METHODOLOGICAL NOTES

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Finite value of the bare charge and the relation of the fine structure constant ratio for physical and bare charges to zero-point oscillations of the electromagnetic field in a vacuum

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Contents

Abstract. **The duality of four-dimensional electrodynamics and the theory of a two-dimensional massless scalar field leads to a functional coincidence of the spectra of the mean number of photons emitted by a point-like electric charge in $3+1$ dimensions and the spectra of the mean number of scalar quanta pairs emitted by a point mirror in $1+1$ dimensions. The spectra differ only by the factor $e^2/\hbar c$ (in Heaviside units). The requirement that the spectra be identical determines unique values of the point-like charge $e_0 = \pm\sqrt{\hbar c}$ and its fine structure constant $\alpha_0 = 1/4\pi$, which have all the properties required by Gell-Mann and Low for a finite bare charge. The Dyson renormalization constant $Z_3 \equiv \alpha/\alpha_0 = 4\pi\alpha$ is finite and lies in the range $0 < Z_3 < 1$, in agreement with the Källén–Lehmann spectral representation sum rule for the exact Green's function of the photon. The value of Z_3 also lies in a very narrow interval $\alpha_{\rm L} < Z_3 \equiv \alpha/\alpha_0 = 4\pi\alpha < \alpha_{\rm B}$ between the values $\alpha_{\rm L} = 0.0916$ and $\alpha_{\rm B} = 0.0923$ of the parameters defining the shifts $E_{\rm L,B} = \alpha_{\rm L,B}\hbar c/2r$ of the energy of zero-point fluctuations of the electromagnetic field in cubic and spherical resonators with the cube edge length equal to the sphere diameter, $L = 2r$. In this case, the cube is circumscribed about the sphere. That the difference between the coefficients $\alpha_{\rm L,B}$ is very small can be explained by the general property of all polyhedra circumscribed about a sphere: despite the difference between their shapes, they share a topological invariant, the surface-to-volume ratio $S/V = 3/r$, the same as for the sphere itself. Shifts of the energy of zero-point oscillations in such resonators are also proportional to this invariant: $E_{\rm L,B} = \alpha_{\rm L,B}\hbar cS/6V$. On the other hand, the shifts $E_{\rm L,B} = \alpha_{\rm L,B}\hbar c/2r$ of the energy of zero-point oscillations of the electromagnetic field essentially coincide with the energy of the mean squared fluctuations of the volume-averaged electric and magnetic fields in resonators, equal to $Z_3\hbar c/2r$ in order of magnitude. It hence follows that $\alpha_{\rm L,B} \approx Z_3$, as it should for the coefficients α_γ of the shifts $E_\gamma = \alpha_\gamma\hbar c/2r$ in other resonators γ circumscribed about a sphere. The closeness of $\alpha_{\rm L}$ and $\alpha_{\rm B}$ to the Z_3 factor is confirmed by the Källén–Lehmann spectral representation and agrees with asymptotic conditions relating the photon creation amplitudes for free and interacting vector fields.**

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1. Introduction

It is known that, at a very high energy of a charge and large momenta transferred by the charge to other charges or fields, the charge itself increases. The charge that determines the amplitude of the emission of soft photons to infinity increases simultaneously. These photons do not change the current that emitted them, and therefore the emission and absorption of such photons is described by the vacuum-to-vacuum amplitude

$$\langle 0_+|0_-\rangle^J = \exp\left[\frac{\rm i}{\hbar}\,W(J)\right],$$

which depends on a fixed external classical current $J(x)$. According to Schwinger and his theory of sources [1], this amplitude provides a comprehensive description of the many-particle emission and absorption processes for particles obeying the Bose–Einstein statistics.

The doubled imaginary part of the action W divided by Planck's constant $\hbar$ is equal to the average number $\bar{N}$ of particles emitted by the source over the entire time. In turn, $\bar{N}$ is an integral over the spectrum of the mean number $\mathrm{d}\bar{n}_k$ of emitted quanta,

$$\bar{N} = \frac{2\,\mathrm{Im}\,W}{\hbar} = \int \mathrm{d}\bar{n}_k\,,$$

each of which has the momentum $\hbar\mathbf{k}$ and energy $\hbar\omega$ determined by the wave vector $k^{\alpha} = (\mathbf{k}, k^0)$, where $k^0 = \omega/c$. Such is the direct relation between a very important semiclassical quantity and the central object in quantum physics, the quanta of a source field.

In the series of studies [2–9] initiated by the author in collaboration with Nikishov, we discovered a functional coincidence of the spectra of the mean number of photons (in another version, scalar quanta) emitted by a point electric (scalar) charge in $3+1$ dimensions with the spectra of the mean number of pairs of scalar (spinor) massless quanta emitted by a point mirror in $1+1$ dimensions. The spectra, which are functions of two variables and functionals of the common trajectory of the charge and the mirror, differ only by the factor $e^2/\hbar c$ (in Heaviside units), because the boundary condition set on the mirror is purely geometric and does not contain any parameters. It is natural to call such a functional coincidence a holographic duality [8].

The requirement that $e^2/\hbar c = 1$ leads to unique values of the point charge $e_0 = \pm\sqrt{\hbar c}$ and its fine structure constant $\alpha_0 = 1/4\pi$. Very importantly, these values satisfy the general properties stated by Gell-Mann and Low [10] for a *finite* bare charge:

(1) α_0 is independent of the fine structure constant α;

(2) $\alpha_0 > \alpha$;

(3) the bare charge distribution density is described by the spatial delta function times e_0: $e_0\delta(\mathbf{x})$.

The requirement $e^2/\hbar c = 1$ follows from the holographic principle of bare charge quantization proposed by the author [8], according to which the quanta and pairs emitted by the charge and the mirror respectively propagating in four-dimensional space and on a two-dimensional surface embedded into it must have coinciding spectra. The duality is due to the integral connection of the causal Green's functions for $3+1$ and $1+1$ dimensions and the relation between the current (charge) densities in $3+1$ dimensions and scalar products of scalar (spinor) massless fields in $1+1$ dimensions [8].

The relations underlying the duality, as well as the Green's functions themselves, are of purely geometric origin. The values found for the bare charge, $e_0 = \pm\sqrt{\hbar c}$, and the corresponding fine structure constant $\alpha_0 = 1/4\pi$ satisfying Gell-Mann and Low's properties for a finite bare charge also have a purely geometric origin, which is natural, because the spectra are calculated using solutions of wave equations for massless fields with zero boundary conditions and time-like curves for the trajectories of a point charge and a mirror.

The discovered duality connects wave processes in four-dimensional and two-dimensional Minkowski spaces. It is in such spaces of even dimension that the propagation of massless waves obeys the Huygens geometric principle [11]. In addition, electrodynamics in $3+1$ dimensions and the theory of a massless scalar field in $1+1$ dimensions, related by duality, admit conformal symmetry.

In his lecture at the University of New South Wales, Dirac referred to the fine structure constant α and the ratio of the proton mass to the electron mass and noted that such constants are likely to be made of simple quantities like 4π [12]. Dirac's insight would seem truly amazing had he meant not α but α_0, the fine structure constant of the bare charge, which, according to the duality under discussion, is exactly equal to $1/4\pi$. The difference between these two quantities is due to the polarization of the vacuum by a point electric charge, and their ratio $\alpha_0/\alpha = 10.90\ldots$ is the permittivity of the vacuum. The inverse ratio $\alpha/\alpha_0 = Z_3$ is the most important of the three renormalization constants introduced by Dyson [13]. The constant Z_3 also has the meaning of the probability that an excitation of the electromagnetic field $\mathbf{A}(x)$ coupled to a dynamic current $j_\mu(x)$ propagates as a free zero-mass photon in the vacuum. Such an interpretation was given to Z_3 by Källén [14] and to a similar quantity Z for a scalar field by Lehmann [15], who obtained spectral representations of the exact Green's functions for electromagnetic and scalar fields.

These representations are discussed in Section 3. It is preceded by Section 2, where the exact Green's functions are regarded as vacuum expectation values of interacting operator fields. The exact causal Green's function arises as the vacuum expectation value of the chronological product of fields. Following Schwinger [16, 17], the radiative (Coulomb) gauge is used for the electromagnetic field $A_\mu(x)$, which is convenient in view of a straightforward probabilistic interpretation of its components and which ensures the three- and four-dimensional transversality of the radiation field.

The central expression in these two sections is the spectral representation for the causal Green's function

$$\mathcal{G}(x-x') = Z_3 D_F(x-x') + \int_{0^+}^{\infty} \mathrm{d}m^2\,\rho(m^2)\,\Delta_F(x-x', m)$$

and the sum rule

$$Z_3 + \int_{0^+}^{\infty} \mathrm{d}m^2\,\rho(m^2) = 1\,,$$

where Z_3 has the meaning of the probability that an on-shell photon is produced by a weak external current $J_\mu(x)$ coupled to fluctuations of the vacuum field $\mathbf{A}(x)$, and its propagation from the location of its production to the location of its absorption by the current $J_\mu(x')$ is then governed by the propagator $D_F(x-x')$. At the same time, $\rho(m^2)\,\mathrm{d}m^2$ has the meaning of the probability that a virtual photon with mass m^2 is produced by the same current $J_\mu(x)$ coupled to vacuum fluctuations of the $j_\mu(x)$ current, and its subsequent propagation until its absorption by the current $J_\mu(x')$ is governed by the propagator $\Delta_F(x-x')$.

A positive energy shift of zero-point fluctuations of the electromagnetic field in spherical and cubic cavities was first obtained by Boyer [18] and Lukosz [19], as well as in [20–25]. Its finiteness means that this shift is observable (see [26, 27]).

The analytic expression obtained in [19] for the energy shift of zero-point field oscillations in a cubic cavity is discussed in detail in Section 4. We focus there on high-frequency divergences inherent in the energy density u of zero-point field oscillations and their removal from the sum of those eigenfrequencies of the cavity that determine the finite shift Δu of the energy density in the cavity. The divergences are removed by introducing a form factor into the energy density to cut off high frequencies and by discarding terms that grow with the cutoff frequency.

In Section 5, the energy shift of zero-point oscillations in a cubic cavity is represented in terms of the values of the Riemann and Epstein analytic zeta functions. Such a representation avoids the divergences arising from the direct summation of the energy spectrum of zero-point fluctuations of the electromagnetic field in the vacuum.

A comparison of the shifts E_{B} (Boyer) and E_{L} (Lukosz) of the energy of zero-point oscillations in spherical and cubic cavities shows that they practically coincide if the edge length of the cube coincides with the diameter of the sphere, $L = 2r$. A possible reason for the weak dependence of the shift on the cavity shape is described in Section 6. In the case $L = 2r$, the cube is circumscribed about the sphere, and all polyhedra circumscribed about a sphere, despite the variety of their shapes, share the exact same surface-to-volume ratio with the sphere itself: $S/V = 3/r$. It is a topological invariant of these polyhedra. On the other hand, the characteristic scale of the energy shift in the cavities is chosen as $\hbar c/2r$, which is proportional to their topological invariant.

In Section 7, we discuss the relation between the coefficients α_{L} and α_{B} that determine the shifts in the energy of zero-point oscillations in cubic and spherical cavities, on the one hand, and the Z_3 factor on the other hand. The coefficients α_{L} and α_{B} characterize the interaction of the current on the conducting shell of the cavity with vacuum fluctuations of the electromagnetic field during the formation of the cavity. As a result, standing electromagnetic waves appear inside it, carrying the energy $\alpha_{\mathrm{L,B}}\hbar c/2r$. The Z_3 factor characterizes the coupling of the external current $J_\mu(x)$ to the same field fluctuations in the vacuum that give rise to on-shell photons.

The closeness of α_{L} and α_{B} to the Z_3 factor is confirmed by the Källén–Lehmann spectral representation in (3.1) and (3.12) and agrees with asymptotic conditions (3.11) and (8.2).

In Section 8, we discuss the measurability of the energy shift of zero-point fluctuations of an electromagnetic field in a cavity. The shift appears as a result of the work of external forces during the adiabatic formation of the cavity and the currents on its conducting shell, which interact with the field fluctuations. The interaction leads to the formation of on-shell waves in the cavity, with the energy determined by the work of external forces.

Cavities whose shells are circumscribed about a sphere and are characterized by the topological invariant $S/V = 3/r$ are discussed in detail in the Conclusion. Calculating the energy shift of zero-point field oscillations in one more such cavity, for example, of a cylindrical shape, could entirely clarify the dependence of the shift on the cavity shape. We show that, in cavities circumscribed about a sphere, the excess pressure p and the volume are related by the adiabat $pV^{4/3} = \mathrm{const}\,\hbar c$, which is similar to the black-body radiation adiabat.

We show that the functional coincidence of the spectra of photons and pairs is completely determined by holographic duality (9.12) between propagators in four-dimensional and two-dimensional spaces and the purely geometric relation (9.22) between the Fourier transform of the velocity 2-vector of the charge and the mirror to Bogoliubov's β coefficient that determines the spectrum of pairs of scalar quanta. Therefore, the holographic principle of bare charge quantization leading to the value $e_0^2/\hbar c = 1$, $\alpha_0 = 1/4\pi$ can be formulated as a requirement that the spectra of photons and pairs coincide or that the same mean total number of photons and pairs be emitted by the charge and the mirror over their entire shared trajectory. In both cases, the spectra coincide because of their functional coincidence.

We show that the resulting value of the bare charge, being finite, satisfies all three conditions of Gell-Mann and Low [10], and the ratio of the squares of the physical and bare charges, equal to the Dyson Z_3 factor, lies in a very narrow interval $\alpha_{\mathrm{L}} < Z_3 = \alpha/\alpha_0 = 4\pi\alpha < \alpha_{\mathrm{B}}$ between the purely geometric constants α_{L} and α_{B} that define the finite energy shifts of zero-point oscillations of the electromagnetic field in cubic and spherical cavities. We show that the remarkable proximity of the coefficients $\alpha_{\mathrm{L,B}}$ to the finite value $Z_3 = 4\pi\alpha$ is not accidental but is a consequence of the fundamental equal-time commutation relations for the interacting operator fields, in particular, a consequence of the Källén–Lehmann spectral representation for the exact Green's function of the photon.

Throughout the paper, we use the metric $g_{\alpha\beta} = \mathrm{diag}\,(1, 1, 1, -1)$, the natural system of units $\hbar = c = 1$, and the Heaviside units for charge.

2. Exact Green's functions as vacuum expectations of interacting operator fields

We follow the concise presentation of the subject by Schwinger [16, 17] and the notation used there.

