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# Laminar submerged jets of incompressible fluid at large Reynolds numbers

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## Contents



**Abstract.** **Fundamental theoretical studies on laminar axisymmetric submerged jets are considered. The problems associated with jets with a flow rate through the initial cross section and with swirling jets are investigated. Some erroneous results of the theory of laminar jets have been discovered and corrected.**

**Keywords:** submerged jet, laminar jet, swirling jet, invariants, momentum flux

## 1. Introduction

The theory of laminar jets in incompressible fluids has been developed in numerous papers. Among the researchers who have made fundamental contributions to the development of this theory should be mentioned G Schlichting, L D Landau, N A Slezkin, V I Yatseev, M A Gol'dshtik, N I Yavorsky, L G Loytsyansky, Yu B Rumer, J Squire, and many others.

This study has two aims. The first is to collect fundamental papers on the theory of laminar axisymmetric jets. The second aim is perhaps more important than the first. It concerns the following point: although many well-known fluid dynamicists have contributed to the creation of the theory of jets, this branch of fluid dynamics, perhaps more than any other, abounds with erroneous statements, many of which, over time, are taken as truths by the scientific community. This situation is aggravated by the fact that errors on the subject permeate well-known monographs and widely cited papers. Exposing such erroneous results and replacing them with rigorous ones is the main objective of this paper.

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## 2. Axisymmetric nonswirling jet

### 2.1 Solution for a jet in the boundary layer approximation

It is well known that a stationary flow of viscous, incompressible fluid satisfies the Navier–Stokes equations

$$(\mathbf{u}\nabla)\mathbf{u} + \frac{1}{\rho}\,\nabla p = \nu\Delta\mathbf{u}\,, \quad \operatorname{div}\mathbf{u} = 0\,, \tag{1}$$

where $\mathbf{u}$ is the fluid velocity vector, $p$ is the pressure, $\rho$ is the density, and $\nu$ is the kinematic viscosity coefficient. However, the first analytical solutions for jets were obtained in the boundary layer approximation for large Reynolds numbers Re, and only then their analogs were found in the framework of the Navier–Stokes equations. In 1933, Schlichting published a short paper in the prestigious journal *ZAMM* (*Zeitschrift für Angewandte Mathematik und Mechanik*) — only four pages long, in which he gave a solution for an axisymmetric jet and sketched the solution for a plane jet [1] (see also [2]). A final solution for the plane jet was given by W Bickley [3].

In the axisymmetric case, one considers a jet emerging from a circular orifice into a domain filled with the same fluid. A thin jet is studied. Its longitudinal velocity component is much larger than the transverse one, but the longitudinal velocity gradients are much smaller than the transverse ones. The solution is sought in the boundary layer approximation, assuming that the flow in a thin jet can be described in the leading approximation of equations (1) expanded at large Reynolds numbers. In the framework of this leading approximation, the contribution with the second derivative along the longitudinal direction is omitted in terms that depend on the viscosity in the longitudinal projection of the equations; the transverse component of the motion equation for a nonswirling jet reduces to the equivalence of the transverse pressure gradient to zero. The pressure change in the longitudinal direction is defined by the pressure change outside the boundary layer [4]. As the jet is thin, the gradient outside it is zero in the leading approximation. The pressure in the jet is then constant, and the boundary layer equations take the form

$$u_x \frac{\partial u_x}{\partial x} + u_r \frac{\partial u_x}{\partial r} = \frac{v}{r} \frac{\partial}{\partial r}\left( r \frac{\partial u_x}{\partial r}\right), \tag{2}$$

$$\frac{\partial (u_x r)}{\partial x} + \frac{\partial (u_r r)}{\partial r} = 0, \tag{3}$$

with boundary conditions

$$u_r = 0, \quad \frac{\partial u_x}{\partial r} = 0 \quad (r = 0), \quad u_x \to 0 \quad (r \to \infty). \tag{4}$$

Here, $x$, $r$ are the coordinates of the cylindrical reference frame, $u_x$, $u_r$ are the components of velocity in this frame, the $x$-axis is directed along the jet, and the point $x = 0$ coincides with the jet origin point. Equation (2) with account for (3) can be rewritten as

$$\frac{\partial (u_x^2 r)}{\partial x} + \frac{\partial (u_x u_r r)}{\partial r} = v \frac{\partial}{\partial r}\left( r \frac{\partial u_x}{\partial r}\right),$$

which leads to the momentum flux conservation in any transverse cross section of the jet

$$J = 2\pi\rho \int_0^{\infty} u_x^2 r \, \mathrm{d}r = \text{const}. \tag{5}$$

If we assume that the jet originates from a point, then, because of the equal dimensions of $\sqrt{J/\rho}$ and the kinematic viscosity coefficient, and also because of the lack of a characteristic linear scale in this problem, the problem solution can be sought in a self-similar form. The index of self-similarity is determined from the condition of momentum flux conservation (5)

$$u_x = \frac{v}{x} \frac{F'}{\eta}, \quad u_r = \frac{v}{x}\left( F' - \frac{F}{\eta}\right), \quad \eta = \frac{r}{x}.$$

The function $F(\eta)$ is related to the stream function $\psi$ by the relation $F = \psi / vx$, $(\ )' \equiv \partial/\partial\eta$, where

$$u_x = \frac{1}{r} \frac{\partial \psi}{\partial r}, \quad u_r = -\frac{1}{r} \frac{\partial \psi}{\partial x}.$$

From (2), it follows that

$$-\left( \frac{FF'}{\eta}\right)' = \left( F'' - \frac{F'}{\eta}\right)'. \tag{6}$$

Taking into account the boundary conditions (4) and the expression for the momentum flux (5), the problem solution becomes

$$F = \frac{\xi^2}{1 + \xi^2/4}, \quad u_x = \frac{3}{8\pi} \frac{K}{vx} \frac{1}{(1 + \xi^2/4)^2},$$
$$u_r = \frac{1}{4x} \sqrt{\frac{3K}{\pi}} \frac{\xi - \xi^3/4}{(1 + \xi^2/4)^2}, \tag{7}$$
$$\xi = \sqrt{\frac{3K}{16\pi}} \frac{1}{v} \eta = \sqrt{\frac{3K}{16\pi}} \frac{r}{vx}, \quad K = \frac{J}{\rho}.$$

Mass flow rate per unit time in the jet section

$$Q(x) = 2\pi\rho \int_0^{\infty} u_x r \, \mathrm{d}r = 8\pi\rho v x. \tag{8}$$

The jets can carry different momentum fluxes which, as follows from (7), define all jet characteristics except for the flow rate.

The jet width increases linearly with $x$. The flow rate (8) across the jet cross section increases accordingly with distance from the initial cross section. This means that the jet entrains the surrounding fluid. This also implies that solution (7), (8) cannot be realized in a bounded domain if its size is comparable to the transverse size of the jet, for example, when the jet is generated in thin channels.

At the initial cross section, although the longitudinal velocity is infinite, the flow rate is equal to zero. Thus, the Schlichting jet is characterized only by the momentum flux (5) for a zero initial flow rate.

Note that the radial velocity is positive for small $\xi$. This is because the part of the jet close to the axis is slowed down as $x$ increases. Beginning from some $\xi$ ($\xi = 2$), the radial velocity becomes negative. The jet, entraining, accelerates the fluid in the direction of the $x$-axis.

### 2.2 Solution for jets in the framework of Navier–Stokes equations. Landau jet

The publication of [1] was followed one year later by [5] by N A Slezkin (see also [6]), where results for nonswirling axisymmetric flows were obtained, namely, the conical axisymmetric solutions of the Navier–Stokes equations were found. In this article, the full Navier–Stokes equations are used for an incompressible fluid in spherical coordinates $R, \theta, \varphi$. The velocity components are $u_R$ and $u_\theta$. The third velocity component is zero. In the spherical reference frame, the stream function is introduced as follows:

$$u_R = \frac{1}{R^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{R \sin\theta} \frac{\partial \psi}{\partial R}. \tag{9}$$

A solution of the Navier–Stokes equation is sought for the case where the stream function can be written as

$$\psi = R f_S(\tau), \tag{10}$$

where $\tau = \cos\theta$.

