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'Anomalous' dissipation of a paraxial wave beam propagating along an absorbing plane

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Abstract. **We present solutions of the Leontovich–Fock parabolic equation which describe the structure of a paraxial quasi-optical wave beam in a medium with strong spatial inhomogeneity of the absorption coefficient transverse to the direction of beam propagation. Using as an example the simplest reference problem of a beam propagating along an infinitely narrow plane absorbing layer, it is shown that, in this case, the energy dissipation and refraction of the beam affect each other, so that they should be considered simultaneously, while the application of the perturbation theory can lead to qualitatively incorrect results.**

Keywords: paraxial wave beam, resonant dissipation, diffraction, parabolic wave equation

1. Introduction

The problem which is considered in this note has emerged as the result of generalization of the experience in modeling electron cyclotron resonance heating of high-temperature plasma in large magnetic traps used in studies of controlled thermonuclear fusion. The characteristic dimensions of such devices are about several meters or tens of meters. The plasma is heated by radiation with the wavelength of several millimeters, which enters the setup as rather broad (paraxial) quasi-optical wave beams which freely propagate through the weakly inhomogeneous plasma up to the region of electron–cyclotron resonance (ECR), where they are fully absorbed in a linear regime.

A typical schematic of microwave plasma heating in a tokamak is given in Fig. 1a. The region of radiation absorption is 'spread' along the surfaces of constant absolute value of magnetic field B, its characteristic thickness being determined by the temperature of the electrons and inhomogeneity of B. This thickness is usually small compared to the beam transverse size and the scale of plasma inhomogeneity but large compared with the wavelength of heating radiation. A rigorous description of problems of this kind, based on the direct solution of Maxwell's equations with due regard to the details of geometry and dielectric medium response, as needed for interpreting experiments, calls for unreasonably high computational resources, which is why it is feasible at present only for rather compact setups [1–6]. This has prompted the development and wide use of asymptotic methods, first and foremost, the approximation of geometrical optics (ray tracing) [7–11] and its generalizations for quasi-optical beams which propagate in a plasma while preserving their self-similar form (aberration-free quasi-optics, beam-tracing) [12–14]. Numerical codes that rely on these methods describe and predict the results of most experiments on EC plasma heating so accurately that they are used as standard tools which do not require additional verification [13–23] (see also review [24]). A more elaborate treatment is resorted to only in special problems involving linear [25–28] and nonlinear [29] wave interaction; in all other cases, the deviation from the standard paradigm of geometrical optics is generally perceived by specialists as a small refinement of well-known and

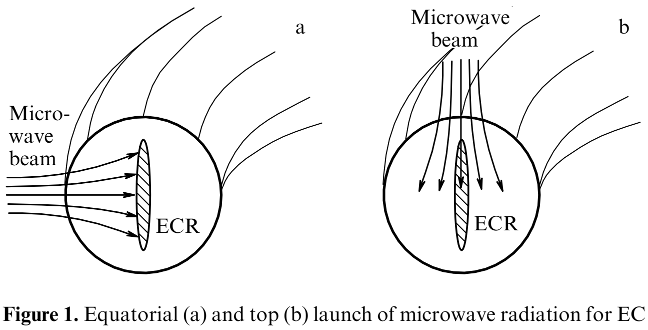

Figure 1. Equatorial (a) and top (b) launch of microwave radiation for EC plasma heating in a tokamak.

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simple physics. Meanwhile, modern experiments are increasingly beginning to use the schemes of EC heating in which geometrical optics and aberration-free quasi-optics are not applicable. The reason is either the formation of caustics in the vicinity of the absorption domain or strong variation in the resonance dielectric medium response [30–32].

An important example of the latter case is given in Fig. 1b, which presents the so-called top launch scheme, in which a microwave beam propagating along the surface of a constant magnetic field is cut by a relatively narrow region of strong absorption into two halves in the longitudinal plane. Such a scheme is realized in both toroidal [33–36] and open [37, 38] magnetic traps. In the framework of standard ray geometrical optics, the beam is identified with a finite set of rays, and the fraction of power absorbed in the beam is determined by the number of rays that get into the absorption region [33]. Clearly, for a *narrow* absorption region, the statement that the ray, i.e., a *broad* beam with a locally flat front, 'hits' this region is rather uncertain. In the approximation of aberration-free quasioptics, absorption is commonly computed for the central ray [13]. In our case, the applicability of this approach is very limited, because absorption which is strongly inhomogeneous in the transverse beam section rapidly destroys the beam self-similar structure. A generalized quasioptical description, which rigorously accounts for inhomogeneous dissipation and diffraction in resonance media with spatial dispersion, is available and rather well advanced [39–43], however there is a substantial set of papers (we refrain from explicitly mentioning them) which use the methods of geometrical optics and aberration-free quasioptics in computations of ECR heating with a top launch.

The primary goal of this study is to present a simple example which transparently demonstrates the limitations of traditional methods as applied to the solution to the problem with a pointed absorption profile in the transverse wave beam section. As a result, a new, as it seems to us, reference problem has appeared with quite unexpected physics, which might be interesting not only to specialists working with microwave plasma heating, but also to a wider circle of readers who in one way or another are interested in wave processes in dissipative linear media.

This paper has the following structure. In Section 2, we formulate the basic model — the Leontovich–Fock parabolic equation [44] describing diffraction of a two-dimensional wave beam in an isotropic medium with an absorbing (infinitely thin) plane. Sections 3 and 4, which are largely of a technical character, present the derivation of the general solution to the initial value problem and the asymptotic form of this solution for small and large propagation distances. In Section 5, which is the main one, the method from the previous sections is applied to explore the dissipation of a Gaussian beam, and the exact solution is compared to the results of the approximate methods mentioned above. Section 6 briefly discusses the generalization of our results to the case of three-dimensional beams with the transverse cross-section bounded in two directions and to media with anisotropic and gyrotropic dielectric responses. The conclusions (Section 7) summarize the results.

2. Parabolic equation with an absorbing plane

Let us consider a stationary wave equation for the field $u \propto \exp(-\mathrm{i}\omega t)$ in an isotropic medium with complex dielectric permittivity ε,

$$\Delta u + k_0^2 \varepsilon u = 0\,, \qquad k_0 = \frac{\omega}{c}\,. \tag{1}$$

To make the treatment as simple as possible, we assume that $\operatorname{Re}\varepsilon$ does not depend on the coordinates and that all the absorption is concentrated in a plain layer with zero thickness,

$$\operatorname{Re}\varepsilon = \text{const}\,, \qquad k_0^2 \operatorname{Im}\varepsilon = 2\nu\,\delta(y)\,. \tag{2}$$

Absorption is associated with $\nu > 0$. This assumption models the situation described in the Introduction when the resonance absorption takes place in a layer which is narrow in comparison to the beam transverse aperture. From a practical standpoint, such a problem statement is not fully consistent, since, by virtue of the Kramers–Kronig relations, the resonance absorption $\operatorname{Im}\varepsilon(\omega)$ inevitably leads to a perturbation in the Hermitian component $k_0^2 \operatorname{Re}\varepsilon(\omega) \sim \nu/y$ localized in the same domain. This effect does not modify the qualitative conclusions made here, and we therefore stay with the simplest model (2).

