivity, C is the heat capacity per unit volume, and V and L are the speed of sound and the free path of phonons in the case of nonmetallic solids and viscous liquids, or the speed of particles and their free path in the case of gases and low-viscosity fluids. For liquids, thermal conductivity is determined by the sum of the contributions of phonons and particles. For metals in solid and liquid states, thermal conductivity is largely determined by the motion of free electrons. Physically, the thermal conductivity is simply the proportionality coefficient between the heat flux J and the temperature gradient in a steady state (the Fourier equation):

$$J = \lambda \frac{\partial T}{\partial x}\,.$$

As for the fluidity of fluids, where two characteristics are important, the dynamical and kinematic viscosity, describing heat conductance involves two equally important quantities, the thermal conductivity λ and the thermal diffusivity χ. The latter is equal to the thermal conductivity divided by the heat capacity per unit volume, and describes nonstationary processes of heat transfer (cooling and heating regimes in terms of the heat conduction equation in the absence of internal heat sources):

$$\frac{\partial T}{\partial t} = \chi \frac{\partial^2 T}{\partial x^2}\,.$$

The dependence of the thermal conductivity of solids on temperature and pressure is determined by the corresponding behavior of the heat capacity, the speed of sound, and the phonon free path. In the low-temperature region, thermal conductivity passes through a maximum associated with the competition between an increase in heat capacity and a decrease in the free path. The thermal conductivity of liquids is typically lower than the thermal conductivity of crystals due to shorter free paths. At pressures above critical, the thermal conductivity of all fluids as a function of temperature, similarly to viscosity, passes through a minimum [10] (Fig. 4). As in the case of viscosity, this minimum occurs simply because the shortest possible free path of both phonons and particles corresponds to the interparticle distance a. Below the Frenkel line in the region of a dense fluid, the speed of sound V and the phonon free path L decrease with increasing temperature; above the line in the quasigaseous region, the thermal velocity of particles and their free path increase with temperature. The speed of sound becomes comparable to the thermal speed of particles also near the Frenkel line [10]. The heat capacity of fluids decreases monotonically with increasing temperature and decreasing density from $3k_B$ per particle near the melting

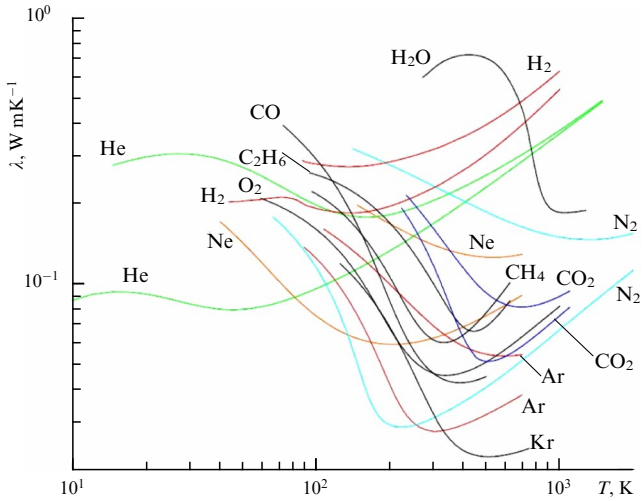

Figure 4. Experimental thermal conductivity of noble, molecular, and reticulate fluids (NIST database https://webbook.nist.gov/chemistry/fluid). Data for Kr, O_2, H_2O, CH_4, C_2H_6, and CO are shown for the respective pressures $P = 30$ MPa, 30 MPa, 70 MPa, 20 MPa, 20 MPa, and 20 MPa. For Ar, Ne, He, N_2, H_2, and CO_2, data are shown at two pressure values: 20 and 100 MPa for Ar, 50 and 300 MPa for Ne, 20 and 100 MPa for He, 10 and 500 MPa for N_2, 50 and 100 MPa for H_2, and 30 and 90 MPa for CO_2. Higher-pressure minima are higher than the lower-pressure minima for each fluid. Details are given in [8].