By inserting (10) into the Navier–Stokes equations and eliminating the pressure in the standard way, it can be found that the function $f_S(\tau)$ satisfies the equation

$$f_S f_S''' + 3 f_S' f_S'' - v\left[ f_S^{IV}(1 - \tau^2) - 4 f_S''' \tau\right] = 0. \tag{11}$$

Equation (11) is equivalent to the following equation:

$$\left[ \frac{f_S^2}{2} - v(1 - \tau^2) f_S' - 2v\tau f_S\right]''' = 0,$$

or, equivalently, to the Riccati equation [7]

$$\frac{f_S^2}{2} - v(1 - \tau^2) f_S' - 2v\tau f_S = C_0 + C_1\tau + C_2\tau^2. \tag{12}$$

It took another nine years 'to turn' a special case of equation (12) into an equation describing a submerged, axisymmetric, nonswirling jet in an incompressible fluid within the framework of the Navier–Stokes equations. Such a jet in spherical coordinates was considered by L D Landau [8] (see also [4]). The jet satisfies the condition that through any closed surface around the origin of the coordinates flows the same total momentum flux as that injected by the jet source into the surrounding fluid. This condition determines the velocity decay law. Similar to (9), (10), the velocities are written in the form

$$u_R = \frac{1}{R}\, F(\theta)\,, \quad u_\theta = \frac{1}{R}\, f_L(\theta)\,. \tag{13}$$

The steady-state Navier–Stokes equations for an incompressible fluid (1) can be rewritten in a Cartesian reference frame as

$$\frac{\partial u_i}{\partial x_i} = 0\,, \quad \frac{\partial \Pi_{ij}}{\partial x_i} = 0\,, \quad \Pi_{ij} = \rho u_i u_j + p\delta_{ij} - \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Similarly, the components of the momentum density flux tensor can be written in a spherical reference frame. From the condition of axial symmetry and the absence of swirl, the components of the momentum flux density $\Pi_{R\varphi}$ and $\Pi_{\theta\varphi}$ are equal to zero. Further, Landau selects the 'physically correct' solution from the variety of solutions, as noted in Refs [9, 10]. He assumed that the components of the tensor $\Pi_{\varphi\varphi} = p - 2\mu(u_R + u_\theta \cot\theta)/R$ and $\Pi_{\theta\theta} = p + \rho u_\theta^2 - 2\mu(u_R + \partial u_\theta/\partial\theta)/R$ are also zero, and, consequently, that the component $\Pi_{R\theta}$ is zero, since in this case

$$\sin^2\theta\, \Pi_{R\theta} = \frac{1}{2}\frac{\partial}{\partial\theta}\left[\sin^2\theta(\Pi_{\varphi\varphi} - \Pi_{\theta\theta})\right].$$

Such a choice allows the boundary conditions on the jet axis to be satisfied: $u_\theta = 0$, and $u_R$ is bounded, or, in other words, the velocity field is regular everywhere except at the jet source point. Only the tensor component $\Pi_{RR}$ remains nonzero. We have

$$\frac{1}{\rho}(\Pi_{\theta\theta} - \Pi_{\varphi\varphi}) = \frac{1}{R^2}\left(f_L^2 + 2\nu f_L \cot\theta - 2\nu\,\frac{\partial f_L}{\partial\theta}\right).$$

Hence,

$$f_L^2 + 2\nu f_L \cot\theta - 2\nu\,\frac{\partial f_L}{\partial\theta} = 0\,. \tag{14}$$

Comparing (13) with (9), (10) and (12) with (14), we find that the function $f_S = -f_L \sin\theta$ in this case satisfies equation (12) with constants $C_0 = C_1 = C_2 = 0$ as before. Later, V I Yatseev proved that solution (12) is regular everywhere except for the jet source point only when all constants are zero [9]. Equation (14) is of the first order; its solution

$$f_L = -\frac{2\nu\sin\theta}{A - \cos\theta} = -\frac{2\nu\sqrt{1-\tau^2}}{A - \tau} \tag{15}$$

depends on only one constant $A$. Other jet characteristics will naturally depend on this constant. In this case,

$$F = 2\nu\left[\frac{A^2 - 1}{(A - \tau)^2} - 1\right]. \tag{16}$$

From the above relationships, it follows that the velocity field will be regular for all angles $\theta$ only if $A > 1$ (jets flowing in the direction of the semi-axis $\theta = 0$ are considered).

The pressure variation is quadratic in the viscosity coefficient and has a singularity when approaching the jet source:

$$p = p_\infty + \frac{4\rho\nu^2}{R^2}\,\frac{A\tau - 1}{(A - \tau)^2}\,. \tag{17}$$

The jet momentum flux

$$K = 16\pi\nu^2 A\left[1 + \frac{4}{3(A^2 - 1)} - \frac{A}{2}\ln\frac{A+1}{A-1}\right]. \tag{18}$$

In 1951, H B Squire [11] obtained an equation which coincides with (12), carried out its analysis, and then found a solution which coincides with that in [8]. Unfortunately, he does not cite Slezkin, Landau, or Yatseev in his work. Ref. [11] also considers the evolution of a heat source placed at the jet source on the background of the solution obtained for the velocity field.

### 2.3 Correspondence between solutions

Let us now find a condition for which the solution (13), (15), and (16) of the Navier–Stokes equations coincides with solution (7) of the boundary-layer equations in the leading approximation. This condition will be naturally satisfied if the jet is sufficiently thin, i.e., if the region adjacent to its axis, where the longitudinal velocity is of the order of the maximum value at the axis, has a radial size which is much smaller than the longitudinal one. According to (7), this will happen if $\sqrt{K} \gg \nu$ or if the characteristic number $\mathrm{Re} = \sqrt{K}/\nu \gg 1$. From (18), it follows that this is only possible if $A$ is close to 1. The behavior of streamlines and their change when $A$ tends to 1 can be found in the well-known monograph [12]. We write $A = 1 + \delta$, where $\delta \ll 1$. Then,

$$K = 16\pi\nu^2\left(\frac{2}{3}\,\delta^{-1} + \frac{1}{2}\ln\delta + O(1)\right),$$

or, in the leading approximation (we neglect $\ln\delta$ versus $\delta^{-1}$),

$$\delta = \frac{32\pi}{3}\,\frac{\nu^2}{K}\,. \tag{19}$$

The velocity components (7) in the cylindrical reference frame are related to components (13) in spherical coordinates by the relations $u_x = u_R\cos\theta - u_\theta\sin\theta$, $u_r = u_R\sin\theta + u_\theta\cos\theta$. In this case, in the section $x = \mathrm{const}$, we have $x = R\cos\theta$, $r = R\sin\theta$. In the cylindrical reference frame, the solution of the Navier–Stokes equation is written in the form

$$u_x = \frac{2\nu\cos\theta}{x}\left[\frac{(A^2 - 1)\cos\theta}{(A - \cos\theta)^2} + \frac{1 - A\cos\theta}{A - \cos\theta}\right], \tag{20}$$

$$u_r = \frac{\nu\sin(2\theta)}{x}\,\frac{A\cos\theta - 1}{(A - \cos\theta)^2}\,. \tag{21}$$

Inserting $A = 1 + \delta$, $\cos\theta \approx 1 - \theta^2/2$ into the last relation and taking into account (19), we obtain in the leading approximation ($\delta \to 0$, $\theta \to 0$) the relations for $u_x$, $u_r$ which coincide with those in formula (7).

It might seem that the coincidence of velocities in the leading approximation in the main part of the jet ensures also the coincidence of other quantities, but this is not the case in reality. The flow rate paradox is mentioned in Refs [13–15].

