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METHODOLOGICAL NOTES

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Vavilov–Cherenkov radiation and radiation energy loss

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Contents

Abstract. **The radiative energy loss of a relativistic charge in media with photon absorption and scattering is discussed. The solution to the Tamm problem in a medium with absorption is given. Expressions for the radiation loss are given with the inclusion of multiple scattering of the radiating charge.**

Keywords: Vavilov–Cherenkov radiation, radiative energy loss

1. Introduction

Vavilov–Cherenkov radiation (VCR) [1] is one of the most striking effects of the electrodynamics of continuous media. Its theoretical interpretation in terms of the emission of light by a uniformly moving charge with a speed exceeding the speed of light in the medium was given by I E Tamm and I M Frank [2]. The spectrum of the average radiation energy from a unit charge trajectory, $\mathrm{d}^2\bar{\mathit{\Delta}}/\hbar\,\mathrm{d}\omega\,\mathrm{d}x$, for example, an electron, is given by the well-known Tamm–Frank formula:

$$\frac{\mathrm{d}^2\bar{\mathit{\Delta}}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x} = \hbar\omega\frac{\mathrm{d}^2\bar{N}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x} = \frac{\alpha\omega}{c}\left(1 - \frac{1}{\beta^2 n^2}\right), \quad (1)$$
$$\cos\theta = \frac{1}{\beta n} < 1\,.$$

Here, $\bar{\mathit{\Delta}}$ and $\bar{N}$ represent the emitted energy and the number of photons, and the relation $\mathrm{d}\bar{\mathit{\Delta}} = \hbar\omega\,\mathrm{d}\bar{N}$ is valid in a narrow frequency range $(\omega, \omega + \mathrm{d}\omega)$. The speed of the charge in units of the speed of light is $\beta = v/c$, and the medium is characterized by the refractive index n. The radiation is directed forward along the charge motion at a fixed angle θ relative to the direction of the particle's velocity. The emitted energy and the number of photons are distributed according to the generalized and ordinary Poisson distributions, respectively, and formula (1) includes their mean values in this sense. Uniformity of motion is implied in the section of the trajectory which is much longer than the radiation length, formally infinite in Ref. [2]. Note that the condition of speed constancy is approximate, since, for a relativistic charge with Lorentz factor $\gamma \gg 1$, the relative change in speed is about γ^2 times less than the relative change in energy, and the loss of speed can be ignored. Formula (1) is valid in a transparent medium, when radiation is observed far away from the charge trajectory. In real media, however, there is absorption and scattering of photons; therefore, from a methodological point of view, it is of interest to consider a generalization of formula (1) for these cases, including the effect of multiple charge scattering.

In this paper, proceeding from the Umov–Poynting theorem, a generalization of the VCR for the case of absorbing and scattering media will be formulated in terms of radiative energy loss. Several methodologically interesting options for the trajectory of charge motion will be considered, including motion with multiple scattering.

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2. Umov–Poynting theorem and radiative energy loss of a relativistic charge

The Umov–Poynting theorem [3] makes it possible to clarify what can be meant by an analogue of VCR in absorbing media. It is usually formulated as a balance of changes in the energy of the electromagnetic field due to the work done by external currents minus the flow of field energy through the surface of the volume under consideration. The Umov–Poynting theorem is determined by the ratio in which, for our purposes, the work of external currents is moved to the left side, and the field energy, to the right (the system of Gauss units is used, $\alpha = e^2/(\hbar c) \simeq 1/137$):

$$-\mathbf{j}^e\mathbf{E} = \frac{1}{4\pi}\left\{\mathbf{E}\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\frac{\partial\mathbf{B}}{\partial t}\right\} + \operatorname{div}\mathbf{S}\,, \quad (2)$$

where $\mathbf{E}$ and $\mathbf{D}$ are the electric field and induction, respectively, and $\mathbf{H}$ and $\mathbf{B}$ are the magnetic field and induction. The density of the current external to the medium, for simplicity of a point charge e, $\mathbf{j}^e = e\mathbf{v}\delta(\mathbf{r} - \mathbf{v}t)$ ($\mathbf{r}$ is the spatial position of the charge at the point in time t, and δ is the Dirac delta function). The Umov–Poynting vector is given by the relation

$$\mathbf{S} = \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}\,. \tag{3}$$

Expression (2) defines the energy loss of an external charge per unit volume of the medium per unit time. This loss is distributed between the change in the energy of the electromagnetic field in a unit volume surrounding the charge and the energy flow of the electromagnetic field through the surface of this volume. It is the calculation of the flow of the Umov–Poynting vector through a cylindrical surface surrounding a uniformly moving charge that led the authors of Ref. [2] to relation (1).