point to $(3/2)k_B$ in the 'gaseous' limit of high temperatures and/or low fluid densities. Near the Frenkel line, the heat capacity is close to $2k_B$ [10]. The minimum value of thermal conductivity is given by an expression similar to expression (6) for viscosity,

$$\lambda_{\min} \sim \frac{k_B \omega_0}{a}\,. \tag{15}$$

Using atomic units for the minimal thermal conductivity, we have

$$\lambda_{\min} \sim \frac{k_B m^{5/2} e^6}{M^{1/2} \hbar^5}\,, \tag{16}$$

where, along with the fundamental constants, the Boltzmann constant also appears. The value of $\lambda_{\min}$ in accordance with (16) is approximately 10 W (m K)$^{-1}$, which is 2 to 3 orders of magnitude higher than the actual values. As in the case of the 'atomic' dynamical viscosity, this is associated with an overestimated energy and an underestimated interparticle distance compared to real values.

We now consider thermal diffusivity. We can write

$$\chi \sim VL\,. \tag{17}$$

For the same reasons as for thermal conductivity, the thermal diffusivity of all fluids passes through a minimum (Fig. 5).

For the minimal thermal diffusivity, we have

$$\chi_{\min} \sim \omega_0 a^2\,. \tag{18}$$

When using atomic units, we obtain the same expression as for the kinematic viscosity (!):

$$\chi_{\min} \sim \frac{\hbar}{(Mm)^{1/2}}\,. \tag{19}$$

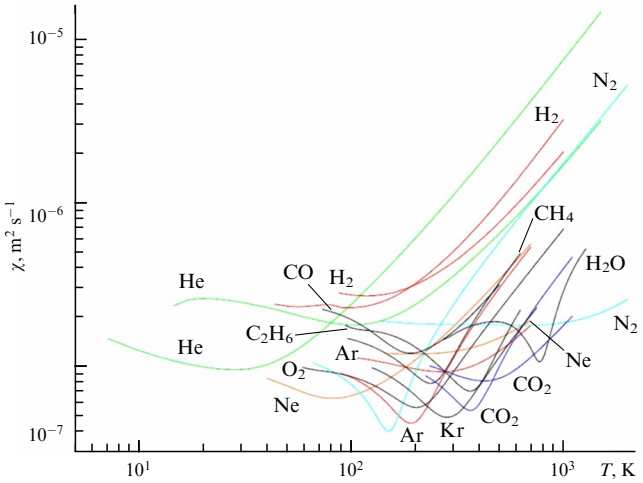

Figure 5. Experimental thermal diffusivity of fluids. Same fluids as in Fig. 4 are shown under the same conditions. Details are given in [8].

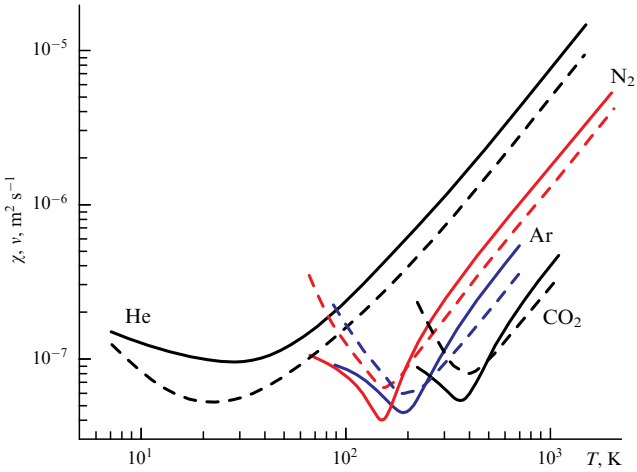

Figure 6. Experimental curves for thermal diffusivity (solid lines) and kinematic viscosity (dashed lines) for He (20 MPa), N_2 (10 MPa), Ar (20 MPa), and CO_2 (30 MPa) (NIST database https://webbook.nist.gov/chemistry/fluid). Details are given in [8].

In contrast to thermal conductivity, the numerical value of the minimum thermal diffusivity, 10^{-7} m^2 s^{-1}, is in very good agreement with the experimental values (see Fig. 5). The reason is the almost exact compensation of the overestimated value of ω_0 and the underestimated value of a in expression (18).