The solution is sought as a product of an envelope and carrying harmonic along the x coordinate,

$$u = U(x,y)\exp(\mathrm{i}kx)\,, \qquad k = k_0\sqrt{\operatorname{Re}\varepsilon}\,.$$

The paraxial approximation is obtained if, after inserting this presentation into (1), we omit the term $\partial^2 U/\partial x^2$, assuming that the dependence $U(x)$ is smooth on the wavelength scale $2\pi/k$ [44–48]. As a result, taking into account (2), we obtain

$$2\mathrm{i}k\,\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial y^2} + 2\mathrm{i}\nu\,\delta(y)U = 0\,. \tag{3}$$

We are interested in problem solutions for $x > 0$ which correspond to an initial distribution $U_0(y)$ given at $x = 0$ with a finite intensity

$$U(0,y) = U_0(y)\,, \quad I_0 = \int_{-\infty}^{+\infty} \left|U_0(y)\right|^2 \mathrm{d}y < \infty\,. \tag{4}$$

We will also assume that the intensity remains finite for all $x > 0$, i.e., that $\int_{-\infty}^{+\infty} |U(x,y)|^2\,\mathrm{d}y < \infty$ and $U(x,\pm\infty) \to 0$.

For $y \neq 0$, equation (3) coincides with the standard parabolic equation in a homogeneous medium; the presence of the absorbing layer is only seen in the kink (the jump of the derivative) in the function U. Indeed, integrating (3) over an arbitrarily small vicinity of $y = 0$ and assuming that U is continuous, we find an additional condition:

$$\frac{\partial U(x,+0)}{\partial y} - \frac{\partial U(x,-0)}{\partial y} = -2\mathrm{i}\nu U(x,0)\,. \tag{5}$$

Using condition (5), we can derive the energy transfer equation in the direction x of beam propagation. Considering equation (3) for $y < 0$ and $y > 0$, we find

$$2\mathrm{i}k\left(U^*\,\frac{\partial U}{\partial x} + U\,\frac{\partial U^*}{\partial x}\right) = U\,\frac{\partial^2 U^*}{\partial y^2} - U^*\,\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}\left(U\,\frac{\partial U^*}{\partial y} - U^*\,\frac{\partial U}{\partial y}\right).$$

Integrating this expression over all y except for point $y = 0$ and assuming that the integrals on the left-hand side are uniformly converging, we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}x} = -\frac{2\nu}{k}\left|U(x,0)\right|^2,$$
$$I(x) = \int_{-\infty}^{-0}\left|U(x,y)\right|^2 \mathrm{d}y + \int_{+0}^{+\infty}\left|U(x,y)\right|^2 \mathrm{d}y\,, \tag{6}$$

where $I(x)$ is the beam intensity. Granted, this same expression can be readily derived directly from (3) by integrating the relationship

$$2ik\left(U^*\frac{\partial U}{\partial x} + U\frac{\partial U^*}{\partial x}\right) = \frac{\partial}{\partial y}\left(U\frac{\partial U^*}{\partial y} - U^*\frac{\partial U}{\partial y}\right) - 2i\nu(U^*U + UU^*)\,\delta(y)$$

over $y \in (-\infty, +\infty)$ and considering $I(x) = \int_{-\infty}^{\infty}|U(x,y)|^2\,\mathrm{d}y$. The agreement between the results of the two approaches confirms the correctness of using boundary condition (5) for matching two 'nondissipative' problems for $y < 0$ and $y > 0$.

The energy transfer equation warrants the uniqueness of the problem solution for $\nu > 0$. Consider two solutions U_1 and U_2 that correspond to the same initial conditions (4). By virtue of (6),

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}|U_1 - U_2|^2\,\mathrm{d}y \leqslant 0 \Rightarrow \int_{-\infty}^{\infty}|U_1 - U_2|^2\,\mathrm{d}y = 0 \Rightarrow U_1 \equiv U_2\,.$$

Similarly one can conclude that, if condition (6) is valid, the intensity $I(x)$ stays finite automatically for all $x > 0$.

3. Green's function for the parabolic equation

From a mathematical standpoint, the formulated problem is ill defined, because boundary condition (5) contradicts initial condition (4) in a general case. Solving such problems using expansions in orthogonal functions commonly results in diverging series or in integrals that require special regularization methods [49]. One can avoid these additional difficulties by using an integral representation for the solution to the initial value problem based on Green's functions. In such an approach, a formal order of operations is exchanged — the summation is performed first and then the expansion of the initial distribution in orthogonal functions. This leads to a substantial simplification of the solution to our problem.

The standard solution of a parabolic equation without dissipation will now be recalled (see, e.g., [47, § 24, p. 257] or [48, Ch. 17, p. 431]). We introduce

$$U(x,y) = \int_{-\infty}^{\infty} U(x,\kappa)\exp(i\kappa y)\,\mathrm{d}\kappa\,,$$
$$U_0(\kappa) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U_0(y')\exp(-i\kappa y')\,\mathrm{d}y'\,. \tag{7}$$

From equation (3), it follows that, for $\nu = 0$,

$$2ik\,\frac{\partial U(x,\kappa)}{\partial x} - \kappa^2 U(x,\kappa) = 0 \Rightarrow U(x,\kappa) = U_0(\kappa)\exp\left(\frac{\kappa^2 x}{2ik}\right).$$

Inserting this solution into (7) and performing integration over κ, we obtain the general solution

$$U(x,y) = \int_{-\infty}^{\infty} U_0(y')\,G(x, y - y')\,\mathrm{d}y'\,, \tag{8}$$

where $G(x,y)$ is a Green's function,

$$G(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(i\kappa y + \frac{\kappa^2 x}{2ik}\right)\mathrm{d}\kappa = \sqrt{\frac{k}{2\pi i x}}\exp\left(\frac{iky^2}{2x}\right). \tag{9}$$

It can be readily seen that

$$\lim_{x\to 0} G(x,y) = \delta(y)\,,$$
$$\int_{-\infty}^{\infty} G(x, y - y')\,G^*(x, y - y'')\,\mathrm{d}y = \delta(y' - y'')\,.$$

The first property ensures that the initial condition is observed for $x = 0$, and the second one ensures that the beam intensity is preserved, $I(x) \equiv I_0$. The main idea of the analysis below is that, in the presence of dissipation, the solution can be composed of two nondissipative solutions of the form (8) matched at the boundary $y = 0$.

We begin with the simplest case when the initial distribution of the beam envelope is symmetric in y: $U_0(y) = U_0(-y)$. The solution $U(x,y)$ is then also even in y and can be sought only in the region $y > 0$. Boundary condition (5) in this case is transformed into

$$\frac{\partial U(x,0)}{\partial y} = -i\nu U(x,0)\,. \tag{10}$$

The general solution in the half-space $y > 0$ is written by analogy with (8) as

$$U(x, y > 0) = \int_{-\infty}^{\infty} V(y')\,G(x, y - y')\,\mathrm{d}y'\,, \tag{11}$$

where $V(y')$ is some function unknown thus far. For $y' > 0$, the function $V(y')$ must coincide with U_0 to ensure that the initial condition is observed; for $y' < 0$, we have the freedom to define this function so that the boundary condition is valid for $y = 0$ and all x. It can readily be seen that

$$\frac{\partial U}{\partial y} = \int_{-\infty}^{\infty} V(y')\frac{\partial}{\partial y}G(x, y - y')\,\mathrm{d}y' = \int_{-\infty}^{\infty}\frac{\partial V(y')}{\partial y'}\,G(x, y - y')\,\mathrm{d}y'\,,$$

so that boundary condition (10) takes the form

$$\int_{-\infty}^{\infty}\left\{\frac{\partial V(y')}{\partial y'} + i\nu V(y')\right\}G(x, y')\,\mathrm{d}y' \equiv 0\,. \tag{12}$$

Further, we do the following: we guess the function $V(y')$ which ensures the validity of (12) and use the uniqueness of the initial problem solution. Since G is even in y', for condition (12) to be valid for all x, it suffices that the function in curly brackets is odd. This condition uniquely defines