If the medium is absorbing, then it is necessary to not consider the flow of the Umov–Poynting vector, but directly the energy loss of the external charge, i.e., the left side of relation (2). This loss represents the work of the charge against the electric field created by it and the medium at the point where the charge is located (Landau's method [4]). In order to single out the part in this loss that corresponds to the VCR in a transparent medium, we more with the help of the Fourier transform to the space of frequencies and wave vectors $\mathbf{k}$; for example, for an electric field, we have

$$\mathbf{E}(\mathbf{r},t) = \int\!\!\int \frac{\mathrm{d}\mathbf{k}\,\mathrm{d}\omega}{(2\pi)^4}\,\mathbf{E}(\mathbf{k},\omega)\exp\left[\mathrm{i}(\mathbf{kr}-\omega t)\right], \tag{4}$$

the inverse transformation being given by integrals over an infinitely large spatial volume in units of the radiation wavelength and a time interval much longer than the reciprocal frequency of the radiation (formally, the limits can be assumed to be infinite). The medium is assumed to be isotropic, homogeneous, and nonrelativistic to the extent that magnetism is allowed to exist.

By dividing $\mathbf{E}(\mathbf{k},\omega)$ into components parallel and perpendicular to the wave vector, we define the average radiative energy loss, i.e., the loss for the production of photons in the medium as work against the electric field component perpendicular to the wave vector,

$$\mathbf{E}_\perp(\mathbf{k},\omega)\,\mathbf{k} = 0\,.$$

Such a definition corresponds to the concept of a photon in a medium as an excitation of the medium, whose wave vector and the electric field are perpendicular. The average radiative energy loss in the entire space of nonzero current densities and for the entire time of their existence is given by the following expression:

$$\bar{\Delta}_\perp = -\frac{2}{(2\pi)^4}\int_0^\infty \mathrm{d}\omega \int_{K_3} \mathrm{d}\mathbf{k}\,\mathrm{Re}\left(\mathbf{j}^*(\mathbf{k},\omega)\mathbf{E}_\perp(\mathbf{k},\omega)\right). \tag{5}$$

The real part of the integrand and the complex conjugate current density of the particle arose from the contribution of integration over negative frequencies. We express the electric field using the Maxwell equations and the continuity equation in terms of the current density of the external charge to obtain

$$\mathbf{E}_\perp = 4\pi\mathrm{i}\,\mu\,\frac{\omega}{c^2}\,\frac{k^2\mathbf{j}-(\mathbf{k}\,\mathbf{j})\,\mathbf{k}}{k^2\left(k^2-\epsilon\mu\,\omega^2/c^2\right)}\,. \tag{6}$$

Here, μ and ϵ are, respectively, the magnetic and dielectric permittivities of the medium, which depend on the frequency and, in the general case, on the wave vector. This choice of response functions is the most common, but not the only one [5], but for our purposes it will be quite sufficient. Substituting formula (6) into expression (5), we express the radiation loss in terms of the current density and the response functions of the medium:

$$\bar{\Delta}_\perp = -\frac{2}{(2\pi)^4}\int_0^\infty \mathrm{d}\omega\int_{K_3}\mathrm{d}\mathbf{k} \times \mathrm{Re}\left[4\pi\mathrm{i}\mu\,\frac{\omega}{c^2}\,\frac{(\mathbf{k}\,\mathbf{k})(\mathbf{j}\,\mathbf{j}^*)-(\mathbf{k}\,\mathbf{j})(\mathbf{k}\,\mathbf{j}^*)}{k^2(k^2-\epsilon\mu\,\omega^2/c^2)}\right]. \tag{7}$$

For a point external charge e moving along the trajectory $\mathbf{r}(t)$ with speed $\mathbf{v}(t)$, the current density is expressed as $\mathbf{j}(\mathbf{r},t) = e\mathbf{v}(t)\,\delta[\mathbf{r}-\mathbf{r}(t)]$. Its Fourier component is of the form

$$\mathbf{j}(\mathbf{k},\omega) = e\int_{-\infty}^{\infty}\mathrm{d}t\,\mathbf{v}(t)\exp\left(\mathrm{i}\omega t-\mathrm{i}\mathbf{kr}(t)\right), \tag{8}$$

where the integration is carried out over the time domain of nonzero velocity. This expression allows us to pass to the intensity of average radiation loss, as well as to the frequency-angular spectrum of the average number of photons emitted by a moving charge per unit time [6]:

$$\frac{\mathrm{d}^3\bar{N}_\perp(t)}{\hbar\,\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}\Omega} = \frac{1}{\hbar\omega}\,\frac{\mathrm{d}^3\bar{\Delta}_\perp(t)}{\hbar\,\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}\Omega} = \frac{\alpha}{2\pi^3\hbar c} \times \mathrm{Im}\left\{\int_0^\infty \frac{\mu(\omega)\,\mathrm{d}k}{k^2-\epsilon(\omega)\mu(\omega)\,\omega^2/c^2} \times \int_{-\infty}^{\infty}\mathrm{d}\tau\left(k^2\mathbf{v}(t+\tau)\mathbf{v}(t)-\omega^2\right) \times \exp\left[\mathrm{i}\omega\tau-\mathrm{i}\mathbf{k}\big(\mathbf{r}(t+\tau)-\mathbf{r}(t)\big)\right]\right\}, \tag{9}$$

where Ω is the solid angle, and the radiation polar angle θ is measured from the direction of the velocity $\mathbf{v}(t)$, i.e., $\mathrm{d}\mathbf{k} = k^2\,\mathrm{d}k\,\mathrm{d}\Omega = 2\pi k^2\,\mathrm{d}k\,\mathrm{d}\cos\theta$. Expression (9), obtained by the method of classical electrodynamics, disregards quantum recoil effects. The time dependence of the velocity will be required below when considering radiation from a finite trajectory and a multiply scattered charge.

