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Multiparton distribution functions in quantum chromodynamics

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Contents

Abstract. The structure of hadrons (protons) and the dynamics of their interaction are usually studied in collisional experiments by exploring hard single parton–parton scattering described in terms of structure functions (single-particle distributions). Completely new and unique information comes from the selection and analysis of events in which two (or more) hard parton scatterings concurrently occur in a single $\bar{p}p$ (Tevatron; FermiLab, USA) or pp (LHC; CERN, Switzerland) collision. The simulation of such double (multiple) parton scatterings involves two-parton (multiparton) distribution functions. Properties of these functions, which may be extracted from quantum chromodynamics, are reviewed.

Keywords: multiparton interactions, double parton scatterings, collinear approach

1. Introduction

As the energy in hadron–hadron collisions increases, the role of hard multiparton interactions, which are an important and significant part of the background in the search for new physics signals at the Large Hadron Collider (LHC), is significantly enhanced due to the strong growth of the density of parton fluxes. The inclusive cross section of a hard process is usually calculated under the assumption that, in a collision, along with many soft interactions, only one hard interaction occurs due to its relatively low probability. Nevertheless, hadron–hadron collisions are possible, in which two (or more) different pairs of partons undergo hard scattering. These double parton scatterings have been theoretically studied for many years, starting from the early days [1–23] of the parton model. The current state of the theory of multiparton interactions and the results of many years of research have been reviewed in the recent monograph [24], which contains an extensive bibliography. The study of double, triple, and *n*-parton scatterings enables the extraction of completely new and unique information about the unknown three-dimensional spatial distribution of partons in hadrons and momentum, flavor, and color correlations in the hadron wave function.

Double parton scatterings were initially observed by the AFS (Axial Field Spectrometer) [25] and UA2 (Underground Area 2) [26] collaborations at CERN and later by the CDF (Collider Detector at Fermilab) [27, 28] and D0 [29, 30] collaborations at Tevatron with statistics sufficiently large for a primary analysis and study; later, they were observed at the Protvino-based accelerator [31]. As anticipated, the LHC energy and luminosity enabled the observation [24] of events with hard multiple scatterings in numbers that are significantly larger than those in the aforementioned experiments. The contribution from double parton scattering has now been reliably measured and separated [24] in a number of processes that contain in the final state heavy quarks (c, b), quarkonia ($J/\psi, \Upsilon$), jets, and gauge bosons (γ, W, Z) (see, for example, some recent results of the collaborations ATLAS (A Toroidal LHC ApparatuS) [32–34], CMS (Compact Muon Solenoid) [35–37] and LHCb (Large Hadron Collider beauty) [38–40], and monograph [24]). Triple parton scattering has not yet been observed

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experimentally, but the estimated cross sections for the production of heavy quarks (c, b) via this mechanism are sufficiently large to be measurable in pp-collisions [41] and pA-collisions [42] at the LHC and at the energies of the Future Circular Collider (FCC) [43, 44]. In the future, both three D-mesons [45] and heavy quarkonia ($3\,\mathrm{J}/\psi$) [46] may be observed.

We primarily review the properties of multiparton distribution functions that may be extracted from quantum chromodynamics (QCD) in a collinear approach with a focus on results presented in insufficient detail in monograph [24] (evolution equations and their explicit solutions). This brief review pursues the essentially practical goal of familiarizing readers (as requested by experimentalists) with the main tools employed in this phenomenological area. The text is organized as follows. Section 2 contains formulas for calculating the inclusive cross section of n-parton scattering in a factorization approach. The cases of double and triple parton scatterings that are of most interest for phenomenology are discussed in Section 3. The QCD evolution of multiparton distribution functions is reviewed in Section 4. Section 5 contains formulas for calculating inclusive cross sections of double and triple parton scatterings taking into consideration the evolution. Overall impacts on phenomenology are discussed in Section 6. Section 7 contains some conclusions and a discussion of future prospects.

2. Cross sections of *n*-parton scattering in hadron collisions

The inclusive cross section $\sigma^{\mathrm{NPS}}_{\mathrm{hh}'\to a_1\ldots a_n}$ of production of n hard particles in n independent hard parton scatterings in an $\mathrm{hh}'\to a_1\ldots a_n$ hadron–hadron collision may be represented in a factorization approach as a convolution of generalized n-parton distribution functions and elementary parton cross sections [47]:

$$\begin{aligned}
&\sigma^{\mathrm{NPS}}_{\mathrm{hh}'\to a_1\ldots a_n}\\
&\quad=\frac{c}{n!}\sum_{i_1,\ldots,i_n,i'_1,\ldots,i'_n}\int\Gamma_{\mathrm{h}}^{i_1\ldots i_n}(x_1,\ldots,x_n;\mathbf{b_1},\ldots,\mathbf{b_n};Q_1^2,\ldots,Q_n^2)\\
&\quad\times\hat\sigma_{a_1}^{i_1i'_1}(x_1,x'_1,Q_1^2)\ldots\hat\sigma_{a_n}^{i_ni'_n}(x_n,x'_n,Q_n^2)\\
&\quad\times\Gamma_{\mathrm{h}'}^{i'_1\ldots i'_n}(x'_1,\ldots,x'_n;\mathbf{b}_1-\mathbf{b},\ldots,\mathbf{b}_n-\mathbf{b};Q_1^2,\ldots,Q_n^2)\\
&\quad\times\mathrm{d}x_1\ldots\mathrm{d}x_n\,\mathrm{d}x'_1\ldots\mathrm{d}x'_n\,\mathrm{d}^2b_1\ldots\mathrm{d}^2b_n\,\mathrm{d}^2b\,.
\end{aligned}\tag{1}$$

