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On the question of a classical analog of the Fano problem

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Contents

Abstract. A system of interacting classical oscillators is discussed, which is an analog of the quantum-mechanical problem of a discrete energy level interacting with an energy quasi-continuum of states considered by Fano. The limiting transition to the continuous spectrum and the possible connection of this problem with the generation of coherent phonons are analyzed.

Keywords: Fano interference, coherent phonons

1. Introduction

This article discusses searches for a classical mechanical oscillatory system, which, under the effect of external periodic forces, would feature a spectral dependence of the oscillation amplitude squared that coincides in appearance with the Fano resonance line-shape and consists of classical elements, each of which would be a classical analog of the corresponding quantum-mechanical element of the Fano model. We begin with a brief explanation as to why suchlike attempts are of interest from various points of view.

1.1 History of the issue and description of the Fano problem

The history of Fano resonance, named after Enrico Fermi's student Hugo Fano, who endeavored to explain the asymmetric shape of experimentally observed absorption lines, is very remarkable. The Fano problem arose from the theoretical description of the characteristic asymmetric absorption lines of noble gases in the ultraviolet region of the spectrum, which appear against the background of a wide nonresonant absorption continuum [1]. It is of interest to note that, in the archive left after Fermi's death, a complete solution was discovered, which he had found before laying out the problem (Fano himself often talked about this [2, 3]). However, when Fano finished the calculations and asked Fermi whether a joint article could be prepared, Fermi replied that an article could be written, but it would be sufficient to extend appreciation to him. Later, the approach of [1] was successfully developed to describe the scattering of an electron beam by helium atoms [4]. It turned out that the minimum in the dependence of the scattering cross section on the energy of incident electrons, i.e., suppression of scattering, can be explained by the quantum-mechanical effect of the destructive interference of the amplitudes of an electron wave not absorbed by an atom and a wave absorbed with the formation of a metastable excited state of the atom, which, after some time, decays again into a continuous spectrum of states. In other words, the states of the discrete and continuous spectra can affect each other, and the problem can be formulated as follows: how will the states of the discrete and continuous spectra alter if some small interaction is switched on, which transfers particles from the discrete state to the continuous-spectrum state.

Fano himself formulated the problem to be solved as follows [4]. Consider an atomic system with a set of zeroth-approximation states, and among these states, one state (φ), which belongs to a discrete configuration, and a continuum of states $\psi_{E'}$. Each of these states is nondegenerate, since the degeneracy is removed by choosing an adequate set of quantum numbers. The problem is to diagonalize the part of the energy matrix that refers to the subset of states φ and $\psi_{E'}$. The elements of this part of the energy matrix, which form

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a square matrix, are denoted as

$$\langle\varphi|H|\varphi\rangle = E_\varphi\,,$$
$$\langle\psi_{E'}|H|\varphi\rangle = V_{E'}\,,$$
$$\langle\psi_{E''}|H|\psi_{E'}\rangle = E'\delta(E''-E')\,.$$

It is assumed that the discrete energy level E_φ lies within the continuum of values E'.

Such a formulation of the problem turned out to be so general that it soon became clear that the Fano problem arises in various fields of physics. It was found that the appearance of asymmetric Fano lines is a common phenomenon in such diverse areas of research as nuclear [5], atomic [6–8], and solid-state [9–12] physics, as well as chemistry [13]. Fano resonances are of interest for their generality and still spur lively discussions [14–16], while the original work [4] is now one of the most frequently cited publications in the history of physics [2, 3].

It was believed for a long time that Fano interference is a purely quantum-mechanical phenomenon that has no classical analog, since interference phenomena cannot be observed in the scattering of classical particles; however, spectral resonance lines, the shape of which resembles the Fano resonance line shape, also arise in considering classical systems represented by mechanical [17–19] or equivalent electric oscillators [20, 21]. It is the formal resemblance of the spectra of classical and quantum-mechanical systems that stimulated the search for a classical problem similar to the Fano problem. It turned out that a characteristic asymmetric resonance line shape can be obtained even in a system of only two interconnected oscillators [17]. In this case, it is fundamentally important that one of the oscillators have nonzero damping. Despite the fact that consideration [17] is very instructive, it is difficult to recognize a system of two coupled oscillators as analogous to one consisting of a discrete state that interacts with a continuum of other states.

The first attempt to explore the classical problem, which is completely analogous to Fano's quantum-mechanical problem, should, apparently, be considered the work of Riffe [18]. He succeeded in obtaining a Fano resonance line shape for a classical system, very similar to the original quantum-mechanical one, which consists of a classical oscillator connected with a quasi-continuum of other classical oscillators. However, Riffe failed to obtain a rigorous solution to the classical problem; in particular, normal oscillations of a coupled system of oscillators have not been found. It should be noted that the Fano problem was formulated in quantum mechanics for a conservative system, while in the classical analog [18] a fundamental role is played by the damping of oscillators, due to which the time reversal symmetry is violated and equations of motion become nonconservative [22].

1.2 Physical meaning and mathematical aspects

Interest in searching for links between the classical and quantum-mechanical descriptions of the world goes back to Bohr's correspondence principle, and Fano's problem looks very attractive in this respect. Analysis of a classical analog of quantum-mechanical phenomena is methodically useful and has often been presented in *Physics–Uspekhi*'s notes [23–25]. The situation in which one oscillator interacts with an ensemble of oscillators of a different nature is very common in solid state physics and in considering the interaction of radiation with matter. At the same time, many collective excitations in solids are formed as a result of the coordinated motion of a large number of charge carriers, and are essentially quasi-classical states with a fairly well-defined oscillation phase.

Interest in the search for analogies between the Fano problem and classical oscillatory systems was partly stimulated by recent advances in the physics of coherent phonons, including the synchronous vibrational excitation of a crystal lattice due to the action of a short laser pulse whose duration is less than the oscillation period of atoms [26–28]. The spectra of these oscillations feature, under certain conditions, a characteristic asymmetric resonance line shape, which resembles the Fano resonance one [26]. It should be noted that asymmetric line profiles of spontaneous Raman scattering in crystals, which were also observed earlier under conditions of continuous-wave laser excitation, were interpreted as a result of the interaction of a phonon with an electron continuum [29]. Time-resolved experiments made it possible to directly measure the phase of atomic vibrations, and the question naturally arose about the relation of this phase to the mechanism of excitation of coherent vibrations [18, 19, 21]. An attempt was made in [18] to carry out a consistent classical treatment of the Fano problem, in which the Fano model is used to explain the dependence of the initial phase of coherent phonons excited by an ultrashort light pulse in a silicon crystal on the doping level of this crystal with donor impurities [30]. The classical equations of motion are derived in this case from the quantum Hamiltonian of the system based on the assumption that a coherent state of the vibrational degree of freedom of the crystal is excited, with which the electronic degrees of freedom, which are also in coherent states, are weakly coupled. As a result, the coherent phonon is replaced by a classical harmonic oscillator, which is assumed to be weakly coupled to the quasi-continuum of classical harmonic oscillators representing the electronic degrees of freedom of the crystal. The author of [18] succeeded in finding a relationship between the initial phase of coherent vibrations of a crystal lattice after its excitation by an ultrashort laser pulse and the Fano parameter of the spectral line of this vibration in the spectrum of spontaneous Raman scattering of light.

The approach developed by Fano can be used to describe the dispersion of quasiparticles, which was noticed by the authors of [31] in considering the dispersion of exciton polaritons in a microcavity. In this case, the same result is obtained as when the standard apparatus of Green's functions is used. It turns out that the analysis of the resonant interaction of an electromagnetic wave in such a microcavity with an ensemble of two-level systems in a random potential [32] leads to an equation for the energy eigenvalues coinciding with the secular equation for interacting classical oscillators, which directly indicates that the semiclassical consideration is well justified. A similar situation arises in considering the interaction of light with an ensemble of atoms in light echo problems, when the energy of a light wave is transferred to an ensemble of local oscillators.

As noted above, from a mathematical point of view, the Fano problem is reduced to finding the eigenstates of a quantum-mechanical Hamiltonian, which has both a discrete and a continuous spectrum, with the inclusion of a weak resonant coupling between the states of the discrete and continuous spectra [4]. The solution to the problem is the superposition of the initial states of these spectra. When such states are excited, the matrix elements of the excitation of a discrete level and continuum are summed with different phases. This leads to the emergence of a characteristic frequency dependence of the system excitation, now called the Fano resonance line shape or Fano interference, which differs significantly from the Breit–Wigner dependence (Lorentzian form of the spectral line). The problem is thus reduced to the study of a system of equations in partial derivatives. As for the classical analogies mentioned above, they all explore systems of ordinary differential equations. This circumstance is discussed below.

1.3 Remaining problems

Despite the formal resemblance between the spectral profile of classical and quantum-mechanical resonances, a strict analogy between these cases has not yet been established. Currently known classical systems that exhibit the spectral dependence of the resonance, resembling the Fano resonance, are described by Hamiltonians that are not classical analogs of the quantum-mechanical Hamiltonian of the Fano problem. The formal resemblance of the resonance spectral profile is obviously not a sufficient basis for drawing an analogy between quantum and classical problems. Attempts to model the Fano problem on classical oscillators that perform finite motion encounter problems arising from the conservative nature of the system: the effect of harmonic forces at the resonant frequency of such a system leads to an unlimited increase in the oscillation amplitude. The oscillation phase of an oscillator excited by an external harmonic force is known to change by a jump when the frequency of the exciting force passes through the resonant frequency. It is only possible to obtain a smooth phase change, as is the case in the Fano problem, by introducing nonzero damping. As a result, classical approaches have to consider dissipative systems with damping, while the Fano quantum-mechanical problem does not contain damping. The introduction of damping turns out to be fundamental for classical calculations, since it is included in expressions for the Fano parameter, and the transition to the zero damping limit yields a Lorentz-symmetric resonance spectral profile. Nevertheless, although for the above reasons no strict analogy can be set between the Fano problem and a system of interacting classical oscillators, finding out the conditions under which such a system demonstrates a spectral resonance profile resembling the Fano profile is of undoubted interest.