3. Radiation energy loss of a uniformly moving charge

In the case of uniform motion with velocity $\mathbf{v}$ ($\mathbf{r}(t) = \mathbf{v}t$), the integral over τ in expression (9) leads to the following result:

$$\int_{-\infty}^{\infty}\mathrm{d}\tau\left(k^2v^2-\omega^2\right)\exp\left[\mathrm{i}(\omega-\mathbf{kv})\tau\right] = 2\pi k^2v^2\sin^2\theta\,\delta(\mathbf{kv}-\omega)\,.$$

Then, the frequency-angular spectrum per unit time of the average number of photons emitted by the charge will assume the form

$$\frac{\mathrm{d}^3\bar{N}_\perp(t)}{\hbar\,\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}\Omega} = \frac{\alpha}{2\pi^3\hbar c}\,\mathrm{Im}\left[\int_0^\infty\frac{\mu(\omega)2\pi k^2v^2\sin^2\theta\,\delta(\mathbf{kv}-\omega)\,\mathrm{d}k}{k^2-\epsilon(\omega)\mu(\omega)\,\omega^2/c^2}\right],$$

and, in order to subsequently obtain the angular dependence, the delta function can be used to integrate modulo the wave vector:

$$k = \frac{\omega}{v\cos\theta}\,, \quad \delta\,(\mathbf{kv}-\omega) \to \frac{1}{v\cos\theta}\,,$$

$$\mathrm{d}\Omega = 2\pi\,\mathrm{d}\cos\theta = \pi\frac{\mathrm{d}\cos^2\theta}{\cos\theta}\,.$$

Finally, we have ($\mathrm{d}x = v\,\mathrm{d}t$)

$$\frac{\mathrm{d}^3\bar{N}_{\perp}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta} = \frac{\alpha}{\hbar c}\,\mathrm{Im}\left[\frac{\mu\tan^2\theta}{\pi(1-\epsilon\mu\beta^2\cos^2\theta)}\right].$$

To expand the imaginary part, we introduce the complex variable $\xi = \epsilon\mu = \xi_1 + \mathrm{i}\xi_2$, where $\epsilon = \epsilon_1 + \mathrm{i}\epsilon_2$, $\mu = \mu_1 + \mathrm{i}\mu_2$. Then,

$$\frac{\mathrm{d}^3\bar{N}_{\perp}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta} = \frac{\alpha}{\hbar c}\left[\mu_1\sin^2\theta + \frac{\mu_2\tan^2\theta}{\xi_2\beta^2}\left(1-\xi_1\beta^2\cos^2\theta\right)\right] \times \frac{\Gamma_\xi}{\pi\left[(\cos^2\theta-\cos^2\theta_\xi)^2+\Gamma_\xi^2\right]}\,, \tag{10}$$

$$\cos^2\theta_\xi = \frac{\xi_1}{\beta^2|\xi|^2}\,, \quad \Gamma_\xi = \frac{\xi_2}{\beta^2|\xi|^2}\,,$$

where $\xi_1 = \epsilon_1\mu_1 - \epsilon_2\mu_2$, $\xi_2 = \epsilon_1\mu_2 + \epsilon_2\mu_1$, $|\xi|^2 = (\epsilon_1^2+\epsilon_2^2)\times(\mu_1^2+\mu_2^2)$.

In a nonmagnetic medium ($\mu = 1$), relation (10) reduces to

$$\frac{\mathrm{d}^3\bar{N}_{\perp}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta} = \frac{\alpha}{\pi\hbar c}\,\frac{(1-\cos^2\theta)\Gamma}{(\cos^2\theta-\cos^2\theta_0)^2+\Gamma^2}\,, \tag{11}$$

where

$$\cos^2\theta_0 = \frac{\epsilon_1}{\beta^2|\epsilon|^2}\,, \quad \Gamma = \frac{\epsilon_2}{\beta^2|\epsilon|^2}\,. \tag{12}$$

Expression (11) was obtained by U Fano in Ref. [7] in a slightly different form based on the quantum mechanical approach. One can see that the radiative energy loss of a uniformly moving charge in a medium experiences a Lorentzian angular broadening proportional to the imaginary part of the permittivity. Figure 1 illustrates this broadening in Freon C_6F_{14} for two values of radiation energy: $\hbar\omega = 6.8$ eV (dashed curve) and $\hbar\omega = 7.7$ eV (solid curve), and it is assumed that $v \sim c$ [6]. This broadening, being proportional to the ratio of the emission wavelength to the absorption length, is extremely small in the visible and ultraviolet regions of the spectrum and is usually not visible against the background of chromatic aberration due to the frequency dispersion of the emitted light, which is especially pronounced in the ultraviolet domain.

