

METHODOLOGICAL NOTES

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Unitarity relation and unitarity bounds for scalars with different sound speeds

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Contents

Abstract. **We consider a theory which contains massless scalar fields with different sound speeds. For these theories we derive unitarity relations for partial wave amplitudes of $2 \to 2$ scattering, with explicit formulas for contributions of two-particle intermediate states. We also obtain unitarity bounds both in the most general case and in the case considered in the literature for the speed of sound, equal to unity. We illustrate our unitarity relations by explicit one-loop calculation to the first nontrivial order in couplings in a simple model of two scalar fields with different sound speeds. Obtained unitarity bounds can be used to estimating the strong coupling scale of a pertinent effective field theory (EFT).**

Keywords: unitarity, quantum field theory, scalar field, cosmology

1. Introduction

Scalar-tensor theories of gravity with nontrivial scalar kinetic terms and/or nonminimal couplings are commonly used to construct models of inflation [1–5] as well as novel cosmological models such as genesis [6–11] and bounce [12–20]. Such theories are also popular in modeling dark energy, e.g., Refs [21–29]. In Ref. [30] the authors discuss corresponding constraints on the dark energy models in the framework of the scalar-tensor theories after gravitational-wave event GW170817.

In these scalar-tensor theories, perturbations about nontrivial backgrounds often propagate with 'sound speeds' different from the speed of light and, moreover, perturbations of different types (e.g., scalar vs tensor in the cosmological context) have different sound speeds. Another feature is that some constructions involve time-dependent couplings which are dangerously large during certain time intervals. An example is the Horndeski theory [31], whose subclass allows genesis and bounce [1] with 'strong gravity in the past' (effective Planck mass tends to zero as $t \to -\infty$) [36]; in this way, one evades the no-go theorem of Refs [37, 38]. In [39], the authors also evade the latter no-go theorem but in a slightly different manner than in [36].

An important parameter in an effective quantum field theory (QFT) is the energy scale of strong coupling or, in other words, the maximum energy below which the effective QFT description is trustworthy. Scalar-tensor theories of gravitation, especially those featuring large couplings, are not exceptional in this regard. While the strong coupling energy scale can often be qualitatively estimated by naive dimensional analysis, more accurate estimates are obtained using unitarity bounds that follow from general unitarity relations. This motivates us to derive unitarity relations and unitarity bounds in theories with different sound speeds of different perturbations.

In this paper, we consider theories with several *scalar* fields; theories of particles with spin can be treated in a similar way.[2] We also study theories in *flat* space–time and with a *trivial* background; this treatment is also expected to be relevant for nontrivial backgrounds, since the classical description of a background is legitimate provided that its classical energy scale is well below the quantum strong coupling scale, in which case the space–time dependence of the background is expected to be negligible when evaluating the quantum scale. Indeed, in [40], the results of this paper are used in order to evaluate the quantum energy scale of strong coupling in the model of Horndeski bounce.

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[1] Note that models of genesis and bounce often require violation of the Penrose theorem [32] and, in particular, violation of the null energy condition (NEC) [33, 34] (see also review [35]). The Horndeski theory allows violation of the NEC.

[2] Bosonic perturbations with spin can often be reduced to effective scalar perturbations at the expense of violation of Lorentz invariance, which occurs in nontrivial backgrounds anyway.

An adequate approach to unitarity relations and unitarity bounds makes use of Partial Wave Amplitudes (PWAs) (see, e.g., Refs [41–44]). We follow this approach in our paper. We aim at a self-contained presentation and give detailed derivations, even though many steps follow closely the analyses existing in the literature. In this sense this paper may serve as a pedagogical mini-review of the subject, with the novelty lying in the fact that we consider different sound speeds of different excitations.

This paper is organized as follows. We derive in Section 2 the general unitarity relations for PWAs of $2 \to 2$ scattering, paying special attention to two-particle intermediate states. In Section 2, we also derive the unitarity bounds. To this end, in Section 2.1 we describe the class of theories we are dealing with. We then study separately the cases of a pair of distinguishable particles in the intermediate state (Section 2.1) and a pair of identical particles (Section 2.2). Unitarity bounds are derived in Section 2.3. We give an illustrative example in Section 2.4, where we explicitly check the validity of the unitarity relation to the leading nontrivial order in a simple model of two real scalar fields. The Appendix is dedicated to the time-reversal invariance and its consequence for PWA's symmetry.

2. Unitarity relation

In this section, we proceed in the spirit of Ref. [41] and obtain the unitarity relation for $2 \to 2$ scattering processes in theories with scalar fields ϕ_i whose sound speeds u_i are different. In further analysis, we consider massless case. The quadratic action reads

$$S = \sum_i S_{\phi_i}\,, \quad S_{\phi_i} = \int \mathrm{d}^4 x \left(\frac{1}{2}\,\dot{\phi}_i^{\,2} - \frac{1}{2}\,u_i^2 (\nabla \phi_i)^2 \right).$$

The linearized equation of motion for ϕ_i is

$$\ddot{\phi}_i - u_i^2 \Delta \phi_i = 0\,,$$

and its solution can be written as follows:

$$\phi_i(\mathbf{x}, t) = \int \frac{\mathrm{d}\mathbf{k}_i}{(2\pi)^3}\, \frac{1}{\sqrt{2E_{k_i}}} \left[a_{\mathbf{k}_i} \exp\left(-\mathrm{i}E_{k_i} t + \mathrm{i}\mathbf{k}_i \mathbf{x}\right) + a^{\dagger}_{\mathbf{k}_i} \exp\left(\mathrm{i}E_{k_i} t - \mathrm{i}\mathbf{k}_i \mathbf{x}\right) \right],$$

where

$$E_{k_i} = u_i\, k_i\,, \tag{1}$$

and the operators $a_{\mathbf{p}_i}$ and $a^{\dagger}_{\mathbf{p}_i}$ obey the standard commutational relation

$$[a_{\mathbf{k}'_i}, a^{\dagger}_{\mathbf{k}_j}] = (2\pi)^3 \delta^{(3)}(\mathbf{k}'_i - \mathbf{k}_j)\delta_{ij}\,, \tag{2}$$

and all other commutators equal zero. We define the one-particle state as follows:

$$|\mathbf{p}_i\rangle \equiv \sqrt{2E_{p_i}}\, a^{\dagger}_{\mathbf{p}_i}|0\rangle\,,$$

so that one has the standard relation

$$\langle 0 | \phi_i(\mathbf{x}, t) | \mathbf{p}_j \rangle = \exp\left(-\mathrm{i}E_{p_j} t + \mathrm{i}\mathbf{p}_j \mathbf{x}\right)\delta_{ij}\,,$$

while the normalization of this state is given by

$$\langle \mathbf{p}'_j | \mathbf{p}_i \rangle = (2\pi)^3 \sqrt{2E_{p'_j} 2E_{p_i}}\, \delta^{(3)}(\mathbf{p}_i - \mathbf{p}'_j)\delta_{ij}\,. \tag{3}$$