$V(y')$:

$$V(y') = \begin{cases} U_0(y')\,, & y' \geqslant 0\,, \\ W(-y')\,, & y' < 0\,, \end{cases}$$

where the function of positive argument $W(y)$ is defined by the differential equation

$$W' - \mathrm{i}\nu W = U_0' + \mathrm{i}\nu U_0\,.$$

The initial condition for W follows from the continuity of V at zero; a jump of this function would imply a delta-type singularity in the derivative of $V(y')$, which does not satisfy the requirement that the function be odd. Using $W(0) = U_0(0)$, we express $W(y)$ in an explicit form:

$$W(y) = U_0 + 2\mathrm{i}\nu \int_0^y U_0(y') \exp\big(\mathrm{i}\nu(y-y')\big)\,\mathrm{d}y'.$$

As a result, the general solution (11) to the problem with a symmetric initial distribution can be written as the sum of formal solution (8) and the dissipative correction,

$$U(x, y>0) = \int_{-\infty}^{\infty} U_0(y') G(x, y-y')\,\mathrm{d}y' + \int_0^{\infty} \big(W(y') - U_0(y')\big) G(x, y+y')\,\mathrm{d}y'\,.$$

Exchanging the order of integration, the second term in the last expression can be written in the form

$$\Delta U = 2\mathrm{i}\nu \int_0^{\infty}\int_0^{y'} U_0(y'') \exp\big(\mathrm{i}\nu(y'-y'')\big) \times G(x, y+y')\,\mathrm{d}y''\,\mathrm{d}y' = 2\mathrm{i}\nu \int_0^{\infty} U_0(y') \int_{y'}^{\infty} \exp\big(-\mathrm{i}\nu(y'-y'')\big) \times G(x, y+y'')\,\mathrm{d}y''\,\mathrm{d}y'\,.$$

The inner integral over y'' can be expressed through the complementary error function $\operatorname{erfc} z$,

$$\Delta G = 2\mathrm{i}\nu \int_{y'}^{\infty} \sqrt{\frac{k}{2\pi \mathrm{i} x}} \exp\left(\mathrm{i}\nu(y''-y') + \frac{\mathrm{i}k}{2x}(y+y'')^2\right) \mathrm{d}y'' = \mathrm{i}\nu \exp\left(-\mathrm{i}\nu(x+y) - \frac{\mathrm{i}\nu^2 x}{2k}\right) \operatorname{erfc}\left(\frac{\nu x + ky + ky'}{\sqrt{2\mathrm{i}kx}}\right). \quad (13)$$

Let us recall that our solution is valid for $y > 0$. In order to obtain a symmetric solution in the negative half-plane, it suffices to make a replacement $y \to |y|$ in the expression for ΔU. As a result, for a symmetric beam, we finally get

$$U(x, y) = \int_{-\infty}^{\infty} U_0(y') G(x, y-y')\,\mathrm{d}y' + \int_0^{\infty} U_0(y') \Delta G\big(x, |y|, y'\big)\,\mathrm{d}y'\,. \quad (14)$$

The first term automatically preserves the even dependence on y, and the modulus sign can be suppressed.

A solution for an arbitrary initial distribution can be constructed in an analogous way. The easiest way of doing this lies in splitting the initial distribution in its symmetric and antisymmetric parts,

$$U_0(y) = U_0^+(y) + U_0^-(y)\,,$$

$$U_0^{\pm}(y) = \frac{U_0(y) \pm U_0(-y)}{2}\,,$$

and looking for respective solutions $U^{\pm}(x, y)$ of equation (3) individually. Obviously, these solutions will preserve the symmetry of the initial distribution. In the case of an initial distribution that is continuous at zero, the dissipation is absent for the odd solution, $U_0^-(0) = U^-(x, 0) = 0$, so that the nondissipative formula (8) can be applied for $U^-(x, y)$. For the even solution $U^+(x, y)$, we resort to the just obtained solution (14). As a result, we obtain the following most general result for the full field $U = U^+ + U^-$:

$$U(x, y) = \int_{-\infty}^{\infty} \big(U_0^+(y') + U_0^-(y')\big) G(x, y-y')\,\mathrm{d}y' + \int_0^{\infty} U_0^+(y') \Delta G\big(x, |y|, y'\big)\,\mathrm{d}y' = \int_{-\infty}^{\infty} U_0(y') G(x, y-y')\,\mathrm{d}y' + \frac{1}{2}\int_0^{\infty} \big(U_0(y') + U_0(-y')\big) \Delta G\big(x, |y|, y'\big)\,\mathrm{d}y'\,. \quad (15)$$

Note that the second (dissipative) term is always even in the transverse coordinate, while the first one reflects the symmetry of the initial distribution and is therefore not necessarily even.

4. Asymptotic representations

For $x \to 0$, the first term in (14) and (15) tends to the initial distribution U_0. We check that the additional dissipative term related to the contribution of ΔG is indeed vanishing. For $x \to 0$, the expression for ΔG (13) contains a strongly oscillating multiplier $\exp(\mathrm{i}k(y+y'')/2x)$, which makes the main contribution at the minimum of $|y + y''|$, i.e., at the lower limit $y'' = y'$. This allows us to assume $\mathrm{i}\nu(y''-y') \to 0$; hence,

$$\Delta G \approx \mathrm{i}\nu \operatorname{erfc}\left[\frac{k(y+y')}{\sqrt{2\mathrm{i}xk}}\right].$$

When computing ΔU, we change the variable as $\xi = y\sqrt{k/2\mathrm{i}x}$,

$$\Delta U(x, y) = \int_0^{\infty} U_0^+(y') \Delta G\,\mathrm{d}y' \approx \mathrm{i}\nu\sqrt{\frac{2\mathrm{i}x}{k}} \int_0^{\infty} U_0^+\left(\xi'\sqrt{\frac{2\mathrm{i}x}{k}}\right) \operatorname{erfc}(\xi+\xi')\,\mathrm{d}\xi' \approx \mathrm{i}\nu\sqrt{\frac{2\mathrm{i}x}{k}}\, U_0(0)\Phi(\xi)\,, \quad (16)$$

where $\Phi(\xi) = \int_0^{\infty} \operatorname{erfc}(\xi+\xi')\,\mathrm{d}\xi'$, it being the function which takes the maximum value $\Phi(0) = 1/\sqrt{\pi}$ in the absorption region and decays as $\Phi(\xi) \approx \exp(-\xi^2)/(2\xi^2\sqrt{\pi}) \sim x/(ky^2)$ at the periphery. We thus found that, for small x, the correction ΔU has a narrow distribution over y with a width of the order $\sqrt{x/k}$ and an amplitude that decays as $\sqrt{x}$

on approaching the initial section. The nonanalytical decay law for the amplitude occurred because of inconsistency between the initial and boundary conditions. One can verify that, if U_0 satisfies both (4) and (5) from the very beginning, the dependence $\sqrt{x}$ in the general solution for U disappears, because the contribution from ΔU is exactly compensated by the contribution from the jump of the derivative of U_0 in the nondissipative term $\int U_0 G\,\mathrm{d}y'$.