We consider the simplest (positive-frequency) Green's function associated with a vector operator field $A_\mu(x)$ coupled to a dynamic current $j_\mu(x)$. This Green's function can be derived from the vacuum expectation value of the unordered product of field operators

$$\left\langle A_\mu(x)A_\nu(x')\right\rangle = \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \exp\left[\mathrm{i}p(x - x')\right] \mathrm{d}m^2 \times \eta_+(p)\delta(p^2 + m^2)A_{\mu\nu}(p)\,, \tag{2.1}$$

where the factor $\eta_+(p)\delta(p^2 + m^2)$ restricts the spectrum to states with a mass $m \geqslant 0$ and positive energy. The spectral components of vacuum expectation value (2.1) are given by the invariant positive-frequency functions

$$\Delta^{(+)}(x, m) = \int \frac{\mathrm{d}^4 p}{(2\pi)^3} \exp\,(\mathrm{i}px)\eta_+(p)\delta(p^2 + m^2) = \int \frac{\mathrm{d}^3 p \exp\,(\mathrm{i}px)}{(2\pi)^3 2p^0}\,, \quad p^0 = \sqrt{m^2 + \mathbf{p}^2}\,, \tag{2.2}$$

which are eigenfunctions for the operator ∂_α^2 with the eigenvalue m^2. The requirement of nonnegative definiteness of the matrix $A_{\mu\nu}(p)$ is satisfied for fields in the radiation gauge, which leads to the structure

$$A^R_{\mu\nu}(p) = B(m^2)\left[g_{\mu\nu} - \frac{(p_\mu n_\nu + p_\nu n_\mu)(np) + p_\mu p_\nu}{p^2 + (np)^2}\right], \tag{2.3}$$

introducing a gauge-dependent asymmetry between space and time. In the Lorentz frame where quantization is performed, a unit time-like vector n_μ is directed along the time axis: $n^\mu = (0, 0, 0, 1)$. In that frame, the nonvanishing components of the matrix in square brackets are given by

$$[\ ]_{\mu\nu} = \delta_{ij} - \frac{p_i p_j}{\mathbf{p}^2} \quad \text{for} \quad \mu = i, \nu = j\,; \quad i,j = 1,2,3\,,$$
$$[\ ]_{\mu\nu} = -\frac{p^2}{\mathbf{p}^2} \qquad \text{for} \quad \mu = \nu = 0\,. \tag{2.4}$$

Another useful representation is

$$[\ ]_{\mu\nu} = \sum_{\lambda=1}^{2} e_\mu(\lambda) e_\nu(\lambda) - \frac{p^2 n_\mu n_\nu}{p^2 + (np)^2}\,, \tag{2.5}$$

which corresponds to the radiation field and the Coulomb field. Here, $e_\mu(\lambda)$, $\lambda = 1,2$, are space-like unit vectors of polarization, which are orthogonal to each other and to the vectors p_μ and n_μ.

The real nonnegative function $B(m^2)$ satisfies the sum rule

$$\int_0^\infty \mathrm{d}m^2\, B(m^2) = 1\,, \tag{2.6}$$

which fully expresses all fundamental equal-time commutation relations for the interacting operator fields.

Field equations lead to a similar structure of the vacuum expectation of the product of current density operators $\langle j_\mu(x) j_\nu(x')\rangle$, with the nonnegative definite matrix

$$j_{\mu\nu}(p) = m^2 B(m^2)(p_\mu p_\nu - g_{\mu\nu} p^2) \tag{2.7}$$

replacing $A^R_{\mu\nu}(p)$. An important corollary of the appearance of the factor m^2 is that the zero mass $m = 0$ is not in the spectrum of vacuum fluctuations of the current density. Fluctuations of the current vector determine $B(m^2)$ for $m > 0$ but leave the possible delta-function contribution at $m = 0$ not fully fixed:

$$B(m^2) = B_0 \delta(m^2) + B_1(m^2)\,. \tag{2.8}$$

The nonnegative constant B_0 is then fixed by the sum rule

$$1 = B_0 + \int_0^\infty \mathrm{d}m^2\, B_1(m^2)\,. \tag{2.9}$$

It follows that the spectrum of vacuum fluctuations of the vector A_μ consists of two parts. One part, corresponding to $m > 0$, is directly related to current density fluctuations, and the other part, corresponding to $m = 0$, can be associated with a pure radiation field, which is transverse in both three and four dimensions and is not related to a current.

We consider the vacuum expectation of the T-product of fields $A_\mu(x)$ and $A_\nu(x')$,

$$\mathcal{G}_{\mu\nu}(x - x', n) = \mathrm{i}\big\langle T A_\mu(x) A_\nu(x')\big\rangle$$
$$= \mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^3} \exp\left[\mathrm{i}\mathbf{p}(\mathbf{x} - \mathbf{x}') - \mathrm{i}p^0|t - t'|\right] \mathrm{d}m^2$$
$$\times\, \eta_+(p)\delta(p^2 + m^2) A^R_{\mu\nu}(p)\,. \tag{2.10}$$

The invariant function

$$\mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^3} \exp\left[\mathrm{i}\mathbf{p}\mathbf{x} - \mathrm{i}p^0|t|\right]\eta_+(p)\delta(p^2 + m^2)$$
$$= \mathrm{i}\int \frac{\mathrm{d}^3 p}{(2\pi)^3 2\sqrt{m^2 + \mathbf{p}^2}} \exp\left[\mathrm{i}\mathbf{p}\mathbf{x} - \mathrm{i}\sqrt{m^2 + \mathbf{p}^2}\,|t|\right]$$
$$= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \frac{\exp(\mathrm{i}px)}{p^2 + m^2 - \mathrm{i}\varepsilon}\,, \tag{2.11}$$

which is a spectral component of the vacuum expectation value in (2.10), is well known. This is the causal Green's function (Feynman propagator). It was introduced by Stueckelberg in [28–30] and denoted by $\Delta^c(x)$ there. Schwinger and Feynman use the notations $\Delta_+(x,m)$ [1, 31] and $I_+(x,m)$ [32, 33] for it, and other authors [34, 35] use the notation $\Delta_F(x,m)$ or $\Delta^f(x,m)$ [36, 8]. It follows from representation (2.11) that this function is even in x^μ and satisfies the equation

$$(-\partial_\alpha^2 + m^2)\Delta_+(x,m) = \delta_4(x)\,. \tag{2.12}$$

In calculating physical amplitudes in quantum field theory, the propagator $\mathcal{G}_{\mu\nu}$, Eqn (2.10), is always placed between conserved currents that are sources of the electromagnetic field. In particular, in the action

$$W = \frac{1}{2c}\int \mathrm{d}^4x\, \mathrm{d}^4x'\, J^\mu(x) \mathcal{G}_{\mu\nu}(x - x', n) J^\nu(x')\,, \tag{2.13}$$

which defines the vacuum-to-vacuum amplitude $\langle 0_+ 0_-\rangle^J = \exp(\mathrm{i}W/\hbar)$ for the external current $J^\mu(x)$, the current conservation relation $\partial_\mu J^\mu(x) = p_\mu J^\mu(p) = 0$ results in the vanishing of n^μ-dependent terms in the matrix $A^R_{\mu\nu}$. As a result, the propagator $\mathcal{G}_{\mu\nu}(x,n)$ reduces to a manifestly Lorentz-invariant expression

$$\mathcal{G}_{\mu\nu}(x) = g_{\mu\nu}\mathcal{G}(x) = g_{\mu\nu}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \exp(\mathrm{i}px)\mathcal{G}(p)\,,$$
$$\mathcal{G}(p) = \int_0^\infty \mathrm{d}m^2 \frac{B(m^2)}{p^2 + m^2 - \mathrm{i}\varepsilon}\,. \tag{2.14}$$

Using the function $B(m^2)$ in (2.8), we obtain the Källén–Lehmann spectral representation [14, 15] for $\mathcal{G}(x)$ in both momentum

$$\mathcal{G}(p) = \frac{B_0}{p^2 - \mathrm{i}\varepsilon} + \int_0^\infty \frac{\mathrm{d}m^2 B_1(m^2)}{p^2 + m^2 - \mathrm{i}\varepsilon} \tag{2.15}$$

and coordinate

$$\mathcal{G}(x) = B_0 D_F(x) + \int_0^\infty \mathrm{d}m^2\, B_1(m^2)\Delta_F(x,m) \tag{2.16}$$

representations.

Schwinger emphasizes that one must clearly distinguish between c-number gauge functions and operator gauge functions, because different operator gauges do not have the same justification at the quantum level. Each coordinate system has a unique operator gauge characterized by three-dimensional transversality (radiative gauge), which allows a standard operator construction in a positive-norm vector space, with a probabilistic physical interpretation. If the theory is formulated in terms of vacuum expectation values of chronological operator products, i.e., Green's functions, then the freedom of formal gauge transformations can be restored [37]. The Green's functions in other gauges have a more complicated operator realization and, generally speaking, do not have the positivity properties available in the radiative gauge.

An alternative to the Källén–Lehmann representation is given by [17]

$$\mathcal{G}(p) = \left[p^2 - \mathrm{i}\varepsilon + (p^2 - \mathrm{i}\varepsilon)\int_0^\infty \frac{\mathrm{d}m^2\, s(m^2)}{p^2 + m^2 - \mathrm{i}\varepsilon}\right]^{-1}, \quad (2.17)$$

where the function $s(m^2)$ is real and nonnegative.

If we require that $m = 0$ appear as an isolated mass value in the physical spectrum of states, then we must have

$$s(0) = 0 \quad \text{and} \quad \int_\delta^\infty \mathrm{d}m^2\, \frac{s(m^2)}{m^2} < \infty \quad \text{for} \quad \delta \to 0\,. \quad (2.18)$$

Only in that case does the Green's function have a pole at $p^2 = 0$:

$$p^2 \sim 0: \quad \mathcal{G}(p) \sim \frac{B_0}{p^2 - \mathrm{i}\varepsilon}\,, \quad 0 < B_0 < 1\,. \quad (2.19)$$

Under these condition, the spectral function

$$B(m^2) = B_0\delta(m^2) + B_1(m^2)$$

is related to $s(m^2)$:

$$B_0 = \left[1 + \int_0^\infty \frac{\mathrm{d}m^2}{m^2}\, s(m^2)\right]^{-1}, \quad (2.20)$$

$$B_1(m^2) = \frac{s(m^2)/m^2}{\left(1 + \mathrm{V.p.}\displaystyle\int_0^\infty \frac{\mathrm{d}m'^2\, s(m'^2)}{m'^2 - m^2}\right)^2 + \left(\pi s(m^2)\right)^2}\,. \quad (2.21)$$

The physical interpretation of the functions $B(m^2)$ and $s(m^2)$ can be deduced from the relation between the Green's function and the vacuum-to-vacuum amplitude in the presence of sources. For sufficiently weak external currents $J_\mu(x)$,

$$\langle 0|0\rangle^J = \exp\left[\frac{1}{2}\,\mathrm{i}\int \mathrm{d}^4x\, \mathrm{d}^4x'\, J^\mu(x)\mathcal{G}(x - x')J_\mu(x')\right] \quad (2.22)$$

$$= \exp\left[\frac{1}{2}\,\mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\, J^\mu(p)\mathcal{G}(p)J_\mu(p)\right]. \quad (2.23)$$

The probability that the vacuum is preserved despite the perturbation by an external current is

$$\left|\langle 0|0\rangle^J\right|^2 = \exp\left[-\int \frac{\mathrm{d}^4p}{(2\pi)^4}\, J^\mu(p)^*J_\mu(p)\, \mathrm{Im}\,\mathcal{G}(p)\right], \quad (2.24)$$

where, according to (2.14),

$$\mathrm{Im}\,\mathcal{G}(p) = \pi\int_0^\infty \mathrm{d}m^2\, B(m^2)\delta(p^2 + m^2)\,, \quad (2.25)$$

and the integral in the exponent is twice the imaginary part of the action. After integration over p^0, it takes the compact form

$$2\,\mathrm{Im}\,W = \int \frac{\mathrm{d}^3p\, \mathrm{d}m^2}{(2\pi)^3 2p^0}\, B(m^2)J^\mu(p)^*J_\mu(p)\,, \quad (2.26)$$

where $p^0 = \sqrt{m^2 + \mathbf{p}^2}$, $\mathbf{p}$, and m are the energy, momentum, and mass of the excitations produced in the vacuum by the external current.

We next recall that the excitation spectrum consists of an isolated excitation with zero mass $m = 0$ and a continuous spectrum of states with masses $m > 0$ (see (2.8)). Then,

$$2\,\mathrm{Im}\,W = B_0\int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\, J^\mu(p)^*J_\mu(p) + \int_0^\infty \mathrm{d}m^2\, B_1(m^2)\int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\, J^\mu(p)^*J_\mu(p)\,, \quad (2.27)$$

where the contraction of currents

$$J^\mu(p)^*J_\mu(p) = \left|\mathbf{J}(p)\right|^2 - \left|J^0(p)\right|^2 \quad (2.28)$$

is positive both in the first term, where the vector p^μ is isotropic, $p^2 = 0$, and in the second term, where it is time-like, $-p^2 = m^2 > 0$.

Indeed, it follows from the current conservation condition

$$\mathbf{pJ} - p^0J^0 = 0 \quad (2.29)$$

that in the first case we have $J_3 = J^0$ in the reference frame where $p_1 = p_2 = 0$ and $p_3 = p^0$, and the contraction reduces to the contribution of the current components transverse to the wave vector $\mathbf{p}$: $|J_1|^2 + |J_2|^2 > 0$. In the second case, the same condition implies that $J^0 = 0$ in the frame where $\mathbf{p} = 0$ and $p^0 = m$, and the contraction reduces to the contribution of three components of the current: $|\mathbf{J}|^2 > 0$. Thus, zero-mass excitations are transverse not only in the four-dimensional sense but also in the three-dimensional sense, and are not related to a current. The Lorentz invariance of the contraction of currents and of the on-shell integration measure $-p^2 = m^2 \geqslant 0$ preserves the invariance and positive definiteness of integrals that depend only on m^2 and parameters that determine the external current. Evidently, $2\,\mathrm{Im}\,W > 0$ for the excitation current, and, hence, the vacuum-to-vacuum probability is less than unity.

Finally, we note that, when divided by the Planck constant, the expression $2\,\mathrm{Im}\,W/\hbar$ becomes dimensionless and represents the mean number of on-shell photons (the first term in (2.27)) and the mean number of on-shell $\mathrm{e}^+\mathrm{e}^-$ pairs and photons (second term) emitted by the current over the entire time.

A nontrivial example of the first term in $2\,\mathrm{Im}\,W$ in (2.27) is Schott's formula [38] for the radiation from a charge moving along a circle in a magnetic field with a frequency $\omega_H = v/r$. In this case, the integral

$$\frac{1}{\hbar}\int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0}\, J^\mu(p)^*J_\mu(p) = \sum_{n=1}^\infty \frac{I_n}{\hbar\omega_n}\, T = 2\alpha_0\omega_H\sum_{n=1}^\infty \left[\beta J_n'(2n\beta) - \frac{1 - \beta^2}{\beta}\int_0^{n\beta} \mathrm{d}(n\xi)\, J_{2n}(2n\xi)\right] T$$

is proportional to the time T and, after multiplication by B_0, becomes the renormalization-invariant quantity $B_0\alpha_0 = \alpha$ that determines the mean number of photons emitted over the entire time.