In the ith one-particle sector, we have

$$\mathbb{1} = \int \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_{p_i}}\, |\mathbf{p}_i\rangle\langle \mathbf{p}_i|\,.$$

The S-matrix and T-matrix are related in the standard way,

$$S = \mathbb{1} + \mathrm{i}T\,,$$

and we extract from T the overall δ-function of 4-momentum conservation:

$$T = (2\pi)^4 \delta^4\left(\mathcal{P}^{\mu\prime} - \mathcal{P}^{\mu}\right) M\,, \tag{4}$$

where $\mathcal{P}^{\mu} = \sum p^{\mu}_{\mathrm{in}}$ and $\mathcal{P}^{\mu\prime} = \sum p^{\mu}_{\mathrm{out}}$ are total 4-momenta of the initial and final states, respectively.

Now, we consider an initial state

$$|\psi, \beta\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\, a^{\dagger}_{p_1} a^{\dagger}_{p_2}|0\rangle\,, \tag{5}$$

with two particles of momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, and a final state $|\psi', \beta'\rangle$, with two particles of momenta $\mathbf{p}'_1$ and $\mathbf{p}'_2$. Notation β refers to the types of the two particles, $\beta = \{\phi_i, \phi_j\}$, while the notation ψ is shorthand for the pair of momenta, $\psi = \{\mathbf{p}_1, \mathbf{p}_2\}$. Thus,

$$|\psi, \beta\rangle = |\phi_i, \mathbf{p}_1\rangle \otimes |\phi_j, \mathbf{p}_2\rangle\,.$$

In Eqn (5), we do not explicitly indicate the type of particle to simplify formulas and write $a^{\dagger}_{p_1} \equiv a^{\dagger}_{ip_1}$, etc.

Our purpose is to derive the unitarity relation for the partial wave amplitudes.

2.1 Distinguishable particles

Let us begin with the case of distinguishable particles in a pair $\beta = \{\phi_i, \phi_j\}$. In the next section, we consider the case of identical particles.

The scalar product of states $|\psi', \beta'\rangle$ and $|\psi, \beta\rangle$ is

$$\langle \psi', \beta' | \psi, \beta \rangle = (2\pi)^6\, 2E_{p_1} 2E_{p_2} \delta^{(3)}(\mathbf{p}'_1 - \mathbf{p}_1)\delta^{(3)}(\mathbf{p}'_2 - \mathbf{p}_2)\delta_{\beta'\beta}\,. \tag{6}$$

This follows from the one-particle state normalization (3). In what follows, we consider the center-of-mass frame of the two-particle system. In this frame, we denote $\mathbf{p} \equiv \mathbf{p}_1 = -\mathbf{p}_2$, $p \equiv |\mathbf{p}| = |\mathbf{p}_1| = |\mathbf{p}_2|$. Let $\hat{\mathbf{p}} = \mathbf{p}/p$ be the unit vector along $\mathbf{p}$ and θ, ϕ be the corresponding angles. We now replace the variables $\mathbf{p}_1$, $\mathbf{p}_2$ in (6) by $\mathcal{P}^{\mu} \equiv p^{\mu}_1 + p^{\mu}_2$, θ, and ϕ, where we have in mind that in the vicinity of the center-of-mass frame we have $\mathcal{P}^{\mu} \approx (E, 0)$, where $E = (u_{1\beta} + u_{2\beta})\, p$ and $u_{1\beta} \equiv u_i$, $u_{2\beta} \equiv u_j$ are sound speeds of the two particles in the pair $\beta = \{\phi_i, \phi_j\}$. For the volume element, we have

$$\mathrm{d}^3\mathbf{p}_1\, \mathrm{d}^3\mathbf{p}_2 = \mathrm{d}^3\boldsymbol{\mathcal{P}} p^2\, \mathrm{d}p\, \mathrm{d}\hat{\mathbf{p}} = \frac{p^2}{u_{1\beta} + u_{2\beta}}\, \mathrm{d}^4\mathcal{P}^{\mu}\, \mathrm{d}\hat{\mathbf{p}}\,,$$

which gives

$$\delta^{(3)}(\mathbf{p}_1' - \mathbf{p}_1)\delta^{(3)}(\mathbf{p}_2' - \mathbf{p}_2)\delta_{\beta\beta'}$$
$$= \frac{u_{1\beta} + u_{2\beta}}{p^2}\,\delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu})\delta^{(2)}(\hat{\mathbf{p}}' - \hat{\mathbf{p}})\delta_{\beta\beta'}\,,$$

and hence

$$\langle\psi', \beta'|\psi, \beta\rangle = (2\pi)^6\, 4u_{1\beta}u_{2\beta}(u_{1\beta} + u_{2\beta})\,\delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu})$$
$$\times\,\delta^{(2)}(\hat{\mathbf{p}}' - \hat{\mathbf{p}})\delta_{\beta\beta'}\,. \tag{7}$$

