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Light diffraction in a plane-parallel layered structure with the parameters of a Pendry lens

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Contents

Abstract. The solution of Maxwell's equations for a plane electromagnetic wave and a Gaussian beam propagating in a Pendry lens has been obtained. The mathematical form of the solution explains details of image formation in such a structure. It is shown that not only the plane wave but also the Gaussian beam in this case is characterized by the absence of diffraction, so the Gaussian beam does not expand when propagated in a multilayer Pendry lens of any size.

Keywords: Maxwell's equations, metal–dielectric piecewise homogeneous medium, Pendry lens, Gaussian beam diffraction

1. Introduction

A classical optical element or a system cannot produce an exact image of an object: the angular distance between the points of the object (measured from the position of the image), distinguishable in the image, is bounded. The criterion limiting the ability to distinguish between two points in the image is named after Rayleigh and is also referred to as the diffraction limit. The limit is due to light wave diffraction. A wave bounded in space is subject to diffraction. It is no longer a plane wave, but a wave beam, which may include components with imaginary wave numbers that correspond to evanescent waves. Such waves decay fast in space and do not reach the image plane. This is why the image is not precise: it lacks the contribution from the decaying waves. According to the Rayleigh criterion, object details that are smaller than half the wavelength cannot be discerned in the image.

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Advances in such branches of nano-optics as nanophotonics and nanoplasmonics observed over the last several decades have allowed researchers to surpass the diffraction limit and obtain an image of object details that are much smaller than the wavelength. This achievement is linked to studies of plasmon waves and the creation of composite media called metamaterials [1–13]. The existence of structures where a wave has negative refraction was known by 1967, but the fundamental contribution to the theory of media with negative refraction, also called Veselago media or left media, was made by V G Veselago in a study published in 1967 [14] on the peculiar electrodynamics of media with negative refraction. In 2000, V G Veselago's idea was advanced by the British physicist J Pendry [15], who showed that, in this case, the diffraction limit intrinsic to conventional lenses can be surpassed, and coined the term 'perfect lens' for respective optical elements. Thus, diffraction is not manifested in these optical elements [15–17].

A three-layer structure is considered as a model of a superlens, in which the two outer layers are dielectric half-spaces and the middle layer is made of metal or a metamaterial. A Veselago superlens, with the parameters of the outer and middle layers, respectively, $\varepsilon_1 = 1$, $\mu_1 = 1$, $\varepsilon_2 = -1$, $\mu_2 = -1$, is distinguished from a Pendry superlens, in which layers are characterized by the material parameters $\varepsilon_1 = 1$, $\mu_1 = 1$, $\varepsilon_2 = -1$, $\mu_2 = 1$. The experimentally studied Pendry superlens has the form of a thin silver plate (film) sandwiched between dielectric half-spaces. The experiment was carried out for plane-polarized monochromatic waves with a frequency that corresponds to a dielectric permittivity of -1 on the dispersion curve of silver [18]. The resolution in the experiment was $1/5$ of the light wavelength, whereas the resolution of a refractive lens is bounded by half of the light wavelength.

A superlens is not a proper lens in the sense that it does not collect or scatter a plane-parallel light beam. There is no notion of 'focal depth.' For this reason, in contrast to a refractive lens, a superlens allows one to obtain the image of

an object not only in the focal plane but also in any other plane parallel to the film. Each of these planes can be viewed as the planes placed on both sides of an illuminated object in the absence of a film. If light illuminates the outer film boundary, as in experiment [18], the image corresponds to the observation position on the closest film boundary.

Such media as metamaterials (the Veselago lens) or metals (the Pendry lens) allow the propagation of inverse waves with oppositely directed phase and group velocities [19]. Since the phase of an evanescent wave has a large imaginary part, which defines wave damping, purely mathematically (both in traditional computations of the image in a superlens and in computations in this work), the inverse count of the wave phase leads to the effect of wave amplification. In order to form a correct object image, the wave fields must be amplified on the optical path in a metamaterial (metal) just as much as they decay on the optical path in a dielectric.

A theoretical explanation of the Pendry lens phenomenon encompasses an analytical derivation of the amplification law for evanescent waves in metal layers of the metal–dielectric structure and a proof of the absence of diffraction in the lens for a spatially bounded wave. Let us consider how the solution to this problem is reflected in the scientific literature. In order to theoretically explain the effect of super-resolution of a three-layer metal–dielectric structure, the amplitude and phase of an electromagnetic wave transmitted by the metal film were computed [20–25]. Based on the expressions obtained, an estimate was given of the minimum distance distinguishable in the image. The result leads to a practically important conclusion: metals with minimum losses are preferable as the film material.

The dispersion equation corresponds to the excitation of interfacial plasmon waves on the film surface, which confirms numerical computations [26]. Moreover, determining the field inside the middle layer can rigorously confirm the existence of surface waves on the film surfaces. For multilayer media, it is particularly important to know the field inside the structure, at least at the interfaces between the layers.

Computation of light transmission in a multilayer aperiodic structure in the framework of traditional matrix theory does not give a universal instrument for a precise analytical study of how varying parameters of the structure affect its optical properties, because the product of a large number of matrices is unwieldy for analysis. For a layered structure with super-thin layers, the equations of the matrix theory are simplified, and a new exact theory, enabling computations of the phase and amplitude of a light wave inside a multilayer structure, can help in the analysis of the optical properties of such a medium. Also, the generalization of the theory to planar layered media and its extension to similar structures with a cylindrical and spherical symmetry can be useful for the design of three-dimensional structures with metal nano-inclusions.

Thus, at the present stage of theoretical studies dealing with the optical properties of layered media, one needs new, simple, and effective instruments which can also be applied to structures with a different symmetry. This work describes the algorithm of a computational method intended for multi-component media with plane, cylindrical, and spherical symmetries.