We now transform the vacuum-to-vacuum amplitude (2.22) for an external current into a function of the vacuum response to a vector potential determined by an external current,

$$A_\mu(p) = \mathcal{G}(p) J_\mu(p)\,. \tag{2.30}$$

We note that, due to current conservation, this potential is transverse in four dimensions: $p_\mu A^\mu(p) = 0$. Using the alternative representation for the Green's function in (2.17), we obtain the amplitude

$$\langle 0|0\rangle^J = \exp\left[\frac{1}{2}\,\mathrm{i}\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, A^\mu(p)^* \mathcal{G}(p)^{*\,-1} A_\mu(p)\right] \tag{2.31}$$

and the probability

$$\left|\langle 0|0\rangle^J\right|^2 = \exp\left[-\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, A^\mu(p)^* A_\mu(p)\,\mathrm{Im}\,\mathcal{G}(p)^{*\,-1}\right] \tag{2.32}$$

that the vacuum is preserved despite the perturbation.

Because

$$\mathrm{Im}\,\mathcal{G}(p)^{*\,-1} = -\pi p^2 \int_0^\infty \mathrm{d}m^2\, s(m^2)\,\delta(p^2 + m^2)\,, \tag{2.33}$$

$$-p^2 A^\mu(p)^* A_\mu(p) = -\frac{1}{2}\, F^{\mu\nu}(p)^* F_{\mu\nu}(p) = \left|\mathbf{E}(p)\right|^2 - \left|\mathbf{H}(p)\right|^2, \tag{2.34}$$

it follows that

$$\left|\langle 0|0\rangle^J\right|^2 = \exp\left[-\int \frac{\mathrm{d}^3 p\, \mathrm{d}m^2}{(2\pi)^3 2p^0}\, s(m^2)\left(-\frac{1}{2}\right) F^{\mu\nu}(p)^* F_{\mu\nu}(p)\right], \tag{2.35}$$

where $p^0 = \sqrt{m^2 + \mathbf{p}^2}$. This shows that $s(m^2)$ is a measure of the probability that the external field produces an excitation in the vacuum, endowing it with on-shell energy and momentum $m^2 = -p^2 > 0$.

It is noteworthy that (2.35) contains no contribution from zero-mass states. This is because the electromagnetic field $F_{\mu\nu}(x)$ is spatially separated from its source, the current $J_\mu(x)$, and is not a source of real photons in and of itself, although it can be made of them (a plane wave) or of virtual photons (a Coulomb field).

A plane-wave field of an arbitrary spectral composition is described by the tensor

$$F_{\mu\nu}(x) = f_{\mu\nu} F(\xi)\,, \qquad \xi = k_\alpha x^\alpha\,, \qquad k_\alpha^2 = 0\,, \tag{2.36}$$

where k_α is an isotropic vector, $f_{\mu\nu}$ is a constant antisymmetric tensor, and $F(\xi)$ is an arbitrary function. Such a field has zero invariants, $f_{\mu\nu} f^{\mu\nu} = {}^*f_{\mu\nu} f^{\mu\nu} = 0$, and a traceless energy–momentum tensor that is Maxwellian in form,

$$T_{\mu\nu} = k_\mu k_\nu F^2(\xi)\,, \qquad T^\mu_\mu = 0\,. \tag{2.37}$$

Such a field does not polarize the vacuum, does not interact with the vacuum current $j_\mu(x)$, and does not give rise to nonlinear vacuum effects. This fundamental result was obtained by Schwinger [39].

The plane-wave field is a purely geometric object, although it inherits a nongeometric dimension (erg cm^{-3})$^{1/2}$ from its source (and detector), the current density $J_\mu(x) = (\rho\mathbf{v}/c, \rho)$, which has the dimension of charge density, i.e., (erg cm^{-5})$^{1/2}$.

3. Källén–Lehmann representation and its relation to the renormalized photon propagator

We now return to the Källén–Lehmann spectral representation with the aim to relate it to the renormalized photon propagator. Instead of Schwinger's B_0 and $B_1(m^2)$, we use the more common notation Z_3 and $\rho(m^2)$, going back to Dyson, Källén, and Lehmann. In the coordinate and momentum representations, the exact Green's function is then given by

$$\begin{aligned} \mathcal{G}(x) &= Z_3 D_F(x) + \int_{0^+}^\infty \mathrm{d}M^2\, \rho(M^2)\,\Delta_F(x, M)\,, \\ \mathcal{G}(k^2) &= \frac{Z_3}{k^2 - \mathrm{i}\varepsilon} + \int_{0^+}^\infty \mathrm{d}M^2\, \frac{\rho(M^2)}{M^2 + k^2 - \mathrm{i}\varepsilon}\,. \end{aligned} \tag{3.1}$$

In their fundamental monograph on quantum electrodynamics [40], Bialynicki-Birula and Bialynicka-Birula comment on the Källén–Lehmann representation as follows.

"The expansion of the photon propagator into a sum of the free photon propagator D_F and the integral of the propagator Δ_F over M^2 has a clear physical interpretation. One can say that in a theory with interaction, perturbations of the electromagnetic field travel in space–time from point x_1 to point x_2, $x = x_1 - x_2$ as a combination of zero-mass propagation determined by $Z_3 D_F$ and propagation with a continuously distributed mass, determined by the function $\rho(M^2)$. The renormalization constant Z_3 determines the weight with which the component describing the free propagation with zero mass enters the superposition. In a theory without mutual interaction of photons and charges, only the free component and the constant $Z_3 = 1$ appear in the photon propagator.

An additional argument in favor of exactly this interpretation of the obtained representation for the propagator is the sum rule

$$1 = Z_3 + \int_{0^+}^\infty \mathrm{d}M^2\, \rho(M^2)\,, \tag{3.2}$$

whence it follows that in accordance with fundamental properties of quantum theory, Z_3 and $\rho(M^2)\,\mathrm{d}M^2$ can be given the meaning of probability of the different propagation types."

Using the sum rule, we write the propagator $\mathcal{G}(k^2)$ as

$$\mathcal{G}(k^2) = \frac{1}{k^2 - \mathrm{i}\varepsilon}\left[1 - \int_{0^+}^\infty \mathrm{d}M^2\, \frac{\rho(M^2) M^2}{M^2 + k^2 - \mathrm{i}\varepsilon}\right]. \tag{3.3}$$

At the same time, the exact *renormalized* photon propagator is usually represented as [8, 10, 34, 35]

$$\begin{aligned} \mathcal{G}_R(k^2) &= \frac{1}{k^2 - \mathrm{i}\varepsilon}\, d_R\left(\frac{k^2}{m^2}, \alpha\right), \\ d_R(x, \alpha) &= \frac{1}{1 + \alpha\pi_R(x, \alpha)}\,, \qquad x = \frac{k^2}{m^2}\,, \end{aligned} \tag{3.4}$$

with the normalization

$$d_R(0, \alpha) = 1\,, \quad \text{i.e.,} \quad \pi_R(0, \alpha) = 0 \quad \text{at} \quad k^2 = 0\,. \tag{3.5}$$

It was shown by Dyson [13, 41] that the exact photon Green's function times the charge squared is a renormalization-invariant quantity, i.e.,

$$\alpha_0\, \mathcal{G}(k^2, \alpha_0) = \alpha\, \mathcal{G}_R(k^2, \alpha)\,, \tag{3.6}$$

where $\alpha = e^2/4\pi\hbar c$ and $\alpha_0 = e_0^2/4\pi\hbar c$ are the fine structure constants of the physical and bare charges. The right-hand side of the last expression is usually written as

$$\alpha\mathcal{G}_R(k^2, \alpha) = \frac{1}{k^2 - \mathrm{i}\varepsilon}\,\alpha\, d_R\left(\frac{k^2}{m^2}, \alpha\right), \tag{3.7}$$

and the dimensionless scalar function $\alpha d_R(k^2/m^2, \alpha)$ is called the effective interaction parameter, the renormalization-invariant charge, the running coupling constant, etc.

Using renormalization invariance relation (3.6) and representations (3.3) and (3.4) with $k^2 = 0$, together with sum rule (3.2), we obtain

$$e_0^2 Z_3 = e^2\,. \tag{3.8}$$

Resorting again to (3.6), (3.3), and (3.4), but this time with $k^2 \to \infty$, we obtain

$$e_0^2 = e^2 d_R(\infty, \alpha)\,. \tag{3.9}$$

Of the four quantities e^2, e_0^2, Z_3, and $d_R(\infty, \alpha)$ entering formulas (3.8) and (3.9), only e^2 is known (experimentally), with $\alpha = 1/137.036$. In our work summarized in [8, 9], a duality relation was found between the four-dimensional electrodynamics and two-dimensional quantum theory of a massless scalar field, which allowed determining the value of a bare point charge. Its fine structure constant turned out to be $\alpha_0 = 1/4\pi$. In what follows, we present additional arguments supporting the obtained purely geometric value of α_0, based on the conformal invariance of four-dimensional electrodynamics and the two-dimensional theory of a massless scalar field.

The merit of the Källén–Lehmann representation is that it is obtained without the use of the perturbation theory. As Dyson proved, the divergences that arise in quantum electrodynamics in each order of the perturbation theory, after their regularization, form a series of three renormalization factors relating all nonrenormalized physical quantities to the renormalized ones. The terms in the series for each renormalized physical quantity are finite, and the regularized terms in the series of each renormalization factor allow treating it as a constant. In particular (and especially), of fundamental value is the factor denoted as Z_3 by Dyson, which relates the bare charge e_0 to the physical charge e and the corresponding fine structure constants:

$$e^2 = Z_3 e_0^2\,, \qquad \alpha = Z_3 \alpha_0\,.$$

Källén and Lehmann provided the Z_3 constant with yet another, probabilistic interpretation. They found the sum rule

$$Z_3 + \int_{0^+}^{\infty} \mathrm{d}m^2\, \rho(m^2) = 1$$

for Z_3 and $\rho(m^2)\,\mathrm{d}m^2$, quantities that have the meaning of the photon propagation probabilities with zero and nonzero masses.

The derivation of the sum rule relies only on the assumptions about the spectrum of eigenvalues of the energy–momentum operator P_μ, namely,

$$\begin{aligned} &(1)\ \ -P^2 \equiv P_0^2 - \mathbf{P}^2 \geqslant 0\,, \quad P^0 \geqslant 0\,, \\ &(2)\ \ P^0|0\rangle = 0\,, \quad \mathbf{P}|0\rangle = 0\,. \end{aligned} \tag{3.10}$$

The probabilistic interpretation of the Z_3 constant is that $\sqrt{Z_3}$ determines the amplitude for a one-particle state to be created by the field $\mathbf{A}(x)$ from a vacuum,

$$\begin{aligned} \langle p\lambda|\mathbf{A}(x)|0\rangle &= \sqrt{Z_3}\langle p\lambda|\mathbf{A}_{\mathrm{in}}(x)|0\rangle \\ &= \sqrt{Z_3}\langle p\lambda|\mathbf{A}_{\mathrm{out}}(x)|0\rangle = \sqrt{Z_3}\,\frac{1}{\sqrt{(2\pi)^3 2p^0}}\,\exp(\mathrm{i}px)\mathbf{e}(\lambda) \end{aligned} \tag{3.11}$$

(see (2.5)). Because the field $\mathbf{A}$ also gives rise to many-particle states, $\sqrt{Z_3} < 1$. The value $\sqrt{Z_3} = 0$ is then excluded, because many-particle states are determined using the free field operator $\mathbf{A}_{\mathrm{in}}(x)$, which produces only one-particle states when acting on the vacuum.

Thirring figuratively describes the vacuum as a state in which there are no 'dressed' particles, but there are 'half-dressed' and 'undressed' particles that are not eigenstates of the energy–momentum operator P_μ. If we express P_μ in terms of the operators of creation and absorption of 'undressed' particles, then the eigenstates of P_μ are given by superpositions of various numbers of 'undressed' particles, i.e., 'clouds of virtual particles' [36, Sec. 14].

An on-shell photon is an eigenstate of P_μ with zero charge and mass, but, unlike a vacuum, this state has nonzero energy and momentum, and spin equal to unity. This excited state therefore propagates in a vacuum at the speed of light. As already mentioned, a real photon does not interact with a vacuum and freely propagates between the locations of its production and absorption by real external currents $J_\mu(x)$ and $J_\mu(x')$.

But, as we have seen, the propagation of a more complex electromagnetic excitation of the field $\mathbf{A}(x)$ coupled to the current $j_\mu(x)$ is described by the exact Green's function

$$\mathcal{G}(x) = Z_3 D_F(x) + \int_{0^+}^{\infty} \mathrm{d}m^2\, \rho(m^2) \Delta_F(x, m)\,, \tag{3.12}$$

consisting of two terms.

The production of a real photon is due to the interaction term in the Lagrangian, with the classical current $J_\mu(x)$ added to the current operator $j_\mu(x)$ of vacuum fluctuations. It primarily interacts with vacuum fluctuations of the $A^\mu(x)$ field, which is the field of 'undressed' particles. If the energy of the $J_\mu(x)$ source is sufficient for producing some 'dressed' particle from an 'undressed' one, then the formation of the missing part of the 'clothes' by the current can be regarded as a real process, and the probability of emitting a photon by the source is the same as for free fields, but this probability must also be multiplied by the probability of finding a 'dressed' particle among the states that constitute an 'undressed' particle.

4. Zero-point fluctuations of the electromagnetic field in a vacuum and in resonant cavities

We consider two ideally conducting planes pressed against each other. If we move them apart, creating a vacuum gap between them, then the fluctuation currents flowing along the interface are forced to split in half into parallel currents separated by the gap. But parallel currents attract and resist gap formation, which is consistent with the energy conservation law and Lenz's rule.

According to the thermodynamic relation

$$\mathrm{d}E = T\,\mathrm{d}S - p\,\mathrm{d}V\,, \tag{4.1}$$

under an adiabatic ($\mathrm{d}S = 0$) increase in the volume of the gap, the work of external forces is spent to increase the internal energy of the vacuum, and hence $\mathrm{d}E = -p\,\mathrm{d}V > 0$. This means that $-p > 0$: the pressure inside the gap is negative. DeWitt [42] remarks on this: "Maxwell would be very happy with this result. It almost makes one believe in the existence of the ether."

We now consider the formation in a vacuum of a cubic cavity resonator made of six ideally conducting square faces with an edge length L. We assume that these faces are first located far from each other, placed in pairs on the axes of a Cartesian coordinate system symmetrically about its center (which is the center of the cavity to be assembled). Fluctuations of the electromagnetic field in the vacuum exert equal pressure on the inner and outer (relative to the center) sides of each face while they are far apart. But as the faces approach each other, the pressure on them from the inside increases compared to the pressure from the outside due to more and more multiple reflections of the fluctuating electromagnetic field. At the instant the assembly is completed, the excess of pressure and energy density inside the cavity over the values outside becomes constant and maximum.