We present here an analysis of the motion of a system of coupled classical mechanical oscillators, which is largely equivalent to the system considered in [18]. A new element, when compared with [18], is the discussion of the properties of normal oscillations of such a system. An analog of the diagonalization of the quantum-mechanical Hamiltonian for a classical oscillatory system is the determination of its normal oscillations. We tried to track a normal oscillations system under an unlimited increase in the number of oscillators it contains. A problem arises then, which is related to the unlimited increase in the amplitude of oscillations of a classical conservative system when it is excited at the normal oscillation frequencies. In expressions similar in meaning to quantum-mechanical ones, in this case, irremovable divergences appear, and it is not possible to determine the Fano parameter. This problem was avoided in [18] by introducing phenomenological damping for oscillators, which has no analog in the original Fano problem. Note that in Fano's first study [1], a calculation was carried out for a restricted quantum system, which, as its size increases, transforms into a system with a discrete level and a continuous spectrum. Similar divergences are eliminated in such an approach due to the normalization of the wave functions, which changes with the size of the system. It is physically clear that the reason for the divergence in a limited conservative system in the process of its resonant excitation is the conservation of energy in it, which continuously increases as a result of the action of a resonant exciting force in the absence of dissipation. It seems reasonable to assume that, in the transition to an unrestricted system in which the energy can go to infinity, a method to solve the problem of the mentioned divergences without introducing additional damping should be found. This possibility is discussed in Section 2 after a detailed analysis of the classical model of coupled local oscillators, and it is shown that the latter cannot be an analog of the Fano problem for an arbitrarily high spectral density of oscillators that form a quasi-continuum.

2. Model of coupled mechanical oscillators. Study of the secular equation for two coupled mechanical oscillators

We begin with a description of the model of coupled classical harmonic oscillators, which is similar to that considered in [18] but does not contain damping. We consider the mechanical system shown in Fig. 1. A cart of mass M connected by a spring with stiffness K to a fixed wall can perform one-dimensional motion along the guides. By means of weak springs with stiffness μ_i, this cart is connected to a set of $2N+1$ spring pendulums attached to the opposite fixed wall (the mass of each ball is m_i and the spring stiffness is k_i). The formulas for the kinetic and potential energy of such a system have the form

$$T = \frac{1}{2} M\dot{X}^2 + \frac{1}{2}\sum_{i=-N}^{N} m_i \dot{x}_i^2\,,$$
$$\Pi = \frac{1}{2}\,KX^2 + \frac{1}{2}\sum_{i=-N}^{N} k_i x_i^2 + \frac{1}{2}\sum_{i=-N}^{N}\mu_i (x_i - X)^2$$
$$= \frac{1}{2}\left(K + \sum_{i=-N}^{N}\mu_i\right) X^2 + \frac{1}{2}\sum_{i=-N}^{N}(k_i+\mu_i)x_i^2 - \sum_{i=-N}^{N}\mu_i x_i X\,. \tag{1}$$

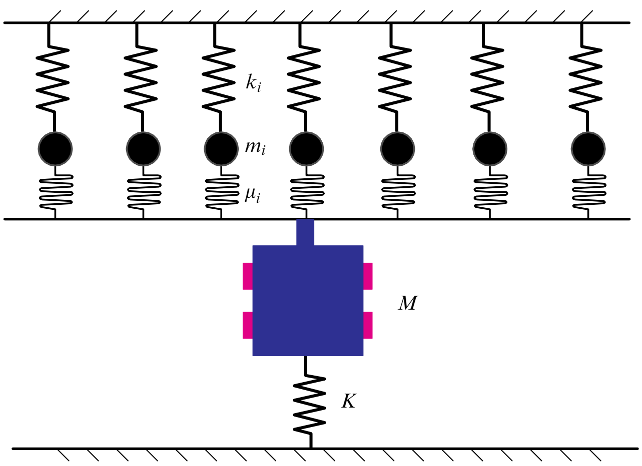

Figure 1. (Color online.) Schematic representation of coupled harmonic oscillators (top view). Cart of mass M located on a horizontal plane is connected by weak springs with a set of spring oscillators. Cart can move along rails under the action of a spring with stiffness K.

From these formulas, the Lagrange equations for the motion of the system follow:

$$\begin{aligned}
&m_N\ddot{x}_N + (k_N + \mu_N)x_N - \mu_N X = 0\,,\\
&m_{N-1}\ddot{x}_{N-1} + (k_{N-1} + \mu_{N-1})\,x_{N-1} - \mu_{N-1}X = 0,\\
&\qquad\ldots,\\
&m_0\ddot{x}_0 + (k_0 + \mu_0)x_0 - \mu_0 X = 0\,,\\
&M\ddot{X} + \left(K + \sum_{i=-N}^{N}\mu_i\right)X - \sum_{i=-N}^{N}\mu_i x_i = 0\,,\\
&\qquad\ldots,\\
&m_{-N}\ddot{x}_{-N} + (k_{-N} + \mu_{-N})x_{-N} - \mu_{-N}X = 0\,.
\end{aligned} \tag{2}$$

To find the normal oscillations, we present the coordinates of the spring pendulums and the cart in the form

$$\begin{aligned}
&x_i = u_i \sin(\omega t + \alpha)\,,\\
&X = U\sin(\omega t + \alpha)\,.
\end{aligned} \tag{3}$$

The values u_i and U are the displacement amplitudes of individual spring pendulums and the cart, respectively. Given that the amplitude can be both positive and negative, the phase of the normal oscillation α is defined in the interval $(0, \pi)$. Substituting formulas (3) into the equations of motion (2), we obtain a system of algebraic equations for finding the amplitudes of normal vibrations:

$$\begin{aligned}
&(k_N + \mu_N - m_N\lambda)u_N - \mu_N U = 0\,,\\
&\qquad\ldots,\\
&\left(K + \sum_{i=-N}^{N}\mu_i - \lambda M\right)U - \sum_{i=-N}^{N}\mu_i u_i = 0\,,\\
&\qquad\ldots,\\
&(k_{-N} + \mu_{-N} - \lambda m_{-N})u_{-N} - \mu_{-N}U = 0\,,
\end{aligned} \tag{4}$$

where $\lambda = \omega^2$. We introduce the notation

$$\frac{k_i + \mu_i}{m_i} = \omega_i^2\,, \qquad \frac{\mu_i}{m_i} = c_i\,, \qquad \frac{K + \sum_{i=-N}^{N}\mu_i}{M} = \Omega^2\,.$$

System of equations (4) then takes the form

$$\begin{aligned}
&(\omega_N^2 - \lambda)u_N - c_N U = 0\,,\\
&\qquad\ldots,\\
&(\Omega^2 - \lambda)U - \sum_{i=-N}^{N}\frac{m_i}{M}\,c_i u_i = 0\,,\\
&\qquad\ldots,\\
&(\omega_{-N}^2 - \lambda)u_{-N} - c_{-N}U = 0\,.
\end{aligned} \tag{5}$$

Nonzero solutions to system of equations (5) only exist if its determinant is equal to zero, which yields the secular equation for our system of oscillators:

$$\Omega^2 - \lambda = \sum_{i=-N}^{N}\frac{m_i}{M}\frac{c_i^2}{\omega_i^2 - \lambda}\,. \tag{6}$$

This is an algebraic equation with respect to λ of order $2N + 2$, which is equal to the number of degrees of freedom, whose roots determine the normal oscillations of the system under consideration.

We now proceed to studying secular equation (6). To date, no assumptions have been made about eigenfrequencies and coupling between oscillators. Since the purpose of our consideration is to give a classical analog of the interactions between elementary excitations in solids, it is necessary for such an analogy to be sensible, and it must be assumed that the stiffness values of the springs that connect the spring pendulums with the cart are small in comparison with those of the oscillator springs, $\mu_i \ll k_i, K$. The 'renormalization' of the initial eigenfrequencies of the oscillators due to the interaction is in this case small:

$$\frac{k_i + \mu_i}{m_i} = \omega_i^2 \approx \omega_{i0}^2 = \frac{k_i}{m_i}\,. \tag{7}$$

For the cart, condition (7) leads to a stronger constraint, since it is required that the total stiffness of all links be small compared to the cart spring stiffness,

$$\sum_{i=-N}^{N}\mu_i \ll K\,. \tag{8}$$

It should be noted that the frequencies contained in secular equation (6) are the 'renormalized' frequencies of the original oscillators, i.e., the values altered due to additional elastic coupling between the cart and the oscillators. In referring to oscillator frequencies below, we always mean these 'renormalized' frequencies.