As the next step, we introduce a two-particle state of definite angular momentum in the center-of-mass frame. The reason is that the unitarity relations have a particularly simple form for the partial-wave amplitudes [41–44] (PWAs). The relevant state is given by

$$|l, m, \mathcal{P}^{\mu}, \beta\rangle = \frac{1}{\sqrt{4\pi}}\int \mathrm{d}\hat{\mathbf{p}}\; Y_l^m(\hat{\mathbf{p}})|\psi, \beta\rangle\,, \tag{8}$$

where the integration runs over the unit sphere and Y_l^m is the spherical function,

$$Y_l^m(\hat{\mathbf{p}}) = (-1)^{(|m|-m)/2}\exp(\mathrm{i}m\phi)\sqrt{\frac{(2l+1)}{4\pi}\,\frac{(l-|m|)!}{(l+|m|)!}}\,P_{l|m|}(\cos\theta)\,,$$

where we use the notation $P_{lm}(x) = (-1)^m P_l^m(x)$; the Y_l^m functions obey orthonormality

$$\int \mathrm{d}\hat{\mathbf{p}}\; Y_l^m(\hat{\mathbf{p}})\, Y_{l'}^{m'*}(\hat{\mathbf{p}}) = \delta_{ll'}\delta_{mm'}\,. \tag{9}$$

Let us also note that l is a standard notation for angular momentum and m is its projection on some axis. The scalar product of these states is found from (7):

$$\langle l', m', \mathcal{P}^{\mu\,\prime}, \beta'|l, m, \mathcal{P}^{\mu}, \beta\rangle = 4\pi u_{1\beta}u_{2\beta}(u_{1\beta} + u_{2\beta})$$
$$\times\,(2\pi)^4\delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu})\delta_{ll'}\delta_{mm'}\delta_{\beta\beta'}\,.$$

Thus, the decomposition of the unit operator reads

$$\mathbb{1} = \int \mathrm{d}^4\mathcal{P}\sum_{l,m,\beta}|l, m, \mathcal{P}^{\mu}, \beta\rangle\langle l, m, \mathcal{P}^{\mu}, \beta|\,\frac{1}{N(\beta)} + \ldots\,, \tag{10}$$

where summation runs over all two-particle states and

$$N(\beta) \equiv 2(2\pi)^5 u_{1\beta}u_{2\beta}(u_{1\beta} + u_{2\beta})\,. \tag{11}$$

The ellipsis in (10) stand for terms with multiparticle states. We omit these terms in what follows and comment later on how they affect the unitarity relation.

Let us now write the partial wave amplitude [45, 46],

$$T^{(l)}_{m'\beta';m\beta} = \langle l, m', \mathcal{P}^{\mu\,\prime}, \beta'|T|l, m, \mathcal{P}^{\mu}, \beta\rangle\,.$$

It is given by

$$T^{(l)}_{m'\beta';m\beta} = \frac{1}{4\pi}\int \mathrm{d}\hat{\mathbf{p}}\int \mathrm{d}\hat{\mathbf{p}}'\; Y_l^m(\hat{\mathbf{p}})\, Y_l^{m'*}(\hat{\mathbf{p}}')\langle\psi', \beta'|T|\psi, \beta\rangle\,.$$

Due to rotational invariance, the T-matrix does not vanish only for $m' = m$ and does not depend on m [45, 46]. Thus, we can write

$$T^{(l)}_{m'\beta';m\beta} = \delta_{m'm}\sum_{\tilde{m}=-l}^{l}\frac{T^{(l)}_{\tilde{m}\beta';\tilde{m}\beta}}{2l+1}\,.$$

We now turn to the addition theorem for the spherical harmonics, which states that the l-order Legendre polynomial, depending on angle γ, can be expressed through the production of spherical harmonics as follows [47]:

$$\sum_{m=-l}^{l} Y_l^{m*}(\hat{\mathbf{p}}')\, Y_l^m(\hat{\mathbf{p}}) = \frac{2l+1}{4\pi}\,P_l(\cos\gamma)\,,$$

where $\gamma \equiv \angle(\hat{\mathbf{p}}', \hat{\mathbf{p}})$ is the angle between the two momenta, and arrive at

$$T^{(l)}_{m'\beta';m\beta} = \frac{\delta_{m'm}}{16\pi^2}\int \mathrm{d}\hat{\mathbf{p}}\int \mathrm{d}\hat{\mathbf{p}}'\,P_l(\cos\gamma)\langle\psi', \beta'|T|\psi, \beta\rangle\,,$$

where, again due to rotational invariance, $\langle\psi', \beta'\,|T|\psi, \beta\rangle$ does not depend on angular variables except for γ. Because of this property, it is straightforward to integrate over all angles but γ and obtain

$$T^{(l)}_{m'\beta';m\beta} = \frac{\delta_{m'm}}{2}\int \mathrm{d}(\cos\gamma)\,P_l(\cos\gamma)\langle\psi', \beta'|T|\psi, \beta\rangle\,.$$

Using (4), we obtain

$$T^{(l)}_{m'\beta';m\beta} = (2\pi)^4\delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu})\,\frac{\delta_{m'm}}{2}\int \mathrm{d}(\cos\gamma)\,P_l(\cos\gamma)M_{\beta'\beta}\,.$$

Finally, we define the partial wave amplitude,

$$a_{l,\beta'\beta} = \frac{1}{32\pi}\int \mathrm{d}(\cos\gamma)\,P_l(\cos\gamma)M_{\beta'\beta}\,, \tag{12}$$

and T-matrix can be written as follows

$$T^{(l)}_{m'\beta';m\beta} = 16\pi(2\pi)^4\delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu})\delta_{m'm}\,a_{l,\beta'\beta}\,. \tag{13}$$

Now, we turn to the unitarity relation. The unitarity of the S matrix, $SS^{\dagger} = S^{\dagger}S = 1$, implies

$$T - T^{\dagger} = \mathrm{i}TT^{\dagger} = \mathrm{i}T^{\dagger}T\,.$$

Inserting the unit operator given by (10) into the right-hand side, we find

$$-\,\mathrm{i}\big(T^{(l)}_{m'\beta';m\beta} - T^{(l)*}_{m\beta;m'\beta'}\big)$$
$$= \int \mathrm{d}^4\mathcal{P}''\sum_{m'',\beta''}\frac{1}{N(\beta'')}\,T^{(l)}_{m'\beta';m''\beta''}T^{(l)*}_{m\beta;m''\beta''}\,.$$

We make use of (13) and recall the definition of $N(\beta)$, Eqn (11), to obtain the unitarity relation in terms of PWAs:

$$-\frac{\mathrm{i}}{2}(a_{l,\alpha\beta} - a^*_{l,\beta\alpha}) = \sum_{\gamma}\frac{2}{u_{1\gamma}\,u_{2\gamma}(u_{1\gamma} + u_{2\gamma})}\,a_{l,\alpha\gamma}\,a^*_{l,\beta\gamma}\,,$$

where $u_{1\gamma}$ and $u_{2\gamma}$ are sound speeds of particles in the intermediate state γ.