Integrating expression (11) over the angular variable, $\cos^2\theta$, gives an expression for the spectrum of the average number of photons emitted by a uniformly moving charge from a unit path:

$$\frac{\mathrm{d}^2\bar{N}_{\perp}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x} = \frac{\alpha}{\pi\hbar c}\left[\left(1-\frac{\epsilon_1}{\beta^2|\epsilon|^2}\right)\arg\left(1-\beta^2\epsilon^*\right) + \frac{\epsilon_2}{\beta^2|\epsilon|^2}\ln\left(\frac{1}{|1-\beta^2\epsilon|}\right)\right]. \tag{13}$$

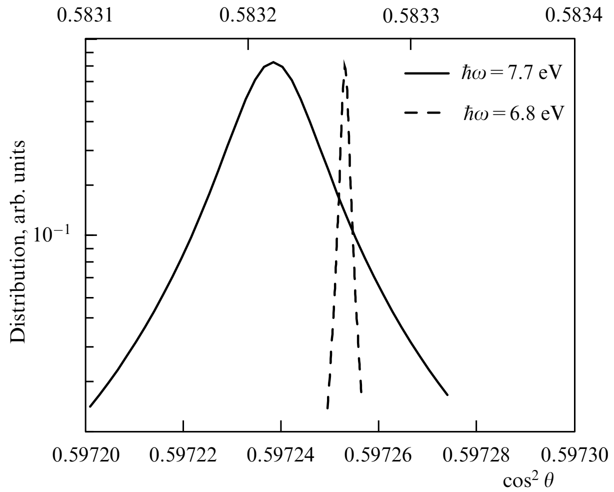

Figure 1. Angular distribution of radiative energy losses of a relativistic charge, $\gamma \gg 1$, in freon C_6F_{14} for radiation energies $\hbar\omega = 6.8$ eV (dashed curve) and $\hbar\omega = 7.7$ eV (solid curve, upper abscissa). Curves are normalized to the same peak value.

Expression (13) was first obtained by E Fermi in Ref. [8], who considered it to be closely related to VCR. Note that expression (13) shows a slow logarithmic increase depending on the Lorentz factor of the relativistic charge, changing in the region $\gamma \sim 10^3$ to a plateau (Fermi plateau), which reflects the density effect noted by E Fermi [9]. Figure 2 illustrates this effect, showing the relative relativistic increase in the ionization loss of charge energy in a 6-cm-thick 93% Ar+7% CH_4 gas mixture at normal pressure and temperature. The curve is the result of calculations using the photoabsorption model of Ref. [10], while the circles are experimental data from Ref. [11] obtained using a drift proportional multiwire chamber. Therefore, the relativistic increase in the ionization loss of a charged particle is entirely determined by the absorption of Cherenkov photons near its trajectory, which makes a contribution to the total losses that grows with increasing Lorentz factor. Note that the photoabsorption model, despite the fact that it does not take into account the spatial dispersion of the permittivity, which is significant near the absorption bands (see, for example, Ref. [4, § 106]), describes the experimental results quite well. The reason is the accuracy of the photoabsorption cross sections used in the model for the outer shells of an atom with an energy of less than 100 eV, which make the main contribution to the ionization energy loss. Therefore, integrally, in terms of average energy losses, the model agrees with the experimental data.

In a medium with random inhomogeneities, for example, in aerogel Cherenkov detectors [12], the electric field and induction, averaged over a volume that includes many inhomogeneities, can be related to the effective permittivity. Being a complex quantity, for a uniformly moving charge it leads to a picture of radiation loss, which is equivalent to an absorbing medium. It is assumed that a volume with a size of the order of the radiation wavelength contains many inhomogeneities. Physically, this is understandable, since both absorbing and scattering media lead to a decrease in photon flux in the direction of their main VCR emission, destroying the coherence inherent in this effect. Compared to detectors based on a homogeneous medium, aerogel Cherenkov detectors show a more pronounced angular aberration of the VCR (which is averaged over a volume that includes

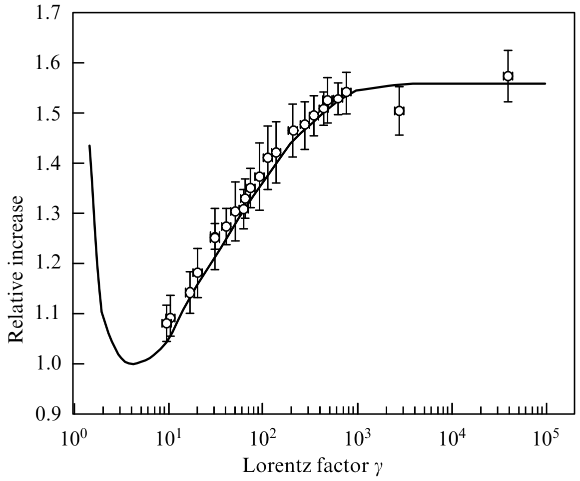

Figure 2. Relative relativistic increase in ionization loss of charge energy in a 6-cm-thick gas mixture of 93% Ar + 7% CH_4 at normal pressure and temperature. Curve is the calculation made according to the photoabsorption model of Ref. [10], circles are the experimental data of Ref. [11] obtained using a drift multiwire proportional chamber.

many inhomogeneities) and a more extended threshold for its occurrence.