The secular equation for two coupled oscillators is easily solved algebraically, but the algebraic approach fails with an increase in the number of oscillators in the system and a corresponding increase in the order of the secular equation. We therefore consider a graphical solution of the secular equation, which provides an easily comprehensible visual way to qualitatively analyze the properties of the considered system of coupled oscillators. Suppose first that we are only dealing with one oscillator coupled with the cart ($N = 0$, $i = 0$). Then, the left side of the secular equation is linear, while the right side is a hyperbolic function of the variable λ:

$$\Omega^2 - \lambda = \frac{m_0}{M}\frac{c_0^2}{\omega_0^2 - \lambda}\,. \tag{9}$$

The case where the frequencies coincide is shown in Fig. 2a. The intersection points of the plots of the left and right parts of the secular equation determine the frequencies of normal oscillations of the system. Equations (5) then assume the following form:

$$\begin{aligned}
&(\omega_0^2 - \lambda)u_0 - c_0 U = 0\,,\\
&(\Omega^2 - \lambda)U - \frac{m_0}{M}\,c_0 u_0 = 0\,.
\end{aligned} \tag{10}$$

It can be easily seen in the equations that, for $\lambda < \Omega^2$, the signs of the amplitudes U and u_0 are the same, i.e., the cart and the spring pendulum move in phase, while in the case of $\lambda > \Omega^2$, the phases of their motion are opposite. The two solutions of the secular equation yield a complete set of normal oscillations, so that any movement can be represented as their superposition. Even this simplest model of coupled oscillators is often used to describe the occurrence of polaritons, elementary excitations in the problem of polarization oscillations of a medium that strongly interact with an electromagnetic field. Interestingly, our mechanical system makes it possible to give a visual representation of the Rabi

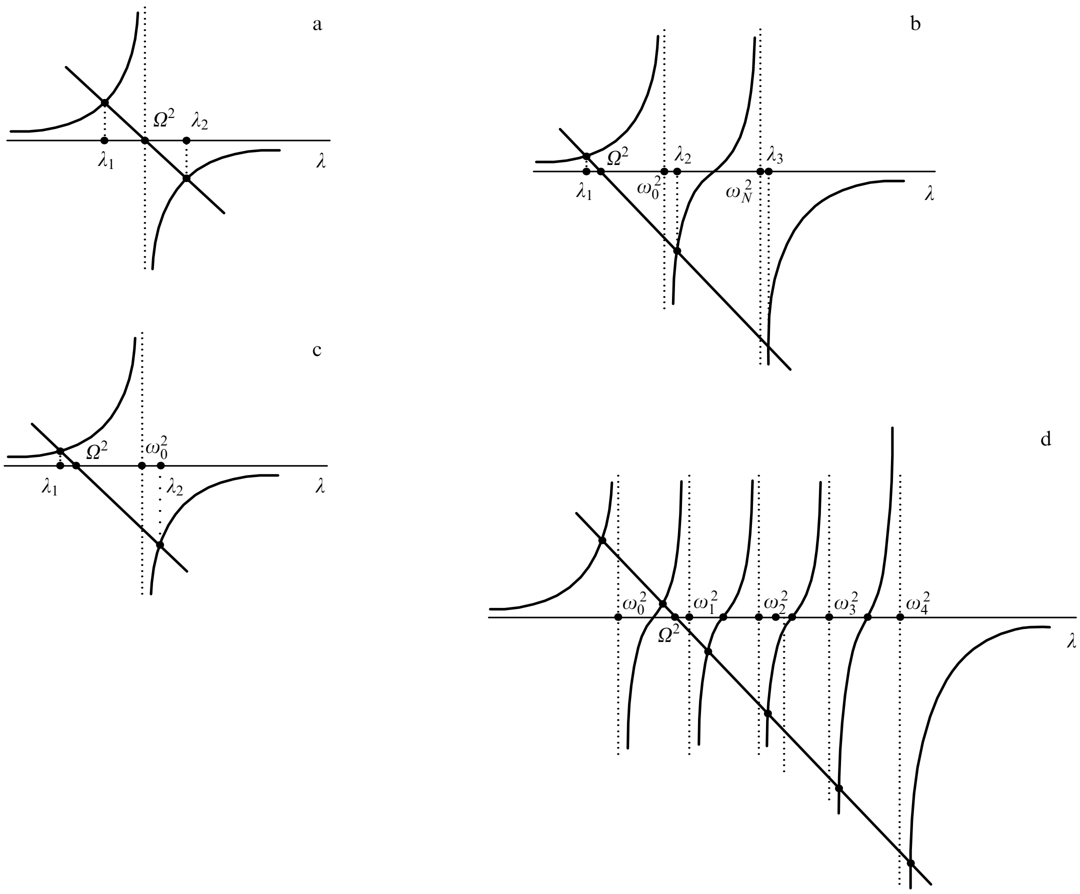

Figure 2. Graphical solution of the secular equation in the problem of two coupled oscillators when (a) frequencies of the spring pendulum and the cart are the same and (b) these frequencies are different, and also if degeneracy is removed when, as dimensions of the system increase, (c) eigenfrequency ω_N of one of the oscillators differs from that of cart Ω and the eigenfrequency ω_0 of other $2N$ oscillators and (d) all eigenfrequencies of the pendulums are different.

frequency. Indeed, let us at the initial moment we deflect only one of the oscillators, for example, a spring pendulum, while the second one (cart) is at rest. This initial state is, obviously, a superposition of the in-phase and anti-phase oscillations of the system, which enter with the same absolute values of the coefficients. The further movement of the system consists of beats with a frequency equal to the difference between the frequencies of normal oscillations, during which all the energy is periodically transferred from the spring pendulum to the cart and back. The frequency of energy exchange between oscillators, which is equal to the difference between the frequencies of normal oscillations, is the mechanical analog of the Rabi frequency [33]. Equations (10) immediately yield

$$\Omega^2 - \lambda = \pm c_0 \sqrt{\frac{m_0}{M}} . \tag{11}$$

Hence, for the frequencies of normal vibrations ω_+ and ω_- and the Rabi frequency Ω_{R}, we obtain

$$\begin{aligned} \omega_+ &= \sqrt{\Omega^2 + c_0 \sqrt{\frac{m_0}{M}}} \approx \Omega + \frac{1}{2} \sqrt{\frac{m_0}{M}} \frac{c_0}{\Omega} , \\ \omega_- &= \sqrt{\Omega^2 - c_0 \sqrt{\frac{m_0}{M}}} \approx \Omega - \frac{1}{2} \sqrt{\frac{m_0}{M}} \frac{c_0}{\Omega} , \\ \Omega_{\mathrm{R}} &= \omega_+ - \omega_- = \sqrt{\frac{m_0}{M}} \frac{c_0}{\Omega} = \frac{1}{\sqrt{m_0 M}} \frac{\mu_0}{\Omega} . \end{aligned} \tag{12}$$

It can be seen that the Rabi frequency is directly proportional to the elasticity of the spring that connects the cart and the spring pendulum.

The case when the oscillation frequencies of the cart and the spring pendulum are different is shown in Fig. 2b. A feature that is new in comparison with the previous case is the predominance of the properties of one of the pendulums in a specific normal oscillation of the system. Indeed, the frequencies of normal oscillations, as can be seen from Fig. 2b, come closer to each other as the frequency difference of the oscillators increases to the frequencies of the corresponding oscillators, and the oscillation amplitude of the oscillator, the frequency of which is approached by the normal oscillation frequency, dominates, as can be seen from Eqns (10). Nevertheless, the oscillators, as before, move in normal oscillation with a lower frequency in phase, and out of phase in an oscillation with a higher frequency. This implies that, by choosing the appropriate superposition, a configuration can be arranged in which one of the pendulums periodically completely transfers its energy to the second one (i.e., it stops moving), while the second oscillator only exchanges part of the energy stored in it.

2.1 Case of identical oscillators

We now proceed to the case of many oscillators coupled with the cart. We start with a degenerate situation, when the frequencies of all oscillators are the same but do not necessarily coincide with the frequency of the cart. We assume for simplicity that all the oscillators and their couplings with the cart are identical. Then, the secular equation takes the form

$$\Omega^2 - \lambda = \frac{1}{\omega_0^2 - \lambda} \sum_{i=-N}^{N} \frac{m_i}{M} c_i^2 = (2N+1) \frac{1}{\omega_0^2 - \lambda} \frac{m_0}{M} c_0^2 . \quad (13)$$

Equation (13), with an accuracy up to notation, coincides with Eqn (9), so all the conclusions made for two oscillators remain valid for the degenerate case as well. It is also physically clear that a large number of spring oscillators, whose frequencies coincide, can be made to oscillate synchronously with each other. Moreover, from the point of view of interaction with the cart, the set of these pendulums behaves like one pendulum, the mass of which is equal to the sum of the masses of the individual oscillators, and the stiffness is the sum of the stiffness values of the springs of these oscillators. If the frequencies of the oscillators coincide with the cart eigenfrequency, a collective Rabi frequency can be introduced [33]. We call these two normal oscillations of the system under consideration, which are analogous to normal oscillations in a two-oscillator system, boundary normal oscillations, in the sense that they determine, as will be clear from what follows, the upper and lower boundaries of the normal oscillation spectrum. The entire set of spring oscillators coupled with the cart performs as a single effective oscillator. Our mechanical analogy describes well the case of multi-atomic Rabi oscillations in a system of identical two-level atoms interacting with a resonant electromagnetic field [32]. The collective Rabi frequency is then proportional to the root of the total number of oscillators. Study [32] is of interest because it considers the general case of resonant interaction of an electromagnetic field with a nondegenerate ensemble of two-level systems with frequencies that are in a certain vicinity of the electromagnetic wave frequency. The equation for the energy eigenvalues derived in [32] coincides with secular equation (6) with an accuracy up to notation, and the behavior of our mechanical system, as will be clear from further consideration, is quite similar to that for a system of quantum mechanical oscillators.