Quantum fluctuations of the electromagnetic field induce circular currents on the conducting surface of each face, running either clockwise or counterclockwise; according to Lenz, they provide the maximum resistance to the approach of the cavity faces. In fact (see the figure), these currents flow in opposite directions on opposite sides of the cavity and near the edges of neighboring faces, and the repulsion of the currents from each other creates an excess of pressure and energy density inside the cavity.

We see from thermodynamic equation (4.1) that, under an adiabatic decrease in the cavity volume, the work of external forces increases the internal energy of the cavity, because the pressure inside the cavity is positive, $p > 0$. By p, we must of course understand the difference between the pressure inside the cavity and in the vacuum in the absence of the cavity.

Below, we follow the remarkable work by Lukosz [19], who obtained an analytic expression for the energy shift of zero fluctuations of the electromagnetic field in a rectangular cavity with conducting walls. For our purposes, it is natural to confine ourselves to a cubic cavity.

Already in the assembly process, the cavity forces the electromagnetic field in the vacuum to satisfy the boundary condition

$$\mathbf{E}_{\mathrm{t}} = 0\,, \qquad \mathbf{H}_{\mathrm{n}} = 0 \tag{4.2}$$

on the ideal conducting surface of the cavity. The subscripts t and n indicate the tangential and normal components. As a result, the electromagnetic field energy inside the cavity becomes greater than the one that was contained in the same volume in the vacuum, before inserting the cavity with its conducting shell. Both these energies are infinite, but their difference, as we now see, is finite and positive.

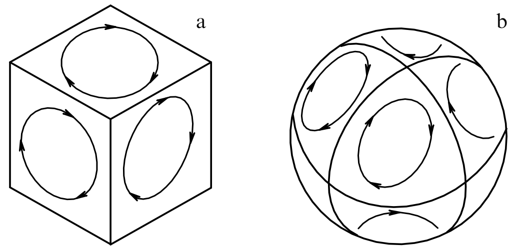

Figure. Circular currents (a) on the faces of a cube and (b) on spherical surfaces of octants can be considered to be made of similar currents on a smaller scale flowing clockwise or counterclockwise and, in accordance with Ampère's rule, mutually repulsive.

The energy of zero-point oscillations summed over the cavity eigenfrequencies

$$\omega_{n_1 n_2 n_3} = c k_{n_1 n_2 n_3} = \frac{\pi c}{L}\sqrt{n_1^2 + n_2^2 + n_3^2}\,, \tag{4.3}$$

after averaging over time (a factor of $1/2$) and taking the double degeneracy of the frequencies with nonzero $n_1 n_2 n_3$ into account (a factor of 2), gives the following expression for the energy density:

$$u = \frac{1}{V}\sum_{n_1 n_2 n_3 \geqslant 0} \frac{1}{2}\,\hbar\omega_{n_1 n_2 n_3}$$
$$= \frac{\pi\hbar c}{VL}\sum_{n_1 n_2 n_3 = 0}^{\infty}{}^{*}\left[1 - (\delta_{n_1}\delta_{n_2} + \delta_{n_2}\delta_{n_3} + \delta_{n_3}\delta_{n_1})\right]\sqrt{n_1^2 + n_2^2 + n_3^2}\,. \tag{4.4}$$

Here, $\sum^*$ means that frequencies with one of the n_i, $i = 1, 2, 3$, vanishing are nondegenerate and must be multiplied by $1/2$. Due to the Kronecker symbols $\delta_{n_i} \equiv \delta_{n_i 0}$, the expression in the square brackets vanishes for modes propagating along the cavity edges, because they do not satisfy boundary condition (4.2) and are not eigenmodes of the cavity.

To find the variation in the vacuum energy density due to the conducting cavity shell, we consider the energy density u' inside a very large cubic cavity with an edge $L' \gg L$. Using (4.4) with L, V, and n_i replaced with their primed counterparts, we obtain a sum over n_1', n_2', and n_3',

$$u' = \frac{\pi\hbar c}{V'L'}\sum_{n_1' n_2' n_3' \geqslant 0}^{\infty}{}^{*}\left[1 - (\delta_{n_1'}\delta_{n_2'} + \delta_{n_2'}\delta_{n_3'} + \delta_{n_3'}\delta_{n_1'})\right]$$
$$\times\sqrt{n_1'^2 + n_2'^2 + n_3'^2}\,,$$

which is equal to $1/8$ times the integral

$$= \frac{\pi\hbar c}{8VL}\iiint_{-\infty}^{\infty}\mathrm{d}n_1\,\mathrm{d}n_2\,\mathrm{d}n_3\left\{1 - \left[\delta(n_1)\delta(n_2) + \delta(n_2)\delta(n_3)\right.\right.$$
$$\left.\left. + \delta(n_3)\delta(n_1)\right]\right\}\sqrt{n_1^2 + n_2^2 + n_3^2}\,, \tag{4.5}$$

extended over the entire space of continuous variables $n_i = (L/L')n_i'$, $i = 1, 2, 3$. They are continuous because, when the discrete n_i' changes by unity, $\Delta n_i' = 1$, their variation $\Delta n_i = (L/L')\Delta n_i'$ becomes arbitrarily small as $L/L' \ll 1$. The Kronecker symbols then also transmute into delta functions. We also note that, when moving from one to eight octants, the points lying on the boundary of two adjacent octants belong to both octants and their contribution is shared in equal parts between these octants. Therefore, the prescription indicated by the asterisk at $\sum$ is implemented automatically.

To work with finite quantities, we introduce a form factor $\exp(-\alpha k_{n_1 n_2 n_3})$, $\alpha \to 0$, which cuts off high frequencies both in the sum for u in (4.4) and in the integral for u' in (4.5). The α parameter has the meaning of the cutoff wavelength. We let

$$f(n_1, n_2, n_3) = \left[1 - (\delta_{n_1}\delta_{n_2} + \delta_{n_2}\delta_{n_3} + \delta_{n_3}\delta_{n_1})\right] \times \sqrt{n_1^2 + n_2^2 + n_3^2}\, \exp(-\alpha k_{n_1 n_2 n_3}) \qquad (4.6)$$

denote the function involving the form factor in the new expression (4.4) for the density u, and pass from the sum in one octant (the first) to the sum in eight octants:

$$\sum_{n_1 n_2 n_3 = 0}^{\infty}{}^{*} f(n_1, n_2, n_3) = \frac{1}{8} \sum_{n_1 n_2 n_3 = -\infty}^{\infty} f(n_1, n_2, n_3)\,. \qquad (4.7)$$

By the Poisson resummation formula, the sum over a discrete variable reduces to the integral over continuous ones:

$$\sum_{n_1 n_2 n_3 = -\infty}^{\infty} f(n_1, n_2, n_3) = \int \mathrm{d}^3 n\, f(\mathbf{n}) \sum_{m_1 m_2 m_3 = -\infty}^{\infty} \exp(2\pi \mathrm{i}\mathbf{mn}) = \int \mathrm{d}^3 n\, f(\mathbf{n}) \left[1 + \sum_{m_1 m_2 m_3 = -\infty}^{\infty}{}' \exp(2\pi \mathrm{i}\mathbf{mn})\right]. \qquad (4.8)$$

Here, $\mathbf{mn}$ is the scalar product of two three-dimensional vectors: $\mathbf{m} = (m_1, m_2, m_3)$ with discrete components and $\mathbf{n} = (n_1, n_2, n_3)$ with continuous components.

The first term in this expression comes from the sum for the vector $\mathbf{m} = 0$. It is identical to the integral for u' in (4.5) with the form factor introduced into it. Therefore, this terms vanishes in the difference $u - u' = \Delta u$:

$$\Delta u = \frac{\pi \hbar c}{8V} \left(-\frac{\partial}{\partial \alpha}\right) \times \left[\sum_{m_1 m_2 m_3 = -\infty}^{\infty}{}' \int \mathrm{d}^3 n\, \exp(-\alpha k_{n_1 n_2 n_3} + 2\pi \mathrm{i}\mathbf{mn}) - 3 \sum_{m = -\infty}^{\infty}{}' \int \mathrm{d}n\, \exp(-\alpha k_n + 2\pi \mathrm{i} mn)\right]. \qquad (4.9)$$

The derivative $-\partial/\partial\alpha$ is used here to simplify the integrands in the triple I_3 and single I_1 integrals in (4.9): they lost the factors $k_{n_1 n_2 n_3} = (n_1^2 + n_2^2 + n_3^2)^{1/2}/L$ and $k_n = \sqrt{n^2}/L$.

Integrating over the angles in a spherical coordinate system where m and n are absolute values of the vectors $\mathbf{m}$ and $\mathbf{n}$, we reduce the triple integral to a tabulated one (see [43, 44]):

$$I_3 = \frac{2}{m} \int_0^{\infty} \mathrm{d}n\, n \exp\left(-\frac{\alpha n}{L}\right) \sin(2\pi mn) = \frac{1}{2\pi^3 L} \frac{\alpha}{\left((\alpha/2\pi L)^2 + \mathbf{m}^2\right)^2}\,.$$

Differenting with respect to α and passing to the limit as $\alpha \to 0$, we obtain

$$\lim_{\alpha \to 0}\left(-\frac{\partial}{\partial \alpha}\right) I_3 = -\frac{1}{2\pi^3 L (m_1^2 + m_2^2 + m_3^2)^2}\,, \quad \mathbf{m}^2 \neq 0\,, \qquad = \frac{24\pi}{L}\left(\frac{L}{\alpha}\right)^4, \quad \mathbf{m}^2 = 0\,, \qquad (4.10)$$

which represents the finite and singular terms, respectively, entering Δu and u'. The single integral in (4.9) also reduces to a tabulated one (see [43, 44]),

$$I_1 = 2 \int_0^{\infty} \mathrm{d}n\, \exp\left(-\frac{\alpha n}{L}\right) \cos(2\pi mn) = \frac{\alpha}{2\pi^2 L \left[(\alpha/2\pi L)^2 + m^2\right]}\,,$$

whence, differentiating with respect to α and passing to the limit as $\alpha \to 0$, we obtain the finite and singular terms

$$\lim_{\alpha \to 0}\left(-\frac{\partial}{\partial \alpha}\right) I_1 = -\frac{1}{2\pi^2 L m^2}\,, \quad m \neq 0\,, \qquad = \frac{2}{L}\left(\frac{L}{\alpha}\right)^2, \quad m = 0\,, \qquad (4.11)$$

respectively, entering Δu and u'.

It follows that the variation in the zero-point energy density inside a cubic cavity is

$$\Delta u = \frac{\pi \hbar c}{8VL}\left[-\frac{1}{2\pi^3} \sum_{m_1 m_2 m_3 = -\infty}^{\infty}{}' (m_1^2 + m_2^2 + m_3^2)^{-2} + \frac{3}{2\pi^2} \sum_{m = -\infty}^{\infty}{}' \frac{1}{m^2}\right] = \frac{\pi \hbar c}{16VL}\left[1 - \frac{1}{\pi^3} \sum_{m_1 m_2 m_3 = -\infty}^{\infty}{}' (m_1^2 + m_2^2 + m_3^2)^{-2}\right]. \qquad (4.12)$$

It is useful to indicate the vacuum energy density u' diverging as $\alpha \to 0$, which drops out from the finite expression for the shift Δu:

$$u' = \frac{\pi \hbar c}{8VL}\left(24\pi\left(\frac{L}{\alpha}\right)^4 - 6\left(\frac{L}{\alpha}\right)^2\right), \quad \alpha \to 0\,. \qquad (4.13)$$

We conclude that the shift in the energy of zero-point oscillations of the electromagnetic field in a cubic cavity can be represented as [19]

$$E_{\mathrm{L}} = \alpha_{\mathrm{L}} \frac{\hbar c}{L}\,,$$

$$\alpha_{\mathrm{L}} = \frac{\pi}{16}\left[1 - \frac{1}{\pi^3} \sum_{m_1 m_2 m_3}^{\infty}{}' (m_1^2 + m_2^2 + m_3^2)^{-2}\right], \qquad (4.14)$$

$$\alpha_{\mathrm{L}} = 0.0916574\ldots\,.$$

This value was confirmed by Mamaev and Trunov [24], who calculated the shift of the vacuum expectation value of the T_{00} component of the energy–momentum tensor of the electromagnetic field.

We quote a similar expression for the shift in the energy of zero-point electromagnetic field fluctuations in a spherical cavity of radius r, first found by Boyer [18]:

$$E_{\mathrm{B}} = \alpha_{\mathrm{B}} \frac{\hbar c}{2r}\,, \qquad \alpha_{\mathrm{B}} = 0.0923531\,. \tag{4.15}$$

The constant $\alpha_{\mathrm{B}} = 0.09$ found by Boyer turned out to be positive, contrary to Casimir's expectations [45]; it was subsequently refined by Davis [20], Balian and Duplantier [21], and Milton, DeRaad, and Schwinger [22]. The numerical value given in (4.15) was obtained in the last study. The analytic expression for α_{B} found there is not as transparent as the one for a cubic cavity.

The dimensionless constants α_{B} and α_{L} are the most important Poincaré-invariant characteristics of zero-point oscillations of the electromagnetic field in a vacuum, measured by spherical and cubic cavities. If $L = 2r$ (the cube is circumscribed about a sphere), then E_{B} and E_{L} differ only in the constants α_{B} and α_{L}.

Several questions arise in connection with these results.

(1) Do the quantities $\hbar c/2r$ and $\hbar c/L$, which have the dimension of energy, correctly express the scale of the expected zero-point energy shifts of the electromagnetic field in spherical and cubic cavities? We adduce physical and geometrical arguments supporting the above scales *for those cavities whose shells are circumscribed about a sphere*. For cubic and spherical cavities, this implies that $L = 2r$.

(2) Why do the parameters α_{L} and α_{B}, which are independent of the cavity size, depend on the cavity shape so weakly, the relative difference between α_{L} and α_{B} being less than 0.8%?

(3) Why are α_{L} and α_{B} more than an order of magnitude less than the values that follow for the lowest (fundamental) eigenfrequencies of cubic and spherical cavities? Each eigenfrequency of the cavity whose shell is circumscribed about a sphere can be associated with a dimensionless number $k_N r$:

$$\omega_N = ck_N = (k_N r)\frac{c}{r}\,, \qquad \frac{1}{2}\,\hbar\omega_N = (k_N r)\frac{\hbar c}{2r}\,.$$

For a spherical cavity, the fundamental frequency corresponds to $k_{\min} r = 2.74$ (E_{110} mode). For a cubic cavity ($L = 2r$), $k_{\min} r = 2.22$ (E_{110} mode) (see [46, Sec. 90] or [47, 48]). It is clear that the physical meaning of α_{B} and α_{L} is quite different.

At the fundamental frequency and other eigenfrequencies, the cavities are excited by an *external* source that transfers the electromagnetic energy to them. The oscillation spectrum of the excited cavity reflects the frequency spectrum of the source, but the frequencies that are close to the eigenfrequencies of the cavity are especially pronounced.

In our case, the cavity is excited because of the appearance of an excess of vacuum energy in its cavity during cavity assembly due to the 'raking' of energy and due to the boundary conditions on an ideally conducting shell imposed on the energy-carrying field.