The traditional matrix theory of spherical and cylindrical waves, which include eigenmodes of spherically and cylindrically symmetric layered media, relies on the Bessel functions, which strongly complicates computations. The theory proposed in this study relies on another solution of the Bessel equation, which follows from the Prüfer transformation [27]. Such a presentation of the result also requires solving a transcendent equation, but computations are simpler, and the layers can be handled in a cycle.

There are static and wave theories of electromagnetic field localization on interfaces in metal-dielectric structures [22]. Interfacial waves on the boundaries between metals and dielectrics are responsible for the localization of static fields at the image point of the Pendry lens. Here, a question arises on the applicability of the term 'overcoming the diffraction limit,' since one is not dealing with a wave. However, the wave theory confirms the localization of the wave field at the point considered [20, 21]. For a Gaussian beam, diffraction needs to be explored with the help of wave theory applied to the Helmholtz equation, because the wave function of the beam is derived from this equation. The present paper presents namely such a solution to the formulated problem.

2. Propagation of a plane-polarized monochromatic wave in a Pendry lens. Data from the literature

The idea of a perfect image was formulated by John Pendry [15] in 2000. To describe the propagation of an electromagnetic wave in a three-layer medium composed of two dielectric half-spaces separated by a thin metal film, Pendry applied the theory of multilayer planar structures [28]. The final equations of this theory are in the matrix form and would lead to rather complex computations if the number of layers were large. However, the characteristics of a three-layer structure can be computed quite easily. Optical super-resolution of such a metal–dielectric structure can be explained by analyzing the reflection and transmission formulas for a thin metal film [28],

$$R = \frac{r + r' \exp(4\mathrm{i}k_z d)}{1 - rr' \exp(4\mathrm{i}k_z d)}\,, \quad T = \frac{tt' \exp(2\mathrm{i}k_z d)}{1 - rr' \exp(4\mathrm{i}k_z d)}\,,$$
$$r = -r' = \frac{k_z' - k_z\varepsilon}{k_z' + k_z\varepsilon}\,, \quad t = \frac{2k_z\varepsilon}{k_z' + k_z\varepsilon}\,, \quad t' = \frac{2k_z'}{k_z' + k_z\varepsilon}\,, \tag{1}$$

where R and T are, respectively, the film reflection and transmission coefficients, k_z is the wave number of the incident wave in the direction transverse to the film, k_z' is the transverse component of the wave vector on the other side of the film, ε is the metal dielectric permittivity, and $2d$ is the film thickness; the film is surrounded by air.

For further computations, we introduce the wave number of an electromagnetic wave in a vacuum k_0:

$$k_0^2 = \frac{\omega^2}{c^2}\,.$$

In the quasistatic limit ($k_0^2 \ll k_x^2 + k_y^2$), when $k_z \approx k_z' \approx \mathrm{i}(k_x^2 + k_y^2)^{1/2}$, the formula for transmission takes the form

$$T = \frac{4\varepsilon^2 \exp(2\mathrm{i}k_z d)}{(1+\varepsilon)^2 + (1-\varepsilon)^2 \exp(4\mathrm{i}k_z d)}\,.$$

The quasistatic limit deals with evanescent waves; their transmission on the exit surface of the film in the limit

$\varepsilon \to -1$ is given by the expression

$$T = \lim_{\varepsilon \to -1} \frac{4\varepsilon^2 \exp(2\mathrm{i}k_z d)}{(1+\varepsilon)^2 + (1-\varepsilon)^2 \exp(4\mathrm{i}k_z d)}$$
$$= \exp(-2\mathrm{i}k_z d) = \exp\left(2d\sqrt{k_x^2 + k_y^2}\, d\right).$$

From this expression, it follows that, at the distance $4d$ from the object plane, the phase of a wave emitted initially from the object plane repeats the phase the wave had in this plane. An evanescent wave decays on the dielectric part of its path and is amplified in the film.

The minimum size of a light spot behind the film was estimated from the following considerations [16]: the spot boundary corresponds to the points where the wave intensity is half of that in the center of the spot. This leads to the relationship

$$\Delta \sim \frac{4\pi d}{\ln(\varepsilon - 1)} = \frac{4\pi d}{\ln \varepsilon''}, \tag{2}$$

where $\varepsilon = \varepsilon' + \mathrm{i}\varepsilon'' = -1 + \mathrm{i}\varepsilon''$, Δ is the minimum spot size behind the film.

The size of a light spot is therefore defined by the imaginary part of metal dielectric permittivity, and can be arbitrarily small if losses in the film are such.

Some other methods of exploring optical properties of the structure considered here are mentioned in the literature. For example, Ref. [23] analyzed transmission formula (1), but conclusions were made for the general case, without considering the quasistatic limit. The uncertainty relation was used, and the result is close to estimate (2).

An estimate of the size that can be distinguished in the plane behind the film is also obtained in Ref. [25], and it has the form of dependence (2). Computations were based on the Fourier–Bessel transform for the field inside and behind the film, while the formula for film optical transmission was also used. The wave field was conditionally split into dynamic and static parts, with which the lens super-resolution is namely linked. There are also studies presenting numerical computations for the field in the structure; they confirm analytical results [27, 29]. The literature also proposes a theoretical analysis of a multilayer structure with the Pendry lens parameters [30, 31] and a cylindrical structure with these parameters [32].

The theory presented in this work contains transmission formula (1), derived directly from Maxwell's equations, as one of its results. As applied to a metal film, its conclusions show that the amplitudes and phases are equal in the planes of an object and its image that are at a distance $4d$ apart and are separated by the metal film. Based on this theory, an analysis of the size that can be distinguished in the image can be carried out by the methods described in Refs [15, 24], which lead to formula (2).

The proposed theory has some advantages over the matrix description of layered structures.

(1) The theory is applicable to harmonic electromagnetic waves with any type of wave front symmetry, satisfying the Helmholtz equation in a homogeneous medium, which also includes axisymmetric beams.

(2) Its formulas do not become more complex if the number of layers is increased [33]. There is no need to expand the material characteristics of a finite-sized structure in a Fourier series, and the solution for infinite periodic structures also preserves a simple form. The solution can be generalized for two- and three-dimensional periodic structures, for example, for a metal film with parallel slots [34, 35].