Flow rate through a section calculated for the Schlichting jet is given by formula (8). If one calculates the flow rate through a sphere surrounding the Landau jet, then naturally a zero discharge is found. But if one calculates the flow rate through the section $x = \text{const}$, it will be found that it is infinite:

$$Q(x) = 4\pi\rho\nu\left[\frac{\sqrt{r^2+x^2}}{A} - \frac{A^2-1}{A^2}\,\frac{x^2}{A\sqrt{r^2+x^2}-x}\right]_{r=0}^{r=\infty}. \tag{22}$$

The point here is that in a Schlichting jet the flow rate is calculated only through a narrow part of the jet where the boundary layer equations are valid, and the outer part, described by Euler's equations, is ignored. To obtain a solution in the outer part of the jet, we turn to formula (7). In the limit $\xi \to \infty$ the radial velocity component

$$u_r \to -\sqrt{\frac{3K}{\pi}}\,\frac{1}{x\xi} = -\frac{4\nu}{r}\,.$$

According to the method of matching asymptotic expansions [16], the flow in the outer domain is generated by the sources located at the axis $r = 0$. Their intensity is obtained from the condition $u_r = -4\nu/r = q/2\pi r$, which gives $q = -8\pi\nu$. Thus, there is a ray $0 \leqslant x < \infty$, $r = 0$, each element $\mathrm{d}x$ of which corresponds to a three-dimensional source with intensity $q\,\mathrm{d}x$. Such a ray of sources gives birth to the axial and radial velocity components

$$u_x(x,r) = -\int_0^\infty \frac{x_1 - x}{4\pi\left[(x_1-x)^2+r^2\right]^{3/2}}\,q\,\mathrm{d}x_1 = \frac{2\nu}{\sqrt{x^2+r^2}}\,, \tag{23}$$

$$u_r(x,r) = \int_0^\infty \frac{r}{4\pi\left[(x_1-x)^2+r^2\right]^{3/2}}\,q\,\mathrm{d}x_1 = -\frac{2\nu}{r}\left(1+\frac{x}{\sqrt{x^2+r^2}}\right). \tag{24}$$

Calculating the flow rate using formula (23), we find

$$Q(x) = 4\pi\rho\nu\sqrt{r^2+x^2}\,\Big|_{r=0}^{r=\infty}. \tag{25}$$

Comparing (25) with (22), and also (23) with (20) and (24) with (21), we conclude that, first, the Schlichting jet gives an infinite flow rate if treated rigorously, and, second, agreement between the characteristics of the Landau jet and the outer flow in the Schlichting jet occurs if one sets $A = 1$ [15]. Although the longitudinal velocity component in the outer inviscid domain is much smaller than in the inner domain, which corresponds to the boundary layer, it still ensures an infinite flow rate.

### 2.4 Broman and Rudenko jet

In 2010, *Physics–Uspekhi* published a paper by G I Broman and O V Rudenko [10], which, first, gave a brief review of the creation of the theory of laminar nonswirling submerged jets, second, noted for the first time the lack of an inviscid limit in the Landau solution, and, third, presented another solution of equations (12) for $C_0 = C_2 = D^2/2$, $C_1 = -D^2$, which generalized the Landau solution [8]

$$f = f_S = -\nu(1-\tau)\left[1+\gamma\,\frac{(1+\tau)^\gamma + G}{(1+\tau)^\gamma - G}\right],$$

where $D$ and $G$ are some constants,

$$\gamma = \sqrt{1+\frac{D^2}{\nu^2}}\,.$$

So, everything is determined by the ratio $D/\nu$, i.e., the quantity $\gamma$. For $D \ll \nu$, we have $\gamma \approx 1$ and

$$f = 2\nu\,\frac{1-\tau^2}{G-1-\tau}\,,$$

which for $G = A + 1$ corresponds to the Landau solution (15). In addition to a submerged jet, another application of this solution was suggested in [10]: "The fact that the Landau jet imparts momentum to the medium without imparting mass allows it to be used to describe streaming caused by sound."

In the case $\nu \to 0$, $\gamma \to \infty$, one finds another limiting solution from which it follows that, in the cylindrical reference frame,

$$u_x = -\frac{D}{x\sqrt{1+\eta^2}}\,, \quad u_r = -\frac{D\left(\sqrt{1+\eta^2}-1\right)}{x\eta\sqrt{1+\eta^2}}\,,$$
$$p = p_\infty - \frac{D^2\rho\left(\sqrt{1+\eta^2}-1\right)}{x^2\eta^2\sqrt{1+\eta^2}}\,, \quad \eta = \frac{r}{x}\,. \tag{26}$$

The authors of Ref. [10] interpret solution (26) as a solution for a jet in an ideal fluid. This solution satisfies Euler's equations and the condition that the radial velocity component be zero and that the axial velocity component be bounded at the jet axis, and also that the velocities decay for $\eta \to \infty$. These conditions explain the choice of the constants $C_0$, $C_1$, and $C_2$ at the beginning of this section.

The equation for a streamline,

$$\frac{\mathrm{d}r}{u_r} = \frac{\mathrm{d}x}{u_x}\,,$$

determines their form,

$$r = r_0\sqrt{1+\frac{2}{r_0}\,x}\,, \tag{27}$$

where $r_0$ is the radial coordinate of the streamline in the section $x = 0$.

The velocity field can be considered either in the whole domain or in the one bounded by a streamline (27). As the fluid is inviscid, any streamline can be replaced by a rigid surface. In this case, the streamline corresponds to a flow in widening or narrowing channels, depending on the sign of $D$.

The velocity field (26) has zero vorticity and is described by the potential

$$\varphi = -D\ln\left|x+\sqrt{x^2+r^2}\,\right|.$$

Consequently, the fields of velocity and pressure (26) satisfy not only the Euler's equations but also the Navier–Stokes equations.

## 3. Axisymmetric jet with a flow rate through the initial cross section

### 3.1 Rumer, Gol'dshtik–Yavorsky, and Loytsyansky solutions

As shown in the previous section, it is possible to find solutions for jets flowing from a point source of momentum.

Such sources are certainly absent, and jets leave orifices of a finite diameter with a finite velocity, i.e., there is a flow rate through the initial jet cross section. An example of such a case is a jet flowing from a submerged cylindrical tube. Here, it is essential that the momentum flux also be conserved in the problem with a nonzero flow rate through the initial cross section. One more invariant in this case is the fluid flow rate through any closed surface containing the source.

The first attempt to obtain a solution of the Navier–Stokes equations for the far field of an axisymmetric jet with nonzero flow rate was made by Yu B Rumer [17]. In this case, a not-self-similar solution was sought in a spherical reference frame in the form of an expansion in integer powers of $1/R$ as a small correction to the Landau solution (13), (15)–(17):

$$u_R = \frac{F(\tau)}{R} + \frac{F_2(\tau)}{R^2} + O\left(\frac{1}{R^3}\right), \quad u_\theta = \frac{f_L(\tau)}{R} + O\left(\frac{1}{R^3}\right),$$
$$p = p_\infty + \frac{4\rho\nu^2}{R^2}\,\frac{A\tau-1}{(A-\tau)^2} + \frac{\rho g_2(\tau)}{R^3} + O\left(\frac{1}{R^4}\right). \tag{28}$$

Solution (28) has a flow rate across the sphere of radius $R$,

$$Q = \rho\int_0^\pi u_R R^2 2\pi\sin\theta\,\mathrm{d}\theta = 2\pi\rho\int_{-1}^1 F_2\,\mathrm{d}\tau\,,$$

which equals the flow rate across the initial jet cross section.

Inserting (28) into the Navier–Stokes equations (1), we find that the functions to be determined satisfy the system of equations

$$-\sqrt{1-\tau^2}\,f_L F_2' - 3FF_2 = 3g_2 + \nu\left[(1-\tau^2)F_2'' - 2\tau F_2'\right],$$
$$g_2' = 2\nu F_2'\,. \tag{29}$$

The solution of the second equation in system (29) can be written as

$$g_2 = 2\nu\left(F_2 - \frac{1}{6}\,C_1\right),$$

where $C_1$ is a constant. The remaining first equation of system (29) has the form ($L$ is the differential operator)

$$LF_2 = (1-\tau^2)F_2'' - 2\left(\tau + \frac{1-\tau^2}{A-\tau}\right)F_2' + 6\,\frac{A^2-1}{(A-\tau)^2}\,F_2 = C_1\,. \tag{30}$$

If we multiply equation (30) by $(A-\tau)^2$ and integrate over $\tau$ from $-1$ to 1, we get

$$C_1 = \frac{9}{2\pi\rho}\,\frac{A^2-1}{3A^2+1}\,Q\,.$$

Thus, if the flow rate across the exit cross section differs from zero, $Q \neq 0$, and also $C_1 \neq 0$.

Reference [17] gives one of the solutions of homogeneous equation (30), which has no singularities in the range $-1 \leqslant \tau \leqslant 1$:

$$h(\tau) = 1 - \frac{3(A^2-1)}{(A-\tau)^2} + \frac{2(A^2-1)^2}{A(A-\tau)^3}\,. \tag{31}$$

The second independent solution of the homogeneous equation

$$q(\tau) = \frac{1}{(1-\tau^2)(A-\tau)^2 h^2} \tag{32}$$

has a singularity on the jet axis and should therefore be discarded.