We note that, for a cylindrical cavity with radius r and generatrix length $L = 2r$, the fundamental frequencies of the E and H modes are very close: $k_{\min} r = 2.420$ for the H_{111} mode and $k_{\min} r = 2.405$ for the E_{010} mode [47, 48]. Thus, all the given values of the fundamental frequencies and their dimensionless values $k_{\min} r$ for these three cavities circumscribed about a sphere are of the same order of magnitude, increase when passing from a cubic cavity to a cylindrical one and from a cylindrical to a spherical one, and exceed the values of the parameters α_{L} and α_{B} by more than an order of magnitude. It is clear that the physical meaning of the parameters α_{L} and α_{B} is entirely different.

To analyze Δu, it is convenient to return to the density u and represent it in the form

$$u = \frac{\pi\hbar c}{8VL}\left\{24\pi\left(\frac{L}{\alpha}\right)^4 - \frac{1}{2\pi^3}\sum_{m_1 m_2 m_3=-\infty}^{\infty}{}' (m_1^2 + m_2^2 + m_3^2)^{-2} - 3\left[2\left(\frac{L}{\alpha}\right)^2 - \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}{}' m^{-2}\right]\right\}. \tag{4.16}$$

The first line shows the contribution to the energy density of *all modes*, including those propagating along the cavity edges, or more precisely, along the x, y, and z axes of the Cartesian system. This contribution consists of two terms given in (4.10): the leading contribution of all modes and frequencies increasing as ω^4 with the cutoff frequency $\omega \sim c/\alpha \to \infty$, and the low-frequency, finite contribution of all modes with frequencies ranging from zero to the fundamental frequency $\sim c/L$ of the cavity. These frequencies are not cavity eigenfrequencies, and therefore the density inside the cavity decreases by this finite low-frequency contribution compared to the density u' outside the cavity. But the leading contribution, which contains non-eigen-modes along with the eigenmodes, overestimates the true density inside the cavity. This overestimation removes the contribution of the second line.

The second line contains the contribution of modes propagating along the x, y, and z axes, modes that are not cavity eigenmodes. This is the contribution of the eigenmodes of three identical one-dimensional cavities of length L, oriented along the x, y, and z axes. The contribution of each of them consists of two terms (4.11): the leading term, increasing like ω^2, cut off at the frequency $\sim c/\alpha \to \infty$, and a finite contribution of modes with frequencies ranging from zero to the fundamental frequency $\pi c/L$ of a one-dimensional cavity. This contribution is subtracted from the leading one, because low frequencies are not eigenmodes of a one-dimensional cavity of length L. But because they are eigenfrequencies of a one-dimensional cavity of length $L' \to \infty$, the finite low-frequency contribution of one-dimensional modes reduces the leading contribution of the second line. Thus, the second line of (4.16) is the contribution of one-dimensional modes with frequencies ranging from the fundamental frequency $\sim \pi c/L$ to the cutoff frequency $c/\alpha \to \infty$ and the negative contribution of one-dimensional low-frequency modes with $\omega \lesssim \pi c/L$. Because the one-dimensional modes are not cavity eigenmodes, the entire contribution of the second line is subtracted from the first-line contribution. The divergent contribution of one-dimensional modes with frequencies in the range from the fundamental frequency $\sim \pi c/L$ to the cutoff frequency $\sim c/\alpha \to \infty$ reduces the leading, divergent contribution of the first line, and the finite contribution of the low-frequency $\omega \lesssim \pi c/L$ of one-dimensional modes increases the cavity energy density.

All this can be seen from representation (4.16) for the density $u = u' + \Delta u$, where u' is given by formula (4.13) with a strongly divergent leading term and a gentler diverging term, whose effect is to decrease the first one.

Similarly, Δu is represented in (4.12) by a positive term, the sum of one-dimensional *low-frequency* modes, which amounts to twice the value of the Riemann zeta function $\zeta(s)$ at $s = 2$, and by a negative term, the triple sum of all *low-*

frequency modes, which amounts to a particular value of the Epstein zeta function $\mathsf{Z}_p(s)$ at $s = 1 + 1/p$ with $p = 3$ [49, Sec. 17.9].

5. Representation of the energy density shift in a cubic cavity in terms of the Riemann and Epstein zeta functions

We focus on the very instructive representation for Δu just obtained at the end of Section 4:

$$\Delta u = \frac{\pi\hbar c}{16VL}\left[\frac{6}{\pi^2}\zeta(2) - \frac{1}{\pi^3}\mathsf{Z}_3\left(\frac{4}{3}\right)\right]. \tag{5.1}$$

It would have emerged if we had not introduced a cutoff form factor into the function $f(n_1, n_2, n_3)$ but still wished to use the Poisson resummation formula. Instead of (4.9), we would then have obtained

$$\Delta u = \frac{\pi\hbar c}{8VL}\left[\sum_{m_1 m_2 m_3=-\infty}^{\infty}{}' \int \mathrm{d}^3 n\,\sqrt{n_1^2 + n_2^2 + n_3^2}\,\exp(2\pi\mathrm{i}\mathbf{mn}) - 3\sum_{m=-\infty}^{\infty}{}' \int \mathrm{d}n\,\sqrt{n^2}\,\exp(2\pi\mathrm{i}mn)\right], \tag{5.2}$$

which, by the Poisson formula, is identically equal to the difference between a triple sum and three one-dimensional divergent sums:

$$\Delta u = \frac{\pi\hbar c}{8VL}\left[\sum_{m_1 m_2 m_3=-\infty}^{\infty} \sqrt{m_1^2 + m_2^2 + m_3^2} - 3\sum_{m=-\infty}^{\infty}\sqrt{m^2}\right]. \tag{5.3}$$

According to Epstein, the divergent sum in a space of p dimensions,

$$\sum_{n_1 n_2 \ldots n_p=-\infty}^{\infty} \sqrt{\sum_{i=1}^{p} n_i^2} = -\frac{1}{2\pi}\,\pi^{-(1/2)(1+p)}\,\Gamma\left(\frac{1}{2}(1+p)\right) \times \sum_{m_1 m_2 \ldots m_p=-\infty}^{\infty}{}' \left(\sum_{i=1}^{p} m_i^2\right)^{-(1/2)(1+p)}, \tag{5.4}$$

is equal to a negative converging sum.

For $p = 1$ and $p = 3$, we have the familiar expressions (see (4.12))

$$\sum_{n=-\infty}^{\infty}\sqrt{n^2} = -\frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}{}' m^{-2},$$
$$\sum_{n_1 n_2 n_3=-\infty}^{\infty}\sqrt{n_1^2 + n_2^2 + n_3^2} = -\frac{1}{2\pi^3}\sum_{m_1 m_2 m_3=-\infty}^{\infty}{}' (m_1^2 + m_2^2 + m_3^2)^{-2}. \tag{5.5}$$

The difference between the divergent sums — the triple one and three one-dimensional ones — is certainly positive. Therefore, also positive is the corresponding difference between the convergent sums, and hence $\Delta u > 0$ (see (4.12)).

We note that the first line in (5.5) is nothing but a relation between a finite value of Riemann's zeta function

$$\zeta(s) = \sum_{m=1}^{\infty} m^{-s}, \qquad \operatorname{Re} s > 1 \tag{5.6}$$

at the point $s = 2$ and the divergent sum naively obtained from the Dirichlet representation (5.6) at the point $s = -1$ that does not belong to the domain $\operatorname{Re} s > 1$. Physicists would call this value nonregularized. We let it be denoted by $\tilde{\zeta}(-1)$. According to advanced mathematics, however, the Riemann zeta function, as an analytic function of s with a pole at $s = 1$, satisfies the functional equation [49, Sec. 17.7]

$$\zeta(1 - s) = \frac{2\Gamma(s)}{(2\pi)^s}\cos\frac{\pi s}{2}\,\zeta(s), \tag{5.7}$$

which analytically relates its values at s and $1 - s$. Therefore, the statement in the first line of (5.5) is equivalent to regularizing the sum $\tilde{\zeta}(-1)$ by removing its divergence and reducing it to a finite value $\zeta(-1)$:

$$\tilde{\zeta}(-1) \to \zeta(-1) = -\frac{1}{2\pi^2}\,\zeta(2).$$

The Epstein zeta function is defined by the p-fold sum [49]

$$\mathsf{Z}_p(s) = \sum_{m_1 m_2 \ldots m_p=-\infty}^{\infty}{}' \left(\sum_{i=1}^{p} m_i^2\right)^{-(1/2)ps}, \qquad \operatorname{Re} s > 1. \tag{5.8}$$

For $p = 1$, it coincides with twice the Riemann zeta function. Being an entire function of s with a pole at $s = 1$, $\mathsf{Z}_p(s)$ satisfies the functional equation

$$\mathsf{Z}_p(1 - s) = \frac{\pi^{(1/2)p(1-s)}\,\Gamma((1/2)ps)}{\pi^{(1/2)ps}\,\Gamma((1/2)p(1-s))}\,\mathsf{Z}_p(s), \tag{5.9}$$

which analytically relates its values at s and $1 - s$.

Formula (5.4) relates the divergent sum obtained naively from representation (5.8) at the point $s = -1/p$ outside the domain $\operatorname{Re} s > 1$ to the finite value $\mathsf{Z}_p(-1/p)$ determined by functional equation (5.9). We let $\tilde{\mathsf{Z}}_p(-1/p)$ denote this diverging sum. Then, formula (5.4) can be regarded as a regularization of the sum $\tilde{\mathsf{Z}}_p(-1/p)$, which amounts to removing its divergence and reducing it to a finite value $\mathsf{Z}_p(-1/p)$ equal to the right-hand side of (5.4), in accordance with functional equation (5.9).

We summarize the foregoing. The series $\sum_{m=-\infty}^{\infty}\sqrt{m^2}$ in (5.3), represented in the form $\sum_{m=-\infty}^{\infty}(\sqrt{m^2})^{-s}$, diverges at $s = -1$, but the doubled Riemann zeta function defined by this series for $s > 1$, being analytically continued to $s = -1$, has the finite value given in (5.5).

Similarly, the series

$$\sum_{m_1 m_2 m_3=-\infty}^{\infty}\sqrt{m_1^2 + m_2^2 + m_3^2}$$

in (5.3), represented as

$$\sum_{m_1 m_2 m_3=-\infty}^{\infty}\left(\sqrt{m_1^2 + m_2^2 + m_3^2}\right)^{-3s},$$

diverges at $s = -1/3$, but the Epstein zeta function $\mathsf{Z}_3(s)$ defined by this series for $s > 1$ (see (5.8)), being analytically continued to the point $s = -1/3$, takes the finite value given in (5.5).

Thus, the same finite result for Δu in (4.12) and (5.1) can be obtained by two different methods. The first amounts to introducing a high-frequency cutoff form factor into the density u and discarding the divergent part dependent on the cutoff wavelength $\alpha \to 0$. It is then asserted (see, e.g., book [25]) that the finite result is independent of the choice of the cutoff function, and its explicit introduction is in fact not necessary.

This is indirectly confirmed by another method of removing divergences from u, where the terms of a linearly divergent series for u are raised to a negative power $-ps$, where s is a parameter and p is the spatial dimension. As a result, the series becomes convergent and determines an analytic function of complex s in the domain $\mathrm{Re}\, s > 1$. This function can be analytically continued to the vicinity of $s = -1/p$, where the original series diverges. However, the value of the analytically continued function at $s = -1/p$ is finite and is the desired result.

Boyer [26] relates the independence of the forces acting on conducting bodies from the high-frequency cutoff to the vector nature of the *electromagnetic field* responsible for the observed phenomena. A similar opinion is held, as we see below, by Lukosz [27].

DeWitt [42], disagreeing with Boyer, shows that a massless *scalar field* in a vacuum would also lead to observable pressures on the walls, bounding it if we assume that it vanishes on the boundary and if we use the *conformally invariant scalar field* with a traceless energy–momentum tensor.

Regardless of whether a massless scalar field exists in Nature or is useful only to theorists to discover symmetry relations between observable physical quantities, it is clear that *conformal invariance* of the quantum theory of massless fields (with spins 1 and 0, in particular) conceals not yet fully understood information, as does the closely related *analyticity* of the functions describing these observable quantities.

At the same time, Todorov [50] stated that, in the region where the momenta of all particles are much greater than their mass, quantum field theory becomes conformal field theory if the effective charge tends to a finite value as the momenta increase.

This opinion is shared by the current author.

Boyer and Lukosz consider the electromagnetic field in a cavity with ideally conducting walls. This field satisfies Maxwell's equations with boundary conditions (4.2), which require that the field be a set of monochromatic waves with the cavity eigenfrequencies. Being a solution of Maxwell's equations, such a field is conformally invariant.

DeWitt draws attention to the fact that finite effects of pressure on the cavity walls must also appear in the 4-dimensional *conformally invariant* theory of a scalar massless field. In that theory, the Lagrangian contains the scalar Ricci curvature instead of the squared mass, $m^2 \to \xi R$, where ξ is a dimensionless parameter depending on the space–time dimension. The energy–momentum tensor obtained by varying the Lagrangian with respect to the metric then contains ξ-dependent terms, even in the zero-curvature limit, and its trace in the 4-dimensional theory vanishes only for $\xi = 1/6$.

However, in this and our preceding papers, we use the duality of 4-dimensional electrodynamics and the *two-dimensional* theory of a massless scalar field, in which $\xi = 0$ and the energy-momentum tensor is traceless. Conformal invariance of the two-dimensional theory of a massless scalar field with a zero boundary condition on the world line of a point mirror is well known (see [51, Sec. 4.4]). Conformal invariance of 4D electrodynamics with the current of a point charge on the world line coinciding with that of the mirror follows from the duality between these two theories, at least as regards their radiation fields. Conformal invariance of Maxwell's equations was proved by Bateman in 1909 (see [40, Sec. 9]).

6. Closeness of the values of α_{L}, α_{B}, and $\alpha/\alpha_0 = 4\pi\alpha$ obtained by nonperturbative methods

In [8], we drew attention to the fact that the value $\alpha_0 = 1/4\pi$ of the fine structure constant of a bare point charge, obtained in [2–9] by a purely geometrical nonperturbative method, is closely related to the parameters α_{B} and α_{L} characterizing the shifts E_{B} and E_{L} of the energy of zero-point oscillations of the electromagnetic field in spherical and cubic cavities. Indeed, the finite value $\alpha_0 = 1/4\pi$ leads to a finite value of the Dyson renormalization factor $Z_3 \equiv \alpha/\alpha_0 = 4\pi\alpha$, which lies in the narrow interval

$$\alpha_{\mathrm{L}} < Z_3 \equiv \frac{\alpha}{\alpha_0} = 4\pi\alpha < \alpha_{\mathrm{B}} \tag{6.1}$$

between α_{L} and α_{B}, which are small compared to unity:

$$\alpha_{\mathrm{L}} = 0.0916574\,, \qquad \alpha_{\mathrm{B}} = 0.0923531\,. \tag{6.2}$$

As we have already noted, the values of α_{L} and α_{B} differ by less than 0.8%, and the values of α_{L} and α/α_0, by less than 0.05%.