We turn now to the case of large x. To compute the asymptotic form of Green's function (13) for $x \to \infty$, we note that the argument of the complementary error function is unboundedly increasing so that we can use the asymptotic expansion [50]

$$\operatorname{erfc} z = \frac{1}{\sqrt{\pi}} \exp(-z^2) \left\{ \frac{1}{z} - \frac{1}{2z^3} + \ldots \right\}, \qquad |\arg z| < \frac{3\pi}{4}, \quad |z| \to \infty. \tag{17}$$

Here, one should keep in mind that, in our case, $z^2 = -\mathrm{i}(vx + ky + ky')^2/(2kx)$ is a purely imaginary number which defines the phase of the exponent, and small corrections to it should be compared to π and not with each other. This is why we will expand only the expression in curly brackets in series in the small parameter, leaving the exponent as it is. Expanding the expression in curly brackets in (17) in powers of small parameter $1/(vx)$ and inserting the result into (13), we find

$$\Delta G = 2\sqrt{\frac{k}{2\pi\mathrm{i}x}} \exp\left(\frac{\mathrm{i}k(y+y')^2}{2x}\right) \left\{ -1 + \frac{k(y+y'+\mathrm{i}/v)}{vx} - \frac{k^2(y+y')(y+y'+3\mathrm{i}/v)}{v^2x^2} + \ldots \right\}.$$

Note that the multiplier featuring in front of the curly brackets is exactly nondissipative Green's function $G(x, y+y')$. For this reason, on insertion into formulas (14) and (15), the first term of the expansion partly compensates the contribution coming from G. In particular, *for symmetric initial distributions* U_0^+, one can readily find that

$$U^+(x,y) = \int_0^\infty U_0^+(y') \big(G(x, |y| - y') - G(x, |y| + y') \big)\,\mathrm{d}y' + 2\int_{|y|}^\infty U_0^+ \big(y' - |y|\big) \left\{ \frac{ky'}{vx} + \frac{\mathrm{i}k}{v^2x} - \frac{k^2y'^2}{v^2x^2} + \ldots \right\} \times G(x, y')\,\mathrm{d}y'. \tag{18}$$

The applicability of this approximation requires that $|z| \ll 1$ in (17), or

$$vx \gg ky', \quad vx \gg \frac{k}{v}, \quad vx \gg ky. \tag{19}$$

The first two conditions bound from below the length of the propagation path. Here, we take into account that the spread $y' \sim a$ is defined by the width a of the initial distribution. On large paths, the beam broadens linearly because of diffraction, so that the characteristic spread $y \sim y'x/(ka^2) \sim x/(ka)$, and the third condition in (19) is reduced to the additional requirement of strong dissipation $va \gg 1$. Furthermore, the third condition is inevitably violated if integration is performed over the whole range of the transverse coordinate y leading, in particular, to diverging values for the full beam intensity I.

However, the constraining third condition in (19) is fulfilled automatically if the field is computed exactly at the axis $y = 0$, i.e., directly in the dissipation domain, which is in a sense a lucky circumstance, since namely this field defines the absorption. For $y = 0$, the first term in (18) becomes zero, i.e., the first term in the expansion for ΔG exactly compensates the nondissipative contribution from G:

$$U(x,0) = 2\int_0^\infty U_0^+(y') \left\{ \frac{ky'}{vx} + \frac{\mathrm{i}k}{v^2x} - \frac{k^2y'^2}{v^2x^2} + \ldots \right\} G(x,y')\,\mathrm{d}y'. \tag{20}$$

Formula (20) is applicable for $x \gg ka/v$, k/v^2 for any $v > 0$. The first two terms on the right-hand side of (20) give the contribution decaying asymptotically as $1/x^{3/2}$, and the second term in curly brackets contains an additional multiplier of the order $1/(va)$. For this reason, the second term is dominant for narrow beams for $va \ll 1$; the third term does not influence the asymptotic form for $x \to 0$, but simplifies the final result in a particular case of a narrow Gaussian beam. For wide beams, on the contrary, the first term is dominant.

Note that expressions (18) and (19) can be simplified by using the equality $\partial G/\partial y' = \mathrm{i}ky'G/x$, but we will not need this equality in the present work.

5. Absorption of a Gaussian beam

With the technique developed in the preceding sections, we are ready to return to the main topic of the present study and to illustrate rather surprising physical details of the problem.

Consider the initial distribution in the form of a Gaussian beam with the net intensity I_0 and width a,

$$U_0(y) = A \exp\left(-\frac{y^2}{2a^2}\right), \qquad A^2 = \frac{I_0}{a\sqrt{\pi}}. \tag{21}$$

Before presenting our solution, we pause to briefly describe the commonly used models of absorption that were mentioned in the Introduction.

In the framework of standard ray geometrical optics, a beam with a flat phase front is identified with a symmetric set of rays which contain only one (in the plane case) central ray $y = 0$ undergoing absorption (infinitely strong). Since the dissipation takes place in a set of measure zero, the absorption of the beam as a whole on a layer of zero thickness is formally absent. The situation is corrected in the framework of the aberration-free quasi-optical approximation. In essence, this phenomenological approach is a variant of the perturbation theory in small parameter vka^2. A rigorous solution (8) of the parabolic equation without dissipation has a self-similar form of a Gaussian beam with complex parameters which vary because of diffraction:

$$U(x,y) = \frac{A}{\sqrt{1+\mathrm{i}\tau}} \exp\left(-\frac{y^2}{2a^2(1+\mathrm{i}\tau)}\right), \qquad \tau = \frac{x}{ka^2}. \tag{22}$$

Here and below, the longitudinal coordinate x will be expressed in units of ka^2, which are natural for problems of diffraction, and all other parameters will be left for now dimensional. For this solution, the full intensity is preserved along the propagation path, $I = I_0$, so, without any loss of generality, it can be assumed that $A^2 = I/a\sqrt{\pi}$. It is further assumed that, in the presence of dissipation, the solution preserves its self-similar form (22), i.e., that the dissipation leads to an additional dependence of the parameters $A(\tau)$ and

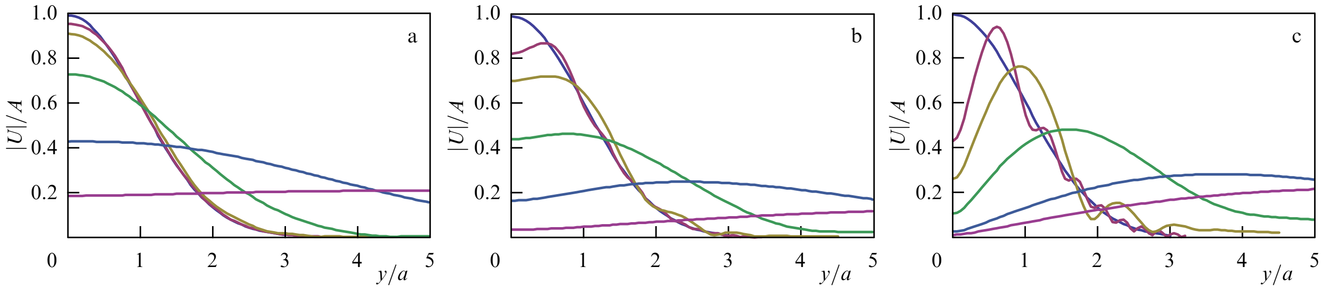

Figure 2. (Color online.) Diffraction of initial Gaussian beam with a flat phase front in the presence of an absorbing plane at $y = 0$ for (a) $\nu a = 0.2$, (b) $\nu a = 1$, and (c) $\nu a = 5$. Plotted profiles of $|U(y)|$ correspond to the consecutive values of marching coordinate $\tau = x/(ka^2) = 0.14$, 0.3, 1.0, 3.0, 10.