One often assumes time reversal invariance, which gives $T^{(l)}_{m'\beta';m\beta} = T^{(l)}_{m\beta;m'\beta'}$ and hence $a_{l,\alpha\beta} = a_{l,\beta\alpha}$ (see Appendix and

Refs [45, 46]). In that case, the unitarity relation reads

$$\mathrm{Im}\, a_{l,\alpha\beta} = \sum_{\gamma} \frac{2}{u_{1\gamma}\, u_{2\gamma}(u_{1\gamma} + u_{2\gamma})}\, a_{l,\alpha\gamma}\, a^*_{l,\gamma\beta}\,.$$

For $u_{1\gamma} = u_{2\gamma} = 1$, this relation coincides with the standard one (see, e.g., Refs [41, 48]).

2.2 Identical particles

We now consider the case of identical particles in a pair β. We again define two-particle states as follows:

$$|\psi, \beta\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\, a^\dagger_{p_1} a^\dagger_{p_2}|0\rangle\,,$$

where $\beta = \{\phi_i, \phi_i\}$, while the commutational relation is still given by (2). In the case of identical particles, the normalization of the two-particle state is different from (6):

$$\langle\psi', \beta'|\psi, \beta\rangle = (2\pi)^6 2E_{p_1} 2E_{p_2}\big(\delta^{(3)}(\mathbf{p}'_1 - \mathbf{p}_1)\delta^{(3)}(\mathbf{p}'_2 - \mathbf{p}_2) + \delta^{(3)}(\mathbf{p}'_2 - \mathbf{p}_1)\delta^{(3)}(\mathbf{p}'_1 - \mathbf{p}_2)\big)\delta_{\beta\beta'}\,. \tag{14}$$

We proceed along the same lines as in Section 2.1. The change of variables in (14) gives

$$\langle\psi', \beta'|\psi, \beta\rangle = (2\pi)^6\, 8u^3_\beta\, \delta^{(4)}(\mathcal{P}^{\mu\prime} - \mathcal{P}^\mu) \times \big(\delta^{(2)}(\hat{\mathbf{p}}' - \hat{\mathbf{p}}) + \delta^{(2)}(\hat{\mathbf{p}}' + \hat{\mathbf{p}})\big)\delta_{\beta\beta'}\,,$$

where $u_\beta \equiv u_i$ is the sound speed of the particle ϕ_i. The states of definite angular momentum are still given by (8), but the scalar product of these states is now

$$\langle l', m', \mathcal{P}^{\mu\prime}, \beta'|l, m, \mathcal{P}^\mu, \beta\rangle = \frac{1}{4\pi}\int \mathrm{d}\hat{\mathbf{p}}\, (2\pi)^2\, 8u^3_\beta\, (2\pi)^4\delta^{(4)}(\mathcal{P}^{\mu\prime} - \mathcal{P}^\mu)\delta_{\beta\beta'} \times \big(Y^m_l(\hat{\mathbf{p}})\, Y^{m'*}_{l'}(\hat{\mathbf{p}}) + Y^m_l(\hat{\mathbf{p}})\, Y^{m'*}_{l'}(-\hat{\mathbf{p}})\big)\,. \tag{15}$$

Since identical scalars always have even[3] l, we consider even l until the end of this subsection without mentioning this explicitly. Making use of the properties of the spherical functions, Eqns (9), and (A.2a), we get

$$\langle l', m', \mathcal{P}^{\mu\prime}, \beta'|l, m, \mathcal{P}^\mu, \beta\rangle = 2\pi\, 8u^3_\beta\, (2\pi)^4\delta^{(4)}(\mathcal{P}^{\mu\prime} - \mathcal{P}^\mu)\delta_{ll'}\delta_{mm'}\delta_{\beta\beta'}\,,$$

and the contribution of two-particle states with identical particles to the decomposition of the unit operator reads

$$\mathbb{1} = \int \mathrm{d}^4\mathcal{P} \sum_{l,m,\beta} |l, m, \mathcal{P}^\mu, \beta\rangle\langle l, m, \mathcal{P}^\mu, \beta|\, \frac{1}{N_{\mathrm{identical}}(\beta)} + \ldots,$$

where

$$N_{\mathrm{identical}}(\beta) \equiv (2\pi)^5\, 8u^3_\beta\,.$$

Note that $N_{\mathrm{identical}}(u_\beta) = 2N(u_{1\beta}, u_{2\beta})|_{u_{1\beta}=u_{2\beta}=u_\beta}$, where N has been introduced in (11), i.e., if all particles have the same sound speed, then the normalization factor N is twice as large for identical particles, as opposed to distinguishable particles. We repeat the calculations in Section 2.1 and find that the contribution to the PWA unitarity relation from intermediate states γ with two identical particles is given by

$$-\frac{\mathrm{i}}{2}\big(a_{l,\alpha\beta} - a^*_{l,\beta\alpha}\big) = \sum_\gamma \frac{1}{2u^3_\gamma}\, a_{l,\alpha\gamma} a^*_{l,\beta\gamma} + \ldots\,.$$

In a T-invariant (i.e., there is the time-reversal invariance) theory one has

$$\mathrm{Im}\, a_{l,\alpha\beta} = \sum_\gamma \frac{1}{2u^3_\gamma}\, a_{l,\alpha\gamma}\, a^*_{l,\gamma\beta} + \ldots\,.$$

The ellipses in formulas above mean the multiparticle state terms. This is consistent with Refs [44, 48]: if all particles have the same sound speed, then the contribution of identical particles in the intermediate state has the extra factor $1/2$, as opposed to distinguishable particles.