(3) The solution can be generalized to layered structures with cylindrical and spherical boundaries between its components [36–39], including combined structures, for example, a metal probe with a pointed end [39].

3. Propagation of a plane-polarized monochromatic wave in a periodic Pendry lens. Theoretical consideration

In this section, we consider an infinite periodic structure with the Pendry lens parameters, shown schematically in Fig. 1. In the structure with two components, the layers of different components (with dielectric permittivities 1 and -1) alternate, and all the layers have the same thicknesses. We turn to Maxwell's equations for a plane-polarized transverse magnetic (TM) wave propagating in such a composite medium.

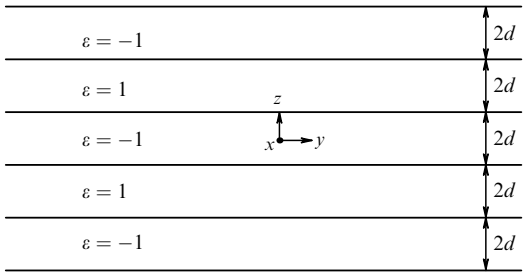

Figure 1. Schematics of plane-parallel layered structure with the Pendry lens parameters.

The coordinate system will be selected in the following way: the z-axis is perpendicular to the structure layers; the xy plane is parallel to the layers and divides the central metal layer into two equal parts. In this section, we will assume that a plane monochromatic electromagnetic wave with TM polarization impinges on the lens; its wave vector lies in the plane xz. Because of polarization, only the y component of the magnetic field is nonzero.

Propagation of an electromagnetic wave is described by the wave equation which is derived from Maxwell's equations for the wave electric and magnetic fields [40, p. 420, formula (88.2)],

$$\Delta \mathbf{H} + \frac{1}{\varepsilon}[\operatorname{grad}\varepsilon \times \operatorname{rot}\mathbf{H}] + \frac{\varepsilon\omega^2}{c^2}\mathbf{H} = 0, \tag{3}$$

where ε is the medium dielectric permittivity,

$$\varepsilon(x,z) = \varepsilon_1 + (\varepsilon_2 - \varepsilon_1)T(x,z),$$

$T(x,z)$ is the Heaviside step function, region 2 is a dielectric with dielectric permittivity ε_2, region 1 is a metal with dielectric permittivity ε_1, and

$$T(x,z) = \begin{cases} 1, & \varepsilon(x,z) = \varepsilon_2, \\ 0, & \varepsilon(x,z) = \varepsilon_1. \end{cases}$$

The equation for the only magnetic field component follows from (3) by taking into account that the dielectric permittivity varies only in the direction z,

$$\frac{\partial^2 H_y}{\partial z^2} - \frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial z}\frac{\partial H_y}{\partial z} = -H_y(\varepsilon k_0^2 - k_x^2). \tag{4}$$

Since we are dealing with a piecewise homogeneous medium with two possible values of dielectric permittivity, we write the squared wave vector as $\varepsilon_m k_0^2 = k_x^2 + k_{zm}^2$, $m = 1, 2$.

Writing equation (4) in such notations, and also adding and subtracting the expression

$$\left(\frac{\varepsilon}{\varepsilon_1}\right)^2 \left(\frac{\varepsilon_1}{\varepsilon H}\,\frac{\partial H_y}{\partial z}\right)^2$$

on its left-hand side, we transform this equation into the form

$$\frac{\varepsilon}{\varepsilon_1}\,\frac{\partial}{\partial z}\left(\frac{\varepsilon_1}{\varepsilon H_y}\,\frac{\partial H_y}{\partial z}\right) + \left(\frac{\varepsilon}{\varepsilon_1}\right)^2 \left(\frac{\varepsilon_1}{\varepsilon H_y}\,\frac{\partial H_y}{\partial z}\right)^2 = -k_z^2\,, \qquad (5)$$

where $k_z = k_{z1}$ in regions with $\varepsilon(x, z) = \varepsilon_1$ and $k_z = k_{z2}$ in regions with $\varepsilon(x, z) = \varepsilon_2$. Such a grouping of multipliers in the terms in equation (5) hints that it is possible to introduce a phase function b in the direction z which is common to all regions of a piecewise homogeneous medium according to the equality

$$\frac{1}{H_y}\,\frac{\partial H_y}{\partial z} = -\frac{\varepsilon}{\varepsilon_1}\,k_{z1}\tan b\,. \qquad (6)$$

Starting from Maxwell's equations for the components of an electromagnetic field, we note that, given the continuity of the function $[\varepsilon_1/(\varepsilon H_y)]\partial H_y/\partial z$ introduced above, the tangent components of electric and magnetic fields are preserved on crossing the interfaces between layers. We will construct the solution so that the function $[\varepsilon_1/(\varepsilon H_y)]\partial H_y/\partial z$ is continuous in the whole space.

Taking into account equation (6), we transform equation (5) into the form

$$k_{z1}\,\frac{\partial \tan b}{\partial z} - \frac{\varepsilon}{\varepsilon_1}\left[(k_{z1}\tan b)^2 + k_{z1}^2 + T(x, z)\left(k_{z2}^2\,\frac{\varepsilon_1^2}{\varepsilon_2^2} - k_{z1}^2\right)\right] = 0\,. \qquad (7)$$

From equation (7), it follows that, in the region filled with medium 1, the derivative of function $b(x, z)$ is

$$\frac{\partial b(x, z)}{\partial z} = k_{z1}\,.$$

For this region, a solution to equation (6) is found by integrating (6), which leads to

$$H_y(x, z) = H_0 \cos b\,.$$

Let us write equation (7) for the region filled with the second medium. For this, we introduce the notation

$$\frac{1}{\beta^2} = \frac{k_{z2}^2}{k_{z1}^2}\,\frac{\varepsilon_1^2}{\varepsilon_2^2}\,.$$