Reference [17] also gives a particular solution of the inhomogeneous equation

$$H(\tau) = C_1 h(\tau)\int_{-1}^{\tau} \frac{\int_1^\xi (A-\eta)^2 h(\eta)\,\mathrm{d}\eta}{(1-\xi^2)(A-\xi)^2 h^2(\xi)}\,\mathrm{d}\xi\,.$$

The general solution to equation (29) is

$$F_2(\tau) = H(\tau) + C_2 h(\tau)\,.$$

Reference [17] repeatedly emphasizes that a regular solution of the Navier–Stokes equations is sought, and yet the fact that the function $H(\tau)$ has a logarithmic singularity at $\tau = -1$, i.e., on the negative jet axis, went unnoticed. Such a singularity in the solution and, respectively, incorrectness of expansion (28) was mentioned in Refs [15, 18]. These papers also point to the reason why the solution turned out to be incorrect — a logarithmic term must be included in the expansion for velocities at large $R$. A strange impression in this respect is left by Ref. [19], which seeks the third approximation of the regular expansion in whole powers of $1/R$, i.e., the next term in the Rumer expansion [17].

Resorting to Refs [15, 18], we write the expressions for velocities and pressure with an accuracy of $o(R^{-2})$ and $o(R^{-3})$, respectively,

$$u_R = \frac{F(\tau)}{R} - \nu\left[u'(\tau)\ln\frac{R}{a} + w(\tau)\right]\frac{1}{R^2}\,,$$
$$u_\theta = \frac{1}{R}\,f_L(\theta) - \nu\,\frac{u(\tau)}{\sqrt{1-\tau^2}}\,\frac{1}{R^2}\,, \tag{33}$$
$$p = p_\infty + \frac{4\rho\nu^2}{R^2}\,\frac{A\tau-1}{(A-\tau)^2} + \nu^2\left[q(\tau)\ln\frac{R}{a} + g(\tau)\right]\frac{1}{R^3}\,,$$

where $a$ is the diameter of the initial cross section of the jet. Expansion (33) also differs from (28) in that the term proportional to $R^{-2}$ appears in the expression for $u_\theta$. Inserting expansion (33) into the Navier–Stokes equation, we have

$$u'(\tau) = Bh(\tau)\,, \quad u(\tau) = B(1-\tau^2)\,\frac{1-A\tau}{A(A-\tau)^2}\,,$$
$$Lw = C + B\left[\frac{6(A^2-1)^2}{(A-\tau)^4} - \frac{8A(A^2-1)}{(A-\tau)^3} + \frac{3(A^2-1)}{(A-\tau)^2} - \frac{6A^2-2}{A(A-\tau)} + 6\right]. \tag{34}$$

Here, $L$ is the differential operator defined in (30) and $h(\tau)$ is defined by expression (31). Two constants $B$ and $C$ are found from the conditions that the flow rate through a sphere of radius $R$ be equal to the flow rate $Q$ at the jet source and that the solution be regular:

$$Q = -\frac{2\pi\rho\nu}{3(A^2-1)}\left\{C\left(A^2+\frac{1}{3}\right) + B\left[9A^2 - 5 + 4A(A^2-1)\ln\frac{A+1}{A-1}\right]\right\},$$

$$C\left[-4A^2+\frac{20}{3}+2\,\frac{(A^2-1)^2}{A}\ln\frac{A+1}{A-1}\right]+B\left[-44A^2\right.$$
$$\left.+\frac{196}{3}-\frac{8}{A^2}+\frac{(A^2-1)(22A^2-18)}{A}\ln\frac{A+1}{A-1}\right]=0\,. \tag{35}$$

Equation (34) has a solution which depends on one constant $c_0$,

$$w(\tau)=c_0h(\tau)+h_1(\tau)\,, \tag{36}$$

where $h_1(\tau)$ is a known function; its unwieldy expression in quadratures can be found in Refs [15, 18]. The second solution of the homogeneous equation (34), which corresponds to (32), is discarded, once again because of singularity.

Free constant $c_0$ in solution (36) does not depend on the flow rate, as the latter is already fixed by the choice of the constants $B$ and $C$. There have been attempts to find a conservation law that would help to determine this constant, i.e., to relate it to some characteristics of the jet source. M A Gol'dshtik and N I Yavorsky [15, 18, 20] linked the constant $c_0$ with the flux of a side component of angular momentum through a surface composed of hemispheres with radii $R$ and $R_0$ and a ring in the plane $y=0$ between the circles $R$ and $R_0$. A drawback of such an approach is that the expressions for the velocity components and the pressure are only known in the far field. Therefore, this invariant does not provide links between the far field of the jet and the characteristics of the source, as we see for integrals of the flow rate and momentum flux. The same drawback is present in the assumption made in Ref. [21] that the appearance of the term $c_0h(\tau)$ in the solution is related to the second term in the expansion of the Landau solution in a series in inverse powers of the distance from the origin of the coordinates, which is displaced relative to the momentum source.

For large numbers Re, as shown in Section 2, the jet characteristics obey the boundary layer equations. A solution for the far field of a jet produced by a finite-size source was obtained by L G Loytsyansky [22] (see also [23, 24]) in a cylindrical reference frame in the form of a series in inverse powers of the coordinate $x$,

$$\begin{aligned}
\psi&=\nu\,\frac{\xi^2}{1+\xi^2/4}\,x-\nu\beta\,\frac{(\xi^2/4)(1-\xi^2/4)}{(1+\xi^2/4)^2}+\ldots\,,\\
u_x&=2\alpha^2\,\frac{1}{(1+\xi^2/4)^2}\,\frac{1}{x}-\frac{\beta\alpha^2}{2}\,\frac{1-(3/4)\xi^2}{(1+\xi^2/4)^3}\,\frac{1}{x^2}+\ldots\,,\\
u_r&=\alpha\sqrt{\nu}\,\frac{\xi-\xi^3/4}{(1+\xi^2/4)^2}\,\frac{1}{x}-\frac{\beta\alpha\sqrt{\nu}}{2}\,\frac{\xi-3\xi^3/4}{(1+\xi^2/4)^3}\,\frac{1}{x^2}+\ldots\,.
\end{aligned} \tag{37}$$

In these formulas, $\xi$ and $K$ are the same as in (7), and $\alpha=(1/4)\sqrt{3K/\pi\nu}$. By comparing (37) and (7), it becomes obvious that the constant $\beta$ must reflect the finite size of the source.

### 3.2 Link of the far-field asymptotic form to the velocity profile in the jet exit cross section

We ask two questions: what characteristics of the velocity profile at the jet source are responsible for the magnitude of the constant $\beta$ and what is the relationship between the solutions found in Refs [18] and [22]?

Loytsyansky calculated the mass flow rate per unit time through the jet cross section and obtained a formula which extends (8),

$$Q(x)=2\pi\rho\int_0^{\infty}u_xr\,\mathrm{d}r=2\pi\rho\nu(4x+\beta)\,. \tag{38}$$

Contrary to the previous opinion [22] that it is impossible to connect the constant $\beta$ to the initial flow rate from the jet source, Ref. [23] incorrectly concludes from equality (38), "...the terms including $\beta$ give a correction for the finite initial jet flow rate. Using this formula and taking $x=0$ we obtain, in the approximation adopted here, the flow rate $Q_0$ in the initial cross section... ." It was therefore assumed that $Q_0=Q(0)$, which would be correct if solution (37) could be continued to the exit cross section of the jet.

The answer to the question of how to determine $\beta$ was found in Refs [25, 26]. It turns out that, in the framework of the boundary layer equations, there is another conservation law for an axisymmetric nonswirling submerged jet, which was first obtained in Ref. [27] in a rather complicated way. We give a simple derivation. A consequence of equations (2) and (3) is the equation

$$\frac{\partial}{\partial x}(u_x^2r)+\frac{\partial}{\partial r}\left(u_xu_rr-\nu r\,\frac{\partial u_x}{\partial r}\right)=0\,.$$

We multiply it by $\psi-\nu x$ and rearrange it into a divergent form,

$$\frac{\partial}{\partial x}\left[u_x^2r(\psi-\nu x)\right]+\frac{\partial}{\partial r}\left[\left(u_xu_rr-\nu r\,\frac{\partial u_x}{\partial r}\right)(\psi-\nu x)+\nu\,\frac{u_x^2r^2}{2}\right]=0\,.$$

This implies the conservation of the quantity

$$E=\int_0^{\infty}u_x^2(\psi-\nu x)\,r\,\mathrm{d}r=\mathrm{const} \tag{39}$$

in any transverse cross section of the jet.