$a(\tau)$ on the longitudinal coordinate. In the first nonvanishing order, $a(\tau) = \text{const}$, and $A(\tau)$ is determined from energy transfer equation (6). Namely this approach is implemented in the quasioptical code TORBEAM which is widely used in computations of EC heating scenarios in tokamaks and stellarators [12–14]. In our problem, this would lead to the transfer equation

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = -\frac{2\nu}{k}\, ka^2 \left|\frac{A}{\sqrt{1+\mathrm{i}\tau}}\right|^2 = -\frac{2\nu a}{\sqrt{\pi}}\,\frac{I}{\sqrt{1+\tau^2}}\,,$$

with the solution

$$I = I_0 \exp\left(-\frac{2\nu a}{\sqrt{\pi}}\,\sinh^{-1}\tau\right). \tag{23}$$

The beam intensity decays exponentially at small distances; in a certain sense, this path interval is described by geometrical optics augmented by the equation of radiation transfer along the ray $\mathrm{d}I/\mathrm{d}x = -2k_0\,\mathrm{Im}\,\sqrt{\varepsilon}\,I$. However, at distances $x \gtrsim ka^2$ comparable to the diffraction length, the decay slows down and now follows a power law, reflecting in this way the effect of diffraction: beam spreading reduces the field on the axis and the absorption becomes weaker. And yet diffraction cannot stop absorption in the aberration-free approximation — on an infinite path all beam energy is transferred to the medium, $I \to 0$ for $x \to \infty$.

The exact solution of the parabolic equation with a dissipative layer is given by formulas (9), (13), and (14) and is expressed, for a Gaussian incident beam (21), through rather unwieldy integrals from the error function, which we will not present here. Figure 2 demonstrates an example of evolving field $U(\tau, \xi)$ for three values of the parameter that governs dissipation, $\nu a \ll 1$, $\nu a \sim 1$, and $\nu a \gg 1$. We note that nonmonotonic profiles of $|U|$ form for any value of dissipation: the larger the dissipation, the earlier this nonmonotonicity takes place in terms of τ.

To determine absorption, we only need to have information on the field on the central beam axis, which can be written in the form

$$U(\tau,0) = \frac{A}{\sqrt{1+\mathrm{i}\tau}}\left\{1 + \mathrm{i}\nu a \int_0^\infty \exp\left(\mathrm{i}\nu a\xi - \frac{\xi^2}{2(1+\mathrm{i}\tau)}\right) \times \mathrm{erfc}\left(\frac{\xi}{\sqrt{2\mathrm{i}\tau(1+\mathrm{i}\tau)}}\right)\mathrm{d}\xi\right\}. \tag{24}$$

We made a transform to the dimensionless transverse coordinate in the integrand and, as the result, the normalized field U/A at the axis is defined by a single dimensionless parameter νa. This same parameter stays in the energy transfer equation, which is written in the form

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = -\frac{2\nu}{k}\, ka^2 \left|U(\tau,0)\right|^2 = -\frac{2\nu a}{\sqrt{\pi}}\,\frac{I_0}{\sqrt{1+\tau^2}}\left|\{\ldots\}\right|^2, \tag{25}$$

where the ellipsis means the expression in curly brackets in (24). Figure 3 displays a numerical solution of equation (25) for different parameters νa. For comparison, it also plots the aberration-free solution (23) and asymptotic approximations, which are considered below.

It can be seen that the aberration-free approximation always overpredicts the absorption, and the error increases with increasing parameter νa. The reason for this overprediction is qualitatively clear. In the aberration-free approximation, the profile is fixed, and energy is collected 'instantaneously' from the entire beam section; for a given full beam intensity, this is the most efficient process, because the field maximum is located exactly in the absorption domain. In the rigorous solution, dissipation 'eats out' the field in the beam center, forming an additional energy flux to the center from the beam periphery. This flux (diffraction) is defined by the diffusion term $\partial_{yy}U$ in the parabolic equation, and hence a finite 'time' $\tau \sim ky^2$ is required for the propagation of perturbation from the beam periphery, which leads to the formation of a nonmonotonic profile. The field maximum drifts further and further from the absorption domain, and, as a result, the absorption decays so fast that on a sufficiently long path the beam intensity ceases to change further. The difference between this limit intensity and the initial beam intensity defines the absorbed power, which depends only on the dissipation parameter νa,

$$\Delta I \equiv I_0 - \lim_{\tau\to\infty} I(\tau,\nu a) = -\int_0^\infty \frac{\mathrm{d}I}{\mathrm{d}\tau}\,\mathrm{d}\tau = \frac{2\nu a I_0}{\sqrt{\pi}}\int_0^\infty \left|\{\ldots\}\right|^2 \frac{\mathrm{d}\tau}{\sqrt{1+\tau^2}}\,.$$

The universal function $\Delta I(\nu a)/I_0$ is plotted in Fig. 4. For small νa, the function $\Delta I(\nu a)/I_0$ grows linearly, and for large νa it decays by the law detailed below. The total absorbed power reaches maximum value $\Delta I \approx 0.72 I_0$ for the optimal value of dissipation parameter $\nu a \approx 0.94$.

Let us develop a quantitative theory that describes the effects we discovered. First, consider the transfer equation on short paths, $\tau \ll 1$. The main contribution to the field at the axis is given by formula (16); for the Gaussian beam, we get

$$U(\tau,0) = \left(1 + \mathrm{i}\nu a\sqrt{\frac{2\mathrm{i}\tau}{\pi}} + \ldots\right)A\,, \tag{26}$$

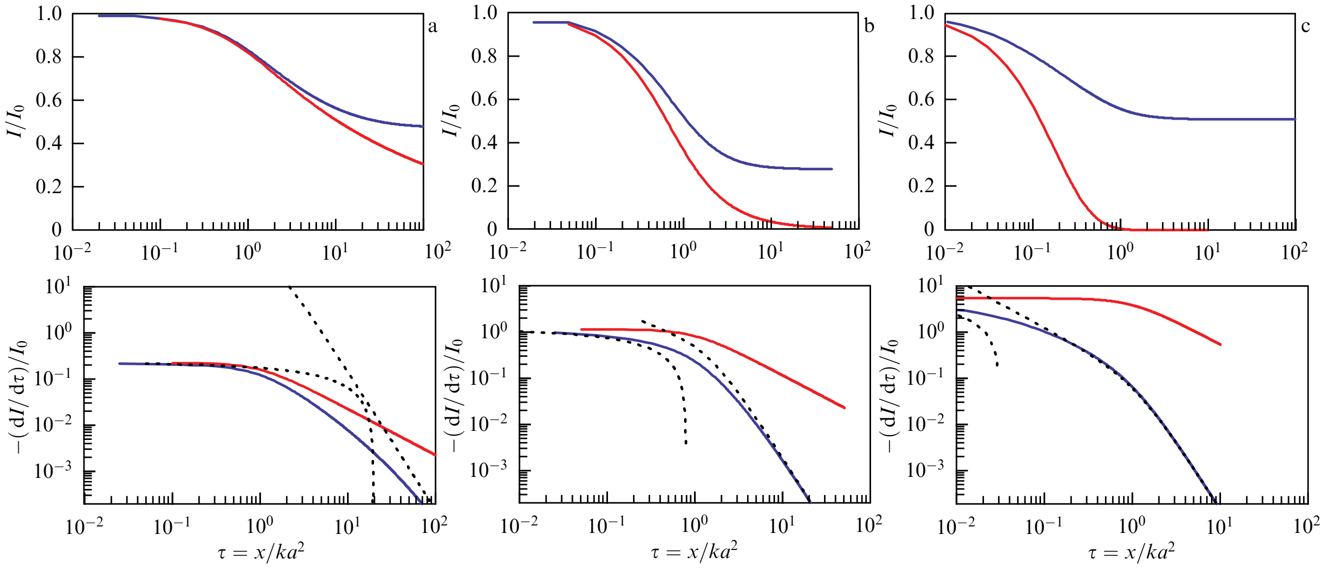

Figure 3. (Color online.) Full intensity $I(x)$ and its derivative $-I'(x)$ along the beam propagation path for the Gaussian initial distribution for (a) $va = 0.2$, (b) $va = 1$, and (c) $va = 5$. Solid blue lines show the numerical solution of transfer equation (25) with rigorous field (24) in the framework of the parabolic approximation, red lines plot solution (23) in the aberration-free approximation, dashed lines show dependences that correspond to asymptotic field representation (26) and (27) for small and large τ, respectively.