Because all three quantities α_{L}, α_{B}, and $\alpha/\alpha_0 \equiv Z_3 \ll 1$ are directly related to the fluctuations of the electromagnetic field $\mathbf{A}(x)$ and the current $j_\mu(x)$ in a vacuum, such a proximity of these values can hardly be considered accidental. Moreover, this proximity only confirms that the finite value $\alpha_0 = 1/4\pi$ is correct, as is the *nonperturbative method* whereby it was obtained. We also note that the parameters α_{L} and α_{B} were found *without any recourse to the perturbation theory*.

Because $Z_3 = e^2/e_0^2$ has the physical meaning of the inverse permittivity of a vacuum, relation (6.1) gives the parameters α_{L} and α_{B} the meaning of approximate inverse permittivities determined by cubic and spherical cavities.

With formula (6.1) rewritten in terms of purely geometrical quantities

$$\alpha_0\alpha_{\mathrm{L}} = \frac{1}{137.101}\,, \qquad \alpha_0\alpha_{\mathrm{B}} = \frac{1}{136.069}\,, \tag{6.3}$$

obtained by nonperturbative and unrelated methods, we arrive at a constraint on the fine structure constant $\alpha = 1/137.036$:

$$\alpha_0\alpha_{\mathrm{L}} < \alpha < \alpha_0\alpha_{\mathrm{B}}\,. \tag{6.4}$$

The analytic expression for $\alpha_0\alpha_{\mathrm{L}}$, whose numerical value differs from the experimental α by less than 0.05%, is

$$\alpha_0\alpha_{\mathrm{L}} = \frac{1}{4\pi}\,\frac{\pi}{16}\left[1 - \frac{1}{\pi^3}\sum_{m_1 m_2 m_3 = -\infty}^{\infty}{}' \left(m_1^2 + m_2^2 + m_3^2\right)^{-2}\right]. \tag{6.5}$$

The sum that appears here (Epstein's zeta function) is equal to 16.53231596... [19]. This formula for $\alpha_0\alpha_{\mathrm{L}}$ suggests that Dirac was not altogether in error after all.

Epstein's zeta functions for three-dimensional and one-dimensional spaces were used in deriving (6.5). The remarkable analytic properties of the Epstein functions allow reducing divergent series to convergent series that have a physical meaning (see functional equation (5.9) and its special case (5.7)).

The qualitative and quantitative relations between α_L, α_B, and α/α_0 discussed above raise the following questions.

(1) Why are the parameters α_L and α_B so weakly dependent on the cavity shape?

(2) Why is α_L less than α_B?

(3) Why is $Z_3 \equiv \alpha/\alpha_0 = 4\pi\alpha$ closer to α_L than to α_B?

Cubic and spherical cavities, despite the difference in their shapes, are characterized at $L = 2r$ by the same topological invariant, the surface-to-volume ratio

$$\frac{S}{V} = \frac{3}{r} \,. \tag{6.6}$$

This global geometric invariant is shared by all polyhedra circumscribed about a sphere (see [9]). But the small difference between the coefficients α_L and α_B is nevertheless caused by the fact that subjecting the zero-point fluctuations of the electromagnetic field to the same local boundary conditions on the conducting shells of the cube and the sphere requires different energy expenditure when these cavities are formed.

In this regard, the parameter α_L turns out to be smaller than α_B because the fulfillment of the boundary conditions by a superposition of plane waves with mutually orthogonal vectors $\mathbf{E}$, $\mathbf{H}$, and $[\mathbf{EH}]$ on the orthogonal adjacent faces of a cube requires less energy than their implementation on the surface of a sphere.

We return to the discussion of these issues in the Conclusion.

7. Relation of the $\alpha_{B,L}$ parameters to the Dyson Z_3 factor: the ratio of the squared physical and bare charges

Our aim in this section is to qualitatively explain why the change in the energy of zero-point oscillations of the electromagnetic field inside a cavity with ideally conducting walls is proportional to the renormalization factor Z_3. For this, we use the operator field theory formalism expounded above to describe quantum mechanical fluctuations of the electromagnetic field in a vacuum. Because the expectation values of the field and current operators are equal to zero in the vacuum, we can surmise that the vacuum expectation values of the squares of these operators are nonzero and can characterize the vacuum fluctuations. Indeed, it can be shown that the squared fluctuations of the field $\mathbf{A}(x)$ and the current $j(x)$ in the vacuum do not vanish, but turn to infinity (see [36, Sec. 7]).

To avoid such divergences, we consider not the operator $\mathbf{A}(x)$ itself but its average

$$\bar{\mathbf{A}} = L^{-3} \int_V \mathrm{d}^3 x\, \mathbf{A}(\mathbf{x}, t) \tag{7.1}$$

over some spatial volume L^3; we also introduce the mean squared fluctuation of the ith component of this averaged $\bar{\mathbf{A}}$:

$$\left\langle (\Delta \bar{A}_i)^2 \right\rangle = \left\langle (\bar{A}_i - \langle \bar{A}_i \rangle)^2 \right\rangle = L^{-6} \int_V \mathrm{d}^3 x\, \mathrm{d}^3 x' \left\langle A_i(\mathbf{x}, t) A_i(\mathbf{x}', t) \right\rangle . \tag{7.2}$$

We note that, because $\bar{A}_i$ is a volume average, the products of the same averages $\bar{A}_i \bar{A}_i$ involve the same time t, just as they involve the same index i (these are the continuous and discrete parameters of the spatial integral for $\bar{A}_i$).

We now use vacuum expectation value (2.1) and formulas (2.2) and (2.3) with $\mu = \nu = i$, $i = 1, 2, 3$, to obtain a spectral representation for the sum of vacuum expectation values:

$$\sum_{i=1}^{3} \left\langle A_i(\mathbf{x}, t) A_i(\mathbf{x}', t) \right\rangle = \hbar c \int_0^\infty \mathrm{d}m^2 \int \frac{\mathrm{d}^3 p \, \exp\left[\mathrm{i}\mathbf{p}(\mathbf{x} - \mathbf{x}') \right]}{(2\pi)^3 2\sqrt{\mathbf{p}^2 + m^2}} \, 2B(m^2) \,. \tag{7.3}$$

Here, we used the fact that the sum

$$\sum_{i=1}^{3} A_{ii}^R(p) = 2B(m^2) \tag{7.4}$$

is independent of p. The momentum-space integral remaining in (7.3) coincides with the real part of the positive-frequency Green's function and with the function $(1/2)\Delta^1(z, m)$ (Hadamard's elementary function) for $z^2 > 0$:

$$\int \frac{\mathrm{d}^3 p \exp(\mathrm{i}\mathbf{p}\mathbf{z})}{(2\pi)^3 2\sqrt{\mathbf{p}^2 + m^2}} = \mathrm{Re}\, \Delta^{(+)}(z, m)\Big|_{z^0 = 0} = \frac{1}{2}\, \Delta^1(z, m)\Big|_{z^0 = 0} \,. \tag{7.5}$$

The Hadamard function is the imaginary part of the causal Green's function (2.11); for $z^2 > 0$, it can be expressed in terms of the Macdonald function, and for $z^2 < 0$, in terms of the Neumann function:

$$\frac{1}{2}\, \Delta^1(z, m) = \frac{m K_1(m\sqrt{z^2})}{4\pi^2 \sqrt{z^2}} \,, \quad z^2 > 0 \,, \qquad = \frac{m N_1(m\sqrt{-z^2})}{8\pi \sqrt{-z^2}} \,, \quad z^2 < 0 \,. \tag{7.6}$$

For $m = 0$ both representations reduce to the function

$$\frac{1}{2}\, D^1(z) = \frac{1}{4\pi^2 z^2} \,, \tag{7.7}$$

which contains the leading singularity of representation (7.6) near the light cone $z^2 = 0$:

$$\frac{1}{2}\, \Delta^1(z, m) = \frac{1}{4\pi^2 z^2} + \frac{m^2}{8\pi^2} \ln \frac{m\sqrt{|z^2|}}{2} + \ldots \,, \quad m\sqrt{|z^2|} \ll 1 \,.$$

Hence, in accordance with (7.2) and (7.3), the sum of mean squared fluctuations of the $\mathbf{A}(\mathbf{x}, t)$ components averaged over the volume $V = L^3$ becomes

$$\begin{aligned} \sum_{i=1}^{3} \left\langle (\Delta \bar{A}_i)^2 \right\rangle &= \hbar c \int_0^\infty \mathrm{d}m^2\, B(m^2) \times L^{-6} \int_V \frac{\mathrm{d}^3 x\, \mathrm{d}^3 x'}{(\mathbf{x} - \mathbf{x}')^2} \frac{m\sqrt{z^2}}{2\pi^2} K_1(m\sqrt{z^2}) \\ &= \hbar c \left\{ B_0 \frac{L^{-6}}{2\pi^2} \int_V \frac{\mathrm{d}^3 x\, \mathrm{d}^3 x'}{(\mathbf{x} - \mathbf{x}')^2} + \int_{0^+}^\infty \mathrm{d}m^2\, B_1(m^2) \times L^{-6} \frac{m^2}{2\pi^2} \int_V \mathrm{d}^3 x\, \mathrm{d}^3 x' \frac{K_1(m\sqrt{\mathbf{z}^2})}{m\sqrt{\mathbf{z}^2}} \right\} , \end{aligned} \tag{7.8}$$

where $\mathbf{z}^2 = (\mathbf{x} - \mathbf{x}')^2$.

We are of course interested in the first term in curly brackets, which is proportional to B_0 and is a purely geometric quantity, depending only on the size and shape of the averaging volume and equal to L^{-2} by order of magnitude. Together with the factor $\hbar c$ but without the summation over the $\mathbf{A}$ field polarizations, it is given in Eqn (2.99) in Thirring's book [36]. This quantity, however, tells us that the electromagnetic field is a free massless field of a standing wave in the selected volume. As regards B_0, we have already mentioned that it is the probability that a photon is produced from the vacuum by the field $\mathbf{A}(x)$; like any probability, it lies between zero and one.

In his book [36], Thirring makes an important remark about *finite-volume averages* of fluctuating vacuum fields.

"However, all the quantities discussed above should not be taken too seriously because they are all space integrals of local variables at precisely fixed time. We have seen that they have infinite root-mean-square fluctuations. Therefore, they cannot be called observables, because any measurement process takes some time, and one should consider local variables integrated over some space–time volume with blurred boundaries. To make the vacuum fluctuations small, one must spread the boundaries of the volume over an area larger than $\hbar/mc$."

Apparently, this remark is also applicable to his formula (2.99),

$$\left\langle (\Delta\bar{\mathbf{A}})^2 \right\rangle = \hbar c L^{-6} \int_V \mathrm{d}V\,\mathrm{d}V'\, \frac{1}{2} D^1(x-x') \sim \hbar c L^{-2}\,,$$

which expresses the mean square of vacuum fluctuations of the expectation value of the electric potential in a spatial volume $\sim L^3$. But the above mean square of the fluctuations is finite and is independent of time, even if it was found only for $t - t' = 0$. This is because $D^1(z)$ is even with respect to z^α, $\alpha = 0, 1, 2, 3$.

We note that the Thirring formula contains the mean square fluctuation of the *free* field $\mathbf{A}(x)$ that *is not coupled* to the vacuum current $j_\mu(x)$. Moreover, it is not summed over the two polarizations of the $\mathbf{A}$ field. Taking the coupling and the two polarizations into account is equivalent to multiplying the Thirring formula by $2B_0$, making it coincident with the first term in (7.8).

If the averaging volume V is chosen as a cube with an edge L, then a purely geometric dimensionless quantity in Minkowski space

$$\frac{L^{-4}}{2\pi^2} \int_V \frac{\mathrm{d}^3x\,\mathrm{d}^3x'}{z^2} \equiv L^{-4} \int_V \mathrm{d}^3x\,\mathrm{d}^3x'\, D^1(z)\,,$$

where $z^2 = \mathbf{z}^2 - z_0^2$, $\mathbf{z} = \mathbf{x} - \mathbf{x}'$, $z^0 = c(t-t')$, can be straightforwardly represented by the triple integral

$$\frac{1}{2\pi^3} \int \frac{\mathrm{d}^3\varkappa}{\varkappa} \left(\frac{\sin\varkappa_1 \sin\varkappa_2 \sin\varkappa_3}{\varkappa_1\varkappa_2\varkappa_3} \right)^2 \cos\left(\frac{2z^0\varkappa}{L} \right), \quad \varkappa = |\boldsymbol{\varkappa}|\,,$$

over the variables $\varkappa_i = k_i L/2$, $i = 1, 2, 3$, where k_i are the component of the wave vector $\mathbf{k}$ in the Fourier representation of $D^1(z)$.

In (7.8), this quantity, accompanied by the factor L^{-2}, features at $z^0 = 0$, where the cosine becomes unity. We use the value of the integral

$$R = \int \frac{\mathrm{d}^3\varkappa}{\varkappa} \left(\frac{\sin\varkappa_1 \sin\varkappa_2 \sin\varkappa_3}{\varkappa_1\varkappa_2\varkappa_3} \right)^2 = 17.70$$

(S L Lebedev, private communication), whence $(1/2\pi^3)R = 0.2854$. But even for $|z^0| < L/2$, the cosine has little effect on the integral, dominated by $\varkappa_i \sim 1$. Averaging over z^0 in the range $-L/2 < z^0 < L/2$ replaces the cosine with its mean,

$$L^{-1} \int_{-L/2}^{L/2} \mathrm{d}z^0 \cos\left(\frac{2z^0\varkappa}{L} \right) = \frac{\sin\varkappa}{\varkappa}\,.$$

Because the mean squared fluctuation of the averaged value of $\mathbf{A}$ in the volume V is finite, the objection regarding its observability can be dispensed with.

The quantity under consideration is finite due to the fact that it is expressed in terms of $D^1(z)$, a function that is even under $z \to -z$. This unique property is shared by the causal Green's function

$$D_+(z) = \bar{D}(z) + \frac{\mathrm{i}}{2} D^1(z)\,,$$

whose imaginary part is $(1/2)D^1(z)$. Therefore, $D_+(z)$ and $\Delta_+(z, m)$ depend only on z^2. Other singular functions in addition depend on the sign of the time z^0. For example, the Pauli–Jordan commutator function [52]

$$\Delta(z, m) = 2\varepsilon(z^0)\,\bar{\Delta}(z, m) = \int \frac{\mathrm{d}^3p\, \exp(\mathrm{i}\mathbf{pz})}{(2\pi)^3 p^0} \sin(p^0 z^0)$$

vanishes at $z^0 = 0$, because it is odd under $z^0 \to -z^0$. Being an invariant function of z^2, it is then equal to zero in the entire space-like domain $z^2 > 0$.