For $x=0$, the invariant $E=\int_0^{r_{\mathrm{out}}}u_x^2(0,r)\psi(0,r)\,r\,\mathrm{d}r$, where $r_{\mathrm{out}}$ is the radius of the jet exit. We assume that the stream function $\psi=0$ for $r=0$, as in formula (37). Let us now consider an arbitrary cross section $x$ of the jet far field. The integral $E$ is zero for the leading term of expansion (37). Taking into account the two terms, $E=2\nu^2\alpha^2\beta/3$. The equality of these two values of the invariant gives the constant $\beta$.

References [25, 26] also give numerical calculations for the initial profile $u_x(0,r)=1-r^n$, $0\leqslant r\leqslant 1$ for $n=1$ and $n=2$. The differences between numerical and asymptotic solutions of (37) from the main term in (37) are shown in Fig. 1.

To answer the second question about the connection between the solutions obtained in Refs [18] and [22], one needs to take the limit $A=1+\delta$, where $\delta$ is determined by relationship (19), as before in the solution (33)–(36). In this case, in the leading approximation, (35) gives

$$B=\frac{9\delta}{8\pi\rho}\,Q\,,\quad C=-\frac{45\delta}{8\pi\rho}\,Q\,.$$

Moreover, a comparison of the leading approximation of the solution for $w(\tau)$ (34), (36) with the second term in expansion (37) points to their correspondence and, as a consequence, to the connection between the constants $c_0$ and $\beta$,

$$c_0=\frac{\nu}{2}\,\beta\,.$$

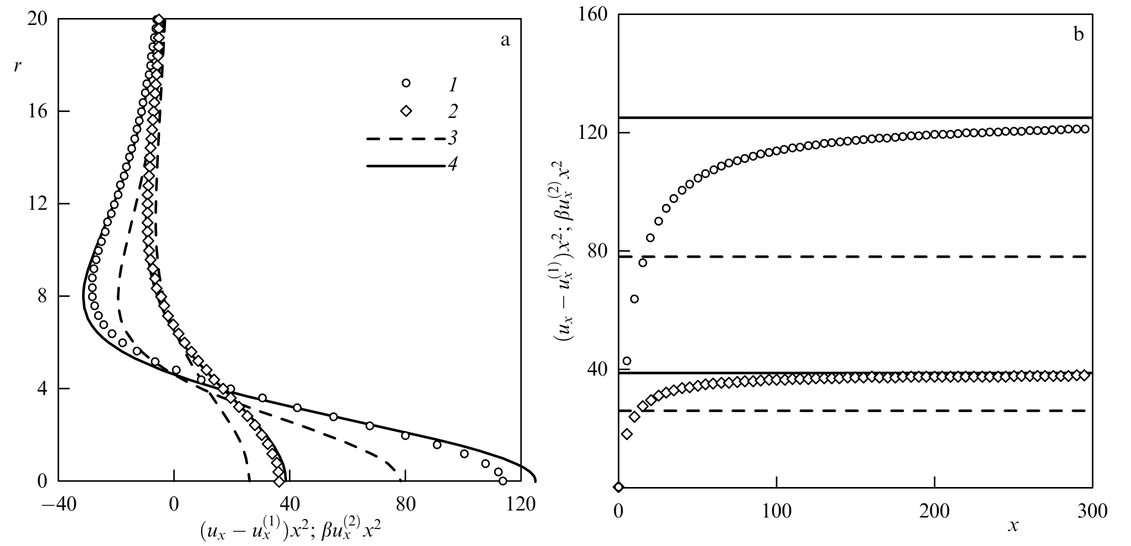


**Figure 1.** Differences between numerical $u_x$ and asymptotic $u_x^{(1)} + \beta u_x^{(2)}$ (37) solutions and the solution $u_x^{(1)}$ (7) for $x = 100$ (a), for $r = 0$ (b): *1* — numerical solution for a parabolic initial velocity profile; *2* — solution for a linear initial velocity profile; *3* — solution (37), (38); *4* — solution (37), (39).

The last expression closes the problem of a jet with a flow rate through its initial cross section for large numbers Re. The problem of determining the constant $c_0$ in other cases has not yet been solved.

## 4. Axisymmetric swirling jet

A swirling jet is obtained by adding some azimuthal velocity to an axisymmetric jet. This immediately raises many questions. What integral quantities are conserved for a swirling jet? What is the flow asymptotic form at an infinite distance, both in the direction of the jet and in the radial direction? Many aspects could possibly depend on the way the jet is created, but the jet starts from a point, and questions about how it is created are outside the scope of the solution.

A historically first solution for a swirling jet was obtained by Loytsyansky in the framework of boundary layer equations in the same study [22] (see also [23]), where the second approximation for an axisymmetric nonswirling jet was used. The problem is considered in a cylindrical reference frame, where only an additional azimuthal velocity component $u_\varphi$ is added. In this case, the pressure is no longer constant and is defined by the azimuthal velocity. The transverse pressure gradient is balanced by the centrifugal force, and, although the pressure outside the jet is constant in the leading approximation, the appearance of the transverse pressure gradient gives rise to its longitudinal gradient. In the equations for the axial and azimuthal velocity components, the terms with second derivatives in the longitudinal direction $\partial^2 u_x/\partial x^2$ and $\partial^2 u_\varphi/\partial x^2$ are discarded:

$$\begin{aligned}
&u_x \frac{\partial u_x}{\partial x} + u_r \frac{\partial u_x}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_x}{\partial r}\right),\\
&\frac{\partial p}{\partial r} = \frac{\rho u_\varphi^2}{r},\\
&u_x \frac{\partial u_\varphi}{\partial x} + u_r \frac{\partial u_\varphi}{\partial r} + \frac{u_r u_\varphi}{r} = \nu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (u_\varphi r)}{\partial r}\right),\\
&\frac{\partial (u_x r)}{\partial x} + \frac{\partial (u_r r)}{\partial r} = 0.
\end{aligned} \tag{40}$$

From the first and last equations of system (40), it follows that

$$\frac{\partial}{\partial x}\left[r(p + \rho u_x^2)\right] + \frac{\partial}{\partial r}\left[r\rho\left(u_x u_r - \nu\frac{\partial u_x}{\partial r}\right)\right] = 0.$$

Here, $p$ is the pressure difference with respect to the pressure at a point at infinity. Integrating the last equation transverse to the jet, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty r(p + \rho u_x^2)\,\mathrm{d}r + \left[r\rho\left(u_x u_r - \nu\frac{\partial u_x}{\partial r}\right)\right]_{r=0}^{r=\infty} = 0.$$

At the jet axis $u_r = 0$, $\partial u_x/\partial r = 0$. Assuming that for $r \to \infty$ the velocity component $u_x$ decays no more slowly than $r^{-1}$, we find that the second term is zero. Then, in any transverse cross section $x = \text{const}$, the total momentum flux is conserved,

$$J = 2\pi\int_0^\infty (p + \rho u_x^2)\, r\,\mathrm{d}r = \text{const}. \tag{41}$$

From the third and last equations of system (40), it can be deduced that

$$\frac{\partial}{\partial x}(r^2 u_x u_\varphi) + \frac{\partial}{\partial r}\left\{r^2 u_r u_\varphi - \nu\left[r\frac{\partial}{\partial r}(r u_\varphi) - 2 r u_\varphi\right]\right\} = 0.$$

Integration across the jet leads to the relationship

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty r^2 u_x u_\varphi\,\mathrm{d}r + \left\{r^2 u_r u_\varphi - \nu\left[r\frac{\partial}{\partial r}(r u_\varphi) - 2 r u_\varphi\right]\right\}_{r=0}^{r=\infty} = 0. \tag{42}$$

Two situations are now possible. The first one, which corresponds to the Loytsyansky jet [22], is the case when the velocity components decay sufficiently fast as the coordinate $r$ is increased. In this case, the second term goes to zero, which indicates the appearance of one more invariant for a swirling jet — the angular momentum flux

$$L = 2\pi\rho\int_0^\infty r^2 u_x u_\varphi\,\mathrm{d}r = \text{const}. \tag{43}$$

If, in the limit $r \to \infty$, the velocity components behave as $r^{-1}$ multiplied by a constant that is different for different components, then, instead of (43), one gets another invariant for $r \to \infty$ equal to the azimuthal velocity multiplied by $r$, i.e., the circulation of the azimuthal velocity divided by $2\pi$,

$$\Gamma = u_\varphi r = \text{const}\,. \tag{44}$$

The flow is axisymmetric, so that the quantity $\Gamma$ is independent of the angle $\varphi$. Furthermore, at large Reynolds numbers, the vortical flow in the jet is concentrated in the region corresponding to the boundary layer. According to the Helmholtz theorem on the conservation of the vorticity flux (the circulation of the azimuthal velocity) through any transverse section of such a vortical flow, the quantity $\Gamma$ does not depend on $x$. Other quantities will have well defined asymptotic form. From the second equation of (40), for $r \to \infty$, it follows that

$$\frac{p}{\rho} = -\frac{\Gamma^2}{2r^2}\,.$$

In this case, to ensure a finite flux of total momentum, it is necessary that, for $r \to \infty$, the integrand in formula (41) be zero,

$$u_x = \sqrt{-\frac{p}{\rho}} = \frac{\Gamma}{\sqrt{2}\,r}\,. \tag{45}$$

The axial velocity component at a large distance from the axis is smaller than the azimuthal velocity component by a factor of $\sqrt{2}$.