We now return to the consideration of a degenerate system of oscillators. What are the remaining $2N$ normal oscillations of the system? It is easy to see that the remaining nontrivial solutions to system of equations (5) satisfy the condition

$$U = 0 , \qquad \sum_{i=-N}^{N} \frac{m_i}{M} c_i u_i = 0 . \quad (14)$$

This condition is nothing but the condition of orthogonality of vectors in a space of $2N+1$ dimensions, which defines a $2N$-dimensional plane in this space. The number of independent mutually orthogonal vectors that define this plane is exactly $2N$, which coincides with the number of missing normal oscillations of the system. Physically, these oscillations occur in such a way that the resulting force acting on the cart is always strictly equal to zero. As a result, all the vibrational energy contained in the system is concentrated in the ensemble of spring oscillators and no exchange with the cart occurs. In this case, the oscillation frequency is degenerate and coincides with the common frequency of all spring oscillators (on the plot, these solutions correspond to the point $\lambda = \omega_0^2$). This can be seen by considering the determinant of system of Eqns (5). If the oscillators are identical, all the diagonal elements of this determinant, with the exception of one (which corresponds to the cart oscillation amplitude), simultaneously vanish. Expanding the determinant in rows, it can be verified that it vanishes if at least two of its diagonal elements are equal to zero at the same time. This implies that, if there are at least two spring oscillators with the same eigenfrequencies, then their common frequency becomes the frequency of the normal oscillation of the system. Indeed, two such oscillators can apparently oscillate in antiphase with each other, so that the total force acting on the cart is always equal to zero. The oscillation amplitudes of all other oscillators and the cart can be set equal to zero, and system of equations (5) will be satisfied.

2.2 2*N* identical oscillators and a single oscillator

We now track the changes in the normal oscillations of the system of oscillators if the degeneracy is removed. A qualitative visual analysis of the changes can be carried out by considering the graphical solution of the secular equation. Let the eigenfrequency ω_N of one of the oscillators differ from the common eigenfrequency ω_0 of the remaining $2N$ oscillators and the eigenfrequency of oscillations of the cart Ω; then, the secular equation takes the form

$$\Omega^2 - \lambda = \frac{1}{\omega_0^2 - \lambda} \sum_{i=-N}^{N-1} \frac{m_i}{M} c_i^2 + \frac{m_N}{M} c_N^2 \frac{1}{\omega_N^2 - \lambda} . \quad (15)$$

The emergence of a new pole on the right side leads to an additional discontinuity in its graph and to an additional intersection point, with the linear function representing the left side (Fig. 2c). In this case, the number of degenerate solutions decreases by one. Now, let all the frequencies of the spring oscillators be different and concentrated in a certain interval around the frequency of the cart. None of the eigenfrequencies of the oscillators now coincides with the frequency of any normal oscillation of the system, and the plot of the right side of the secular equation now has $2N+1$ points of discontinuity and $2N+2$ points of intersection of this graph with the linear function of the left side (Fig. 2d). The sign and magnitude of the ith oscillator amplitude in the jth normal oscillation follow from formula (5):

$$(\omega_i^2 - \lambda_j) u_i = c_i U . \quad (16)$$

In all normal oscillations with $\lambda_j < \omega_i^2$, the oscillator oscillates in phase with the cart, while in normal oscillations with $\lambda_j > \omega_i^2$, it oscillates in antiphase with it (Fig. 3). The normal oscillation with the lowest frequency is the in-phase boundary oscillation, and that with the highest frequency, the antiphase boundary oscillation of the ensemble of oscillators and the cart, which is schematically illustrated in Fig. 4.

It can be seen that, although there are no fundamental differences between boundary oscillations in degenerate and nondegenerate systems, an important feature of a degenerate (or close to degenerate) system is that the amplitudes of individual oscillators in boundary normal oscillations in such a system can be close, and this implies the possibility of an almost complete exchange of energy between the cart and spring pendulums in the course of collective Rabi oscillations. A fundamental distinction of a degenerate system is also the existence of normal oscillations, in which the combined interaction between the oscillators and the cart is zero.

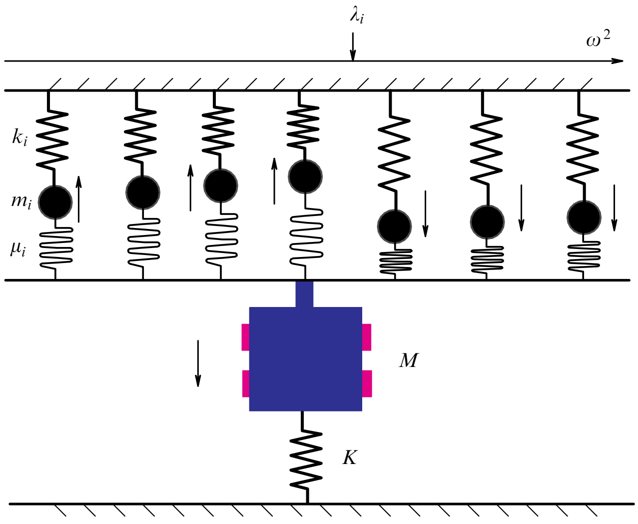

Figure 3. (Color online.) Visual representation of the motion of spring pendulums under excitation of normal oscillation λ_j: pendulums with eigenfrequencies $\omega_i^2 > \lambda_j$ move in phase with the cart, while those with $\omega_i^2 < \lambda_j$ move out of phase.

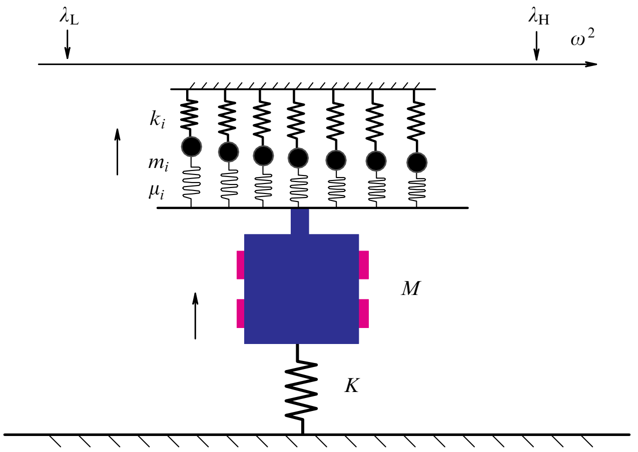

Figure 4. (Color online.) Motion of spring pendulums and the cart under excitation of in-phase boundary oscillation λ_L.

3. Behavior of the system under the effect of external forces

We now consider the behavior of our system under various excitations. First, we consider whether the mechanisms of excitation of coherent phonons can be modeled in this way.

3.1 Physical illustration — coherent phonons

In the physics of coherent phonons, as a result of direct measurements of time-resolved vibrations of a crystal lattice, a new experimentally measurable quantity appears: the initial phase of coherent vibrations. It is well established that the initial phase is different for different crystals, and it is generally associated with the mechanism of excitation of coherent phonons. This initial phase essentially depends on the equilibrium position of the vibrating atoms. When coherent phonons are excited, oscillators are subjected to the short-term impact of the pulsed force. The effect of a short pulsed force on a system in equilibrium (the duration of the action is much shorter than any of the oscillation periods of the pendulums) can be replaced by introducing an initial condition with nonzero initial velocities and further considering the free evolution of such a system. It is clear that, after the end of the laser pulse, the system begins to move away from the equilibrium position, and the initial phase of the oscillations can be either zero or π. Such an initial phase is indeed observed in many crystals transparent at the laser excitation frequency, in which the energy gap width exceeds that of the exciting photons [25, 28]. This excitation of coherent phonons is usually called dynamic and is associated with the dynamic Raman excitation mechanism. In nontransparent crystals, coherent oscillations, on the contrary, usually begin from the position of maximum amplitude, i.e., with an initial phase equal to $\pi/2$ or $-\pi/2$. In this case, as a result of the laser pulse action, the equilibrium position of atoms instantly changes. Such a change in the equilibrium position can be described by a parametric excitation mechanism when, due to the pulse effect, the wall to which the oscillator springs are attached is instantly displaced by a finite distance, or the stiffness of the springs instantly changes. This corresponds to an instantaneous change in the interatomic potential in a crystal, which is usually associated with a high density of photoexcited electrons and screening of interatomic interaction. The atoms then begin to oscillate with zero initial velocity; this excitation is usually called kinematic.

If we compare the Fourier spectra of coherent oscillations under dynamic and kinematic excitation (kinematic or dynamic with respect to an oscillator with a discrete spectrum), the dynamic excitation spectrum only contains the excited-phonon frequency, while the kinematic excitation spectrum also contains a continuum of frequencies due to the stepwise emergence of a finite amplitude. The relationship between the discrete frequency and the continuum in the Fourier spectrum of coherent oscillations should thus depend on the initial phase of the oscillations. In this connection, it is of interest to take a slightly different look at Fano's formula. If the spectrum corresponding to the Fano formula is that of some coherent oscillation, then the time evolution of this oscillation can be determined by performing the inverse Fourier transform of the Fano resonance line shape. The result of such a transformation, carried out by us in [21], is the oscillation of a damped harmonic oscillator with eigenfrequency Ω, the initial phase of oscillations of which is actually determined by the Fano parameter.

By solving the secular equation, it is possible to determine the frequencies of all normal oscillations of coupled oscillators and expand any initial state of the system that arises as a result of the pulsed action into its normal oscillations. The impact of an ultrashort pulse can be modeled, as mentioned above, by the corresponding initial condition: the action of the pulsed force on the cart can be replaced by accelerating it at the zero moment of time to some initial speed, while the action of the excitation pulse on the spring oscillators can be described by setting at the zero moment of time their initial deviation from the equilibrium position. The further evolution of the system can be easily calculated on the basis of expansion into normal vibrations.

We have performed numerical simulations of the system under consideration with the number of spring pendulums equal to 200. After setting the eigenfrequencies and couplings, the secular equation was numerically solved, and the evolution of the system with some initial condition was determined. It turned out that it is not possible to obtain damped cart oscillations with different initial phases, as might be expected based on the coherent phonon excitation model. If at the initial moment the oscillations of the cart are excited, at first their amplitude actually begins to decrease due to the transfer of energy to the spring oscillators and the rapid dephasing of their oscillations, as is usually assumed in the physical picture of a light echo. Later, however, a reverse energy transfer — from the spring oscillators to the cart — is also observed, so the exponential damping of its oscillations over time is not relevant.