Equation (7) for the region filled with medium 2 in compact notation becomes

$$\frac{\partial \arctan\left(\beta \tan b(x, z)\right)}{\partial z} = k_{z2}\,.$$

Integration leads to the following result for the phase function:

$$b(x, z + \Delta z) = \arctan\left\{\frac{1}{\beta}\tan\left[k_{z2}\Delta z + \arctan\left(\beta \tan b(x, z)\right)\right]\right\}.$$

In order to write equation (7) in a form invariant for any homogeneous region, we introduce the function $\tilde{b}(x, z)$ according to the equality

$$\tilde{b}(x, z) = \arctan\left(\beta \tan b(x, z)\right).$$

The wave equation for a wave with TM polarization is satisfied in all regions of the structure if the coordinate z in differentials in this equation is everywhere replaced by the function $\tilde{b}$. This is possible, because the derivatives of function $\tilde{b}$ and the coordinate z multiplied by the wave vector coincide. As a result, we obtain the equation

$$-\frac{\partial \tan \tilde{b}}{\partial \tilde{b}} + \tan^2 \tilde{b} + 1 = 0\,,$$

which is valid in all layers of the structure.

The function $\tilde{b}$ is a universal one, and equations including it are invariant to a change in the components of a plane-parallel structure, in particular, the structure with the Pendry lens parameters. The change in function $\tilde{b}$ on the homogeneous interval of wave propagation coincides with the value $k_{z2}\Delta z$, which in turn coincides with the traditional solution in a homogeneous domain, only, instead of 'matching' at the boundaries of regions, we write the constant in the phase function using the general rule:

$$\tilde{b}(x, z) = \begin{cases} k_{z1}(z)\,, & 0 < |z| < d\,, \\ k_{z2}(z - d) + C_1\,, & d < |z| < 3d\,, \\ k_{z1}(z - 3d) + C_2\,, & 3d < |z| < 5d\,, \\ k_{z2}(z - 5d) + C_3\,, & 5d < |z| < 7d\,, \\ k_{z1}(z - 7d) + C_4\,, & 7d < |z| < 9d\,, \end{cases}$$

$$C_1 = \arctan\left(\beta \tan k_{z1}(d)\right),$$

$$C_2 = \arctan\left(\frac{1}{\beta}\tan\left(k_{z2}(3d) + C_1\right)\right),$$

$$C_3 = \arctan\left(\beta \tan\left(k_{z1}(5d) + C_2\right)\right),$$

$$C_4 = \arctan\left(\frac{1}{\beta}\tan\left(k_{z2}(7d) + C_3\right)\right).$$

The solution to a wave equation written in this notation takes the form

$$H_y(x, z) = H_0 \cos \tilde{b} = \frac{\cos b}{\sqrt{1 + (\beta^2 - 1)\cos^2 b}}\,.$$

We find the film transmission based on the equalities obtained here. The condition for a plasmon resonance or, in other words, the dispersion relation describing the eigenmodes of a three-layer structure, is expressed by the equation

$$1 + (\beta^2 - 1)\cos^2 b = \sqrt{\sin b + \mathrm{i}\beta\cos b}\,\sqrt{\sin b - \mathrm{i}\beta\cos b} = 0\,.$$

We transform this equation as

$$-\sin b + \mathrm{i}\beta\cos b = \frac{\mathrm{i}}{2}\left(\exp(\mathrm{i}b)(1 + \beta) + \exp(-\mathrm{i}b)(\beta - 1)\right) = \frac{\mathrm{i}}{2}(1 + \beta)\exp(-\mathrm{i}b)\left(\exp(2\mathrm{i}b) + \frac{\beta - 1}{\beta + 1}\right).$$

Thus, the plasmon resonance condition becomes

$$\exp(2\mathrm{i}b) + \frac{\beta - 1}{\beta + 1} = \exp(2\mathrm{i}b) + \frac{\beta^2 - 1}{(\beta + 1)^2} = 0\,,$$

where

$$\frac{\beta^2 - 1}{(\beta + 1)^2} = \frac{k_{z1}^2/\varepsilon_1^2 - k_{z2}^2/\varepsilon_2^2}{(k_{z1}/\varepsilon_1 + k_{z2}/\varepsilon_2)^2}$$
$$= \frac{k_0^2/\varepsilon_1 - k_0^2/\varepsilon_2 - (k_x^2 + k_y^2)(1/\varepsilon_1 - 1/\varepsilon_2)(1/\varepsilon_1 + 1/\varepsilon_2)}{k_0^2/\varepsilon_1 + k_0^2/\varepsilon_2 - (k_x^2 + k_y^2)(1/\varepsilon_1^2 + 1/\varepsilon_2^2) + 2(k_{z1}/\varepsilon_1)(k_{z2}/\varepsilon_2)}\,.$$

The following equality holds in the quasistatic limit:

$$\lim_{\substack{k_x, k_y \to \infty \\ \varepsilon_1 = 1 \\ \varepsilon_2 = -1 + \mathrm{i}\varepsilon''}} \frac{\beta^2 - 1}{(\beta + 1)^2} = \frac{\mathrm{i}\varepsilon''}{2}\,.$$

Hence, it follows that

$$\exp(2\mathrm{i}b) = -\frac{\mathrm{i}\varepsilon''}{2}\,.$$

This implies that, in the case under consideration, the field at the lens boundaries may increase without limit if losses in the film material are reduced accordingly.