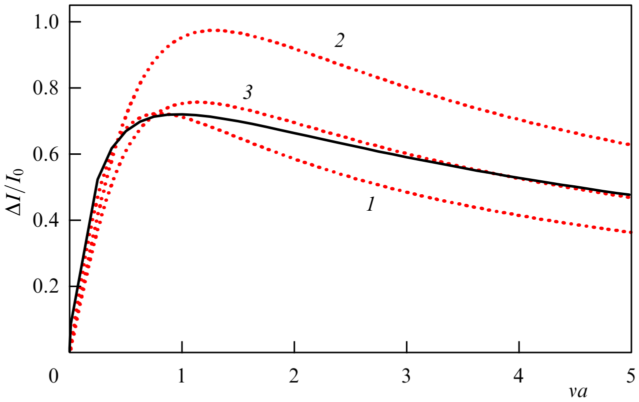

Figure 4. Universal dependence of full power $\Delta I(va)/I_0$ absorbed by an infinitely thin dissipative layer in the case of longitudinal incidence of a Gaussian beam with a flat phase front. Solid line is the exact solution that follows from transfer equation (25) for field (24). Dotted lines show approximate estimates: *1* — $\Delta I_N/I_0$ for $N = 0.4$ by formula (28), *2* — $\Delta I/I_0$ by formula (29), *3* — $0.75\Delta I/I_0$ by formula (29).

and hence

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = -\frac{2vaI_0}{\sqrt{\pi}}\left(1 - \frac{2va}{\sqrt{\pi}}\sqrt{\tau} + \ldots\right).$$

Comparing the last expression with (23), we see that the aberration-free approximation and exact solution coincide for $2va\tau/\sqrt{\pi} \ll 1$ and $2va\sqrt{\tau/\pi} \ll 1$, or, in dimensional variables,

$$x \ll \frac{ka}{v}, \qquad x \ll \frac{k}{v^2}.$$

The second condition, which follows from the exact solution, is commonly more stringent, and namely this one is frequently forgotten when the applicability of the aberration-free method is substantiated. In other words, even though the aberration-free approximation and exact solution give the same dissipation rate for $\tau \to 0$, these solutions diverge fast, since, for finite τ, the change in the dissipation rate in the exact solution takes place with an infinite derivative.

Consider the energy transfer equation for large τ. For a Gaussian beam, formula (20) leads to the result

$$U(\tau, 0) \approx \frac{A}{va}\sqrt{\frac{2\mathrm{i}}{\pi\tau}}\,\frac{1}{1+\mathrm{i}\tau} - \frac{A}{(va)^2(1+\mathrm{i}\tau)^{3/2}}\,. \qquad (27)$$

Formula (27) is valid for $\tau \gg 1/(va), 1/(va)^2$, the first term is dominant for $va \gg 1$ (for strong dissipation and wide beams), and the second one is dominant in the opposite case, for $va \ll 1$. Inserting (27) into the energy transfer equation (25), we find $\mathrm{d}I/\mathrm{d}\tau \propto 1/\tau^3$ for large paths. The integral of absorbed power $\Delta I = -\int(\mathrm{d}I/\mathrm{d}\tau)\,\mathrm{d}\tau$ is converging for $\tau \to \infty$, so that, in the framework of the exact solution of the parabolic equation, the entire supplied power, in contrast to that in the aberration-free approximation, *cannot be* absorbed on the infinite path. However, computation of the limiting absorbed power encounters a complication due to which, for $\tau \to 0$, formula (26) leads to integral ΔI, which is logarithmically diverging at the lower limit. Formally, this is not a contradiction, because approximation (27) is not applicable for small τ.

Advancing further in our computations of ΔI, we consider the case of wide beams, $va \gg 1$, i.e., we leave only the first term in (27). The respective energy transfer equation has the form

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} \approx -\frac{4I_0}{\pi\sqrt{\pi}\,va}\,\frac{1}{\tau(1+\tau^2)}\,.$$

To estimate the absorbed power from below, we integrate this equation over the region $N/va < \tau < \infty$, where N is a sufficiently large number selected so that our asymptotic field presentation remains valid in the integration domain. We find

$$\Delta I \gtrsim \Delta I_N \equiv \frac{4I_0}{\pi\sqrt{\pi}\,va}\int_{N/va}^{\infty}\frac{\mathrm{d}\tau}{\tau(1+\tau^2)} = \frac{2I_0}{\pi\sqrt{\pi}}\,\frac{\ln\left(1+(va/N)^2\right)}{va}\,. \qquad (28)$$

A comparison with the results of numerical computations (the solid line in Fig. 4) shows that inequality (28) is already valid for $N > 0.5$. Thus, we have found that, for large arguments, the absorbed power decays as $\Delta I \sim \ln(\nu a)/(\nu a)$. As is commonly the case, the logarithm of the large number in the result implies that the main contribution to the integral comes from the region of large τ. For this reason, to find the converging integral for the quantity ΔI, we can to a degree arbitrarily continue the integrand, and the final result will be the same with logarithmic accuracy. As the simplest solution, one can fit the value of N such that this quantity is maximally close to ΔI computed above; optimum agreement is achieved for $N = 0.35-0.4$. Figure 4 gives an example of such a fit; it can be seen that $\Delta I \approx \Delta I_{0.4}$. A surprising fact is that theoretical estimate (28), despite being derived for $\nu a \gg 1$, gives a qualitatively correct result also for $\nu a \ll 1$ and in the intermediate domain. In particular, in the region of small parameters, we obtain a linear dependence $\Delta I_N \approx 2I_0 \nu a/(\pi^{3/2}N^2)$; the maximum value of absorbed power $\Delta I_N \approx 0.3/N$ is achieved for $\nu a \approx 2N$.

As an alternative, we consider the second, more physical, way of computing the converging integral for absorbed power. We continue field (27) such that $U(\tau, 0) \to A$ for $\tau \to 0$. To achieve this, it is sufficient to take $(\pi\tau + 2\mathrm{i}/(\nu a)^2)^{1/2}$ instead of $\sqrt{\pi\tau}$ in (27), which will lead to the energy transfer equation

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} \approx -\frac{2\nu a I_0}{\sqrt{\pi}} \frac{1}{\sqrt{1 + (\pi\nu^2 a^2 \tau/2)^2}} \frac{1}{1+\tau^2},$$

which leads to the absorbed power

$$\Delta I = -\int_0^\infty \frac{\mathrm{d}I}{\mathrm{d}\tau}\,\mathrm{d}\tau \approx \frac{2\nu a I_0}{\sqrt{\pi}} \frac{\cosh^{-1}(\pi\nu^2 a^2/2)}{\sqrt{(\pi\nu^2a^2/2)^2 - 1}}. \qquad (29)$$

Note that the last expression is real for all real values of νa; if $\pi\nu^2 a^2/2 < 1$, there are purely imaginary values in both the numerator and denominator, but their ratio gives a real number. For large and small values of the argument in expression (29),

$$\frac{\Delta I}{I_0} \approx \begin{cases} \sqrt{\pi}\,\nu a, & \nu a \ll 1, \\ \dfrac{8}{\pi\sqrt{\pi}} \dfrac{\ln(\nu a)}{\nu a}, & \nu a \gg 1. \end{cases}$$

Figure 4 compares the theoretical dependence given by (29) with the numerical results. Just like the preceding approximate formula, the estimate in (29) gives a result that is qualitatively valid in the entire range of νa. The value $\nu a \approx 1.3$ for which the absorbed power reaches a maximum is close to the numerical result, but the value proper $0.97 I_0$ is substantially higher. For this reason, for practical estimates, one can use formula (29) multiplied by a correcting factor of 3/4. Taking into account this correction, the asymptotic form of $\Delta I(\nu a)$ for large arguments can be taken in the form $\Delta I/I_0 \approx \ln(\nu a)/(\nu a)$, and the numerical factor with the logarithm is very close to one (it is 1.06).