Leaving aside subtle issues related to the frequency measurement accuracy, errors in setting the initial conditions, and the related general predictability of the behavior of a deterministic system, we only note that, in the general case, the energy exchange between the cart and oscillators can turn out to be quite nontrivial in time, which has been confirmed by our computer modeling. The number of oscillators in our numerical simulation is, of course, very small compared to physically interesting situations, when their number can reach values on the order of the Avogadro number, i.e., 10^{23}. One can therefore hope that, with a very large number of oscillators, an exponential damping of oscillations will appear in accordance with the Fourier transform of the Fano formula. Another option to ensure exponential damping of the amplitude of coherent oscillations is to introduce damping for spring oscillators.

In some cases, detailed information about excited normal oscillations is not required. It is apparent, for example, that, under pulsed excitation with the width of the excitation force spectrum greater than the total width of the normal vibration spectrum, virtually all normal vibrations of the system are excited. The evolution after such a pulsed action essentially depends on the ratio of the frequencies of normal oscillations. Indeed, imagine that all frequencies of normal oscillations are multiples of some fundamental frequency, as is the case for oscillations in cavities. Then, the initial state, which arose immediately after the end of the excitation, is periodically repeated. If the pulsed action was only applied to the cart, the energy in the process of evolution would first be transferred to the spring oscillators, but after a time equal to the oscillation period of the fundamental frequency, of which frequencies of normal oscillations are multiple, the spring oscillators would again be stationary, and all the energy would again concentrate in the oscillation of the cart. It can be shown that such a fundamental frequency always exists if the ratio of the frequencies of normal oscillations is represented by rational numbers.

3.2 Effect of harmonic forces on the system

We now consider a system of oscillators to be nondegenerate and assume that it is subject to the effect of harmonic driving forces:

$$
\begin{aligned}
f_i(t) &= f_i(\omega)\sin(\omega t + \alpha_{f_i}), \\
F(t) &= F(\omega)\sin(\omega t + \alpha_F).
\end{aligned} \tag{17}
$$

We only focus on the forced motion of the system and look for solutions also in the form of harmonic functions:

$$
\begin{aligned}
x_i &= u_i(\omega)\sin(\omega t + \alpha_i), \\
X &= U(\omega)\sin(\omega t + \alpha).
\end{aligned} \tag{18}
$$

As in the case of the search for normal oscillations, it can be easily seen that, if the phases of the acting forces are equal, $\alpha_{f_i} = \alpha_F$, only in-phase and antiphase motions of oscillators are possible, which are reflected in the signs of corresponding amplitudes, i.e.,

$$\alpha_i = \alpha = \alpha_{f_i} = \alpha_F.$$

We then arrive at a system of algebraic equations:

$$
\begin{aligned}
&(\omega_i^2 - \omega^2)u_i - c_i U = \frac{f_i}{m_i}, \quad i = -N, \ldots, N, \\
&(\Omega^2 - \omega^2)U - \sum_{i=-N}^{N} \frac{m_i}{M} c_i u_i = \frac{F}{M}.
\end{aligned} \tag{19}
$$

3.2.1 The force only acts on the cart. We consider first the case when all $f_i = 0$, i.e., the force only acts on the cart. We find

$$
\begin{aligned}
u_i &= \frac{c_i U}{\omega_i^2 - \omega^2}, \\
U &= \frac{F}{M} \frac{1}{(\Omega^2 - \omega^2) - \sum_{i=-N}^{N} \frac{m_i}{M} \frac{c_i^2}{\omega_i^2 - \omega^2}}.
\end{aligned} \tag{20}
$$

The equation for the zeros of the denominator of the formula for the cart oscillation amplitude coincides with the secular equation. This amplitude increases in a resonant way as the frequency of the excitation force approaches the frequencies of normal oscillations of the system (Fig. 5a). It is of interest to track the amplitudes of the cart and the ith oscillator as the frequency of the driving force changes in the vicinity of the oscillator resonant frequency ω_i, which includes the frequencies of two adjacent normal oscillations $\omega_- = \omega_i - \delta_{\mathrm{L}}$ and $\omega_+ = \omega_i + \delta_{\mathrm{H}}$. The oscillator oscillation amplitude, according to Eqn (20), is represented as

$$u_i = \frac{F}{M} \frac{c_i}{(\omega_i^2 - \omega^2)\left[(\Omega^2 - \omega^2) - \sum_{i=-N}^{N} \frac{m_i}{M} \frac{c_i^2}{\omega_i^2 - \omega^2}\right]}. \tag{21}$$

The value of u_i, similar to the cart oscillation amplitude, increases resonantly as the frequency of the driving force approaches the frequencies of normal oscillations (Fig. 5b). The cart oscillation amplitude vanishes at the frequency ω_i, while the oscillator amplitude remains finite and equal to $u_i = -F/(m_i c_i)$, which is apparent if Eqn (21) is represented in the form

$$u_i = \frac{F}{M} \frac{c_i}{(\omega_i^2 - \omega^2)\left[(\Omega^2 - \omega^2) - \sum_{k \neq i} \frac{m_k}{M} \frac{c_k^2}{\omega_k^2 - \omega^2}\right] - \frac{m_i}{M} c_i^2}. \tag{22}$$

This effect in a system of two oscillators has been discussed in [17] in connection with the Fano resonance. The vanishing of the oscillation amplitude of one of the two coupled oscillators at a certain frequency of the driving harmonic force by which it is excited was explained by the antiphase action of the second oscillator on it. This result is consistent with the picture presented in Fig. 5, but, in our opinion, it is not directly related to the Fano problem. The oscillator oscillation amplitude vanishes at the eigenfrequencies of other oscillators contained in the system. This leads to a kind of resonant phenomenon: if the frequency of the driving force coincides exactly with the oscillator eigenfrequency, the

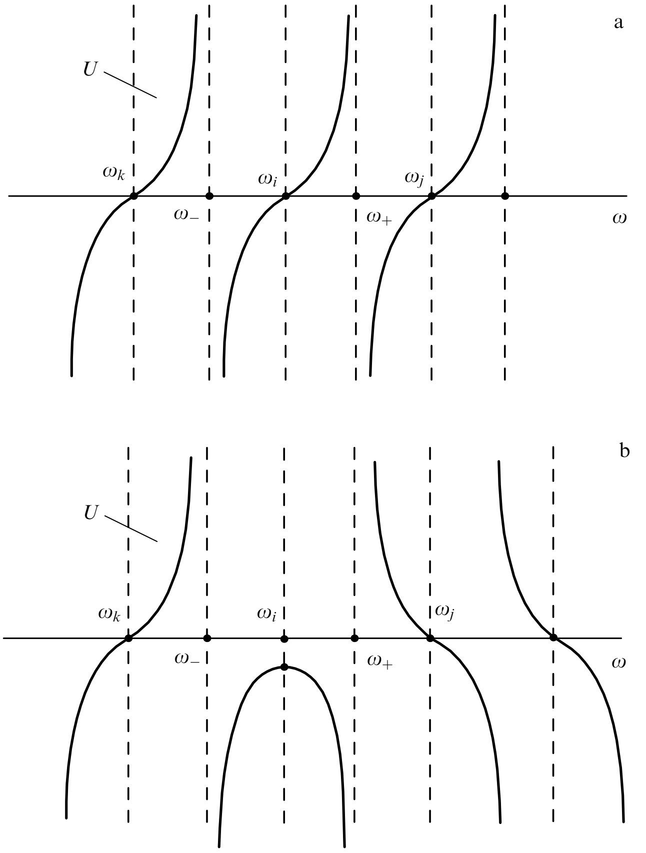

Figure 5. Amplitude of oscillations (a) of the cart and (b) of one of the spring pendulums as a function of the frequency of the harmonic driving force, which varies in the vicinity of resonant frequency ω_i of this pendulum. Force is only applied to the cart.

oscillation amplitudes of all other oscillators and the cart are equal to zero, i.e., only one oscillating oscillator remains in the system in which all the oscillatory energy is concentrated.