2.3 Unitary bound

We combine the results of Sections 2.1 and 2.2 and write the PWA unitarity relation as follows:

$$-\frac{\mathrm{i}}{2}\big(a_{l,\alpha\beta} - a^*_{l,\beta\alpha}\big) = \sum_\gamma g_\gamma a_{l,\alpha\gamma} a^*_{l,\beta\gamma}\,, \tag{16}$$

where

$$g_\gamma = \frac{2}{u_{1\gamma}\, u_{2\gamma}(u_{1\gamma} + u_{2\gamma})} \quad \text{distinguishable}\,, \tag{17a}$$

$$g_\gamma = \frac{1}{2u^3_\gamma} \quad \text{identical}\,, \tag{17b}$$

where Eqns (17a) and (17b) refer to distinguishable and identical particles in the two-particle intermediate state, respectively. We still do not write contributions explicitly due to multiparticle intermediate states. We note in passing that Eqn (16) can be written in the matrix form,

$$-\frac{\mathrm{i}}{2}(a_l - a^\dagger_l) = a_l\, g\, a^\dagger_l\,,$$

where g is the diagonal matrix with matrix elements g_γ.

To obtain the unitary bound, we introduce rescaled amplitudes $\tilde{a}_{l,\alpha\beta}$ via

$$a_{l,\alpha\beta} = \frac{\tilde{a}_{l,\alpha\beta}}{\sqrt{g_\alpha g_\beta}}\,. \tag{18}$$

In terms of the rescaled amplitudes, we write the unitarity relation (16) in a simpler form:

$$-\frac{\mathrm{i}}{2}\big(\tilde{a}_{l,\alpha\beta} - \tilde{a}^*_{l,\beta\alpha}\big) = \sum_\gamma \tilde{a}_{l,\alpha\gamma}\tilde{a}^*_{l,\beta\gamma} + \sum_M A_{l,\alpha M} A^*_{l,M\beta}\,, \tag{19}$$

or in matrix form:

$$-\frac{\mathrm{i}}{2}\big(\tilde{a}_l - \tilde{a}^\dagger_l\big) = \tilde{a}_l\tilde{a}^\dagger_l + A_l A^\dagger_l\,, \tag{20}$$

where we restore the contribution of multiparticle intermediate states M on the right-hand side and denote schematically the (rescaled) amplitude $2 \to M$ by $A_{l,\alpha M}$.

Now, let us introduce Hermitean matrices

$$P_l = -\frac{\mathrm{i}}{2}(\tilde{a}_l - \tilde{a}^\dagger_l)\,,$$

$$Q_l = \frac{1}{2}(\tilde{a}_l + \tilde{a}^\dagger_l)\,.$$

[3] This can be seen also from Eqn (15): the integral on the right-hand side vanishes for odd l.

It can be written through introduced matrices and the rescaled partial amplitude as

$$\tilde{a}_l = Q_l + \mathrm{i}P_l .$$

Then, the unitarity relation reads

$$P_l = P_l^2 + Q_l^2 + A_l A_l^\dagger - \mathrm{i}[P, Q] . \tag{21}$$

We now choose the orthonormal basis in two-particle state space in such a way that the Hermitean matrix P_l is diagonal:

$$P_{l,\alpha\beta} = p_{l,\alpha}\delta_{\alpha\beta} .$$

In other words, this basis consists of those linear combinations of states with two particles of definite types which are eigenvectors of P_l. Then, the diagonal $\alpha\alpha$-component of Eqn (21) is (no summation over α)

$$p_{l,\alpha} = p_{l,\alpha}^2 + (Q_l^2)_{\alpha\alpha} + (A_l A_l^\dagger)_{\alpha\alpha} .$$

Diagonal elements of matrices $Q_l^2 \equiv Q_l Q_l^\dagger$ and $A_l A_l^\dagger$ are nonnegative,[4] so that

$$p_{l,\alpha}^2 - p_{l,\alpha} \leqslant 0 ,$$

and therefore

$$0 \leqslant p_{l,\alpha} \leqslant 1 .$$

To cast this relation into a somewhat more familiar form, we come back to the unitarity relation (20), sandwich it between an arbitrary vector $|\psi\rangle$ of the unit norm, and write, still using the basis of eigenvectors of P_l,

$$\langle\psi|\tilde{a}_l\tilde{a}_l^\dagger|\psi\rangle = \sum_\alpha p_{l,\alpha}|\psi_\alpha|^2 - \langle\psi|A_l A_l^\dagger|\psi\rangle .$$

This gives

$$\langle\psi|\tilde{a}_l\tilde{a}_l^\dagger|\psi\rangle \leqslant 1$$

for all $|\psi\rangle$, and we arrive at the result that

$$\text{no eigenvalues of } \tilde{a}_l\tilde{a}_l^\dagger \text{ are greater than } 1 . \tag{22}$$

Until now, we have been working in full generalities. To the best of our knowledge, previous analyses not only were restricted to the unit sound speed, but also studied somewhat less general situations (see, e.g, Refs [48–51]). Namely, (i) the matrix $\tilde{a}_{l,\alpha\beta}$ was assumed to be symmetric due to T-invariance, $\tilde{a}_{l,\alpha\beta} = \tilde{a}_{l,\beta\alpha}$. Then Q_l and P_l are its real and imaginary parts, respectively. (ii) It was further assumed that P_l and Q_l are simultaneously diagonalizable. This property holds, in particular, when there is just one type of particle and also when the contribution of multiparticle states is negligible in (20): in the latter case, the imaginary part of Eqn (21) gives $[P, Q] = 0$. In this situation, Eqn (22) says that any eigenvalue $\tilde{a}_{\alpha\alpha}$ of matrix $\tilde{a}$ obeys $|\tilde{a}_{\alpha\alpha}| \leqslant 1$. In fact, in this case, one obtains a slightly stronger bound [52]. In the basis of eigenvectors of $\tilde{a}$ (i.e., common eigenvectors of Q and P), one writes the diagonal part of the unitarity relation (19) for each α (no summation over α):

$$\mathrm{Im}\,\tilde{a}_{l,\alpha\alpha} = \tilde{a}_{l,\alpha\alpha}\tilde{a}_{l,\alpha\alpha}^* + \sum_M A_{l,\alpha M}A_{l,M\alpha}^* .$$

Again, the contribution of multi-particle intermediate states is nonnegative, so we arrive at the inequality

$$\mathrm{Im}\,\tilde{a}_{l,\alpha\alpha} \geqslant |\tilde{a}_{l,\alpha\alpha}|^2 .$$

This gives

$$\left(\mathrm{Im}\,\tilde{a}_{l,\alpha\alpha} - \frac{1}{2}\right)^2 + (\mathrm{Re}\,\tilde{a}_{l,\alpha\alpha})^2 \leqslant \frac{1}{4}$$

and, therefore,

$$|\mathrm{Re}\,\tilde{a}_{l,\alpha\alpha}| \leqslant \frac{1}{2} \tag{23}$$

for any eigenvalue of $\tilde{a}$.