In the limit of a transparent medium, when $\Gamma \to 0$, the real permittivity can be expressed in terms of the square of the refractive index, $\epsilon \to \epsilon_1 = n^2(\omega)$. Since the Lorentz line of the angular distribution in expression (11) tends to the delta function in this case, we arrive at the VCR frequency-angular spectrum:

$$\frac{\mathrm{d}^3 \bar{N}_\mathrm{c}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta} = \frac{\alpha}{\hbar c}\left(1 - \frac{1}{\beta^2 n^2}\right)\delta\left(\cos^2\theta - \frac{1}{\beta^2 n^2}\right). \tag{14}$$

This expression explicitly fixes the VCR radiation angle, and subsequent integration over the angle (moreover, the natural angular variable for VCR is $\cos^2\theta$) gives relation (1).

We consider a transparent medium in which the magnetic permeability is not equal to unity, but $\epsilon_2 \to 0$ and $\mu_2 \to 0$. In such a medium, the square of the refractive index is expressed in terms of the dielectric and magnetic permeabilities, $n^2 = \epsilon_1\mu_1$, and the frequency-angular spectrum of Cherenkov photons emitted from a unit path is expressed as

$$\frac{\mathrm{d}^3 \bar{N}_\mathrm{c}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta} = \mathrm{sign}\,(\Gamma_\xi)\,\frac{\alpha\mu_1}{\hbar c}\left(1 - \frac{1}{\beta^2 n^2}\right) \times \delta\left(\cos^2\theta - \frac{1}{\beta^2 n^2}\right). \tag{15}$$

Here, we can consider an interesting case when $\epsilon_1\mu_1 > 0$, but both permeabilities are less than zero, $\epsilon_1 < 0$ and $\mu_1 < 0$ (left media or metamaterials [13, 14]). Then, the radiation follows the group velocity $\partial\omega/\partial\mathbf{k}$, which is opposite to the wave vector $\mathbf{k}$, while the radiation is emitted backwards in the angular range $(\pi/2, \pi)$. Taking into account the sign of $\mathrm{sign}\,(\Gamma_\xi)\mu_1$ allows us to consider not only the emission but also the absorption of Cherenkov photons. Figure 3 shows variants of emission or absorption of Cherenkov photons in conventional media and metamaterials.

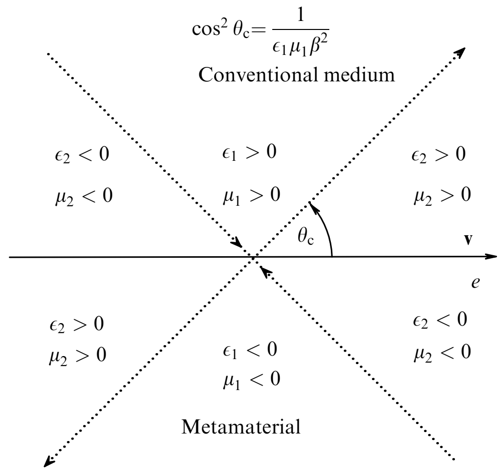

Figure 3. Versions of emission or absorption of Cherenkov photons in conventional materials and metamaterials.

4. Tamm problem in an absorbing medium

Consider the uniform motion of a charge with velocity $\mathbf{v}$ in a finite interval of a trajectory (Tamm's problem [15]). The Fourier component of the current density in this case is of the form

$$\mathbf{j}\,(\mathbf{k}, \omega) = 2e\mathbf{v}\,\frac{\sin\left[(\mathbf{kv} - \omega)\,\tau/2\right]}{\mathbf{kv} - \omega}, \tag{16}$$

where τ is the total time of charge motion (Fig. 4). Repeating the calculations given in Sections 2, 3 (see also Ref. [6]), we obtain the following expression for the frequency-angular spectrum of the average radiation intensity:

$$\frac{\mathrm{d}^3 \bar{\varDelta}_\perp(t)}{\hbar\,\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}\cos\theta} = \frac{\alpha}{\pi}\,\beta^2\omega\sin^2\theta \times \mathrm{Re}\left\{\sqrt{\epsilon}\exp\left[-\mathrm{i}\omega t\,(1 - \beta\sqrt{\epsilon}\cos\theta)\right] \times \frac{\sin\left[\omega\tau\,(1 - \beta\sqrt{\epsilon}\cos\theta)\right]}{1 - \beta\sqrt{\epsilon}\cos\theta}\right\}. \tag{17}$$

Expression (17) is a complete solution to the Tamm problem in an absorbing medium. Its integration over time within $(-\tau/2, \tau/2)$ leads to the frequency-angular spectrum of the average energy radiated from a trajectory interval of length $v\tau$:

$$\frac{\mathrm{d}^2 \varDelta_\perp}{\hbar\,\mathrm{d}\omega\,\mathrm{d}\cos\theta} = \frac{2\alpha}{\pi}\,\beta^2\sin^2\theta \times \mathrm{Re}\left\{\frac{\sqrt{\epsilon}\sin^2\left[(\omega\tau/2)(1 - \beta\sqrt{\epsilon}\cos\theta)\right]}{(1 - \beta\sqrt{\epsilon}\cos\theta)^2}\right\}. \tag{18}$$