The coincidence of the minimal values of the 'atomic' kinematic viscosity and thermal diffusivity is an important and interesting theoretical result. The heat and momentum transfers near the Frenkel line are governed by the same laws. Moreover, the obtained result is in excellent agreement with experimental data (Fig. 6). The dimensionless ratio of ν and χ is called the Prandtl number. In the region of minimal values of ν and χ, this number ranges from 0.7 to 1.5 for different fluids. We note that the temperature dependence of ν and χ is the same in the quasigaseous region above the Frenkel line and entirely different in the low-temperature region (see Fig. 6), which is explained by different mechanisms of diffusive motion and the heat transfer process in a dense liquid [8].

We now consider the processes of heat transfer in metals. As already mentioned, the main carriers of heat in metallic solids and liquids are light and mobile electrons. As a result, the characteristic thermal conductivity of metals is 2 to 3 orders of magnitude higher than the thermal conductivity of dielectrics [12]. For a purely electron system, expression (17) should then involve the speed of electrons and their free path. The minimal electron free path is again of the order of the interparticle distance, while the electron speed is of the order of the Fermi speed. When using atomic units for the minimal thermal diffusivity of the electron system, we obtain another universal value,

$$\chi_{\min} \sim \frac{\hbar}{m} \approx 10^{-4}\ \mathrm{m^2\ s^{-1}}\,. \tag{20}$$

If we now consider the metal as a whole, including both electrons and ions, we can write the thermal diffusivity as $\chi = (\lambda_i + \lambda_e)/(C_i + C_e)$, where i and e refer to the ion and electron subsystems. Because the thermal conductivity of the ion subsystem due to phonons is considerably less (by 2 to 3 orders of magnitude) than due to electrons, while the heat capacity of the ion subsystem, on the contrary, is much greater (by 10 to 100 times) than that of the electron subsystem, we can write the minimal thermal diffusivity of a metal as approximately

$$\chi_{\min} \sim \frac{(C_e/C_i)\hbar}{m}\,. \tag{21}$$

This means that the expression for the minimal thermal diffusivity of metals does not include the mass of the particle, as in (19), but does include the ratio of the electron and ion heat capacity of the metal. At high temperatures near the critical temperatures for metals, this ratio is of the order of 0.1, and $\chi_{\min} \approx 10^{-5}$ m^2 s^{-1}. Interestingly, this simple formula predicts quite accurate values of the thermal diffusivity of liquid metals at very high temperatures (see Table).

Table. Experimental, $\chi^{\exp}$, and theoretical, $\chi^{\mathrm{th}}_{\min}$, minimal values of thermal diffusivity for several liquid metals. Data are also given for density ρ, thermal conductivity λ, total heat capacity C, and electron contribution to heat capacity C_e at the corresponding temperatures.

	T, K	λ, W (m K)$^{-1}$	ρ, kg m^{-3}	C, J (kg K)$^{-1}$	C_e, J (kg K)$^{-1}$	$\chi^{\exp}\times 10^5$, m^2 s^{-1}	$\chi^{\mathrm{th}}_{\min}\times 10^5$, m^2 s^{-1}
Hg	573	11.7	12,500	104	5.2	0.9	0.6
Li	812	30.6	480	4200	210.6	1.5	0.6
In	900	50	6500	250	13.1	3.1	0.6
Sn	773	32.5	6800	257	20.8	1.9	1.0
Sn	1300	—	—	—	—	1.7	1.5
Zr	773	25	5900	473	23.6	0.9	0.6
Zr	2000	—	—	—	—	1.4	1.4
Pb	1300	15	9900	130	19.6	1.2	1.8
Hf	2000	28.8	12,000	191	29.1	1.3	1.8

From (21), we can also obtain a simple expression for the minimum thermal conductivity of metals,

$$\lambda_{\min} \sim \frac{C_e \hbar}{m},$$

but this expression includes a quantity that is difficult to measure and calculate, the electron heat capacity per unit volume.

4. Absolute extrema of viscosity and thermal conductivity in the phase diagram

As we have noted, the kinematic and dynamical viscosities, the thermal conductivity, and the thermal diffusivity go through minima in the fluid state as the temperature increases. We now consider the question of the absolute minima of these quantities in the entire phase diagram of substances.