We additionally explore the resonance condition for a three-layer structure (infinite dielectric half-spaces separated by a thin metal film) where both boundaries, located infinitely far from the film, have the coordinates $z = -h$, $z = h$, $h \to \infty$. The eigenmodes of the structure satisfy the matching condition at the structure boundaries:

$$\begin{aligned} &\tilde{b}(x, d) + k_{z2}(h - d) = \arctan\big(\beta \tan(k_{z1}d)\big) \\ &\quad + k_{z2}(h - d) = \pi m\,, \quad m = 0, 1, 2, \ldots, \quad h \to \infty\,. \end{aligned} \tag{8}$$

We obtain the useful identity

$$\arctan\big[\beta \tan(k_{z1}d)\big] = k_{z2}(d - h)\,. \tag{9}$$

The last identity, with account for wave numbers being complex-valued, expresses the condition of plasmon resonance:

$$\tan(k_{z1}d)\beta = -\lim_{h \to \infty} \tan\big(k_{z2}(d - h)\big) = -\mathrm{i}\,.$$

For a symmetric structure that consists of three layers, we compute the function of the z component of the phase in the plane of the object and image. The condition of phase matching (8) sets the phase difference between the boundaries of an infinite structure equal to $2\pi m$ $(m = 1, 2, \ldots)$. Thus, the boundaries of a three-layer structure, infinitely distant from the film, are in our model the planes with a zero z component of the phase, and we can count the phase from one of them. Let us assume that the phase is increasing in the positive z direction. Then, the z component of the phase will be $k_{z2}(h - 2d)$ in the object plane. In turn, the z component of the phase in the image plane will also be $k_{z2}(h - 2d)$. The values of the phase in planes that coincide with film surfaces are expressed as follows:

$$P_1 = k_{z2}(h - 2d) + k_{z2}d + \varphi_1\,,$$

$$P_2 = k_{z2}(h - 2d) + k_{z2}d + \varphi_2\,,$$

where φ_1 is the additional phase of the ray illuminating a point on the object with respect to the phase of the object location in our model, and φ_2 is the additional phase of the image of that point.

Let us compute the difference between two computational results for the phase on the second film surface:

$$\begin{aligned} 0 &= k_{z2}(h - d) + \varphi_2 \\ &\quad + \arctan\left\{\beta \tan\left[2k_{z1}d + \arctan\left(\frac{1}{\beta}\tan P_1\right)\right]\right\}. \end{aligned}$$

We assume $\varphi_1 = 0$; then,

$$\begin{aligned} &-k_{z2}(h - d) - \varphi_2 \\ &\quad = \arctan\left\{\beta \tan\left[2k_{z1}d + \arctan\left(\frac{1}{\beta}\tan\big(k_{z2}(h - d)\big)\right)\right]\right\}. \end{aligned}$$

We use equality (9),

$$-k_{z2}(h - d) - \varphi_2 = \arctan\Big(\beta \tan(k_{z1}d)\Big)\,.$$

We apply it once more to get

$$-k_{z2}(h - d) - \varphi_2 = -k_{z2}(h - d)\,.$$

The last equality implies that, if phase φ_1 equals zero, φ_2 is also zero. Thus, the wave has the same phases in the planes of the object and image.

For a structure made of $2n + 1$ layers, instead of (9), one needs to apply the dispersion relation for the multilayer structure:

$$\beta \tan(k_{z1}d + C_n) = -\lim_{h \to \infty} \tan\big(k_{z2}(d - h)\big) = -\mathrm{i}\,.$$

Let us discuss this result. We denote the magnetic field in the central plane of the structure as $H_0(x)$. Then, at infinitely distant boundaries of the structure, the field will be $H_0(x)\exp(\mathrm{i}2\pi m) = H_0(x)$. Field amplification at film interfaces under the conditions of plasmon resonance is fully compensated only at infinity. If we place a light source at distance d from the film, we virtually create a wave with phase $k_{z2}(h - 2d)$ at this source location, i.e., the decaying wave is damped with respect to the infinitely distant plane, and this damping, together with the amplification at the film boundary (9), gives the final result for the phase: $-k_{z2}d$. In the image plane, the wave is also damped with respect to an infinitely distant plane, and the phase addition is the same, $-k_{z2}d$. Net amplification equals wave damping on the dielectric interval of the wave optical path in the lens (between the planes of the object and image).

Infinite amplification of the field between regions happens in the metal layers. Within this part of the structure, the field has a high energy density only in narrow subsurface layers, and decays practically to zero inside the metal. Mathematically, this is expressed through the contribution from the arctangent function in the wave phase; under the conditions of plasmon resonance, the cosine of the function $\arctan[(1/\beta)\tan(k_{z1}d + C_n)]$ is infinite at the boundary. In physical terms, the arctangent defines the wave phase on reflection from the metal–dielectric boundary. The cosine of the arctangent therefore describes a wave inside the film upon its reflections from two boundaries, infinitely increasing under the conditions of plasmon resonance.

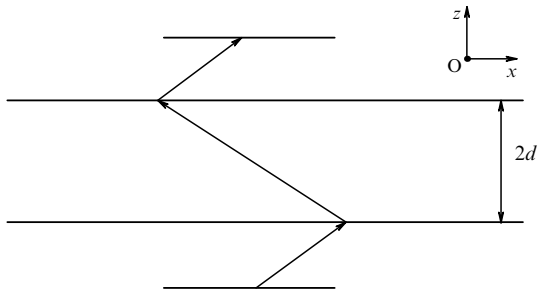

Figure 2. Optical path of a wave emitted by a light source in a Pendry lens.

Let us find the change in the along-lens components of the phase,

$$\frac{\partial a(x,z)}{\partial x} = \frac{\varepsilon_2}{\varepsilon_1}\, k_{x1} = -k_{x1}\,,$$

where $a(x,z)$ is the phase in the direction of the x-axis. Just like the phase component transverse to the lens, the x component of the wave phase decreases in metal because the derivative is negative. Hence, the wave is a reverse one, and the ray direction is changed to the mirror one, which corresponds to negative refraction. The wave optical path in this situation is shown in Fig. 2. If the dielectric permittivities are equal by absolute value, but opposite in sign, the proportions of the object are preserved in the image.

As follows from the result obtained, in the absence of losses, the dielectric permittivity should be equal to -1 in order to get an ideal image. However, in metals, where the real part of dielectric permittivity is negative, there are losses, defining a nonzero imaginary part of dielectric permittivity. The smaller the last quantity, the more precise the image. In this sense, materials with minimum losses are preferable.