3.2.2 The force only acts on one of the spring pendulums. Now let the force acting on only one of the spring pendula be nonzero. The equations assume in this case the following form:

$$
\begin{aligned}
&(\omega_i^2 - \omega^2) u_i - c_i U = \frac{f_i}{m_i}\,, \\
&(\omega_j^2 - \omega^2) u_j - c_j U = 0\,, \quad j \neq i\,, \\
&(\Omega^2 - \omega^2) U - \sum_{k=-N}^{N} \frac{m_k}{M}\, c_k u_k = 0\,.
\end{aligned} \tag{23}
$$

Multiplying each of these equations for oscillators by $(m_k/M)c_k(\omega_k^2 - \omega^2)^{-1}$ and summing up these equations with the equation for the cart oscillation amplitude, we obtain

$$
U\left[(\Omega^2 - \omega^2) - \sum_{k=-N}^{N} \frac{m_k}{M}\, \frac{c_k^2}{\omega_k^2 - \omega^2}\right] = \frac{f_i}{M}\, \frac{c_i}{\omega_i^2 - \omega^2}\,. \tag{24}
$$

It follows from this equation that the cart oscillation amplitude is

$$
U = \frac{f_i}{M}\, \frac{c_i}{(\omega_i^2 - \omega^2)\left[(\Omega^2 - \omega^2) - \sum_{k=-N}^{N} \dfrac{m_k}{M}\, \dfrac{c_k^2}{\omega_k^2 - \omega^2}\right]}\,. \tag{25}
$$

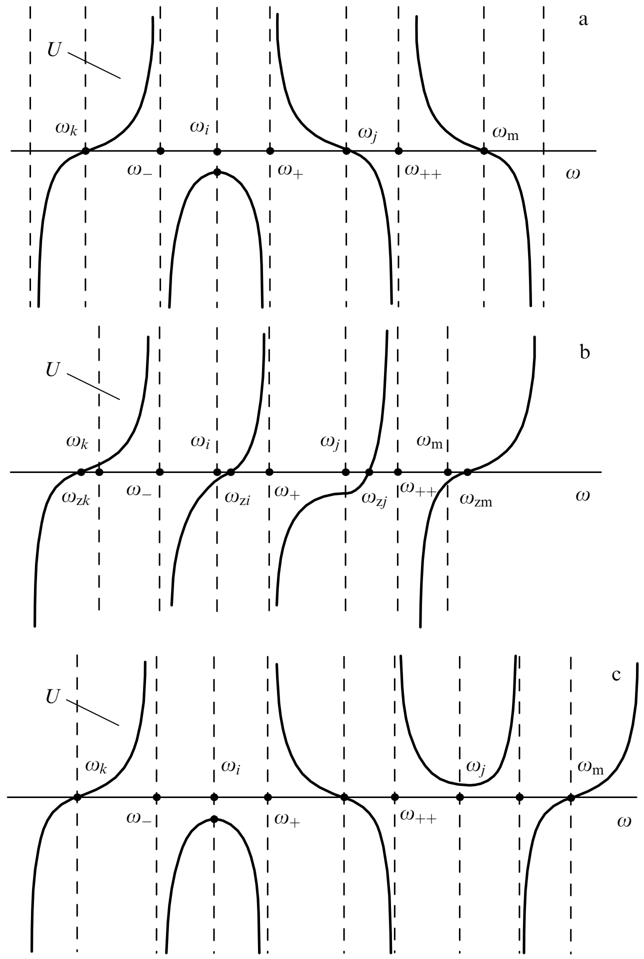

Figure 6. Frequency dependence of the amplitudes of oscillation of spring oscillators and the cart if a harmonic force is only applied to the oscillator with eigenfrequency ω_i. Frequencies ω_-, ω_+, and ω_{++} correspond to normal oscillations, and ω_{zj} and ω_{zi} are frequencies of the normal oscillations of the 'truncated' system of oscillators (see text).

For the oscillator oscillation amplitudes, we obtain

$$
u_j = \frac{f_i}{M}\, \frac{c_j c_i}{(\omega_j^2 - \omega^2)(\omega_i^2 - \omega^2)\left[(\Omega^2 - \omega^2) - \sum_{k=-N}^{N} \dfrac{m_k}{M}\, \dfrac{c_k^2}{\omega_k^2 - \omega^2}\right]}\,, \quad j \neq i\,, \tag{26}
$$

$$
u_i = \frac{f_i}{M}\, \frac{c_i^2}{(\omega_i^2 - \omega^2)^2\left[(\Omega^2 - \omega^2) - \sum_{k=-N}^{N} \dfrac{m_k}{M}\, \dfrac{c_k^2}{\omega_k^2 - \omega^2}\right]} + \frac{f_i}{m_i}\, \frac{1}{\omega_i^2 - \omega^2}\,. \tag{27}
$$

All amplitudes have resonances at the frequencies of the normal oscillations of the system, and Eqn (25) for the cart oscillation amplitude coincides up to replacement $F \to f_i$ with Eqn (21) for the oscillator oscillation amplitude presented in Section 3.2.1. The behavior of all amplitudes with the variation in the excitation frequency in the vicinity of oscillator frequencies ω_i and ω_j, $j \neq i$, is shown in Fig. 6, where ω_+ and ω_- denote, as before, the two frequencies of the normal oscillations adjacent to the frequency of the excited oscillator.

Let us consider the behavior of the oscillators and the cart in more detail. The cart oscillation amplitude (Fig. 6a) has resonances at the frequencies of the normal oscillations of the system. Near the frequencies of normal oscillations that are smaller than frequency ω_i of an oscillator excited by the external force, the cart moves in phase with the exciting force on the lower-frequency side of the resonance and in antiphase with it on the higher-frequency side. For the frequencies of normal oscillations larger than ω_i, the cart movement phase changes to the opposite one. At frequency ω_i, the cart moves in antiphase with the exciting force, and the amplitude of its oscillations is finite:

$$U(\omega_i) = -\frac{f_i}{m_i c_i} \,. \tag{28}$$

The cart oscillation amplitude vanishes at frequencies of other oscillators:

$$U(\omega_j) = 0 \,, \quad j \neq i \,. \tag{29}$$

The oscillator excited by an external force (Fig. 6b) also has resonances at all frequencies of normal oscillations; however, the phase of its movement near these resonances always coincides with the phase of the driving force on the lower-frequency side and is opposite to it on the higher-frequency side of the resonance. The amplitude of this oscillator vanishes at the frequencies given by the equation that follows directly from formula (27):

$$M(\omega_i^2 - \omega^2)^2 \left[(\Omega^2 - \omega^2) - \sum_{k=-N}^{N} \frac{m_k}{M} \frac{c_k^2}{\omega_k^2 - \omega^2} \right] = -m_i c_i^2 (\omega_i^2 - \omega^2) \,. \tag{30}$$

Apart from the singular point $\omega = \omega_i$, there are additional solutions satisfying the equation

$$(\Omega^2 - \omega^2) - \sum_{\substack{k=-N \\ k \neq i}}^{N} \frac{m_k}{M} \frac{c_k^2}{\omega_k^2 - \omega^2} = 0 \,. \tag{31}$$

Equation (31) coincides with the secular equation for the system of oscillators derived from the original system by excluding the oscillator with frequency ω_i from it. The roots of this equation are indicated in Fig. 6 as ω_{zj}. The vanishing of u_i at the frequencies of normal vibrations of the 'truncated' system of oscillators is consistent with the analogous result of the previous case, when the force was applied to the cart. The 'truncated' system performs with respect to the oscillator excited by an external force as a separate oscillatory system with its own resonant frequencies with which the given oscillator interacts; therefore, the vanishing of u_i at the frequencies of the normal vibrations of the 'truncated' system is quite similar to the vanishing in the previous case of the cart amplitude at the frequencies of normal oscillations of the original system. At the singular point ω_i, the amplitude u_i has a finite value, which can most easily be seen directly from the original equations (23). Substituting $\omega = \omega_i$ into the first of the equations, we find $U = -f_i/(m_i c_i)$. This determines the amplitudes of other oscillators:

$$u_j(\omega_i) = -\frac{f_i c_j}{m_i c_i} \frac{1}{\omega_j^2 - \omega_i^2} \,, \quad j \neq i \,,$$

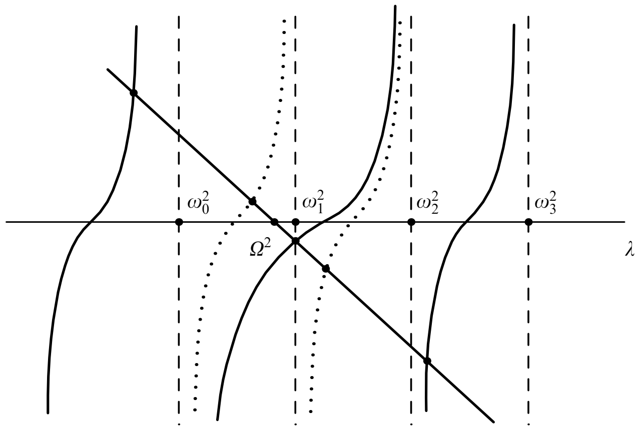

Figure 7. Graphical solution of the secular equation for a nondegenerate system consisting of a cart and oscillators with frequencies ω_0, ω_2, and ω_3 (solid curves). It changes after adding one more oscillator to the system with frequency ω_1 that coincides with the frequency of one of the normal oscillations (dashed curves). In this case, ω_1 is no longer the frequency of the normal oscillation of the system. In the reverse process of removing the oscillator with frequency ω_1 from the system, its frequency again becomes that of the normal oscillation. A slight change in the frequencies of the remaining normal oscillations, which are separated from frequency ω_1, is not shown in the figure.

and the last equation yields

$$u_i(\omega_i) = -\frac{M f_i}{(m_i c_i)^2} \left[(\Omega^2 - \omega_i^2) - \sum_{k \neq i} \frac{m_k}{M} \frac{c_k^2}{\omega_k^2 - \omega_i^2} \right] . \tag{32}$$

This quantity can be positive, negative, or zero if the frequency ω_i turns out to be a root of 'truncated' secular equation (31). The possibility of these situations is illustrated by Fig. 7.

The question can be reformulated as follows: can the eigenfrequency of an oscillator become the frequency of normal oscillations after removing this oscillator from the system? This question should be answered in the affirmative by considering the reverse process. We now add another oscillator to the system of nondegenerate oscillators, the eigenfrequency of which coincides with the frequency of one of the normal oscillations of the system. In the new system, this frequency is no longer the frequency of normal oscillations, which is obvious from Fig. 7. However, the reverse process of removing this oscillator again makes this frequency the frequency of normal oscillations.