Let us mention one more way to estimate $\Delta I(\nu a)$ for small values of the argument, which leads to a linear dependence. One can take the second term in (26), which dominates for $\nu a \ll 1$, and integrate the related energy transfer equation over the domain where the asymptotic field expansion is valid, in this case, $N/(\nu a)^2 < \tau < \infty$. However, the result will not be applicable for large νa and is therefore not so rewarding as the universal approximations $\Delta I(\nu a)$ given above.

6. Some generalizations

We have been exploring the two-dimensional problem, excluding the third coordinate z from consideration. However, the results can be generalized to the case of three-dimensional beams with the transverse aperture bounded in two directions. This can be done most simply if the initial field distribution at $x = 0$ is given in the factorized form $U_0(y)V_0(z)$, for example, a Gaussian beam defined as $A\exp[-y^2/(2a^2) - z^2/(2b^2)]$. We seek a solution to the Helmholtz equation in an isotropic medium in the form

$$u = U(x, y)V(x, z)\exp(\mathrm{i}kx), \qquad k = k_0\sqrt{\mathrm{Re}\,\varepsilon}.$$

Assuming that the dependence of U and V on x is a smooth one on the wavelength scale, we arrive at the reduced equation

$$\left(2\mathrm{i}k\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial y^2}\right)V + U\left(2\mathrm{i}k\frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial z^2}\right) + 2\mathrm{i}\nu\delta(y)UV = 0.$$

This equation is solved by separating the variables. We rewrite the equation as

$$\begin{aligned} &\frac{1}{U}\left(2\mathrm{i}k\frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial y^2} + 2\mathrm{i}\nu\delta(y)\right) = \mu, \\ &\frac{1}{V}\left(2\mathrm{i}k\frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial z^2}\right) = -\mu. \end{aligned} \qquad (30)$$

From the first equation in (30), it follows that μ does not depend on z, and from the second one, that it does not depend on y, so that μ can only depend on x. The terms proportional to $\mu(x)$ are excluded with the help of the standard variable transform

$$U = \tilde{U}\exp\left[(2\mathrm{i}k)^{-1}\int\mu\,\mathrm{d}x\right], \qquad V = \tilde{V}\exp\left[-(2\mathrm{i}k)^{-1}\int\mu\,\mathrm{d}x\right];$$

in this case, for $\tilde{U}$ we obtain equation (3), which we already dealt with, and $\tilde{V}$ will satisfy the one-dimensional parabolic equation without dissipation, with a solution analogous to (8). In other words, since we are interested in $u \propto UV = \tilde{U}\tilde{V}$, from the very beginning, we can simply consider (30) with $\mu = 0$. The beam intensity will be expressed as a product of linear intensities $I(x) = \int|U|^2\,\mathrm{d}y\int|V|^2\,\mathrm{d}z$, and the change in intensity will be only due to the first multiplier (the intensity related to V is preserved). We conclude that the problem for a three-dimensional beam is reduced to the problem of the propagation of a two-dimensional beam, which was considered above.[1]

Generalization to media with anisotropic or gyrotropic dielectric response is more complicated. The wave vector surfaces characteristic of such media do not possess spherical

[1] We note that the method can be used, among other applications, to construct the Green's function of a two-dimensional problem as a solution in the form UV that corresponds to $U_0(y)V_0(z) = \delta(y - y_0)\delta(z - z_0)$. Using a convolution with the two-dimensional Green's function, one can express the general solution for an arbitrary initial condition $u_0(y, z)$ which is not necessarily a factorized one.

symmetry, so that wave propagation in such media is not described by the Helmholtz equation in a general case. In a somewhat broader sense, this reflects the presence of spatial dispersion. However, in a particular case of waves propagating transverse to an external magnetic field, which in fact triggered this note, the proposed model can still be applied.

Let the magnetic field be directed along the coordinate z. Since charged particles undergo Larmor precession around this direction, the coordinates x and y are similar from the viewpoint of dielectric plasma response. For waves in plasma, all azimuthal propagation directions are equivalent, seen by the fact that the solution of the dispersion equation for the jth normal mode in a homogeneous plasma is always expressed in the form [51]

$$k_x^2 + k_y^2 = k_{\perp j}^2(\omega, k_z)\,.$$

For a wave beam in a plasma which is weakly inhomogeneous on the scale of the Debye radius and gyroradius, this representation is equivalent to the Helmholtz equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k_{\perp j}^2(\omega, k_z) u = 0\,.$$

Thus, the problem can be reduced to the one considered by us through the formal replacement $k_0^2 \varepsilon \to k_{\perp j}^2(\omega, k_z)$ or

$$k \to \sqrt{\operatorname{Re} k_{\perp j}^2(\omega, k_z)}\,, \qquad \operatorname{Im} k_{\perp j}^2(\omega, k_z) \to 2\nu\,\delta(y)\,.$$

Importantly, we cannot now assume that the beam is paraxial and neglect the dependence on k_z when determining the carrying wavenumber k in (2) and all subsequent derivations. In a plasma, spatial dispersion not only essentially affects wave refraction but is also responsible for their collisionless absorption [52–54]. Thus, if we are interested in a beam bounded in the z direction, we need first to expand the initial distribution in the Fourier spectrum over k_z, find an independent solution $U(x, y, k_z)$ for each harmonic following the technique presented above, and then combine the contribution from all the harmonics. The qualitative effects discussed above cannot be changed by this procedure. The reader can find a detailed description of resonance microwave radiation absorption in a hot magnetoactive plasma in the framework of the correct (accounting for aberrations) parabolic equation in [31, 40, 42, 55, 56].

7. Conclusions

We have demonstrated that a limit exists for the power that can be absorbed in a thin dissipative layer longitudinally cutting a paraxial wave beam, even if the layer and the beam continue infinitely. This effect, coming from the competition of inhomogeneous dissipation and diffraction of radiation, seemed to us nontrivial and rather universal. A simple reference problem reproducing this effect that is simultaneously analytically treatable has been proposed; the main result is displayed in Fig. 4.

It should be noted that approximate relationships (28) and (29) for the limiting absorbed power are intended mainly for a qualitative understanding of the effect: they must be applied with care to practical estimates. Good quantitative agreement with the numerical results is almost accidental: the formula obtained for large values of νa turned out to work for all values of νa.

We intentionally avoided additional examples in this note, limiting ourselves to the treatment of an initial Gaussian beam with a flat front and a center coinciding with the absorbing plane. Our approach can be applied without modifications to more complex initial problems, for example, to beams with linear and quadratic phase distributions, or beams with their centers displaced off the absorbing plane. Our effect will be preserved in this case; the proposed approximate analytical methods will similarly reproduce the main details of the exact solution, but with a noticeably larger quantitative error related to additional problem parameters.