At present, there are different methods to synthesize metals that satisfy this requirement [41]. The most suitable material is silver. A 'dream' film, according to specialist estimates, should have the parameter $\varepsilon'^2/\varepsilon'' > 2000$. Good parameters are achieved in silver polycrystals grown by certain methods. In polycrystalic samples, because of the presence of grains and different orientation in their boundaries, this parameter is substantially smaller [38].

One of the most important characteristics of samples used to produce plane-parallel optical elements is the 'roughness' of their surface. This is an essential factor, which might increase the propagation distance of surface plasmon-polaritons or the quality of superlenses. A critical value for this parameter in constructions based on a superlens is 1 nm; monocrystals have a roughness of 0.5 nm.

Figure 3a shows the result of wave field computations in an ideal film. We compare it with an analogous result for a film with the parameter $\varepsilon'^2/\varepsilon'' = 20$ plotted in Fig. 3b. The comparison shows that, in the first case, the resonance is expressed more prominently, and the maximum peak width is narrower than in the second case. Since the maxima on the plots are rather narrow, in estimates of lens resolving capability it was assumed that, in this case, the frequency dispersion of dielectric permittivity can be neglected.

Let us compute the minimum size of the light spot in the image. With this aim, we use the uncertainty principle $\Delta x \Delta k \approx 1$. The width of the peak in Fig. 3 divided by the film thickness defines Δk. In the two cases considered, the minimum sizes computed by the plots in Fig. 3 are $\Delta x \approx 30$ nm, $\Delta x \approx 2$ μm, respectively. The wavelength of light that corresponds to the peaks in the plot is ≈ 240 nm for the structure parameters used in computations. According to the Rayleigh criterion, the maximum size for a refractive lens cannot be better than half the light wavelength. Thus, an ideal film should create an image with a resolved size that is eight time smaller than the wavelength, which exceeds the diffraction limit, and a film with the parameters used in Fig. 3b does not have this property. Note that the plots are based on formulas derived from Maxwell's equations in a general case, and the results of computations are not bounded by the quasistatic limit.

As was shown analytically above, the smaller the imaginary part of metal dielectric permittivity, the stronger the field increase at the lens boundaries. The analysis of plots in Fig. 3 leads us to the same conclusion.

Here, we are considering two limiting cases, one with optimal and one with unsuitable film parameters. Real polycrystalline and monocrystalline films have intermediate characteristics. An optimal lens can be made of monocrystalline silver film: it will have improved parameters and high resolution. As is seen, the lens with less suitable parameters has a resolution that does not exceed the diffraction limit (the resolution of a refractive lens). A monocrystalline silver film with a thickness of 30 nm, i.e., with the parameters used for the plot in Fig. 3a, can be produced by several contemporary methods [41].

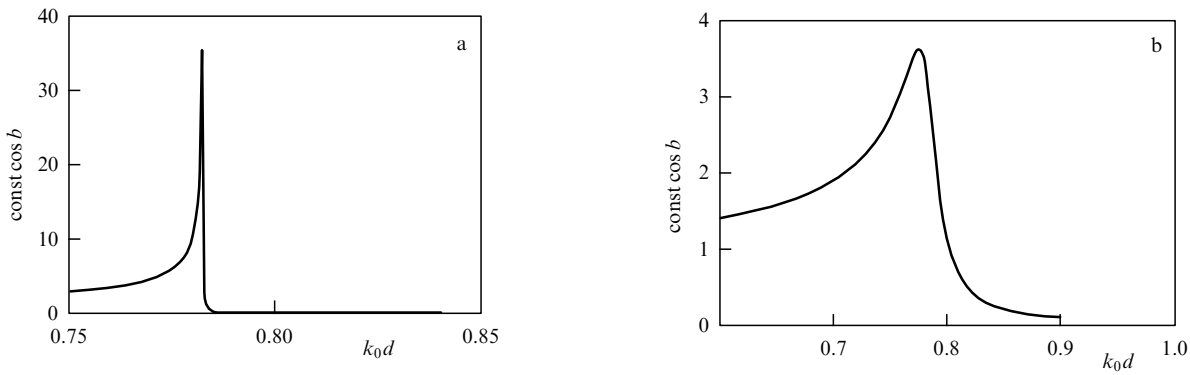

Figure 3. Dependence of relative intensity of an electromagnetic wave at the boundary of a three-layer structure ($H(z) = \text{const}\cos b(z)$) on the wave frequency; structure parameters are: (a) $d = 30$ nm, $\varepsilon_2 = 1$, $\varepsilon_1 = -1 + \mathrm{i}\,0.0005$; (b) $d = 30$ nm, $\varepsilon_2 = 1$, $\varepsilon_1 = -1 + \mathrm{i}\,0.05$. Frequency dispersion of dielectric permittivity was not taken into account.

4. Algorithm for computing the electromagnetic field in a layered symmetric medium

In Section 3, we dealt with the main idea of the method for computing electromagnetic fields in a layered structure with plane-parallel, cylindrical, or spherical symmetry. It can be summarized in the following algorithm.

(1) *Dispersion characteristics are computed and a diagram is constructed.* In layer 1 of a one-dimensional structure which has the dielectric permittivity ε_1 and neighbors the ambient media (ε), the wave phase is

$$b(z) = \arctan\left(\frac{1}{\beta_1}\tan\Delta b_1\right),$$

where

$$\beta_1 = \frac{k}{k_1}\frac{\varepsilon_1}{\varepsilon}, \quad k^2 = \varepsilon k_0^2 - k_\perp^2, \quad k_1^2 = \varepsilon_1 k_0^2 - k_\perp^2.$$

In layer 2 with the dielectric permittivity ε_2, the wave phase is

$$b(z) = \arctan\left(\frac{1}{\beta_2}\tan\left(\Delta b_2 + \arctan\left(\frac{1}{\beta_1}\tan\Delta b_1\right)\right)\right),$$

$$\beta_2 = \frac{k_1}{k_2}\frac{\varepsilon_2}{\varepsilon_1}, \quad k_2^2 = \varepsilon_2 k_0^2 - k_\perp^2.$$