The radial velocity component is defined by the first equation in system (40). For $r \to \infty$,

$$u_r = -\frac{\nu}{r}\,. \tag{46}$$

### 4.1 Loytsyansky jet

We consider in detail the first case, i.e., the Loytsyansky jet, which is characterized by two invariants $J$ (41) and $L$ (43). The ratio $L/J$ has a dimension of length, and, consequently, the solution for the jet characteristics will not be self-similar. Reference [22] constructed a solution at large distances from the jet source. The approach of [22] allows one to consider simultaneously the swirl and the flow rate through the initial cross section. Here, it is important that, in the leading approximation, the problem solution at large distances coincide with the solution for the nonswirling Landau jet.

The solution satisfies equations (40). Boundary conditions (4) are still valid for the velocity components $u_x$ and $u_r$. The azimuthal velocity should be zero at the jet axis, and, for $r \to \infty$, it should decay faster than $r^{-1}$ and maintain integral (43).

As for the nonswirling jet (37), the solution for a swirling jet is sought as a series in inverse powers of the coordinate $x$. Inserting this expansion into system (40) and the boundary conditions, one obtains a solution that coincides with (37) for the components $u_x$ and $u_r$,

$$u_x = 2\alpha^2\,\frac{1}{(1+\xi^2/4)^2}\,\frac{1}{x} - \frac{\beta\alpha^2}{2}\,\frac{1-(3/4)\xi^2}{(1+\xi^2/4)^3}\,\frac{1}{x^2} + \ldots,$$

$$u_r = \alpha\sqrt{\nu}\,\frac{\xi-\xi^3/4}{(1+\xi^2/4)^2}\,\frac{1}{x} - \frac{\beta\alpha\sqrt{\nu}}{2}\,\frac{\xi-3\xi^3/4}{(1+\xi^2/4)^3}\,\frac{1}{x^2} + \ldots,$$

$$u_\varphi = \gamma\,\frac{\xi}{(1+\xi^2/4)^2}\,\frac{1}{x^2} + \ldots,$$

$$p = -\frac{2}{3}\,\rho\gamma^2\,\frac{1}{(1+\xi^2/4)^3}\,\frac{1}{x^4} + \ldots\,. \tag{47}$$

Here, $\gamma = 3\alpha L/16\pi\rho\nu^{3/2}$.

Compared to (37), solution (47) includes the expression for the azimuthal velocity, which decays proportionally to $x^{-2}$. The azimuthal velocity component decays with $x$ faster than the axial velocity component. The ratio of these velocities therefore decreases as $x$ increases.

We only need to check the asymptotic behavior of the velocity components for $r \to \infty$. The radial component decays proportionally to $r^{-1}$ and the azimuthal component decays as $r^{-3}$, providing the required decay of the second term in equation (42). The circulation of the azimuthal velocity along a contour of infinite radius becomes zero.

A solution for a swirling jet with the conservation of the angular momentum flux in the framework of the Navier–Stokes equations has been obtained by M S Tsukker [28]. In this work, the velocities and pressure in a spherical coordinate system are taken as series in inverse powers of $R$:

$$u_R = -\nu\left[\frac{f_1'(\tau)}{R} + \frac{f_2'(\tau)}{R^2} + \ldots\right], \quad u_\theta = -\nu\left[\frac{f_1}{R\sqrt{1-\tau^2}} + \ldots\right],$$

$$u_\varphi = \nu\left[\frac{g_1(\tau)}{R} + \frac{g_2(\tau)}{R^2} + \ldots\right],$$

$$\frac{p-p_0}{\rho} = \nu^2\left[\frac{h_1(\tau)}{R} + \frac{h_2(\tau)}{R^2} + \ldots\right].$$

From the equations and boundary conditions, Tsukker found that the azimuthal velocity in this approximation does not modify the solution which corresponds to the Landau jet

$$h_1 = 0\,, \quad g_1 = 0\,, \quad f_1 = \frac{2(1-\tau^2)}{A-\tau}\,, \quad h_2 = -\frac{4(1-A\tau)}{(A-\tau)^2}\,.$$

In the leading approximation, the zonal velocity is expressed as

$$u_\varphi = \nu\,\frac{\gamma_1(A+1)}{2R^2}\,\frac{\sqrt{1-\tau^2}}{(A-\tau)^2}\,, \tag{48}$$

where $\gamma_1$ is a constant that accounts for the swirl.

The standard transition $A = 1+\delta$, $\cos\theta \approx 1-\theta^2/2$ ($\delta \to 0$, $\theta \to 0$) from solution (48) taking into account (19) to solution (47) of the boundary layer equation gives the connection between the constants $\gamma_1$ and $\gamma$:

$$\gamma_1 = \frac{\sqrt{2}\,\delta^{3/2}}{\nu}\,\gamma\,.$$

### 4.2 Long jet

We now consider the second situation — that of a swirling axisymmetric jet with the invariants (41) and (44) conserved. Such a jet was first considered by R R Long [29] in 1961 in the framework of boundary layer equations, although he had formulated all the necessary equations three years earlier [30].

Since circulation (44) has the same dimension as $\sqrt{J/\rho}$ and the kinematic viscosity coefficient, and additionally there is no characteristic scale in this problem, a self-similar

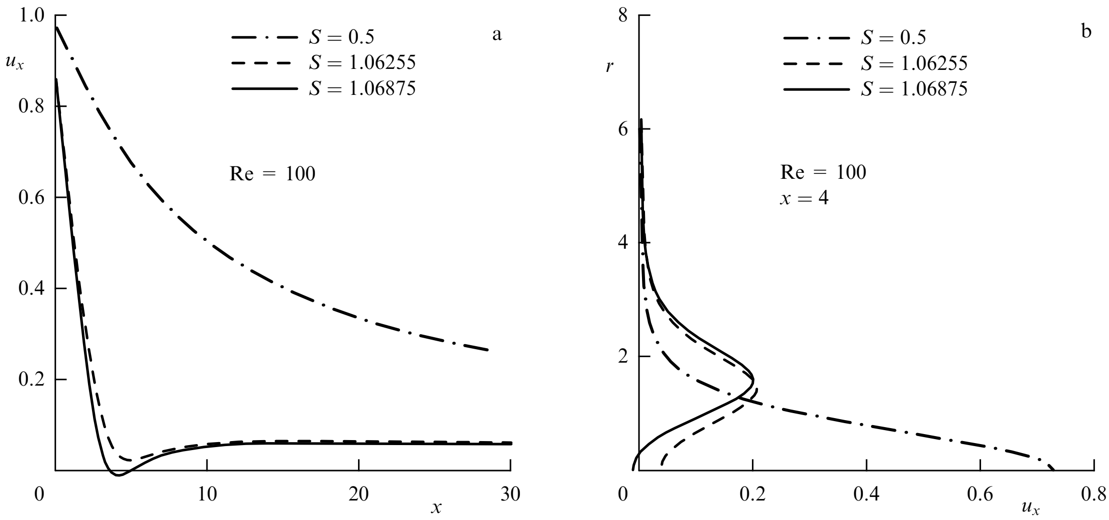


**Figure 2.** Calculated distributions of axial velocity on the jet axis (a) and in the section $x = 4$ (b) for Re = 100, $S = 0.5$, 1.0625, 1.06875. Linear scales are nondimensionalized by the tube radius, and velocities are normalized by the maximum axial velocity in the tube.