Integration of expression (18) over the photon emission angle in the limit $\omega\tau \gg 1$ leads to two contributions to the total spectrum of radiative energy loss. The first is expression (13) multiplied by $v\tau$, which reflects the radiation loss from a segment of uniform charge motion. The second contribution relates to radiative energy loss during instantaneous (over a time much shorter than the reciprocal of the characteristic radiation frequency) acceleration and deceleration at the ends of the trajectory (bremsstrahlung). These contributions,

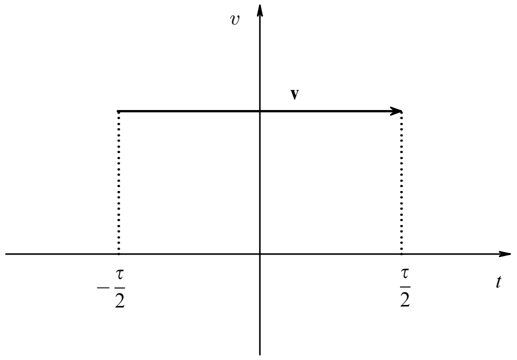

Figure 4. Uniform motion on a finite interval of the trajectory (Tamm's problem).

respectively, are of the form

$$\frac{\mathrm{d}\varDelta_\perp}{\hbar\,\mathrm{d}\omega}=\frac{\alpha}{\pi\beta}\,\mathrm{Re}\left[\frac{1}{\epsilon}\left(\ln\frac{1+\beta\sqrt{\epsilon}}{1-\beta\sqrt{\epsilon}}-2\beta\sqrt{\epsilon}\right)\right]$$

$$=\left[\epsilon\to\epsilon_1=n^2(\omega)\right]$$

$$=\frac{\alpha}{\pi\beta n^2(\omega)}\left(\ln\frac{1+\beta n(\omega)}{\left|1-\beta n(\omega)\right|}-2\beta n(\omega)\right). \qquad (19)$$

Expression (19) for a transparent medium obtained by I E Tamm in Ref. [15] transforms in a vacuum into the well-known expression

$$\frac{\mathrm{d}\varDelta_\perp}{\hbar\,\mathrm{d}\omega}=\frac{\alpha}{2\pi}\,\beta^2\int_0^{\pi}\frac{\sin^3\theta\,\mathrm{d}\theta}{(1-\beta\cos\theta)^2}=\frac{\alpha}{\pi\beta}\left(\ln\frac{1+\beta}{1-\beta}-2\beta\right), \qquad (20)$$

given in Ref. [3].

5. Radiative energy loss of a multiply scattered charge

In a number of experiments using liquid Cherenkov detectors, it is necessary to evaluate the accuracy of reconstructing the vertices of rare, for example neutrino, events by analyzing the amplitudes of light signals of VCR relativistic electrons in photomultipliers (PMs) or LEDs distributed around a liquid (water) radiator. Multiple scattering of electrons in liquid VCR radiators affects the accuracy of reconstructing the kinematics of events [16]. Consideration of the radiation loss of a multiply scattered charge makes it possible to analytically estimate the change in the frequency-angular spectrum of emitted photons [17]. The effect of multiple scattering on the radiation of a relativistic, $\gamma \gg 1$, charge in a medium is usually considered for small radiation angles, on the order of $1/\gamma$, in the X-ray frequency range (see, for example, reviews [18, 19] and references therein).

Here, we dwell on the problem in the optical region, when the radiation angle in a dense medium is $\theta\sim 1$ at small multiple scattering angles compared to unity. Let us consider a charge e, which moves along an arbitrary but close to straight trajectory $\mathbf{r}(t)$ in a homogeneous isotropic nonmagnetic absorbing medium with a complex permittivity $\epsilon=\epsilon_1+\mathrm{i}\epsilon_2$. For simplicity, we confine ourselves to the case of only the frequency dispersion $\epsilon(\omega)$ in the optical range. We consider the frequency-angular intensity spectrum of the emitted photons averaged over all possible trajectories that

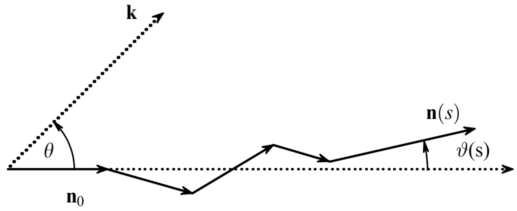

Figure 5. Notation in the treatment of multiple scattering.

are parallel to the unit vector $\mathbf{n}_0$ at a given time (Fig. 5). Then, relation (9) averaged over trajectories gives

$$\left\langle\frac{\mathrm{d}^3\bar{N}_\perp}{\hbar\,\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}\Omega}\right\rangle=\frac{\alpha}{2\pi^3\hbar c}\,\mathrm{Im}\left(\int_0^\infty\frac{\left\langle I(\mathbf{k},\omega)\right\rangle\mathrm{d}k}{k^2-\epsilon(\omega)\omega^2/c^2}\right\}, \qquad (21)$$

$$\left\langle I(\mathbf{k},\omega)\right\rangle=\frac{2}{v}\,\mathrm{Re}\left[\int_0^\infty\mathrm{d}s\exp\left(\mathrm{i}\frac{\omega}{v}s\right)\right.$$

$$\left.\times\int_{4\pi}\mathrm{d}\mathbf{n}\,f(s,\mathbf{k},\mathbf{n})\left(k^2v^2\mathbf{n}\,\mathbf{n}_0-\omega^2\right)\right].$$