Here, $\Gamma_{\mathrm{h}}^{i_1\ldots i_n}(x_1,\ldots,x_n;\mathbf{b}_1,\ldots,\mathbf{b}_n;Q_1^2,\ldots,Q_n^2)$ are the n-parton generalized distribution functions that depend on the fraction of longitudinal momentum $x_1,\ldots,x_n$, transverse coordinates $\mathbf{b}_1,\ldots,\mathbf{b}_n$ of scattering partons $i_1,\ldots,i_n$ at the energy scales $Q_1,\ldots,Q_n$ of hard subprocesses with cross sections $\hat\sigma_{a_1}^{i_1i'_1},\ldots,\hat\sigma_{a_n}^{i_ni'_n}$, and the production of hard particles $a_1,\ldots,a_n$ in the final state. The $c/n!$ factor takes into account the symmetry properties of the final state, i.e., the presence of identical particles. If all particles are identical ($a_1=\ldots=a_n$), $c=1$; if the number of distinguishable particles in the final state increases, $c=2,3,6,\ldots$. For double parton scattering, which is currently of the most interest in phenomenological applications, $c=1$ if $a_1=a_2$ and $c=2$ if $a_1\neq a_2$.

These n-parton distribution functions

$$\Gamma_{\mathrm{h}}^{i_1\ldots i_n}(x_1,\ldots,x_n;\mathbf{b}_1,\ldots,\mathbf{b}_n;Q_1^2,\ldots,Q_n^2)$$

theoretically encode all the information about the three-dimensional structure of a hadron needed to calculate the cross sections for n-parton scatterings, including data on the density of partons in the transverse plane and on all kinds of correlations in both quantum numbers and kinematic variables. The $\Gamma_{\mathrm{h}}^{i_1\ldots i_n}$ functions have in general a very complex structure, so simplified versions are often used in phenomenological calculations. Actually, any cross section of n-parton scattering may be expressed without a significant loss of generality in terms of inclusive cross sections $\sigma^{\mathrm{SPS}}_{\mathrm{hh}'\to a}$ of single-parton scatterings calculated in perturbative QCD with standard ('longitudinal') single distribution functions $D_{\mathrm{h}}^{i}(x;Q^2)$:

$$\sigma^{\mathrm{SPS}}_{\mathrm{hh}'\to a}=\sum_{i_1,i_2}\int D_{\mathrm{h}}^{i_1}(x_1;Q_1^2)\,\hat\sigma_a^{i_1i_2}(x_1,x'_1)\,D_{\mathrm{h}'}^{i_2}(x'_1;Q_1^2)\,\mathrm{d}x_1\,\mathrm{d}x'_1\,.\tag{2}$$

Any n-parton inclusive cross section can be represented in a more accurate formulation as the product of n cross sections of corresponding single-parton scatterings, in each of which one hard particle is produced, normalized to the effective cross section to the power $n-1$:

$$\sigma^{\mathrm{NPS}}_{\mathrm{hh}'\to a_1\ldots a_n}=\frac{c}{n!}\,\frac{\sigma^{\mathrm{SPS}}_{\mathrm{hh}'\to a_1}\ldots\sigma^{\mathrm{SPS}}_{\mathrm{hh}'\to a_n}}{\sigma^{n-1}_{\mathrm{eff,\,NPS}}}\,,\tag{3}$$

where $\sigma_{\mathrm{eff,\,NPS}}$ encodes all unknown information associated with generalized distribution functions. Equation (3) reflects the intuitive idea that the probability of production of hard particles in a particular inelastic hadron–hadron collision should simply be proportional to a product of n probabilities of independent production of each of these particles in the collision under consideration. This probability is normalized to the effective cross section to the power $n-1$ to guarantee the units of measurement required for the final result (3).