In absolute value, the minima of all four quantities (dynamical and kinematic viscosity, thermal conductivity, and thermal diffusivity) slowly increase as the pressure increases, which is associated with an increase in the Debye frequency and with the contribution of pressure to energy (chemical potential). Therefore, for each of the four quantities in the fluid region, there is an absolute minimum at the lowest pressures near the boiling line (on the liquid side) not far from the critical point. It can be assumed that the absolute minimum point is located at the intersection of the Frenkel line with the boiling line (Fig. 7). For the kinematic viscosity, interestingly, this is the absolute minimum, even if we consider all aggregation states of matter. The same is true for the thermal diffusivity, with the exception of the critical point. Indeed, in the gaseous phase for both χ and ν we have $\chi, \nu \sim VL$. If we move from the critical point to lower temperatures along the boiling line, and then from the triple point along the sublimation line (see Fig. 7), then the particle speed decreases as $T^{1/2}$, while the mean free path, being inversely proportional to pressure and density, increases exponentially. Thus, χ and ν increase as the temperature decreases along the boiling–sublimation lines. When moving away from the lines toward lower pressures and densities at a constant temperature, both quantities increase due to an increase in the mean free path, and when moving away from the lines toward higher temperatures at a constant pressure, they also increase due to an increase in the particle speed. Consequently, both quantities have a minimum near the lines (see Fig. 7).

In the solid state, the kinematic viscosity is very high and tends to infinite values. The thermal diffusivity, again, is $\chi \sim VL$, where the speed and free path relate to phonons. The speed of phonons (actually, the speed of sound) increases slightly as the temperature decreases and pressure and density increase, and the phonon free path increases indefinitely with decreasing temperature due to a decrease in phonon–phonon scattering (in real samples, the increase in phonon free path with decreasing temperature is limited by scattering on impurities or sample boundaries). This means that, in the entire region of the existence of a solid phase (crystal or glass), the thermal diffusivity is also higher than at the minimum point for the fluid. Therefore, χ and ν have an absolute minimum at a certain point on the fluid phase diagram for the entire (T, P) plane (see Fig. 7). An exception for the thermal diffusivity is given by the critical point. Many quantities exhibit critical behavior (diverge) in the immediate vicinity of the critical point. Thermal conductivity also increases without bounds near the critical point, but with a small exponent of divergence, while the heat capacity diverges faster [16]. As a result, the thermal diffusivity in the immediate vicinity of the critical point (fractions of a degree) starts decreasing and tends to zero. The viscosity also diverges in the immediate vicinity of the critical point [16]. Moreover, this applies to both dynamical and kinematic viscosity, because the density does not experience any divergence. Thus, the kinematic viscosity ν has a special point of absolute minimum in the entire (T, P) diagram, and the thermal diffusivity χ also has an absolute minimum point in the entire diagram, except in a very small vicinity of the critical point, which is difficult to attain experimentally. In the (T, P) phase diagram, interestingly, the point where the kinematic viscosity has the absolute minimum and the point where it goes to infinity are close to each other (see Fig. 7). As noted above, near the Frenkel line, the kinematic viscosity and thermal diffusivity are close in value (the Prandtl number is close to unity). In the immediate vicinity of the critical point, viscosity starts increasing and thermal diffusivity starts decreasing (the Prandtl number tends to infinity). It is obvious that, in the expressions $\nu, \chi \sim VL$, the effective free path L behaves totally differently near the viscosity and thermal diffusivity critical points. This is obviously due to giant density fluctuations, but a detailed analysis of the inapplicability of simple expressions like (17) for the near-critical region has not yet been carried out.