The last special situation is particularly relevant when it comes to perturbative unitarity and estimating the *strong coupling* scale [49–52] in corresponding EFT, where the strong coupling problem indeed potentially arises. In that case, the multiparticle intermediate states (almost) always make contributions to (20) which are indeed suppressed by extra powers of the couplings, while the matrix $\tilde{a}$ is real at the tree level. Perturbative unitarity then requires that inequality (23) hold for the tree level amplitudes. Note, however, that bounds (22) and (23) are qualitatively the same, even in this situation.

2.4 Example: theory of two real scalar fields

In this section, we show explicitly that the unitarity relation (16) holds at the lowest nontrivial order in a model of two real scalar fields. Note that this model does not have a strong coupling energy scale and is not particularly relevant for cosmological perturbations.[5]

For our simple model, the Lagrangian reads

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left(\dot{\phi}_1^2 - u_1^2(\nabla\phi_1)^2\right) + \frac{1}{2}\left(\dot{\phi}_2^2 - u_2^2(\nabla\phi_2)^2\right) \\ &+ \frac{\lambda_1}{4!}\,\phi_1^4 + \frac{\lambda_2}{4!}\,\phi_2^4 + \frac{\lambda_3}{4}\,\phi_1^2\phi_2^2 ,\end{aligned} \tag{24}$$

where u_1 and u_2 are the two sound speeds. The scalar potential in Eqn (24) is a general fourth-order homogeneous polynomial symmetric under the transformation $\phi_{1,2} \to -\phi_{1,2}$. In this theory, the PWA matrix $a_{\alpha\beta}$ is symmetric due to T-invariance, so the unitarity relation is

$$\mathrm{Im}\,a_{l,\alpha\beta} = \sum_\gamma g_\gamma a_{l,\alpha\gamma}a_{l,\gamma\beta}^*$$

or, in matrix form,

$$\mathrm{Im}\,a_l = \sum_\gamma a_l g a_l^\dagger , \tag{25}$$

where elements of the diagonal matrix g are still given by Eqn (17).

[4] Because, e.g., $0 \leqslant \langle\psi^{(\alpha)}|A_l A_l^\dagger|\psi^{(\alpha)}\rangle = (A_l A_l^\dagger)_{\alpha\alpha}$ for $\psi_\beta^{(\alpha)} = \delta_{\alpha\beta}$.

[5] However, in [40], where the authors consider the model of Horndeski bounce, formula (16) and corresponding unitarity bounds are used to evaluate the quantum energy scale of strong coupling.

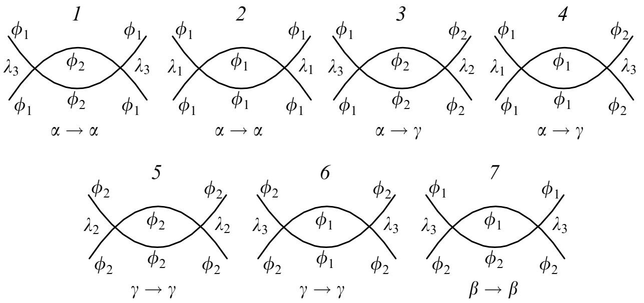

Figure 1. One-loop s-channel diagrams in the theory with Lagrangian (24).

The beginning of the calculation follows textbooks. There are three two-particle states $\alpha = (\phi_1, \phi_1)$, $\beta = (\phi_1, \phi_2)$, and $\gamma = (\phi_2, \phi_2)$ in this theory. The tree-level matrix elements make the matrix

$$M_{\text{tree}} = \begin{pmatrix} M_{\alpha\alpha} & M_{\alpha\beta} & M_{\alpha\gamma} \\ M_{\beta\alpha} & M_{\beta\beta} & M_{\beta\gamma} \\ M_{\gamma\alpha} & M_{\gamma\beta} & M_{\gamma\gamma} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_3 & 0 \\ \lambda_3 & 0 & \lambda_2 \end{pmatrix}.$$

Since these matrix elements do not depend on scattering angle γ, the only nonzero PWA, as given by Eqn (12), is a_0, i.e., scattering occurs in the s wave.

The matrix of these PWAs is given by

$$a_{0,\,\text{tree}} = \frac{1}{32\pi}\int_{-1}^{1} \mathrm{d}(\cos\theta)\, P_0(\cos\theta) M_{\text{tree}} = \frac{M_{\text{tree}}}{16\pi} = \frac{1}{16\pi}\begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_3 & 0 \\ \lambda_3 & 0 & \lambda_2 \end{pmatrix}. \tag{26}$$

As usual, in QFT, the right-hand side of (25) is of order $\lambda_i \lambda_j$, so $\operatorname{Im} a_l$ obtains its lowest-order contribution in one loop. This contribution comes from s-channel diagrams shown in Fig. 1, while t- and u-channel diagrams make no contribution to the imaginary part in one loop.