solution is sought with the quantities taken as

$$\psi = \nu x f(y)\,, \quad u_x = \frac{\Gamma}{r\sqrt{2}}\, f'\,, \quad u_r = -\frac{\nu}{r}\, f + \frac{\Gamma}{x\sqrt{2}}\, f'\,,$$
$$u_\varphi = \frac{\Gamma}{r}\,\lambda(y)\,, \quad \frac{p}{\rho} = -\frac{\Gamma^4}{\nu^2 x^2}\, s(y)\,, \quad y = \frac{r\Gamma}{\nu x\sqrt{2}}\,, \quad (\;)' \equiv \frac{\mathrm{d}}{\mathrm{d}y}\,. \tag{49}$$

Two dimensionless quantities are introduced: $\varepsilon = \nu/\Gamma$, an analog of the inverse Reynolds number, which is assumed to be small for the validity of boundary layer equations, and $M = J/\rho\Gamma^2$, an analog of the inverse squared number of the flow swirl. The solution should depend on these two dimensionless quantities. Inserting (49) into the boundary layer equations (40), we obtain the system of equations

$$\begin{aligned} &f''y - f'(1-f) - 4y^3 s = 0\,,\\ &\lambda^2 + 2y^3 s' = 0\,,\\ &\lambda'' y - \lambda'(1-f) = 0\,. \end{aligned} \tag{50}$$

The boundary conditions on the jet axis ($y = 0$) are $f = 0$, $\lambda = 0$. Boundary conditions at $y \to \infty$ follow from (44)–(46): $f \to 1 + y$, $\lambda \to 1$, $s \to 0$. In Ref. [29], system (50) was solved numerically. It turned out that in such a formulation a solution exists only if $M > 3.65$. Depending on the problem parameters, the axial velocity can be both positive and negative (reverse flows).

This topic was further developed in Refs [31–33]. Reference [31] rejects the possibility of creating such a jet with the strange statement that "the initial flux of the angular momentum of the jet source is zero." In Ref. [32], a solution in a swirling jet is sough with the help of the Navier–Stokes equations in spherical coordinates. The boundary conditions on the jet axis are discussed not only for $x > 0$ but also for $x < 0$. On the negative semi-axis, the same circulation should be given and hence the azimuthal velocity component will be infinite. This boundary condition follows from the Helmholtz theorem and the zero size of the jet at its origin. Thus, a vortex filament with finite circulation should follow the entire negative semi-axis, and a jet is formed at $x = 0$ due to the momentum added to the flow. In order to eliminate the singularity in the momentum flux on the semi-axis $x < 0$, the axial velocity component should be infinite on it. Reference [33] constructs an asymptotic solution for large Reynolds numbers and small swirl. It is the case of large $M$ in the terminology of Ref. [30].

### 4.3 On the realization of swirling jets. Hidden invariants

Since there are two fundamentally different solutions for swirling jets, the question naturally arises as to which will be realized in practice. The same question is posed in Refs [14, 15, 32]. Let us have a closer look at these studies, as not all their statements can be considered correct. In Ref. [15], swirling jets are subdivided into weakly and strongly swirling jets (in all probability, following Refs [31, 34]):

...Its solution was characterizing a 'weak' swirling jet in which the velocity of rotation decayed much faster than the axial or radial ones. This did not agree very well with the experimental data... which, for sufficiently strong jets, showed complicated reverse flows in the near-axial zone, whereas the theory did not predict them. The reason lies in the existence of a hidden invariant... overlooked by previous researchers. This invariant, to be discussed here, makes the problem self-similar... .

Surely, the invariant (we will return to it later) has nothing to do here, nor do the reverse flows. If the jet is a Loytsyansky jet for a small swirl, the increase in swirl will not change the jet type. The asymptotic form of Ref. [22] is the far field one, and the presence of near-axial reverse currents only shifts further the range where it becomes valid for large $x$. The numerical simulations in Refs [35, 36] may serve as proof of this fact. In Ref. [36], it is shown that different flow regimes are realized as the swirl is increased. The first regime is characterized by the fact that the axial velocity component in any section $x = \mathrm{const}$ is maximal at the jet axis and decreases monotonically as $x$ increases (Fig. 2). As the swirl increases, the second regime is realized, in which the behavior of the maximum axial velocity is no longer monotonic with $x$, but the axial velocity remains positive (see Fig. 2). Finally, for an even larger swirl, return currents appear in the vicinity of the axis (Fig. 3). In all the regimes described, in the case of both weak and strong swirls, jets follow the asymptotic solution of [22] in the far field.

We now return to the hidden invariant found in Refs [14, 15, 20, 32]. The invariants in the case of jet flows are usually

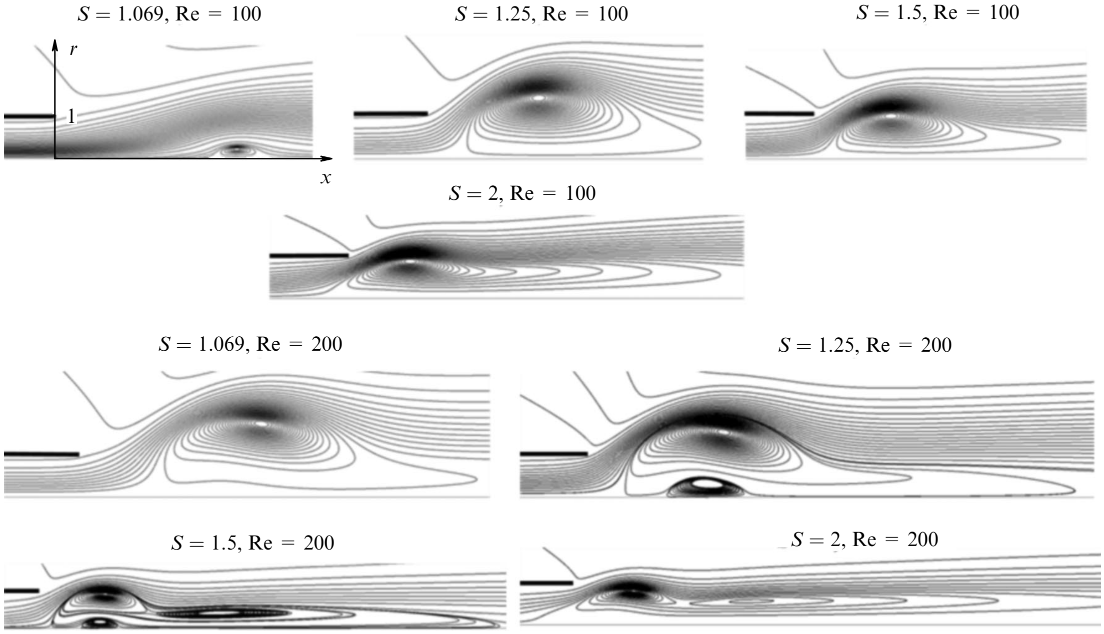


**Figure 3.** Dependence of the size and shape of the recirculation domain on Reynolds number Re and swirl $S$; thick black line depicts the tube wall.

the values of the fluxes of momentum, angular momentum, and other quantities that do not change when integrated over a closed surface around the coordinate origin or the section $x = \text{const}$. A not quite conventional flow invariant of Refs [14, 15, 20, 32] was called a hidden invariant by Gol'dshtik.

The flux of the $j$ component of the momentum in the Cartesian coordinate system through a closed surface $S$

$$J_j = \iint_S (\Pi_{xj} n_x + \Pi_{yj} n_y + \Pi_{zj} n_z)\,\mathrm{d}S\,,$$

where $\mathbf{n}(n_x, n_y, n_z)$ is a unit vector of the outer normal to the surface $S$.