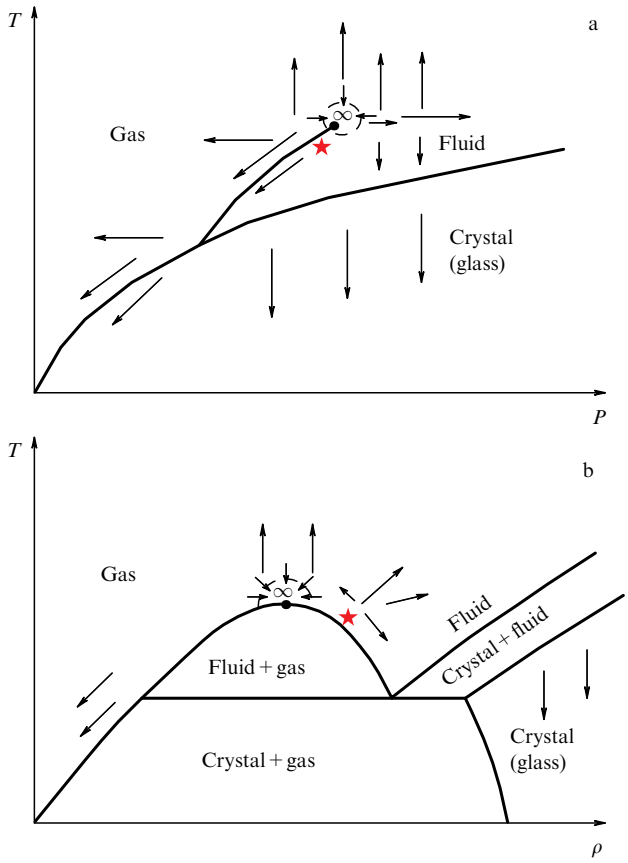

Figure 7. Generalized phase diagrams of substances in (a) pressure–temperature and (b) density–temperature coordinates. Red star shows the position of the absolute minimum of kinematic viscosity in the phase diagrams. Kinematic viscosity increases in the directions of all arrows. Vicinity of the critical point where the viscosity diverges is also indicated.

As regards the other two quantities, the dynamical viscosity and thermal conductivity, the positions of their absolute minima are related only to the region of liquid and fluid. We note that both quantities increase without bounds at the critical point. In the gaseous phase, the dynamical viscosity and thermal conductivity can have arbitrarily small values at low temperatures and densities. The same is true for the thermal conductivity in the solid phase, where it tends to zero due to a decrease in heat capacity. Recall that for real samples the increase in the phonon free path at low temperatures is saturated due to scattering on impurities or sample boundaries.

5. Elastic characteristics and mechanical properties of quasi-one-dimensional and quasi-two-dimensional systems

It was mentioned in the introduction that, when expressed in atomic units, the elastic moduli of substances correspond to the atomic unit of pressure $P \sim m^4 e^{10}/\hbar^8$, which is 2 to 3 orders of magnitude higher than the real moduli of substances. This is primarily due to the small value of the Bohr radius compared to the actual interparticle distances. It can be expected that, for low-dimensional systems, where the atomic unit of length does not enter to the third but to the second or first power, the difference between atomic units and real properties would be noticeably smaller.

We first consider the surface tension σ. In atomic units, we have

$$\sigma \sim \frac{E}{a^2} \sim \frac{m^3 e^8}{\hbar^6}\,, \tag{22}$$

which is approximately 10^3 N m^{-1}.

At first glance, this is a very large value. For example, the surface tension at the water–air boundary is 0.07 N m^{-1}, and at the liquid mercury–air boundary, 0.5 N m^{-1}. Expression (22) actually gives an estimate for the maximum possible elastic moduli in two-dimensional systems. Among the known two-dimensional systems, graphene is the strongest. Direct measurements [17] show that the tensile strength of single-layer graphene reaches 55 N m^{-1}, and the corresponding two-dimensional Young's modulus is 340 N m^{-1}. Consequently, the graphene moduli and ultimate strength are only several times less than the maximum surface tension given by (22).

We can also consider the elastic moduli of one-dimensional systems or the rupture force F. In atomic units, we have

$$F \sim \frac{E}{a} \sim \frac{m^2 e^6}{\hbar^4}\,, \tag{23}$$

which is approximately 40 nN. Among the quasi-one-dimensional atomic chains, carbon systems — carbynes — have also been studied. For carbine chains, direct studies of tensile strength give values of 8–12 nN [18, 19]. This means that expression (23) for the maximum elastic moduli in one-dimensional systems gives a good estimate for the maximum force required to break an atomic chain.

6. Maximum speed of sound in condensed matter

The atomic units of elastic moduli of three-dimensional systems are greatly overestimated compared to the real values. However, if we consider the speed of sound, then the use of atomic quantities gives entirely correct values. Indeed, the speed of sound V can be calculated from the elastic moduli as

$$V = \left(\frac{B}{\rho}\right)^{1/2}, \tag{24}$$

where B is the corresponding modulus (the longitudinal modulus for the longitudinal speed of sound, and shear modulus for the transverse speed of sound).