We begin with the first diagram in Fig. 1. It gives the one-loop contribution to matrix element

$$\mathrm{i}M^{(1)}_{\text{1-loop}} = \frac{\lambda_3^2}{2}\int \frac{\mathrm{d}^4 q}{(2\pi)^4}\left\{\left[\left(\frac{E}{2} - q^0\right)^2 - u_2^2 \mathbf{q}^2 + \mathrm{i}\epsilon\right]\right. \times \left.\left[\left(\frac{E}{2} + q^0\right)^2 - u_2^2 \mathbf{q}^2 + \mathrm{i}\epsilon\right]\right\}^{-1},$$

where E is still the total energy in the center-of-mass frame. Upon rescaling $u_2 \mathbf{q} \to \mathbf{q}$, a textbook calculation gives

$$\operatorname{Im} M^{(1)}_{\text{1-loop}} = \frac{\lambda_3^2}{32\pi u_2^3}.$$

Likewise, diagrams 2–6 in Fig. 1 give

$$\operatorname{Im} M^{(2)}_{\text{1-loop}} = \frac{\lambda_1^2}{32\pi u_1^3}, \quad \operatorname{Im} M^{(3)}_{\text{1-loop}} = \frac{\lambda_2 \lambda_3}{32\pi u_2^3},$$

$$\operatorname{Im} M^{(4)}_{\text{1-loop}} = \frac{\lambda_1 \lambda_3}{32\pi u_1^3}, \quad \operatorname{Im} M^{(5)}_{\text{1-loop}} = \frac{\lambda_2^2}{32\pi u_2^3},$$

$$\operatorname{Im} M^{(6)}_{\text{1-loop}} = \frac{\lambda_3^2}{32\pi u_1^3}.$$

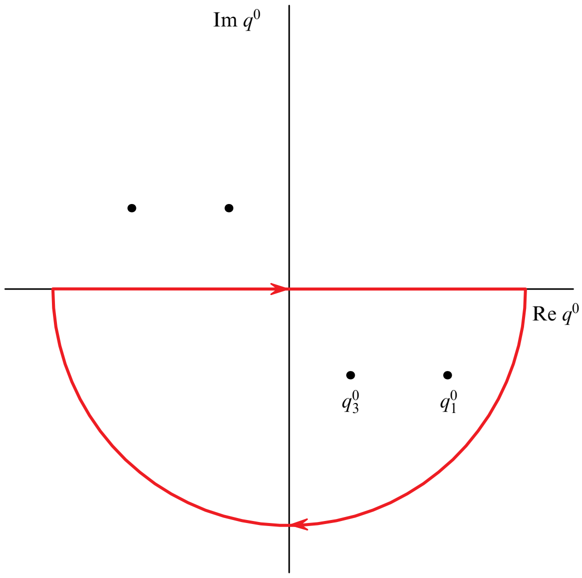

Figure 2. Integration contour relevant to Eqn (27).

We now turn to diagram 7 in Fig. 1. Unlike the others, it has two different particles in the loop. We write

$$\mathrm{i}M^{(7)}_{\text{1-loop}} = \lambda_3^2 \int \frac{\mathrm{d}^4 q}{(2\pi)^4}\left\{\left[\left(\frac{E}{2} - q^0\right)^2 - u_1^2 \mathbf{q}^2 + \mathrm{i}\epsilon\right]\right. \times \left.\left[\left(\frac{E}{2} + q^0\right)^2 - u_2^2 \mathbf{q}^2 + \mathrm{i}\epsilon\right]\right\}^{-1}. \tag{27}$$

There are four poles of the integrand at

$$q^0_{1,2} = \frac{E}{2} \pm u_1 |\mathbf{q}| \mp \mathrm{i}\epsilon,$$

$$q^0_{3,4} = -\frac{E}{2} \pm u_2 |\mathbf{q}| \mp \mathrm{i}\epsilon.$$

Without loss of generality, we assume

$$u_1 \geqslant u_2.$$

Then, it is convenient to close the integration contour as shown in Fig. 2; the poles inside it are at q_1^0 and q_3^0. We integrate over q^0 and get

$$\mathrm{i}M^{(7)}_{\text{1-loop}} = \lambda_3^2 \int \frac{\mathrm{d}^3 q}{(2\pi)^4}(-2\pi\mathrm{i}) \times \left[\frac{1}{2qu_1\big(E + q(u_1 - u_2)\big)\big(E + q(u_1 + u_2)\big)} + \frac{1}{(-2qu_2)\big(-E + q(u_1 + u_2) - \mathrm{i}\epsilon\big)\big(E + q(u_1 - u_2)\big)}\right].$$

The first term in the integrand does not contribute to $\operatorname{Im} M^{(7)}_{\text{1-loop}}$. The imaginary part due to the second term is calculated using the Sokhotski–Plemelj formula,

$$\lim_{\epsilon \to 0^+}\left(\frac{1}{x \pm \mathrm{i}\epsilon}\right) = \mp \mathrm{i}\pi\delta(x) + P\left(\frac{1}{x}\right),$$

where P stands for the principal value. We find

$$\mathrm{Im}\, M^{(7)}_{1\text{-loop}} = \lambda_3^2 \int \frac{\mathrm{d}^3 q}{(2\pi)^3} \frac{1}{2qu_2\big(E + q(u_1 - u_2)\big)} \times \pi\delta\big(-E + q(u_1 + u_2)\big)\,,$$

and, finally,

$$\mathrm{Im}\, M^{(7)}_{1\text{-loop}} = \frac{\lambda_3^2}{8\pi u_1 u_2 (u_1 + u_2)}\,.$$

To sum up, we collect all results in one matrix:

$$\mathrm{Im}\, a_{0,\,1\text{-loop}} = \frac{1}{16\pi}\,\mathrm{Im}\, M_{1\text{-loop}} = \frac{1}{16\pi}\begin{pmatrix} \frac{\lambda_1^2}{32\pi u_1^3} + \frac{\lambda_3^2}{32\pi u_2^3} & 0 & \frac{\lambda_1\lambda_3}{32\pi u_1^3} + \frac{\lambda_2\lambda_3}{32\pi u_2^3} \\ 0 & \frac{\lambda_3^2}{8\pi u_1 u_2 (u_1+u_2)} & 0 \\ \frac{\lambda_1\lambda_3}{32\pi u_1^3} + \frac{\lambda_2\lambda_3}{32\pi u_2^3} & 0 & \frac{\lambda_2^2}{32\pi u_2^3} + \frac{\lambda_3^2}{32\pi u_1^3} \end{pmatrix}. \quad (28)$$

Now, Eqn (17) gives for matrix g in (25)

$$g = \mathrm{diag}\left(\frac{1}{2u_1^3}\,, \frac{2}{u_1 u_2 (u_1 + u_2)}\,, \frac{1}{2u_2^3}\right). \quad (29)$$

Making use of Eqns (26), (28), and (29), we find that

$$\mathrm{Im}\, a_{0,\,1\text{-loop}} = a_{0,\,\text{tree}}\, g\, a_{0,\,\text{tree}}\,,$$

i.e., the unitarity relation (25) is indeed valid to the lowest nontrivial order in the couplings.