It is assumed that the averaging over the previous and subsequent trajectories is the same, which allows us to restrict ourselves to integration over positive values of the trajectory length, $s>0$. Here,

$$f(s,\mathbf{k},\mathbf{n})=\int_{R_3}\mathrm{d}\mathbf{r}\,f(s,\mathbf{r},\mathbf{n})\exp(-\mathrm{i}\mathbf{k}\mathbf{r})$$

represents the Fourier component of the probability density function $f(s,\mathbf{r},\mathbf{n})$, where $\mathbf{n}$ is the unit direction vector of the trajectory for its length s, and the unit vector $\mathbf{n}_0$ corresponds to the time t at $s=0$. It satisfies the following kinetic equation [20]:

$$\frac{\partial f(s,\mathbf{k},\mathbf{n})}{\partial s}+\mathrm{i}\mathbf{k}\mathbf{n}f(s,\mathbf{k},\mathbf{n})$$

$$=N\int_{4\pi}\left[f(s,\mathbf{k},\mathbf{n}')-f(s,\mathbf{k},\mathbf{n})\right]\sigma\left(|\mathbf{n}'-\mathbf{n}|\right)\mathrm{d}\mathbf{n}',$$

with initial condition

$$f(0,\mathbf{k},\mathbf{n})=\delta(\mathbf{n}-\mathbf{n}_0).$$

Here, σ is the elastic cross section responsible for multiple scattering and N is the number of atoms per unit volume. It is assumed that the multiple scattering angle ϑ is much smaller than the radiation angle θ, which allows us to put $\mathbf{kn}\simeq\mathbf{kn}_0$ in a simplified way. Now, the distribution function can be expanded into a series of Legendre polynomials $P_l(\cos\vartheta)$ ($\cos\vartheta=\mathbf{nn}_0$):

$$f(s,\mathbf{k},\mathbf{n})=\sum_{l=0}^{\infty}f_l(s,\mathbf{k})P_l(\cos\vartheta). \qquad (22)$$

The expansion coefficients $f_l(s,\mathbf{k})$ satisfy the following equation:

$$\frac{\partial f_l(s,\mathbf{k})}{\partial s}+(\sigma_l+\mathrm{i}k\cos\theta)f_l(s,\mathbf{k})=0,$$

where the cross sections σ_l are of the form

$$\sigma_l = 2\pi N \int_0^{\pi} \sigma(\chi)\big(1 - P_l(\cos\chi)\big)\sin\chi\,\mathrm{d}\chi\,.$$

The normalized distribution function is expressed by the following sum:

$$f(s,\mathbf{k},\mathbf{n}) = \sum_{l=0}^{\infty}\frac{2l+1}{4\pi}P_l(\cos\vartheta)\exp\big[-(\sigma_l + \mathrm{i}k\cos\theta)\,s\big]\,.$$

Due to the orthogonality of the Legendre polynomials, only the first two terms of this sum contribute to $\langle I(\mathbf{k},\omega)\rangle$. Integration of series (22) over s and extraction of the real part, taking into account $k\cos\theta = \mathbf{kn}_0$, leads to the relation

$$\big\langle I(\mathbf{k},\omega)\big\rangle = -\frac{2\pi\omega^2}{v}\,\delta\left(k\cos\theta - \frac{\omega}{v}\right) + \frac{2k^2 v\,\sigma_1}{(k\cos\theta - \omega/v)^2 + \sigma_1^2}\,.$$

Here, the first term on the right-hand side corresponds to the contribution of the on-average-straight trajectory, and it is associated with the VCR. The second term shows the average effect of multiple scattering on the angular radiation distribution. To simplify calculations in expression (21), we consider the case of an almost transparent medium; then, the contribution from multiple scattering reduces to (instead of k we introduce the complex variable z)

$$\left\langle \frac{\mathrm{d}^3\bar N_{\perp\mathrm{msc}}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}t\,\mathrm{d}\cos^2\theta}\right\rangle = \frac{\alpha}{\hbar c}\,\frac{v\sigma_1}{\pi^2\cos^3\theta}\,\mathrm{Im}\left\{\oint_C g(z)\,\mathrm{d}z\right\},$$

$$g(z) = \frac{1}{2}\,\frac{z^2}{z^2 - a^2}\left[\frac{1}{(z-b)(z-b^*)} + \frac{1}{(z+b)(z+b^*)}\right],$$

$$a^2 = \epsilon\,\frac{\omega^2}{c^2}\,,\quad b = \frac{1}{\cos\theta}\left(\frac{\omega}{v} + \mathrm{i}\sigma_1\right).$$