On the other hand, the speed of sound can be estimated as

$$V = \omega_0 a\,, \tag{25}$$

where the Debye frequency ω_0 can be calculated in accordance with expression (7). In atomic units, expressions (24) and (25) lead to the same result:

$$V = \frac{e^2}{\hbar}\left(\frac{m}{2M}\right)^{1/2}. \tag{26}$$

Noting that $e^2/\hbar = c\alpha$, where α is the fine structure constant, we can rewrite (25) as

$$\frac{V}{c} = \alpha\left(\frac{m}{2M}\right)^{1/2}. \tag{27}$$

Thus, in atomic units, the speed of sound is determined only in terms of the speed of light, the fine structure constant, and the electron-to-atom mass ratio. Despite its simplicity, expression (27) is in reasonable agreement with experimental data for the longitudinal speed of sound of many elementary substances (the ratio of experimental values to theoretical ones in the range of 0.6–2.4). But when viewed 'on average,' the agreement is quite impressive (Fig. 8). Both the averaged values of the speed of sound and the proportionality to the

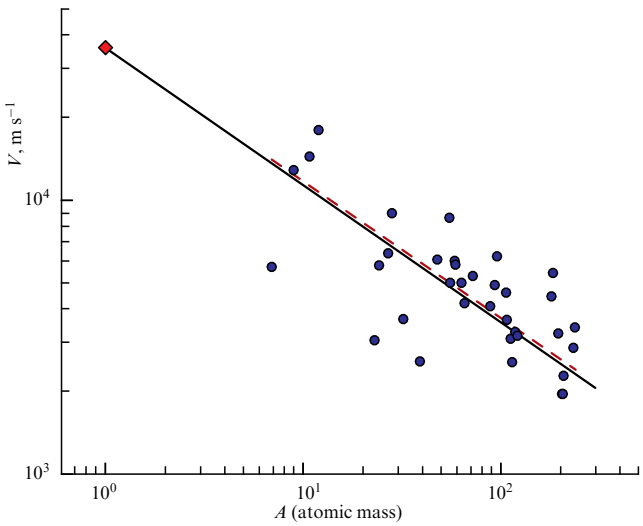

Figure 8. Experimental longitudinal speed of sound in 36 elementary solids (blue dots) [12], depending on the atomic mass. Solid line plots Eqn (27): $V = c\alpha(m/2M)^{1/2}$. Red diamond shows the upper bound for the speed of sound (28). Dashed line corresponds to the interpolation of all experimental data by the least mean squares to a function $\sim M^{-1/2}$. In the order of increasing mass, the following elemental solids are listed: Li, Be, B, C, Na, Mg, Al, Si, S, K, Ti, Mn, Fe, Ni, Co, Cu, Zn, Ge, Y, Nb, Mo, Pd, Ag, Cd, In, Sn, Sb, Ta, W, Pt, Au, Tl, Pb, Bi, Th, and U.

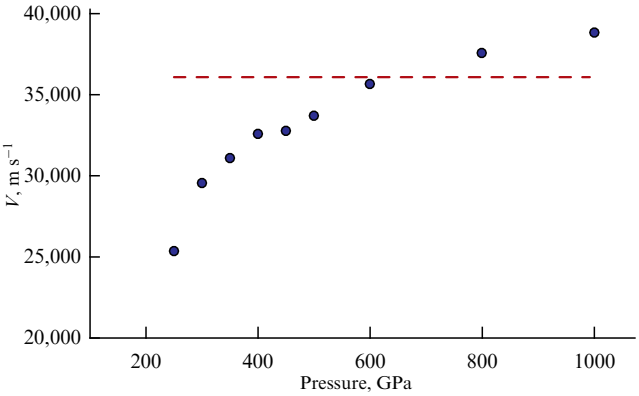

Figure 9. Calculated longitudinal speed of sound of metallic atomic hydrogen depending on pressure. Horizontal dashed line indicates the upper bound for the possible speed of sound at normal pressure in accordance with Eqn (28).

inverse square root of the elemental mass are in good agreement.

The smallest possible atomic mass corresponds to the hydrogen atom. In this case, the maximum 'atomic' speed is attained,

$$V_0 = c\alpha\left(\frac{m}{2M_{\mathrm{p}}}\right)^{1/2} = 36.1\ \mathrm{km\ s^{-1}}\,. \tag{28}$$

This maximum speed is determined only by fundamental constants: the speed of light, the fine structure constant, and the electron-to-proton mass ratio, the last two being the most important dimensionless constants in physics.