In layer 3, the wave phase is

$$b(z) = \arctan\left\{\frac{1}{\beta_3}\tan\left(\Delta b_3 + \arctan\left[\frac{1}{\beta_2}\tan\left(\Delta b_2 + \arctan\left(\frac{1}{\beta_1}\tan\Delta b_1\right)\right)\right]\right)\right\},$$

$$\beta_3 = \frac{k_2}{k_3}\frac{\varepsilon_3}{\varepsilon_2}, \quad k_3^2 = \varepsilon_3 k_0^2 - k_\perp^2,$$

where ε_3 is the dielectric permittivity of the material in the third layer. In other structure layers, the phase increment is written similarly. In the last layer m, the wave phase has the form

$$b(L) = \Delta b_m + \arctan\left[\frac{1}{\beta_m}\tan\left(\Delta b_m + \ldots + \arctan\left(\frac{1}{\beta_1}\tan\Delta b_1\right)\right)\right], \tag{10}$$

$$\beta_m = \frac{k_{m-1}}{k_m}\frac{\varepsilon_m}{\varepsilon_{m-1}}, \quad k_m^2 = \varepsilon_m k_0^2 - k_\perp^2,$$

where ε_m is the dielectric permittivity of the material in layer m and L is the linear size of the structure in the direction perpendicular to the layers.

To compute the component of the wave vector that is perpendicular to the layers, referred to as the 'constant of mode propagation,' we write the dispersion equation

$$\tan b(L) = \frac{\mathrm{i}}{\beta_{m+1}}, \quad \beta_{m+1} = \frac{k_m}{k}\frac{\varepsilon}{\varepsilon_m}. \tag{11}$$

The phase increment inside a homogeneous domain is expressed differently for different symmetries:

— plane-parallel medium (in layer m),

$$\Delta b_m = k_m d_m,$$

where d_m is the layer width;

— cylindrical symmetry for the fundamental mode of the structure (layer m),

$$\Delta b_m = k_m r_m + \frac{\sin^2 b(R)}{2R} - \frac{\sin^2\left[b(R - r_m)\right]}{2R},$$

where r_m is the difference between the outer and inner layer radii and R is the layer radius;

— spherical symmetry for the fundamental mode (layer m),

$$\Delta b_m = k_m \rho_m + \frac{\sin^2 b(P)}{P} - \frac{\sin^2\left[b(P - \rho_m)\right]}{P},$$

where ρ_m is the difference between the outer and inner layer radii and P is the layer radius.

(2) *The magnetic field at a point of a one-dimensional layered structure is given by the formula*

$$H(z) = \mathrm{const}\cos b(z). \tag{12}$$

The above algorithm for computing the wave electromagnetic field in a symmetric layered structure can be used to analyze, design, or optimize optoelectronic devices based on super- and hyper-lenses.

For spherical and cylindrical symmetries, one has to solve a transcendent equation on passing from one layer to the next one, which involves computational costs, but in the limit of small sizes (small layer widths) the theory considered here has rather interesting and important consequences, which include, for example, a formula for high-order modes of spherical symmetry, allowing one to estimate the polarizability of spherical three-layer nanoparticles [36], or a formula describing a decaying dispersion branch in the diagram for multilayer nanowires [37].

5. Gaussian beam diffraction in a Pendry lens

In Sections 3 and 4, we arrived at a conclusion about the high resolution of a Pendry lens based on the computations of evanescent wave decay. In contrast to evanescent waves in a classical optical element, which decay fast, these waves are preserved in full measure in the image plane in the Pendry lens, which is the reason why the diffraction limit can be surpassed. Further, we consider the diffraction of a diverging axisymmetric beam in a plane-parallel structure with the Pendry lens parameters using a Gaussian beam as an example.

Since the beam is axisymmetric, we perform computations in a cylindrical reference frame. We assume that the beam propagation direction is perpendicular to the lens layers. We begin by writing wave equation (3) in the cylindrical reference frame,

$$\Delta\mathbf{H} - \frac{1}{\varepsilon}\left(\frac{\partial\varepsilon}{\partial z}\right)\frac{\partial\mathbf{H}}{\partial z} = -\mathbf{H}\varepsilon k_0^2. \tag{13}$$

We transform this equation by analogy with transformations (4)–(7) of equation (3); the result takes the form

$$-\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \left(\frac{\partial f}{\partial r}\right)^2 - \frac{\partial^2 f}{\partial z^2} - k_z^2\frac{\partial\tan\tilde{b}}{\partial\tilde{b}} + \left(k_z\tan\tilde{b} - \frac{\partial f}{\partial z}\right)^2 + k_z^2 = 0,$$

where

$$H_y(z) = H_0(z)\exp f(r,z)\,, \quad \tan\tilde{b} + \frac{\partial f(r,z)}{\partial k_z z} = \frac{1}{H_y}\frac{\partial H_y}{\partial k_z z}\,.$$

In Section 1, we dealt with the function

$$\tan b = \frac{\varepsilon_2}{\varepsilon H_y}\frac{\partial H_y}{\partial k_z z}\,.$$

The conditions for preserving the tangent field components are fulfilled if this function is continuous, and the function H_y should be continuous too. The beam function will satisfy these conditions if the variable $k_z z$ present in the function $f(r,z)$ for the beam in a homogeneous medium is replaced by the function $(\varepsilon/\varepsilon_2)k_{z2}z$.