Here, integration over z can be carried out over the entire real axis, closing the contour in the upper half-plane with a semicircle with an infinitely long radius. The integral along a semicircle is infinitely small, and the contribution of singularities in the limit of an almost transparent medium is limited by the first-order pole at point a. As a result, we arrive at ($\mathrm{d}x = v\,\mathrm{d}t$)

$$\left\langle \frac{\mathrm{d}^3\bar N_{\perp\mathrm{msc}}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta}\right\rangle = \frac{\alpha}{\hbar c}\,\frac{\beta(n^2\cos^2\theta + \beta^{-2} + \varepsilon^2)}{2n\cos\theta}$$

$$\times\frac{\Gamma_{\mathrm{msc}}}{\pi\big[(\cos^2\theta - \cos^2\theta_{\mathrm{msc}})^2 + \Gamma^2_{\mathrm{msc}}\big]}\,,$$

$$\cos^2\theta_{\mathrm{msc}} = \frac{1-\beta^2\varepsilon^2}{\beta^2 n^2}\,,\quad \Gamma_{\mathrm{msc}} = \frac{2\varepsilon}{\beta n^2}\,,\quad \varepsilon = \frac{c\sigma_1}{\omega}\,.$$

The calculation for a straight-on-average trajectory is carried out immediately and gives

$$\left\langle \frac{\mathrm{d}^3\bar N_{\perp\mathrm{cr}}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta}\right\rangle = -\frac{\alpha}{\hbar c}\,\frac{1}{n^2\beta^2}\,\delta\left(\cos^2\theta - \frac{1}{n^2\beta^2}\right).$$

In the case of negligibly small multiple scattering, $\varepsilon \to 0$, the frequency-angular distribution of the average number of photons per unit length of the trajectory reduces to VCR:

$$\left\langle \frac{\mathrm{d}^3\bar N_{\perp}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta}\right\rangle = \left\langle \frac{\mathrm{d}^3\bar N_{\perp\mathrm{msc}}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta}\right\rangle + \left\langle \frac{\mathrm{d}^3\bar N_{\perp\mathrm{cr}}}{\hbar\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}\cos^2\theta}\right\rangle$$

$$= \frac{\alpha}{\hbar c}\left(1 - \frac{1}{n^2\beta^2}\right)\delta\left(\cos^2\theta - \frac{1}{n^2\beta^2}\right). \qquad (23)$$

The value of the transport cross section σ_1 for small angles of multiple scattering $\vartheta \ll 1$ is determined by its rms angle. The relative broadening of the angular distribution can then be estimated from $\Gamma_{\mathrm{msc}}/\cos\theta_{\mathrm{msc}} \sim 2\varepsilon/n \lesssim 0.1$. This value for typical liquid or solid radiators is much greater than the absorption effect but does not exceed several milliradians in water for electrons with an energy of 1–10 MeV.

6. Discussion

The above examples show that the radiative loss of a relativistic charge in absorbing and scattering media can serve as a generalization of its radiation observed far from the charge in transparency windows. The radiative energy loss of a uniformly moving charge makes it possible to obtain an expression for the frequency-angular spectrum of the average number of photons emitted by a charge from a unit path, which, in the limit of a transparent medium, gives a similar spectrum for VCR. Absorption and/or scattering of photons lead to a Lorentzian broadening of the angular distribution, with $\cos^2\theta$ acting as a natural angular variable, and the relative broadening is determined by the ratio of the emission wavelength to the absorption (scattering) wavelength.

The threshold for the occurrence of VCR in absorbing media experiences spreading dependent on the charge energy, which is experimentally observed as a relativistic increase in the ionization loss of a charged particle. This increase is due to ionization associated with photoabsorption, near the trajectory of the charge, of photons emitted by it in the region of atomic shell energies above the first ionization potential.

Similarly, multiple scattering leads to a Lorentz broadening of the VCR angular spectrum even in a transparent medium, although, strictly speaking, this broadening is determined by bremsstrahlung with an instantaneous change in the direction of velocity due to elastic scattering of the charge by the nuclei of the medium. Within the framework of this paper, we limited ourselves to a consideration of radiation loss for the motion of a charge along an almost straight trajectory, simultaneously obtaining expressions for bremsstrahlung with an instantaneous change in velocity — at the ends of the trajectory in the Tamm problem or during elastic scattering by the nuclei of the medium in the case of multiple scattering.

It is interesting to represent the radiation of a relativistic charge in a medium in terms of the coherence length. For a transparent medium, according to review Refs [21, 22], it is introduced as

$$L(\omega,\theta) = \frac{\pi v}{\omega}\,\frac{1}{1-\beta n\cos\theta}\,. \qquad (24)$$

Its generalization to an absorbing medium is the complex quantity

$$Z(\omega, \theta) = \frac{\pi v}{\omega} \frac{1}{1 - \beta\sqrt{\epsilon}\cos\theta}\,, \tag{25}$$

which is usually defined as a complex radiation formation zone. Its real part in a transparent medium is reduced to the coherence length. Note that, in a transparent medium under the VCR condition, the coherence length (24) becomes infinite, while the real part of expression (25) remains a finite value, which corresponds by the order of magnitude to the absorption (scattering) length of Cherenkov photons. The VCR coherence length acquires a natural physical meaning in real media when absorption, scattering, and/or multiple scattering are taken into account. In the last case, it is determined by the transport length of elastic scattering by the nuclei of the medium, $\sim \sigma_1^{-1}$.

Consideration of radiation loss according to expression (9) is not limited to the above cases. Dipole, undulator, and magnetic breaking radiative loss in an absorbing medium, as well as VCR in absorbing media with spatial dispersion, remain beyond the scope of this paper. They will be considered in other studies.

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