Finally, we would like to note that, for example, for relatively large negative λ_3, relation (22) does not hold. However, this is not a sign of the strong coupling regime. The reason for the violation of (22) is that the potential in Lagrangian (24) for relatively large negative λ_3 is unbounded from below, and the theory becomes physically unacceptable.

3. Conclusions

In this paper, we found PWA unitarity relations (16) in a theory containing massless scalar fields with different sound speeds. We illustrated these relations in a model with Lagrangian (24) to the lowest nontrivial order in the couplings. When written in terms of rescaled amplitudes (18), the unitarity relations have a particularly simple form (19), which is formally the same as in a theory with unit sound speeds.

Using the unitarity relations, we derived the unitarity bounds, which in the most general case have the form (22), and, in the (still quite general) case considered in the literature, reduce to the familiar form (23) (but written in terms of rescaled amplitudes). The latter form is particularly useful for evaluating the quantum strong coupling scale in pertinent EFT. It is worth stressing here that we consider the model with Lagrangian (24) in Section 2.4 only as a clear example of how to calculate unitarity relation (16) for a concrete theory at the lowest nontrivial order. For sure, there is no strong coupling energy scale in the theory (24). However, in [40], we consider another model, Horndeski bounce, which has strong gravity in the past, i.e., effective Planck mass runs to zero. There are two different types of perturbations with, generally speaking, different sound speeds in this model. There, we use formula (23) in order to evaluate the quantum energy scale of strong coupling.

We anticipate, however, that the results of this paper may have applications in other theories where different perturbations about nontrivial backgrounds propagate with different sound speeds. For example, in condensed matter studies, different sound speeds effectively emerge in different setups corresponding to anisotropic media [53–55]. In such media, once unitarity is ensured, a corresponding model may become a useful and interesting tool to analyze condensed matter systems with anisotropic properties (see, e.g., Refs [56, 57]).

Finally, one can turn to unitarity in theories with violated Lorentz invariance. For example, in Ref. [58], the authors develop a systematic approach to the calculation of scattering cross sections in theories with violation of the Lorentz invariance and apply it to compute the probabilities of several astrophysically relevant processes. In this case, it is interesting to understand if unitarity even holds in these kinds of models (see, e.g, Ref. [59], where the author checks the optical theorem for the concrete physical process). If so, one can also calculate the corresponding unitarity relation and unitarity bounds which can be useful for analyzing the cutoff energy scale of the concrete physical theory.

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4. Appendix. Time-reversal invariance and symmetry of S matrix

In this Appendix, we show that T-invariance of the S matrix implies the symmetry of the partial-wave amplitudes,

$$T^{(l)}_{\beta'\beta} = T^{(l)}_{\beta\beta'}\,. \quad (\mathrm{A}.1)$$

T-invariance of the S matrix is invariance under the exchange of initial and final states and sign reversal of all spatial momenta:

$$\langle \mathbf{p}', \beta' |\, S\, | \mathbf{p}, \beta \rangle = \langle -\mathbf{p}, \beta |\, S\, | -\mathbf{p}', \beta' \rangle\,.$$

We make use of this property to write (we work in the center-of-mass frame)

$$\begin{aligned} \langle l, m; \beta' |\, S\, | l, m; \beta \rangle &= \frac{1}{4\pi} \int \mathrm{d}^3\hat{\mathbf{p}}'\, \mathrm{d}^3\hat{\mathbf{p}}\; Y_l^{m*}(\hat{\mathbf{p}}')\, Y_l^m(\hat{\mathbf{p}}) \langle \mathbf{p}', \beta' |\, S\, | \mathbf{p}, \beta \rangle \\ &= \frac{1}{4\pi} \int \mathrm{d}^3\hat{\mathbf{p}}'\, \mathrm{d}^3\hat{\mathbf{p}}\; Y_l^{m*}(\hat{\mathbf{p}}')\, Y_l^m(\hat{\mathbf{p}}) \langle -\mathbf{p}, \beta |\, S\, | -\mathbf{p}', \beta' \rangle \\ &= \frac{1}{4\pi} \int \mathrm{d}^3(-\hat{\mathbf{p}}')\, \mathrm{d}^3(-\hat{\mathbf{p}})\; Y_l^{m*}(-\hat{\mathbf{p}}')\, Y_l^m(-\hat{\mathbf{p}}) \langle \mathbf{p}, \beta |\, S\, | \mathbf{p}', \beta' \rangle\,. \end{aligned}$$

Now, the spherical functions obey

$$Y_l^m(-\hat{\mathbf{p}}) = (-1)^l\, Y_l^m(\hat{\mathbf{p}})\,, \quad (\mathrm{A}.2\mathrm{a})$$

$$Y_l^{m*}(\hat{\mathbf{p}}) = (-1)^m\, Y_l^{-m}(\hat{\mathbf{p}})\,, \quad (\mathrm{A}.2\mathrm{b})$$

so that

$$Y_l^{m*}(-\hat{\mathbf{p}}) = (-1)^{l+m}\, Y_l^{-m}(\hat{\mathbf{p}})\,.$$

This gives

$$\begin{aligned}&\langle lm;\beta'|S|lm;\beta\rangle\\&\quad=\frac{1}{4\pi}\iint \mathrm{d}^3\hat{\mathbf{p}}'\,\mathrm{d}^3\hat{\mathbf{p}}\;Y_l^{-m}(\hat{\mathbf{p}}')\,Y_l^{-m*}(\hat{\mathbf{p}})\langle\mathbf{p},\beta|S|\mathbf{p}',\beta'\rangle\\&\quad=\langle l,\,-m;\,\beta|S|l,\,-m;\,\beta'\rangle\,.\end{aligned}$$

Since these matrix elements are actually independent of m, this proves relation (A.1).

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