Similarly, in the spherical coordinate system, one can write momentum fluxes in the direction of the longitudinal coordinate $x$ and one of the transverse coordinates $y$ through a sphere of radius $R$:

$$J_x = \int_0^{2\pi}\int_0^{\pi} (\Pi_{RR}\cos\theta - \Pi_{R\theta}\sin\theta) R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,,$$

$$J_y = \int_0^{2\pi}\int_0^{\pi} (\Pi_{RR}\sin\theta\cos\varphi + \Pi_{R\theta}\cos\theta\cos\varphi - \Pi_{R\varphi}\sin\varphi) \times R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,.$$

For an axisymmetric jet, integrating over $\varphi$ we obtain

$$J_x = 2\pi \int_0^{\pi} (\Pi_{RR}\cos\theta - \Pi_{R\theta}\sin\theta) R^2 \sin\theta\,\mathrm{d}\theta\,,$$

$$J_y = 0\,.$$

The integral $J_y$ is calculated in [14, 32] not over the entire sphere but over the hemisphere with the 'bottom' (Fig. 4): we denote it $J_y^*$. This integral is universal and should be conserved for any axisymmetric swirling jet for any $R$. Making the erroneous assumption that the integral over the 'bottom' equals zero due to the axial flow symmetry, Gol'dshtik obtained

$$J_y^* = R^2 \int_0^{\pi} \Pi_{R\varphi} \sin\theta\,\mathrm{d}\theta\,. \tag{51}$$

It will be recalled that

$$\Pi_{R\varphi} = \rho u_R u_\varphi - \mu\left(\frac{\partial u_\varphi}{\partial R} - \frac{u_\varphi}{R}\right). \tag{52}$$

A paradoxical situation arises in this case. Due to the conservation of invariant (51) taking into account (52), it is necessary that, for large $R$, the components of the velocity be inversely proportional to the radius $u_R \sim R^{-1}$, $u_\varphi \sim R^{-1}$, which corresponds to the swirling Long jet. Based namely on this, Ref. [14] concludes that, for the given invariants $J_x$ and

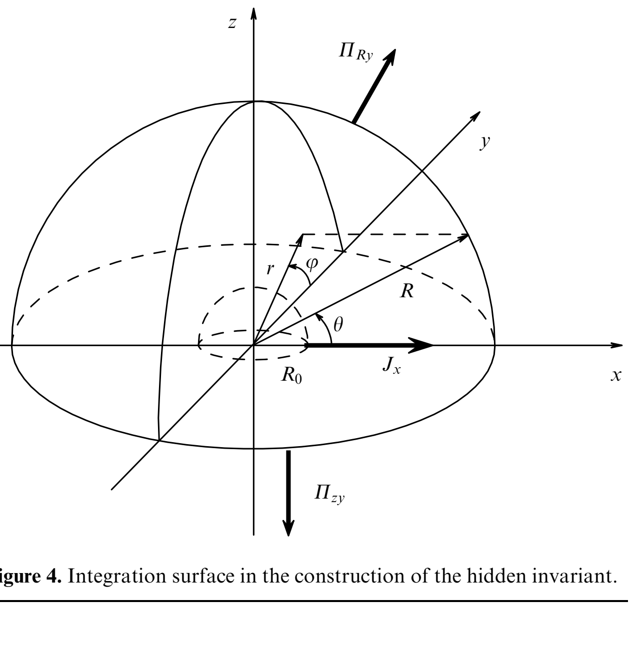


**Figure 4.** Integration surface in the construction of the hidden invariant.

$J_y^*$, the solution should be sought in a self-similar form, and the velocity component should be inversely proportional to the radius.

The error of Refs [14, 32] is corrected in [36], where it is shown that the flux of the $y$-component of the momentum through the 'bottom' of the hemisphere is not zero. In order to take it into account, the flux $J_y$ between two hemispheres with radii $R_0$ and $R$ with the 'bottom,' for $R > R_0$, was calculated in this work. The flux through the inner hemisphere is equal to the flux through the outer hemisphere and the ring region of the 'bottom' between them. The correct universal invariant is given by the expression

$$J_y^{**} = R^2 \int_0^{\pi} \Pi_{R\varphi} \sin\theta \, d\theta + \int_{R_0}^{R} \int_0^{\pi} R(\Pi_{R\varphi} \sin\theta + \Pi_{\varphi\theta} \cos\theta) \, d\theta \, dR \,, \tag{53}$$

$$\Pi_{\varphi\theta} = \rho u_\varphi u_\theta - \frac{\mu}{R} \left( \frac{\partial u_\varphi}{\partial \theta} - u_\varphi \cot\theta \right).$$

Let us now consider possible far asymptotic forms for swirling jets. As was shown above, from the continuity equation and the conservation of the total momentum flux (41) in both the Loytsyansky and Long jets, it follows that, when their solutions are rewritten in spherical coordinates, two velocity components will be inversely proportional to the radius:

$$u_R = \frac{\alpha(\theta)}{R} + \ldots, \quad u_\theta = \frac{\beta(\theta)}{R} + \ldots. \tag{54}$$

We express the azimuthal velocity in the form

$$u_\varphi = \frac{\chi(\theta)}{R^k} + \ldots, \tag{55}$$

where $k$ can take the values 1 (the Long jet) and 2 (the Loytsyansky jet).

We insert (54) and (55) into (53) and take the range in $R$ where these asymptotics hold. For $k = 1$,

$$\frac{J_y^{**}}{\rho} = \int_0^{\pi} f(\theta) \sin\theta \, d\theta + \ln \frac{R}{R_0} \int_0^{\pi} \big( f(\theta) \sin\theta + g(\theta) \cos\theta \big) \, d\theta \,,$$

where $f(\theta) = \alpha\chi + (k+1)\nu\chi$, $g(\theta) = \beta\chi - \nu(\chi' - \chi \cot\theta)$. Integral (53) is conserved if

$$\int_0^{\pi} (f \sin\theta + g \cos\theta) \, d\theta = 0 \,.$$

For $k = 2$,

$$\frac{J_y^{**}}{\rho} = -\frac{1}{R} \int_0^{\pi} g(\theta) \cos\theta \, d\theta + \frac{1}{R_0} \int_0^{\pi} \big( f(\theta) \sin\theta + g(\theta) \cos\theta \big) \, d\theta \,.$$

Integral (53) is conserved when

$$\int_0^{\pi} g \cos\theta \, d\theta = 0 \,.$$

So invariant (53) does not define the power of the decay of the azimuthal velocity.

The second question, for which there is already a correct answer in Ref. [15], is: can the source of the self-similar Long jet be point-like? The answer in Ref. [15] is as follows: "It should be mentioned that the vortex filament differs in principle from a point or spherical source by having $u_\varphi = \infty$ on the semi-axis $\theta = \pi$.... Therefore, the source of the self-similar swirling jet cannot be point-like."

It is not difficult to produce a Loytsyansky jet. In Refs [35, 36], such swirling jets leaving a tube of finite length were created by rotating the inner tube surface. Any other source of swirl can be used instead of the rotating inner surface. As for the Long jet, producing it would be difficult, if not impossible. One of the methods to produce a swirling jet that is close to the Long jet was proposed in Ref. [36]. However, the jet in this work was generated numerically, and an appropriate zonal velocity profile was used as a boundary condition for the incoming flow, which was not specified.

What are the difficulties in generating a self-similar Long jet? First, such a jet should be injected into a vortex filament. In this case, and only in this case, the source of the self-similar jet can be point-like. Ideally, the vortex filament should start from minus infinity and somehow avoid the action of diffusion. A vortex tube can be created, for example, behind a long rectangular wing or behind a propeller with a certain type of blade. In the first case, a vortex formation resembling a vortex tube is shed from the tip of the wing, and, in the second case, it is observed in the butt part of the propeller. In both cases, a zonal velocity close to that from a point vortex can be created. Second, at $x = 0$, the total momentum $J_x$ is injected in the flow, which, according to Refs [32, 33], should have a zero value for $x < 0$. However, in the examples given above, the axial velocity is determined from the condition that the Bernoulli integral hold and not from the condition that the momentum flux be zero, which gives a very different value. The question of the feasibility of the Long jet remains open.

## 5. Conclusions

The theory of laminar jets will soon be 90 years old. During this time, solutions to many problems have been found that are surprisingly elegant. At the same time, some problems remain unsolved. There are no sufficiently satisfactory theories to describe jets emerging from a point source perpendicular or parallel to an infinite plane. Little is known about the mechanics of jet interaction. An open question is whether it is possible to create a swirling jet while conserving circulation along the outer contour of the jet.

The authors of this paper have attempted to critically review studies on the jet theory and to pick out the outstanding ones among them, as well as those with erroneous statements. As far as the latter is concerned, the material presented in this paper should fundamentally change the view on swirling jets and jets with flow rate through the initial cross section. We corrected the expression for the hidden invariant, from which it followed earlier that a swirling jet should tend to a jet with a given circulation at a large distance from its source. This fact was contrary to physical intuition, which says that at infinity a swirling jet should tend to a Landau jet in the leading approximation, and hence the azimuthal velocity should decay faster than the longitudinal velocity. The correct calculation of this invariant indicates the mathematical possibility of the existence of both Loytsyansky and Long jets. For jets with a flow rate through the initial cross section, a correct procedure was presented for finding the connection between the far asymptotic form and the initial velocity profile in the framework of the boundary layer equations.

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