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References

1. Vdovin V *Fusion Sci. Technol.* **59** 690 (2001)
2. Vdovin V L *Plasma Phys. Rep.* **39** 95 (2013); *Fiz. Plazmy* **39** 115 (2013)
3. Köhn A et al. *Plasma Phys. Control. Fusion* **55** 014010 (2013)
4. Hammond K C et al. *Plasma Phys. Control. Fusion* **60** 025022 (2018)
5. Sakharov A S *Plasma Phys. Rep.* **45** 289 (2019); *Fiz. Plazmy* **45** 291 (2019)
6. Aleynikov P, Marushchenko N B *Comput. Phys. Commun.* **241** 40 (2019)
7. Litvak A G et al. *Sov. Tech. Phys. Lett.* **1** 374 (1975); *Pis'ma Zh. Tekh. Fiz.* **1** 858 (1975)
8. Litvak A G et al. *Nucl. Fusion* **17** 659 (1977)
9. Maekawa T et al. *Phys. Lett. A* **69** 414 (1979)
10. Baranov Yu F, Fedorov V I *Sov. J. Plasma Phys.* **9** 391 (1983); *Fiz. Plazmy* **9** 677 (1983)
11. Tereshchenko M et al. *Plasma Phys. Control. Fusion* **55** 115011 (2013)
12. Pereverzev G V *Phys. Plasmas* **5** 3529 (1998)
13. Poli E, Peeters A G, Pereverzev G V *Comput. Phys. Commun.* **136** 90 (2001)
14. Poli E et al. *Comput. Phys. Commun.* **225** 36 (2018)
15. Smolyakova O B et al. *Sov. J. Plasma Phys.* **9** 688 (1983); *Fiz. Plazmy* **9** 1194 (1983)
16. Harvey R W et al. *Nucl. Fusion* **37** 69 (1997)
17. Wolf R C et al. *Nucl. Fusion* **41** 1259 (2001)
18. Günter S et al. *Nucl. Fusion* **45** S98 (2005)
19. Henderson M A et al. *Fusion Eng. Des.* **82** 454 (2007)
20. Bilato R et al. *Nucl. Fusion* **49** 075020 (2009)
21. Poli E et al. *Nucl. Fusion* **53** 01301 (2013)
22. Farina D "GRAY: a quasi-optical ray tracing code for electron cyclotron absorption and current drive in tokamaks", IFP-CNR Internal Report FP 05/1 (2005); https://www.ifp.cnr.it/publications/2005/FP05-01.pdf
23. Tereshchenko M A, Castejón F, Cappa Á "TRUBA user manual", Informes Técnicos CIEMAT, Editorial CIEMAT 1134 (Madrid: Laboratorio Nacional de Fusión por Confinamiento Magnético, 2008); http://www-fusion.ciemat.es/InternalReport/IR1134.pdf
24. Prater R et al. *Nucl. Fusion* **48** 035006 (2008)
25. Laqua H P et al. *Nucl. Fusion* **43** 1324 (2003)
26. Timofeev A V *Phys. Usp.* **47** 555 (2004); *Usp. Fiz. Nauk* **174** 609 (2004)
27. Balakina M A et al. *Radiophys. Quantum Electron.* **49** 617 (2006); *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **49** 686 (2006)
28. Shalashov A G, Gospodchikov E D *Phys. Usp.* **54** 145 (2011); *Usp. Fiz. Nauk* **181** 151 (2011)
29. Gusakov E Z, Popov A Yu *Phys. Usp.* **63** 365 (2020); *Usp. Fiz. Nauk* **190** 396 (2020)
30. Balakin A A et al. *Plasma Phys. Rep.* **34** 486 (2008); *Fiz. Plazmy* **34** 533 (2008)

31. Maj O, Balakin A A, Poli E *Plasma Phys. Control. Fusion* **52** 085006 (2010)
32. Batanov G M et al. *Plasma Phys. Rep.* **39** 882 (2013); *Fiz. Plazmy* **39** 987 (2013)
33. Alberti S et al. *Nucl. Fusion* **45** 1224 (2005)
34. Kwak Jong-Gu et al. *Nucl. Fusion* **53** 104005 (2013)
35. Shimozuma T et al. *Nucl. Fusion* **55** 063035 (2015)
36. Kirneva N A et al. *Vopr. Atom. Nauki Tekh. Ser. Termoyad. Sintez* **44** (3) 24 (2021)
37. Imai T et al. *Fusion Sci. Technol.* **51** (2T) 36 (2007)
38. Gospodchikov E D et al. *EPJ Web Conf.* **149** 03023 (2017)
39. Balakin A A et al. *J. Phys. D* **40** 4285 (2007)
40. Balakin A A, Balakina M A, Westerhof E *Nucl. Fusion* **48** 065003 (2008)
41. Balakin A A *Radiophys. Quantum Electron.* **55** 472 (2012); *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **55** 521 (2012); *Radiophys. Quantum Electron.* **55** 502 (2013); *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **55** 555 (2012); *Radiophys. Quantum Electron.* **55** 556 (2013); *Izv. Vyssh. Uchebn. Zaved. Radiofiz.* **55** 624 (2012) in three parts
42. Balakin A A, Gospodchikov E D, Shalashov A G *JETP Lett.* **104** 690 (2016); *Pis'ma Zh. Eksp. Teor. Fiz.* **104** 701 (2016)
43. Dodin I Y et al. *Phys. Plasmas* **26** 072110 (2019); *Phys. Plasmas* **26** 072111 (2019); *Phys. Plasmas* **26** 072112 (2019)
44. Leontovich M A, Fock V A *J. Phys. USSR* **10** 13 (1946); *Zh. Eksp. Teor. Fiz.* **16** 557 (1946)
45. Fock V A *Problemy Difraktsii i Rasprostraneniya Elektromagnitnykh Voln* (Problems of Diffraction and Propagation of Electromagnetic Waves) (Moscow: Sov. Radio, 1970)
46. Levy M *Parabolic Equation Methods for Electromagnetic Wave Propagation* (London: Institute of Electrical Engineering, 2000)
47. Vaganov R B, Katsenelenbaum B Z *Osnovy Teorii Difraktsii* (Fundamentals of Diffraction Theory) (Moscow: Nauka, 1982)
48. Vainshtein L A *Elektromagnitnye Volny* (Electromagnetic Waves) (Moscow: Radio i Svyaz', 1988)
49. Vladimirov V S *Obobshchennye Funktsii v Matematicheskoi Fizike* (Generalized Functions in Mathematical Physics) (Moscow: Nauka, 1979)
50. Abramowitz M, Stegun I A *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (Washington, DC; New York: United States Department of Commerce, National Bureau of Standards, 1972) Ch. 7
51. Ginzburg V L *Propagation of Electromagnetic Waves in Plasma* (Oxford: Pergamon Press, 1970); Translated from Russian: *Rasprostranenie Elektromagnitnykh Voln v Plazme* (Moscow: Nauka, 1967)
52. Bornatici M *Plasma Phys.* **24** 629 (1982)
53. Alikaev V V et al. "Electron-cyclotron resonance heation of toroidal plasmas", in *High-Frequency Plasma Heating* (New York: AIP, 1991); Translated from Russian: in *Vysokochastotnyi Nagrev Plazmy* (Gorky: IPF AN SSSR, 1983) p. 6
54. Brambilla M *Kinetic Theory of Plasma Waves: Homogeneous Plasmas* (Oxford: Clarendon Press, 1998)
55. Shalashov A G et al. *Phys. Plasmas* **23** 112504 (2016)
56. Shalashov A G et al. *J. Exp. Theor. Phys.* **124** 325 (2017); *Zh. Eksp. Teor. Fiz.* **151** 379 (2017)