At the frequencies of the remaining oscillators, the amplitude of the excited oscillator takes on the values

$$u_i(\omega_j) = \frac{f_i}{m_i} \frac{1}{\omega_i^2 - \omega_j^2} \,, \quad j \neq i \,, \tag{33}$$

whose sign depends on the difference between the oscillator frequencies. The oscillation amplitude of the oscillator with $j \neq i$ is shown in Fig. 6c, from which it can be seen that the amplitudes of such oscillators vanish at frequencies ω_k, $k \neq i, j$ of other spring oscillators and take the following values at frequencies ω_i and ω_j:

$$u_j(\omega_i) = \frac{f_i}{m_i c_i} \frac{c_j}{\omega_i^2 - \omega_j^2} \,, \qquad u_j(\omega_j) = -\frac{f_i}{m_j c_j} \frac{c_i}{\omega_i^2 - \omega_j^2} \,. \tag{34}$$

The behavior of the amplitudes of oscillators with frequencies ω_j, $j \neq i$ near the frequencies of normal oscillations is similar in appearance to the behavior of the amplitude of the cart, but the phase of their motion with respect to the acting force additionally depends on the sign of the quantity $(\omega_j^2 - \omega^2)$.

3.2.3 Concurrent excitation of the cart and the oscillator. Finally, we consider the case where both the cart and the oscillator with eigenfrequency ω_i are excited at the same frequency. This case is of particular interest, since, as shown below, it is precisely such a classical problem that has some similarity with the Fano quantum mechanical problem. The equations then take the form

$$(\omega_i^2 - \omega^2) u_i - c_i U = \frac{f_i}{m_i}\,, \tag{35}$$

$$(\omega_j^2 - \omega^2) u_j - c_j U = 0\,, \quad j \neq i\,, \tag{36}$$

$$(\Omega^2 - \omega^2) U - \sum_{i=-N}^{N} \frac{m_i}{M} c_i u_i = \frac{F}{M}\,, \tag{37}$$

and their solutions are

$$U = \frac{1}{M} \frac{F + f_i \dfrac{c_i}{\omega_i^2 - \omega^2}}{(\Omega^2 - \omega^2) - \displaystyle\sum_{k=-N}^{N} \frac{m_k}{M} \frac{c_k^2}{\omega_k^2 - \omega^2}}\,, \tag{38}$$

$$u_i = \frac{1}{\omega_i^2 - \omega^2} \left\{ \frac{c_i}{M} \frac{F + f_i \dfrac{c_i}{\omega_i^2 - \omega^2}}{(\Omega^2 - \omega^2) - \displaystyle\sum_{k=-N}^{N} \frac{m_k}{M} \frac{c_k^2}{\omega_k^2 - \omega^2}} + \frac{f_i}{m_i} \right\}, \quad \omega \neq \omega_i\,, \tag{39}$$

$$u_j = \frac{1}{\omega_j^2 - \omega^2} \frac{c_j}{M} \frac{F + f_i \dfrac{c_i}{\omega_i^2 - \omega^2}}{(\Omega^2 - \omega^2) - \displaystyle\sum_{k=-N}^{N} \frac{m_k}{M} \frac{c_k^2}{\omega_k^2 - \omega^2}}\,. \tag{40}$$

The element which is new in comparison with the case of excitation of only one oscillator is the emergence of new zeros in the plots and a change in the position of the old ones, as shown in Fig. 8.

The new zero of the cart amplitude is given by the formula

$$F + f_i \frac{c_i}{\omega_i^2 - \omega^2} = 0\,. \tag{41}$$

Equation (41) yields

$$\omega_U = \sqrt{\omega_i^2 + \frac{f_i}{F} c_i}\,. \tag{42}$$

This zero always exists at a reasonable ratio of forces, since, in our model, $c_i \ll \omega_i^2$; however, depending on the sign of the ratio of the forces, options $\omega_U < \omega_i$ or $\omega_U > \omega_i$ are possible. The value $U(\omega_i) = -f_i/(m_i c_i)$ remains the same as in the case of $F = 0$ and does not depend on the position of the new zero. The same zero also appears in the plot of u_j, as follows from Eqn (40). No new zeros appear in the graph u_i, while at frequency ω_U all the vibrational energy again, as in the case of excitation of only one cart (see Section 3.2.1), turns out to be concentrated in one oscillator. The 'old' zeros of u_i are somewhat shifted, which can be easily seen from the equation for the zeros u_i, which now takes the form

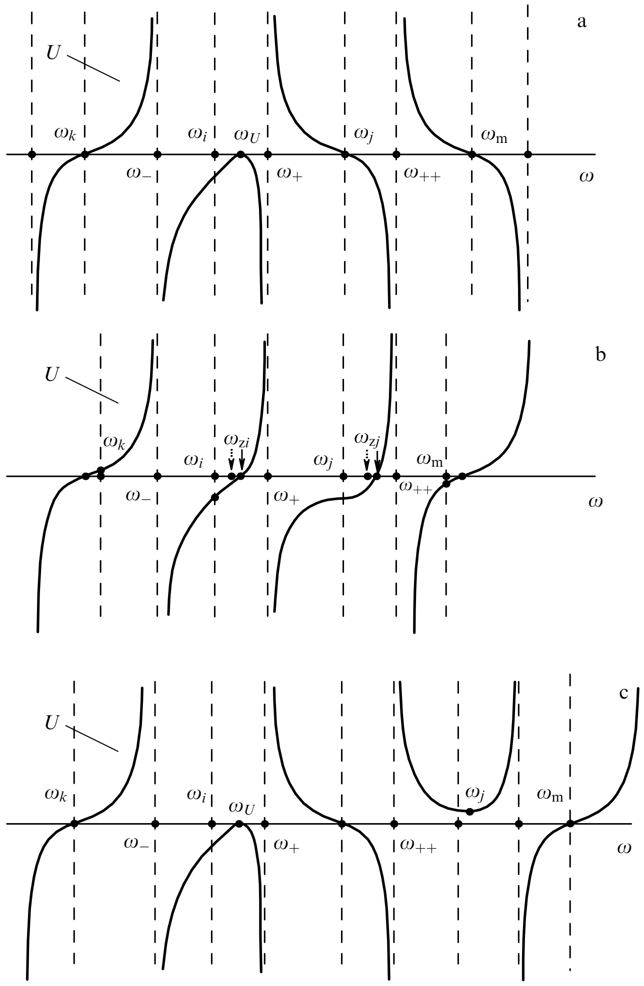

Figure 8. Concurrent excitation of a cart and an oscillator with eigenfrequency ω_i by harmonic forces with the same phase and frequency ω. In Fig. b, the left arrow points to the previous position of zero, while the right arrow indicates the new one. Compared to Fig. 6, a new zero of the amplitude of the cart and other oscillators appears in the vicinity of ω_i, and the position of the zeros of oscillator ω_i changes.

$$(\Omega^2 - \omega^2) - \sum_{\substack{k=-N \\ k \neq i}}^{N} \frac{m_k}{M} \frac{c_k^2}{\omega_k^2 - \omega^2} = -\frac{m_i c_i}{M} \frac{F}{f_i}\,. \tag{43}$$

This equation coincides with the secular equation of the system, from which the oscillator with frequency ω_i has been removed, and the cart's eigenfrequency has been changed:

$$\tilde{\Omega}^2 = \Omega^2 + \frac{m_i c_i}{M} \frac{F}{f_i}\,. \tag{44}$$

Examining the graphical solution of the secular equation displayed in Fig. 2d, it is easy to see that the shift of the cart's eigenfrequency leads to a shift in the same direction of all the roots of this equation.

Similarly to the previous case, it is easy to obtain from the initial equations the amplitude of an oscillator excited by an external force at its eigenfrequency:

$$u_i(\omega_i) = -\frac{Mf_i}{(m_i c_i)^2}\left[(\Omega^2 - \omega_i^2) - \sum_{k\neq i}\frac{m_k}{M}\frac{c_k^2}{\omega_k^2 - \omega_i^2}\right] - \frac{F}{m_i c_i}\,. \tag{45}$$

4. Limiting transition to a continuous spectrum of oscillators

Having gained a fairly detailed understanding of the behavior of our conservative system under its excitation by external periodic forces, we now proceed to a search for the limiting transition to a continuous spectrum of oscillators. We choose the frequencies of the oscillators as multiples of some fundamental frequency $\nu = \Omega/(2N+1)$, so the equality $\omega_i = (2N+1+i)\nu$ holds. In order not to violate our initial assumption (8) about the weakness of the coupling between the cart and the ever-increasing number of oscillators, it is necessary to require fulfillment of the condition $\sum_{i=-N}^{N} c_i \ll \Omega^2$. We assume for simplicity that all the coupling coefficients are the same and decrease with an increase in the number of oscillators in the system: $c_i = c = V/(2N+1)$, $V \ll \Omega^2$. Since the value of V is limited, the transition to an infinite number of oscillators results in the coupling coefficients tending to zero.

The solution to the Fano problem gives formulas for the eigenvectors of an unrestricted system with a continuous spectrum. Our system of coupled oscillators always remains localized in space, but we can try, by increasing the number of oscillators, to bring their frequency spectrum as close to continuous as possible. Figure 2 shows that the frequencies of normal oscillations of our nondegenerate system are located in an alternating way between the frequencies of neighboring oscillators, so, with an increase in the frequency density of oscillators (the number of oscillators per unit frequency interval), the frequency density of normal oscillations also increases.