If we consider the fluctuations not of the field $\mathbf{A}(x)$ but of the electric field $\mathbf{E}(x) = -\partial\mathbf{A}/c\,\partial t$, then, for its average over a volume V, we can approximately set $\bar{\mathbf{E}} \approx (\omega/c)\bar{\mathbf{A}}$, where ω is some effective fluctuation frequency of the fields $\mathbf{A}$ and $\mathbf{E}$ inside V. It is close to cL^{-1}, and hence $\mathbf{E} \sim L^{-1}\bar{\mathbf{A}}$. As a result, we obtain the mean squared fluctuation of the average of $\mathbf{E}$ over the volume V in the form

$$\sum_{i=1}^{3} \left\langle (\Delta\bar{E}_i)^2 \right\rangle = \hbar c Z_3 \frac{L^{-8}}{2\pi^2} \int_V \frac{\mathrm{d}^3x\,\mathrm{d}^3x'}{(\mathbf{x}-\mathbf{x}')^2} + \ldots\,. \tag{7.9}$$

Here and hereafter, we switch to the notation Z_3 and $\rho(m^2)$ instead of Schwinger's B_0 and $B_1(m^2)$.

The purely geometric factor depending on the shape of the volume is now of the order of L^{-4}. The resulting quantity, which has the dimension erg cm^{-3}, can be considered (when multiplied by 1/2) the energy density of the fluctuating *electric field* in the volume V. Obviously, the energy density of the fluctuating *magnetic field* in the volume V is the same, because $\mathbf{H}(x) = \mathrm{rot}\,\mathbf{A}(x)$ (see [46, Sec. 90] for a direct proof of the equality of these densities).

Summing both densities and multiplying by the volume, we obtain the estimate

$$\hbar c Z_3 \frac{L^{-5}}{2\pi^2} \int_V \frac{\mathrm{d}^3x\,\mathrm{d}^3x'}{(\mathbf{x}-\mathbf{x}')^2} \sim Z_3 \frac{\hbar c}{L} \tag{7.10}$$

for the energy of a fluctuating electromagnetic field in the volume $V = L^3$. Comparing this energy with the shift of the zero-point electromagnetic field energy in spherical and cubic cavities,

$$E_{\mathrm{B,L}} = \alpha_{\mathrm{B,L}} \frac{\hbar c}{L}\,, \tag{7.11}$$

we see that, regardless of the shape of the averaging volume and the shape of the cavity volume, the coefficients $\alpha_{\mathrm{B,L}}$ are *proportional* to Z_3. This means that the striking closeness of the dimensionless coefficients $\alpha_{\mathrm{B,L}}$ to the finite value $4\pi\alpha$ that we obtained for the constant $Z_3 = \alpha/\alpha_0 = 4\pi\alpha$ [8, 9] is not accidental but is a corollary of fundamental equal-time and non-equal-time commutation relations for the interacting operator fields. In particular, it is a consequence of the Källén–Lehmann spectral representation for the exact Green's function of the photon.

We return to formula (7.10), which *estimates* the energy of the fluctuating electromagnetic field in a volume V, and write it for a cube using the value of the integral

$$\frac{L^{-4}}{2\pi^2}\int_{L^3}\frac{\mathrm{d}^3x\,\mathrm{d}^3x'}{(\mathbf{x}-\mathbf{x}')^2} = \frac{R}{2\pi^3} = 0.2854$$

found by Lebedev. Formula (7.10) then takes a more concrete form:

$$\hbar c Z_3\,\frac{L^{-5}}{2\pi^2}\int_{L^3}\frac{\mathrm{d}^3x\,\mathrm{d}^3x'}{(\mathbf{x}-\mathbf{x}')^2} = Z_3\times 0.2854\,\frac{\hbar c}{L}\,. \tag{7.12}$$

Because the finite value $Z_3 \equiv 4\pi\alpha$ that we obtained is very close to $\alpha_{\mathrm{B,L}}$, the estimated energy (7.12) inside a cube with 'blurred boundaries' is 3.5 times less than the *exact* energy inside spherical and cubic cavities with perfectly conducting walls (7.11):

$$E_{\mathrm{B,L}} = \alpha_{\mathrm{B,L}}\,\frac{\hbar c}{L}\,.$$

The spectra of eigenoscillations of the electromagnetic field inside the cavities, the eigenfrequencies, the degenerate modes, and other features are here determined by boundary conditions (4.2) on the conducting boundaries. The work of external forces during the formation of the cavity was needed for the vacuum fluctuations to satisfy the boundary conditions, and this work, in accordance with the energy conservation law, increased the energy of zero-point oscillations of the field *inside the cavity* compared with the energy *in the same volume in the absence of the cavity*.

We note that the difference between the approximate estimate of the field energy *in a spherical cavity* and estimate (7.12) for a cubic cavity amounts to replacing the number 0.2854 with $0.2962 = (3/4\pi)(6/\pi)^{1/3}$. The right-hand side of (7.12) then becomes

$$Z_3\cdot 0.2962\,\frac{\hbar c}{L} = Z_3\cdot 0.3675\,\frac{\hbar c}{2r}\,, \tag{7.13}$$

because the length L, which makes the variables dimensionless, is the cube root of the volume, $L = V^{1/3}$.

It follows that the approximate estimates in (7.12) and (7.13) are about three times lower than the exact energy values. We emphasize that, in contrast to exact calculation of the energy shift of zero-point oscillations in cavities, its approximate estimate in the same volume with 'blurred boundaries' is not plagued with divergences and does not require their removal.

8. Shift of the zero-point energy of the electromagnetic field in a cavity as a measurable physical quantity

Concerning the calculation of the Casimir energy, DeWitt [42] writes, referring to the layer of a vacuum between two ideally conducting parallel planes:

"Using the thermodynamic law (4.1) for pressure in a Casimir vacuum, I required that the conductors move slowly. If I allowed the presence of some noticeable acceleration, then the conductors would begin to emit photons and the entropy inside the layer would begin to increase. At first, it may seem surprising that by accelerating a neutral conductor, one can produce photons, but one will immediately remember that the surface layers of a real conductor carry currents. Free electrons near the surface interact with quantum fluctuations of the electromagnetic field in the same way as they interact with the classical field, and form (microscopic — VR) currents of just such a magnitude that guarantee standard (macroscopic — VR) boundary conditions. Because the boundary conditions are sufficient to determine all (macroscopic — VR) physical phenomena outside the conductor, there is no need at all to refer to currents, as was done now."

Indeed, the classical boundary conditions on a perfectly conducting cavity shell are sufficient to explain all the physical processes of propagation and reflection of electromagnetic waves inside the cavity. But there is a considerable difference between the excitation of the eigenfrequencies of a cavity placed in the field of an external classical source, whose spectrum and power are always bounded, and the excitation of eigenfrequencies of a cavity placed in a vacuum with a constantly fluctuating quantum electromagnetic field, whose energy spectrum

$$\frac{\hbar\omega}{2}\,\frac{V4\pi\omega^2\,\mathrm{d}\omega}{(2\pi c)^3}\,2 \tag{8.1}$$

differs drastically from the spectrum of classical sources. Such a spectrum would lead to the energy inside the cavity volume V diverging as ω^4, had that energy been an observable quantity.

In fact, the observable quantity here is the *change* in the zero-point energy of the electromagnetic field in the cavity due to the emergence and repeated reflection of electromagnetic waves from the conductive walls of the cavity during its formation, as described above.

As a result, excessive zero-point energy and pressure appear in the assembled cavity as compared to their values in the same volume in the absence of the cavity.

This energy and pressure are carried by quite real, observable electromagnetic standing waves that satisfy the known boundary conditions (4.2) due to a self-consistent coupling to microcurrents given by the displacement currents on a perfectly conducting surface. If a wave incident on the surface has a nonzero tangential component of the electric field, then a magnetic field normal to the surface appears in accordance with the Maxwell equation $\operatorname{rot}\mathbf{H} = \partial\mathbf{E}/c\,\partial t$. In turn, this magnetic field, according to another Maxwell equation $\operatorname{rot}\mathbf{E}' = -\partial\mathbf{H}/c\,\partial t$, creates an opposite electric field $\mathbf{E}' = -\mathbf{E}$ on the conducting surface, and $\partial\mathbf{E}'/\partial t$ creates the field $\mathbf{H}' = -\mathbf{H}$, and condition (4.2) is thus restored.

The fields $\mathbf{E}$ and $\mathbf{H}$ appearing here are nothing but the fields $\sqrt{Z_3}\mathbf{E}_{\mathrm{in}}$ and $\sqrt{Z_3}\mathbf{H}_{\mathrm{in}}$ that found themselves inside the cavity and became on-shell fields as a result of the adiabatic compression of the electromagnetic field fluctuations in the vacuum in the course of cavity formation. They received energy due to the work of compressing forces against the light pressure force. Experts in waveguides and cavity resonators call the process of the appearance of these energy-carrying waves the 'raking' (compression) of vacuum fluctuations (see [53, 54]). They are related to interacting operator fields by the asymptotic conditions

$$\mathbf{A}(\mathbf{x},t) \to \sqrt{Z_3}\,\mathbf{A}_{\substack{\text{in}\\ \text{out}}}(\mathbf{x},t) \quad \text{at} \quad t \to \mp\infty\,,$$
$$\langle 0|\mathbf{A}(x)|p\lambda\rangle = \sqrt{Z_3}\langle 0|\mathbf{A}_{\substack{\text{in}\\ \text{out}}}(x)|p\lambda\rangle\,, \tag{8.2}$$

and are uniquely determined by the matrix elements of these fields between the vacuum and one-photon states (see [34, Sec. 111 and 112], [35, Sec. 5.1.2]).

Thus, the change in the zero-point energy of electromagnetic oscillations in the cavity has become an observable physical quantity. Here is what Lukosz writes on the subject [27].

"The work required to change the size or shape of a cavity can be measured. Due to the law of conservation of energy, it is equal to the change in zero-point energy shift. Therefore, the change in zero-point energy shift is an observable and hence a finite quantity. However, the cavity must be physically realizable, at least in principle. The assumed perfect conductivity of the boundary for all frequencies seems to be an acceptable extrapolation from experience, consistent with the fundamental laws of physics."

9. Conclusion

We return to the question of why the dimensionless parameters α_{B} and α_{L}, and together with them the energy shifts E_{B} and E_{L} of zero-point oscillations of the electromagnetic field in spherical and cubic cavities, are so close, their relative difference being less than 0.8% (see (4.15) and (4.14)).

Because the radius r of the spherical cavity and the edge L of the cubic cavity are related as $L = 2r$, the cube is circumscribed about the sphere. At the same time, it can be shown [9] that, for any polyhedron circumscribed about a sphere, the ratio of its surface area S to its volume V is exactly the same as for the sphere itself,

$$\frac{S}{V} = \frac{3}{r}\,. \tag{9.1}$$

Moreover, this formula also applies to a truncated cone and a cylinder circumscribed about a sphere. The surfaces of these polyhedra consist of pieces that have zero Gaussian curvature. They are homeomorphic to the surface of the inscribed sphere and are characterized by topological invariant (9.1). We let γ denote cavities with topological invariant (9.1).

In passing from a three-dimensional to a d-dimensional space, the right-hand side of (9.1) expressing the ratio of the generalized S and V must be replaced with d/r. In particular, a cube in d dimensions has $2d$ faces, each L^{d-1} in area, whence $S/V = 2dL^{d-1}/L^d = 2d/L = d/r$.

It would be very interesting to find a polyhedron Γ circumscribed about a sphere whose ideally conducting surface shifts the energy of the vacuum electromagnetic fluctuations by the quantity

$$E_\Gamma = \alpha_\Gamma \frac{\hbar c}{2r}\,, \tag{9.2}$$

with the parameter α_Γ just equal to $4\pi\alpha$, i.e., $\alpha_0\alpha_\Gamma = \alpha$. The symmetry of such a surface would be indicative of a certain intrinsic symmetry of zero-point electromagnetic oscillations in the vacuum.

Each *polyhedron circumscribed about a sphere*, in addition to the general topological invariant (9.1), also has its own topological invariant, the numbers of faces f, edges e, and vertices v, which satisfy the Euler identity $f - e + v = 2$. This invariant, taken together with (9.1), characterizes a class of equivalent maps of a polyhedron onto a sphere, and is called a homotopy invariant (see Sec. 1.3 in [57]).

Unfortunately, the energy shift of zero-point oscillations of the electromagnetic field caused by a cylindrical cavity whose conducting surface is circumscribed about a sphere (a cylinder of radius r with the generatrix length $L = 2r$) has not yet been calculated. A cylindrical cavity was considered in detail in Feynman's lectures [55]. In book [56], moreover, Feynman notes that the fine structure constant can be related to the zeros of the Bessel functions.

Indeed, the eigenfrequencies of a cylindrical cavity whose shell is circumscribed about a sphere are given by (see [48])

$$\omega^E_{mnp} = \frac{c}{r}\sqrt{\nu^2_{mn} + \left(\frac{\pi p}{2}\right)^2}\,,$$
$$\omega^H_{mnp} = \frac{c}{r}\sqrt{\mu^2_{mn} + \left(\frac{\pi p}{2}\right)^2} \tag{9.3}$$

for E- and H-waves. Here, m, n, and p are integers, and ν_{mn} and μ_{mn} are zeros of the Bessel functions and their derivatives:

$$J_m(\nu_{mn}) = 0\,, \qquad J'_m(\mu_{mn}) = 0\,. \tag{9.4}$$

It is possible that a cylindrical cavity with the shell circumscribed about a sphere is the same cavity Γ in which the energy shift of zero-point oscillations of the electromagnetic field is determined by formula (9.2) with the parameter α_Γ just equal to $4\pi\alpha$.

Because the ratio $S/V = 3/r$ is the same for any cavity γ whose shell is circumscribed about a sphere, the zero-point energy shift E_γ can be represented as the work of external forces against the overpressure force inside the cavity over the length r,

$$E_\gamma = \alpha_\gamma \frac{\hbar c}{2r} = \alpha_\gamma \frac{\hbar c}{6}\frac{S_\gamma}{V_\gamma} = \frac{E_\gamma}{3V_\gamma} S_\gamma r = p_\gamma S_\gamma r\,, \tag{9.5}$$

where $p_\gamma = E_\gamma/3V_\gamma$ is the difference between pressures inside and outside the cavity.

Because the energy shift E_γ and the pressure shift p_γ of zero-point fluctuations of the electromagnetic field inside a cavity γ are

$$E_\gamma = \alpha_\gamma \frac{\hbar c}{2r}\,, \qquad p_\gamma = \frac{E_\gamma}{3V_\gamma}\,, \tag{9.6}$$

the pressure and volume of the cavity γ satisfy the relation

$$p_\gamma V_\gamma^{4/3} = \frac{1}{3}\,\alpha_\gamma \left(\frac{V_\gamma}{V_{\text{L}}}\right)^{1/3} \hbar c\,, \qquad V_{\text{L}} = L^3 = 8r^3\,, \tag{9.7}$$

similar to the black-body radiation adiabat

$$pV^{4/3} = \frac{1}{4}\left(\frac{45}{4\pi^2}\right)^{1/3} S^{4/3}\hbar c = \frac{\pi^2}{45}\left(\frac{\pi^2 N}{2\zeta(3)}\right)^{4/3}\hbar c\,, \tag{9.8}$$

ensuring the constancy of the entropy S and the mean number of photons N (see [58, Sec. 63]).