Inserting the expression $(1/H_y)(\partial H_y/\partial k_z z)$ into (13), we arrive at the equation

$$-\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + \mathrm{i}\,\frac{k_z^2}{k_{z2}}\,\frac{\varepsilon_2}{\varepsilon}\,\frac{\partial f\left[(\varepsilon/\varepsilon_2)k_{z2}z\right]}{\partial z} = 0\,, \quad (14)$$

where k_z is the wave vector in the structure, $k_z = k_{z1}$ in the first medium, and k_{z2} in the second medium. From Eqn (13) written for a homogeneous medium, the wave function of a Gaussian beam is derived [42]:

$$H_y(r,z) = \exp\left(f(r,z) + \mathrm{i}\sqrt{\varepsilon}\,k_0 z\right) = \exp\left(\frac{\sqrt{\varepsilon}k_0^2 r^2}{\mathrm{i}k_0 z + \mathrm{const}} + \mathrm{i}\sqrt{\varepsilon}\,k_0 z\right). \quad (15)$$

Analysis of Eqns (14) and (15) leads to the conclusion that, in a layered structure with Pendry lens parameters, the wave function of the Gaussian beam is expressed in the form

$$H_y(r,\tilde{b}) = \exp\left(f(r,\tilde{b}) + \mathrm{i}\tilde{b}\right) = \exp\left(\frac{\varepsilon k_0^2 r^2}{\mathrm{i}(\varepsilon/\varepsilon_2)k_{z2}z + \mathrm{const}} + \mathrm{i}\tilde{b}\right). \quad (16)$$

This solution corresponds to a mirror reflection of the beam phase on the nearest and furthest lens boundaries. The solution can be generalized to a beam incident at an angle to the lens, when the beam wave vector has the z and x components. This conclusion is based on the following factors.

First, the increment of the along-beam phase between the lens surfaces is $-2k_{z2}d$, whereas the change in the wave phase in each of two dielectric intervals of its path between the planes of the object and image is $k_{z2}d$.

Second, the ray propagates in the positive z direction and in the direction opposite to the initial one along the x-axis (k_x changes its sign at the lens boundaries). In the lens, it passes a distance along the x-axis that is equal to the sum of the distances passed at the dielectric interval of the path.

Third, since the angle of the beam to the z-axis in metal equals in absolute value the angle in a dielectric, but is opposite in sign, the radial coordinate in oblique systems of coordinates is mirror-symmetric with respect to the lens surface in media 1 and 2.

The negative increment of coordinates z and x defines not only a wave with the opposite phase but also the opposite count for the along-beam coordinate in the function f. If the beam was diverging before the lens, it starts to narrow in the lens, increasing the curvature radius of the wave front, and vice versa. The narrowing before the object position at a distance from it not larger than d is reflected in the lens region with a coordinate from the interval $(0, d)$. A schematic picture of the beam propagation through the lens is shown in Fig. 4. As follows from Fig. 4, the beam has the same characteristics in the planes of the object and image.

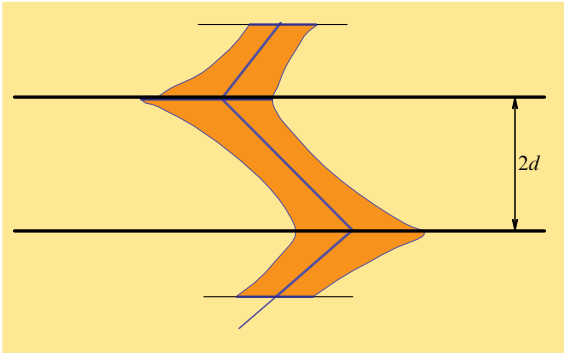

Figure 4. (Color online.) Gaussian beam propagation in a Pendry lens.

The picture presented for the Gaussian beam corresponds to formulas (15) and (16), because they were derived not only for evanescent waves in the static limit, as in Ref. [29], but for all wave types, including propagating ones. Figure 4 depicts the change in the entire beam in the film; the derivation of formulas (15) and (16) corresponds to the wave theory of electromagnetic field localization. Here, the following condition is at work: the optical path between a point in the object plane and its image has to be equal to zero, and it should be observed for each point in the object plane.

Since in the structure studied here there is no phase accumulation from one cell to another, function $\tilde{b}$, distinct from coordinate z, varies only within cells, but remains unchanged on translations between the same points of different cells. For this reason, the transverse beam size does not change on the optical path; a Gaussian beam does not experience diffraction in a layered structure with Pendry lens parameters.

A layered structure with the Pendry lens parameters creates a ray image, i.e., translates it through the space. A similar structure with other relationships between geometric and material parameters can be used, for example, to focus a beam; in this case, one will need to determine the position of the beam relative to the lens. A diverging beam will be focused to the size of its nearest narrowing. This focusing will be precise, without distortions characteristic of a conventional lens.

6. Conclusions

We theoretically explored the propagation of a plane wave and Gaussian beam in a periodic structure with Pendry lens parameters, and also considered a particular case when the structure is modeled by a metal plate of sub-wavelength thickness placed in air. The parameters of the components of the media — the dielectric permittivity and magnetic permeability — are -1 and 1, respectively, and the layers have the same thickness of $2d$. The final formulas allow computing the wave intensity and phase at any point of the structure. They explain the character of the image in a Pendry lens.

At positions placed at the half-width of the lens transverse size from the lens (metal layer of the structure) boundaries, the direction, intensity, and phase of the wave coincide. Hence, for waves with a resonance wave vector that leave the plane of an object located at distance d from the lens boundary, an image is created at the symmetric position relative to the lens, which fully repeats the initial wave.

Relying on the original wave theory, the Helmholtz equation for a Gaussian beam in a superlens was explored. The result explains the absence of diffraction in the Pendry lens for a Gaussian beam and allows computations of all parameters for the beam in the lens, its phase, and the distribution of amplitudes in its cross section.

A Gaussian beam in a Pendry lens changes (narrows, widens) its cross section only within one layer; in the next layer, it experiences the opposite change (widening, narrowing). In such a structure, there is no macroscopic accumulation of the along-beam phase and, as a consequence, diffractive widening in this case is absent.

We give the algorithm to compute dispersion and also distributions of electric and magnetic fields of eigenmodes for plane, cylindrical, and spherical multicomponent layered media (10)–(12). Theoretical conclusions give an instrument for exploring multilayer structures with various symmetries, which may serve as a basis for plasmon devices.

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