An analog of the equation for the energy eigenvalues of the finite-dimensional Fano problem [1] for our system is the secular equation

$$(\Omega^2 - \omega_\lambda^2) - \sum_{k=-N}^{N}\frac{c^2}{\omega_k^2 - \omega_\lambda^2} = 0\,, \tag{46}$$

in which, for simplicity, we set all the masses of the spring oscillators the same and equal to the mass of the cart and explicitly indicate the frequency of the normal oscillation ω_λ. Due to the multiplicity of frequencies for our system, we have

$$\sum_{k=-N}^{N}\frac{c^2}{\omega_k^2 - \omega_\lambda^2} = \frac{\pi c^2}{\nu^2}\sum_{k=-N}^{N}\frac{1}{\pi(2N+1+k)^2 - \pi(\omega_\lambda/\nu)^2} = -\frac{\pi c^2}{\nu^2}\cot\pi\left(\frac{\omega_\lambda}{\nu}\right)^2, \tag{47}$$

and the secular equation takes the form

$$(\Omega^2 - \omega_\lambda^2) + \frac{\pi c^2}{\nu^2}\cot\pi\left(\frac{\omega_\lambda}{\nu}\right)^2 = 0\,. \tag{48}$$

This equation is similar to the one for energy eigenvalues obtained in the first study by Fano in considering the interaction between a discrete atomic state and a quasi-continuum of states [1],

$$E = \frac{q^2\pi}{\tau}\cot\frac{E\pi}{\tau}\,, \tag{49}$$

if the replacements $\omega_\lambda^2 \to E$, $c \to q$, and $\nu^2 \to \tau$ are made and Ω^2 is chosen as the reference point for energy, as was done in Fano's study. (It should be noted here that the symbol q is used in [1] to denote a quantity proportional to the matrix element of the transition between the discrete state and quasi-continuum states, which is quite similar to the coupling factor c, rather than the parameters of the Fano resonance line shape.) The value ν is the frequency step and τ is the energy step of the quasi-continuum.

Apparently, the eigenstates of the Fano problem with a given energy E correspond to the normal oscillations of our system with a given frequency ω_λ. Then, the amplitude of the wave function will correspond to the amplitude of the normal oscillation and the square of the modulus of the wave function, i.e., probability, to the square of the amplitude of the normal oscillation, i.e., spectrum power. However, the square of the modulus of the wave function is subject to a limitation associated with normalization, while the square of the amplitude of the normal oscillation increases without limit when the system is excited at its resonant frequency. As a result, the frequency dependence of the oscillator oscillation amplitudes will contain discontinuities at the frequencies of normal oscillations, despite the increase in the spectral density of the number of oscillators. When such a system is excited at the frequencies of normal oscillations, an unlimited increase in the oscillation amplitude will be observed, while, for an arbitrarily small deviation from this frequency, the oscillation amplitude will be finite. Thus, the oscillator oscillation amplitude does not tend to a finite limit with an unlimited increase in the spectral density of normal vibrations. The introduction of small damping for oscillators radically changes the situation, and the transition to the limit of the continuous spectrum becomes possible. Damping in our classical model plays the same role as normalization of the wave function in the quantum problem.

In the case of a frequency continuum, excitation always occurs at the eigenfrequency of the state included in the continuum. If the system is excited only at a frequency that coincides with one of the eigenfrequencies of the oscillators it contains (Eqns (35)–(37) with $\omega = \omega_i$), and in this case we are interested in the oscillation amplitude of the excited oscillator (45), then

$$u_i(\omega_i) = -\frac{Mf_i}{(m_i c_i)^2}\left[\Omega^2 - \omega_i^2 + R(\omega_i)\right] - \frac{F}{m_i c_i} = u_{0c}(q+\epsilon)\,. \tag{50}$$

Here,

$$R(\omega_i) = -\sum_{k\neq i}\frac{m_k}{M}\frac{c_k^2}{\omega_k^2 - \omega_i^2}\,, \tag{51}$$

and coupling coefficients are assumed to be weakly dependent on frequency ω_i, so $R(\omega_i) \neq 0$ (if the coupling constants do not depend on frequency in transiting to the continuous spectrum, the principal value integral vanishes). Equation (50) contains quantities similar to those used by Fano in [4]:

$$\epsilon = \frac{1}{c_i}\left[\omega_i^2 - \Omega^2 - R(\omega_i)\right]\,, \tag{52}$$

$$q = -\frac{F}{f_i}\frac{m_i}{M}\,, \tag{53}$$

and the amplitude of excitation of the continuum states $u_{0c} = (M/m_i^2)(f_i/c_i)$. We obtain a result equivalent to the numerator in the Fano formula:

$$\left(\frac{u_i(\epsilon)}{u_{0c}}\right)^2 = (q+\epsilon)^2 \,. \tag{54}$$

It can be seen that, if

$$q + \epsilon = 0 \,, \tag{55}$$

oscillator oscillations cannot be excited. This result coincides with the condition of the minimum in the Fano formula.

We now show that, if damping of quasi-continuum oscillators is introduced, it is also possible to reproduce the resonance denominator of the Fano formula. To this end, we introduce in Eqn (39) phenomenological damping γ, which is the same for all spring oscillators:

$$u_i = \frac{1}{\omega_i^2 - \omega^2 - \mathrm{i}\gamma\omega} \times \left[\frac{c_i}{M} \frac{F + f_i \dfrac{c_i}{\omega_i^2 - \omega^2 - \mathrm{i}\gamma\omega}}{(\Omega^2 - \omega^2) - \sum\limits_{k=-N}^{N} \dfrac{m_k}{M} \dfrac{c_k^2}{\omega_k^2 - \omega^2 - \mathrm{i}\gamma\omega}} + \frac{f_i}{m_i} \right] . \tag{56}$$

We introduce the dimensionless energy ϵ as was done above. In calculating the small correction $R(\omega_i)$ to the resonant frequency of the cart, we neglect the damping and focus on the case $\omega = \omega_i$:

$$u_i(\omega_i) = \frac{1}{\mathrm{i}\gamma\omega_i} \left\{ \frac{1}{M} \frac{F - f_i\, c_i/(\mathrm{i}\gamma\omega_i)}{\epsilon - (m_i/M)\, c_i/(\mathrm{i}\gamma\omega_i)} - \frac{f_i}{m_i} \right\} = u_{0c} \frac{q+\epsilon}{1 + \mathrm{i}\,\epsilon\gamma\omega_i M/(m_i c_i)} \,. \tag{57}$$

For the modulus of the ratio of amplitudes squared, we have

$$\left(\frac{u_i(\epsilon)}{u_{0c}}\right)^2 = \frac{(q+\epsilon)^2}{1 + \left[\gamma\omega_i M/(m_i c_i)\right]^2 \epsilon^2} \,. \tag{58}$$

Thus, we see that, with the introduction of small damping for oscillators, an approximate result is obtained, which is equivalent, in fact, to the quantum mechanical Fano formula. The physical meaning of damping is to take into account the energy that leaves the system of oscillators to infinity. If only the relaxation of a discrete metastable state is of interest, phenomenological damping can be introduced instead of considering a continuous spectrum of states. In the extended formulation of the problem, which includes a continuous spectrum states, phenomenological damping is no longer required, since the part of the system that provides relaxation of the discrete state is explicitly included in the consideration. Both in the quantum and in the classical problem, the excitation energy must go to infinity due to infinite motion. It is this energy behavior that models the phenomenological damping of the quasi-continuum oscillators. This damping should scale with the increasing number of oscillators, similar to scaling of the coupling coefficients. For damping quasi-continuum oscillators, the amplitude of normal oscillations becomes finite, and with an unlimited increase in the spectral density of oscillators, the amplitudes of oscillations will tend to the same limit as the frequencies come close to each other. If, when scaling the coupling coefficients, we set

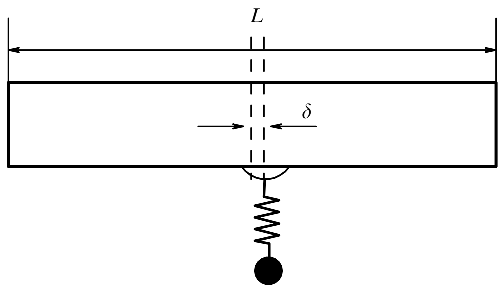

Figure 9. Classical oscillatory system that consists of an acoustic cavity of length L, in which sound vibrations can be excited, and a local spring oscillator connected to the resonator by means of a flexible membrane. As L increases, the system can provide energy escape to infinity, while remaining conservative.

$$\frac{\gamma\omega_i M}{m_i c_i} = 1 \,, \;\; \text{i.e.,} \;\; \gamma = \frac{m_i c_i}{\omega_i M} \,, \tag{59}$$

the damping will decrease in proportion to the decrease in the coupling coefficients, and formula (58) will coincide with the Fano formula.

Our model does not contain the part of the system that features a truly continuous spectrum, i.e., performs infinite motion. For the normal oscillations of our system to transform into those of the Fano problem, it is not enough to make the frequency of normal oscillations arbitrarily dense; it is also necessary to enable infinite motion, i.e., allow the energy that enters the system to go to infinity. The classical model, which is adequate for the Fano problem, could apparently be represented by a local oscillator interacting with a waveguide of finite length, which increases indefinitely in transiting to the case of a continuous spectrum, which is schematically shown in Fig. 9. It should provide the transition of normal oscillations described by standing waves to normal oscillations of traveling waves.

5. Conclusion

We examined in detail a system of interacting classical oscillators and showed that under certain conditions it is in many ways, albeit not everywhere, similar to the quantum mechanical system of a discrete energy level interacting with an energy quasi-continuum of states. The limiting transition to the case of a continuous spectrum and the possible connection of the problem under study to the generation of coherent phonons by ultrashort laser pulses were analyzed. We have shown that, from a fundamental point of view, the introduction of phenomenological damping does not make the problem considered above completely equivalent to the Fano problem, although in practical terms the analogy established in [18] may turn out to be useful. The classical analogy with the Fano problem can be valid, however, not in particle mechanics, but in continuum mechanics, i.e., the mechanics of waves: it is a local oscillator weakly coupled to external traveling waves. For a more detailed review of specific aspects of Fano resonance, we refer the reader to the articles, reviews, and monographs cited above.

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