Atomic metallic hydrogen is unstable at normal pressure. *Ab initio* calculations of the speed of sound were carried out for a hypothetical phase of metallic hydrogen (structure $I4_1/amd$) in a wide range of pressures (see details in [9]). At such high pressures, the PV contribution to the thermodynamic potential and binding energy of a substance is very large, and it is not entirely legitimate to use just the atomic energy value for estimates. Nevertheless, agreement between the calculated speed of sound and the theoretical 'atomic' limit is quite good (Fig. 9).

Both factors that reduce the maximum speed of sound compared to the speed of light are associated with the electromagnetic nature of interaction in condensed matter. The fine structure constant characterizes just the strength of electromagnetic interaction. The electron-to-proton mass ratio also occurs naturally because the interaction is effected by electrons and the sound wave propagates through the system of ions (protons). It is interesting that the speed of sound in hadronic matter, where both small factors are absent, can be comparable to the speed of light, $V = c/\sqrt{3}$ [20].

7. 'Quantum' values of 'classical' quantities and the anthropic principle

It is known that many fundamental constants cannot be varied to any significant degree without evoking fundamental changes in physical processes in the Universe, including the existence of visible matter, the synthesis of elements inside stars, and the origin of life [1]. The most important dimensionless fundamental constants, the ratio of the proton and electron masses and the fine structure constant, must also lie within very narrow limits, both individually and in relation to each other. These constants determine the thermonuclear fusion of elements, primarily carbon, an important element for life [1]. Without Hoyle's 'carbon resonance,' as well as without 'oxygen resonance,' there would have been practically no elements heavier than hydrogen and helium in the Universe. Often, in connection with the very fine tuning of fundamental constants that is necessary for the appearance of elements and the origin of life, the anthropic principle is invoked [1]. The results obtained on the relation of viscosity and thermal conductivity to fundamental constants are also important for the viability of many biological processes. For example, water and water-based liquids are of paramount importance for all organisms. The expressions for the minimum viscosity leave a fairly narrow window for possible changes in Planck's constant. The same is true for the thermal conductivity of biological objects. Similarly, the effect of the electron mass on the minimum viscosity and thermal conductivity also defines a rather narrow window for its possible value. Of course, the constraints on fundamental constants and their combinations in high-energy physics are much more stringent than those discussed above, the windows allowed for variation not exceeding a few percent. At the same time, stringent constraints in high-energy physics are imposed only on certain combinations of constants. The constraints associated with minimum viscosity and thermal conductivity impose new bounds, in particular, on the value of the ratio $\hbar/m^{1/2}$, which is also important for the anthropic principle.

8. Conclusions

For a number of macroscopic quantities, we have briefly reviewed recent estimates of their extreme values derived from fundamental constants. For 'kinetic' quantities such as the dynamical and kinematic viscosities, thermal conductivity, and thermal diffusivity, such estimates have apparently been made for the first time. The use of the atomic scale to analyze the properties of condensed matter has proven to be very fruitful. For quantities as such the kinematic viscosity and thermal diffusivity, the presence of the minimum possible value has been established, determined only by the Planck constant, the electron mass, and the mass of the particle (atom, ion, or molecule). Moreover, the resulting 'universal' expression for the minima of these quantities gives not only the correct order of magnitude but also good quantitative agreement with experimental values. The same coincidence is observed for the speed of sound estimated from the fundamental constants. The observed coincidences appear to be a consequence of a nonaccidental compensation of various factors that would make our simplified analysis more intricate. A nontrivial fact is the presence of the absolute minimum point at finite temperatures and pressures for the kinematic viscosity in the phase diagram of any substance. It is also interesting to note the role of Planck's constant, special among all fundamental constants, as the minimum quantum of angular momentum, which determines the minimum possible viscosity and thermal diffusivity. The root cause of the minimum values of these macroscopic quantities is the apparently 'microscopic' Heisenberg uncertainty relation. This is unsurprising: the reason we do not fall through the floor also has to do with the Heisenberg uncertainty relation. Quantum mechanics is often closer than it appears.

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