The adiabat of an ultrarelativistic Fermi gas has a similar structure (see [58, Sec. 61]).

The similarity between formula (9.7) and the adiabat of black-body radiation allows asserting that the dimensionless constant on the right-hand side of (9.7) is the entropy (raised to the power of 4/3) of zero-point oscillations of the electromagnetic field inside the cavity γ. But here it depends on the shape of the cavity resonator and is finite at zero temperature. We note that the volume ratio $V_\gamma/V_{\rm L}$ is a geometric constant for a cavity γ whose shell is circumscribed about a sphere. For a cube, a cylinder, and a sphere, this ratio is 1, $\pi/4$, and $\pi/6$.

The adiabat relating the pressure of zero-point oscillations of the electromagnetic field inside a cavity γ with its volume, Eqn (9.7), allows interpreting the energy E_γ of zero-point oscillations of the field inside the cavity as the work $\alpha_\gamma \hbar c/2R$ during its formation plus the work E during adiabatic ($\mathrm{d}S = 0$) compression that reduces the radius of the inscribed sphere from R to r with the cavity shape preserved:

$$E = \int_R^r \mathrm{d}E = \int_R^r (T\,\mathrm{d}S - p\,\mathrm{d}V) = -\int_R^r p\,\mathrm{d}V$$
$$= -\frac{1}{3}\,\alpha_\gamma \left(\frac{V_\gamma}{V_{\rm L}}\right)^{1/3} \hbar c \int_R^r V_\gamma^{-4/3}\,\mathrm{d}V_\gamma = \alpha_\gamma\,\frac{\hbar c}{2r} - \alpha_\gamma\,\frac{\hbar c}{2R}\,. \quad (9.9)$$

This work E was referred to in Lukosz's remark above.

In [2–8], a duality relation was established between the emission of photons by a *point* electric charge accelerated in $3+1$ dimensions and the production of pairs of scalar quanta by an accelerated *point* mirror in $1+1$ dimensions.

The duality is underlain by the fundamental relation between causal functions in d- and $(d-2)$-dimensional Minkowski spaces:

$$\Delta_d^f(x,\mu) = \frac{1}{4\pi}\int_{\mu^2}^\infty \mathrm{d}m^2\,\Delta_{d-2}^f(x,m)\,, \quad d \geqslant 4\,. \quad (9.10)$$

The causal function satisfies the equation

$$(-\partial_\alpha^2 + \mu^2)\,\Delta_d^f(x,\mu) = \delta_d(x) \quad (9.11)$$

and depends only on the invariant x^2, which makes it a purely geometric object. In our case, $d = 4$ and the function $\Delta_4^f(x,\mu)$ on the left-hand side of (9.10) describes the propagation of a particle with mass μ that can have any (say, arbitrarily small or even zero) value. On the right-hand side, the function $\Delta_2^f(x,m)$ describes the propagation of a 'particle' with a mass m that takes all possible values in the range $\mu \leqslant m < \infty$. Such a 'particle' is made of a pair of two massless particles flying apart in opposite directions, whose doubled frequencies coincide with the $k_\pm = k^0 \pm k^1$ components of the photon momentum: $2\omega = k_+$ and $2\omega' = k_-$. The masslessness of the particles making up a massive pair in $1+1$ dimensions is a fundamental purely geometric feature of the duality under discussion.

We consider the holographic duality relation (9.10) between the causal Green's functions for four-dimensional and two-dimensional spaces:

$$\Delta_4^f(z,\mu) = \frac{1}{4\pi}\int_{\mu^2}^\infty \mathrm{d}m^2\,\Delta_2^f(z,m)\,, \quad (9.12)$$

$$\Delta_2^f(z,m) = \mathrm{i}\int_{-\infty}^\infty \frac{\mathrm{d}\theta}{4\pi}\,\exp\left[\mathrm{i}m(z^1\sinh\theta - |z^0|\cosh\theta)\right]. \quad (9.13)$$

Both Green's functions featured here are invariant purely geometric objects. They are expressed in terms of the variables z^α with the dimension of length and the parameters μ and m with the dimension of inverse length. We assume that the μ parameter is arbitrarily small.

For a timelike trajectory of the charge and the mirror,

$$x^\alpha(\tau) = \left(x^1(\tau), x^0(\tau)\right), \quad \tau \text{ is the proper time}\,, \quad (9.14)$$

while the two-dimensional vector $z^\alpha = x^\alpha(\tau) - x^\alpha(\tau')$ is timelike and lies in the (x,t) plane of the four-dimensional Minkowski space, as does the $x^\alpha(\tau)$ trajectory itself.

If we use the timelike vector z^α for such a trajectory in formulas (9.12) and (9.13), then $\mathrm{Im}\,\Delta_4^f(z,\mu)$ can be replaced with

$$\mathrm{Im}\,\Delta_4^f(z,\mu) = \mathrm{Re}\int \mathrm{d}\omega_k\,\exp(\mathrm{i}k_\alpha z^\alpha)$$
$$= \mathrm{Re}\int_{\mu^2}^\infty \frac{\mathrm{d}m^2}{4\pi}\int_{-\infty}^\infty \frac{\mathrm{d}\theta}{4\pi}\,\exp(\mathrm{i}kz)\,, \quad (9.15)$$

where we use the variables

$$k^0 = \omega + \omega' = m\cosh\theta\,, \quad k^1 = \omega - \omega' = m\sinh\theta\,, \quad (9.16)$$

characterizing the state of a photon emitted by the charge and the state of a pair of scalar quanta emitted by the mirror.

We also quote the invariant measures of the number of photon states,

$$\mathrm{d}\omega_k = \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0} = \frac{\mathrm{d}k_+\,\mathrm{d}k_-}{(4\pi)^2}\,\frac{\mathrm{d}\varphi}{2\pi}\,, \quad (9.17)$$

and the number of states of the pair,

$$\frac{\mathrm{d}k_+\,\mathrm{d}k_-}{(4\pi)^2} = \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2} = \frac{\mathrm{d}m^2\,\mathrm{d}\theta}{(4\pi)^2}\,. \quad (9.18)$$

Here,

$$k_+ = k^0 + k^1 = 2\omega = m e^\theta\,,$$
$$k_- = k^0 - k^1 = 2\omega' = m e^{-\theta}\,. \quad (9.19)$$

We note that, due to the azimuthal symmetry, the angle φ is a cyclic variable, of which the functions in the integrands are independent.

We now multiply (9.15) by the invariant product $\dot{x}^\alpha(\tau)\dot{x}_\alpha(\tau')$ of two-dimensional velocity and integrate over τ and τ':

$$\iint \mathrm{d}\tau\,\mathrm{d}\tau'\,\dot{x}^\alpha(\tau)\dot{x}_\alpha(\tau')\,\mathrm{Im}\,\Delta_4^f(z,\mu)$$
$$= \int_{\mu^2}^\infty \frac{\mathrm{d}m^2}{4\pi}\int_{-\infty}^\infty \frac{\mathrm{d}\theta}{4\pi}\,\left|u^\alpha(k)\right|^2$$
$$= \int \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2}\left|\beta_{\omega'\omega}^{\rm B}\right|^2 = \bar{N}^{\rm B}\,, \quad (9.20)$$

where we have a purely geometric 2-vector,

$$u_\alpha(k) = \int_{-\infty}^\infty \mathrm{d}\tau\,\dot{x}_\alpha(\tau)\,\exp\left(-\mathrm{i}kx(\tau)\right). \quad (9.21)$$

The squared modulus of this vector coincides with the squared modulus of Bogoliubov's β coefficient,

$$\left|u_\alpha(k)\right|^2 = \left|\beta_{\omega'\omega}^{\rm B}\right|^2\,, \quad (9.22)$$

which determines the spectral density of pairs of scalar quanta (Bose pairs for brevity) emitted by the mirror in $1+1$ dimensions (see formulas (3.19) and (3.21) in [8]). We note that the Bogoliubov coefficient $\beta^{\mathrm{B}}_{\omega'\omega}$ is determined by the scalar product of the solutions $\phi_{\mathrm{in}\,\omega'}$ and $\phi_{\mathrm{out}\,\omega}$ of the wave equation for a massless scalar field (see (2.9) in [8]), has the dimension of length, and is a purely geometric quantity depending on the frequencies and trajectory parameters, which have the dimensions of powers of length and time. This is unsurprising, because the quantum theory of a massless scalar field does not contain Planck's constant.

Because the number of states (9.19) of a bosonic pair is also a geometric invariant, the mean total number $\bar{N}^{\mathrm{B}}$ of Bose pairs emitted by the mirror along the entire trajectory over the entire time, determined by the last integral in (9.20), is an invariant geometric dimensionless functional of the mirror trajectory. At the same time, the left-hand side of (9.20) is related to the photon propagator in $3+1$ dimensions,

$$\mathrm{Im}\iint \mathrm{d}\tau\,\mathrm{d}\tau'\,\dot{x}^{\alpha}(\tau)\dot{x}_{\alpha}(\tau')\Delta_4^f(z,\mu) = \int \mathrm{d}\omega_k\,\left|u_{\alpha}(k)\right|^2, \quad (9.23)$$

and it is tempting to compare it with the mean total number $\bar{N}^{(1)}$ of photons emitted by a charge moving along the same trajectory as the mirror,

$$\bar{N}^{(1)} = \frac{2}{\hbar}\,\mathrm{Im}\,W^{(1)} = \frac{1}{\hbar c}\int \mathrm{d}\omega_k\,\left|j_{\alpha}(k)\right|^2 \quad (9.24)$$

(see Eqn (3.2) in [8]). In our case, the charge has a very large transferred momentum (acceleration) and can therefore be considered point-like and unscreened. In this state, its value e_0 is greater than the value e at small transferred momenta. The soft photons emitted by it to infinity do not affect the charge trajectory, which can be considered fixed. The distribution of photons is described by the propagator $\Delta_4^f(z,\mu)$, and their mean total number $\bar{N}^{(1)}$ and spectrum, by formula (9.24), where $j_{\alpha}(k)$ is the Fourier transform of the charge current density. For the trajectories $x^{\alpha}(\tau)$ of the charge and the mirror coinciding in the (x,t) plane, the Fourier transforms of the 2-vectors of the current density and velocity are proportional to each other: $j_{\alpha}(k) = e_0 u_{\alpha}(k)$.

The holographic principle of quantization of the bare (point) charge, formulated as the requirement that the mean total numbers of photons and Bose pairs emitted by the charge and the mirror coincide, $\bar{N}^{(1)} = \bar{N}^{\mathrm{B}}$,

$$\begin{aligned} \bar{N}^{(1)} &= \frac{e_0^2}{\hbar c}\int \mathrm{d}\omega_k\,\left|u_{\alpha}(k)\right|^2, \\ \bar{N}^{\mathrm{B}} &= \int \frac{\mathrm{d}\omega\,\mathrm{d}\omega'}{(2\pi)^2}\left|\beta^{\mathrm{B}}_{\omega'\omega}\right|^2, \end{aligned} \quad (9.25)$$

then leads to the bare charge quantization $e_0^2 = \hbar c$, and the coincidence of the spectra then follows from here and the holographic duality relation (9.12) between propagators in four- and two-dimensional spaces, together with the non-trivial connection (9.22) between the Bogoliubov coefficient and the 2-vector u_{α} (also see (3.19) in [8]).

Hence, for the same trajectory of the charge and the mirror, the spectra of photons and pairs coincide if, given the point-like nature of the sources, we consider the charge to be bare with the fine structure constant $\alpha_0 = 1/4\pi$. The discovered duality can therefore be regarded as a holographic principle of bare charge quantization. The result $\alpha_0 = 1/4\pi$ satisfies all three conditions obtained by Gell-Mann and Low [10] for a finite bare charge (see p. 469). Moreover, it is precisely because of the value $\alpha_0 = 1/4\pi$ that the ratio $\alpha/\alpha_0 = 4\pi\alpha$ of the fine structure constants of the physical and bare charges, which has the physical meaning of the Dyson Z_3 factor (or the inverse permittivity of a vacuum), lies between two geometric constants, $\alpha_{\mathrm{L}} < Z_3 \equiv \alpha/\alpha_0 = 4\pi\alpha < \alpha_{\mathrm{B}}$, that determine the shifts $E_{\mathrm{L,B}} = \alpha_{\mathrm{L,B}}\,\hbar c/2r$ of zero-point oscillation energies of the electromagnetic field in cubic and spherical cavities. In this paper, we have shown that, regardless of the shape of the cavity, the coefficients $\alpha_{\mathrm{L,B}}$ are proportional to Z_3. This means that the remarkable closeness of the coefficients $\alpha_{\mathrm{L,B}}$ to the finite value $4\pi\alpha$ that we obtained for the constant $Z_3 \equiv \alpha/\alpha_0 = 4\pi\alpha$ is not accidental, but follows from the fundamental equal-time commutation relations for interacting operator fields. In particular, it is a consequence of the Källén–Lehmann spectral representation for the exact Green's function of the photon.

Indeed, the spectral distribution of the energy of zero-point oscillations of the electromagnetic field in a vacuum is well established. It is determined by the mean energy $(1/2)\hbar\omega$ of zero-point oscillations with the frequency ω (Planck) times the number of quantum states of photons in a volume V with the frequencies between ω and $\omega + \mathrm{d}\omega$ (Rayleigh and Jeans):

$$\frac{1}{2}\,\hbar\omega\,\frac{V4\pi\omega^2\,\mathrm{d}\omega}{(2\pi c)^3}\,2\,.$$

Introducing a form factor to cut off high frequencies, we integrate the spectral distribution of zero-point oscillations in a cavity with conducting walls and a volume V over frequencies. Condition (4.2) on the conducting boundary then leaves only those frequencies that are eigenfrequencies of the cavity and have the corresponding multiplicities. Letting the cutoff frequency tend to infinity and omitting the terms that diverge with the cutoff frequency and relate to the energy of zero-point oscillations of the field in the vacuum volume V, we obtain a finite shift of the zero-point oscillation energy in the cavity, Eqn (7.11). For cubic and spherical cavities, the constants $\alpha_{\mathrm{L,B}}$ coincide with an accuracy better than 0.8% with the ratio α/α_0 of the fine structure constants of the physical and bare charges. This means that the ratio $\alpha/\alpha_0 \equiv Z_3$ is proportional to the observed energy shift of zero-point oscillations of the electromagnetic field, at least in cubic and spherical cavities.

The correctness of this fundamental result is confirmed by the method of estimating the energy of mean squared fluctuations of the electromagnetic field averaged over a volume with 'blurred boundaries,' by the spectral representation of the exact Green's functions with its 'sum rule,' and also by asymptotic conditions (3.11) and (8.2) for in- and out-fields.

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