The quantity $\sigma_{\mathrm{eff,\,NPS}}$ in Eqn (3) may be estimated using reasonable standard approximations. It is natural to assume that the n-parton distribution functions may be represented as the product of the longitudinal and transverse components:

$$\begin{aligned}
&\Gamma_{\mathrm{h}}^{i_1\ldots i_n}(x_1,\ldots,x_n;\mathbf{b}_1,\ldots,\mathbf{b}_n;Q_1^2,\ldots,Q_n^2)\\
&\quad=D_{\mathrm{h}}^{i_1\ldots i_n}(x_1,\ldots,x_n;Q_1^2,\ldots,Q_n^2)\,f(\mathbf{b}_1)\ldots f(\mathbf{b}_n)\,,
\end{aligned}\tag{4}$$

where $f(\mathbf{b}_1)$ is the transverse density of partons in a hadron, which is often assumed to be independent of the type of parton. Next, ignoring the longitudinal momentum correlations

$$D_{\mathrm{h}}^{i_1\ldots i_n}(x_1,\ldots,x_n;Q_1^2,\ldots,Q_n^2)=D_{\mathrm{h}}^{i_1}(x_1;Q_1^2)\ldots D_{\mathrm{h}}^{i_n}(x_n;Q_n^2),\tag{5}$$

the effective cross section may be represented in terms of the integral of the powers of the hadron–hadron overlap function by the impact parameter $\mathbf{b}$:

$$\sigma_{\mathrm{eff,\,NPS}}=\left(\int\mathrm{d}^2b\,T^n(\mathbf{b})\right)^{-1/(n-1)}\,,\tag{6}$$

where

$$T(\mathbf{b})=\int f(\mathbf{b}_1)\,f(\mathbf{b}_1-\mathbf{b})\,\mathrm{d}^2b_1\tag{7}$$

with the fixed normalization $\int T(\mathbf{b})\,\mathrm{d}^2b=1$.

3. Cross sections of double and triple parton scattering in hadron–hadron collisions

General expression (1) in the case of double parton scattering in hadron–hadron collisions $\mathrm{hh}' \to a_1 a_2$, which is of importance for phenomenological applications, takes the form

$$\begin{aligned}\sigma^{\mathrm{DPS}}_{\mathrm{hh}'\to a_1 a_2} = \frac{c}{2}\sum_{i,j,k,l}\int \Gamma^{ij}_{\mathrm{h}}(x_1,x_2;\mathbf{b}_1,\mathbf{b}_2;Q_1^2,Q_2^2)\\ \times\,\hat{\sigma}^{ik}_{a_1}(x_1,x_1',Q_1^2)\,\hat{\sigma}^{jl}_{a_2}(x_2,x_2',Q_2^2)\\ \times\,\Gamma^{kl}_{\mathrm{h}'}(x_1',x_2';\mathbf{b}_1-\mathbf{b},\mathbf{b}_2-\mathbf{b};Q_1^2,Q_2^2)\\ \times\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_1'\,\mathrm{d}x_2'\,\mathrm{d}^2b_1\,\mathrm{d}^2b_2\,\mathrm{d}^2b\,. \qquad (8)\end{aligned}$$

Original equations (3) and (6) for $n = 2$ may be used to represent the inclusive cross section of double parton scattering as a product of independent single inclusive cross sections

$$\sigma^{\mathrm{DPS}}_{\mathrm{hh}'\to a_1 a_2} = \frac{c}{2}\,\frac{\sigma^{\mathrm{SPS}}_{\mathrm{hh}'\to a_1}\sigma^{\mathrm{SPS}}_{\mathrm{hh}'\to a_2}}{\sigma_{\mathrm{eff,\,DPS}}}\,, \qquad (9)$$

where the effective cross section of double parton scattering (6), which normalizes the product of single scatterings, has in the factorization approach a simple geometric representation:

$$\sigma_{\mathrm{eff,\,DPS}} = \left[\int \mathrm{d}^2b\,T^2(\mathbf{b})\right]^{-1}. \qquad (10)$$

Similarly, in the case of triple parton scattering, it follows from general expression (1) for the process $\mathrm{hh}' \to a_1 a_2 a_3$ [48] that

$$\begin{aligned}\sigma^{\mathrm{TPS}}_{\mathrm{hh}'\to a_1 a_2 a_3}\\ = \frac{c}{3!}\sum_{i,j,k,l,m,n}\int \Gamma^{ijk}_{\mathrm{h}}(x_1,x_2,x_3;\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3;Q_1^2,Q_2^2,Q_3^2)\\ \times\,\hat{\sigma}^{il}_{a_1}(x_1,x_1',Q_1^2)\,\hat{\sigma}^{jm}_{a_2}(x_2,x_2',Q_2^2)\,\hat{\sigma}^{kn}_{a_3}(x_3,x_3',Q_3^2)\\ \times\,\Gamma^{lmn}_{\mathrm{h}'}(x_1',x_2',x_3';\mathbf{b}_1-\mathbf{b},\mathbf{b}_2-\mathbf{b},\mathbf{b}_3-\mathbf{b};Q_1^2,Q_2^2,Q_3^2)\\ \times\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_1'\,\mathrm{d}x_2'\,\mathrm{d}x_3'\,\mathrm{d}^2b_1\,\mathrm{d}^2b_2\,\mathrm{d}^2b_3\,\mathrm{d}^2b\,. \qquad (11)\end{aligned}$$

This cross section may be represented as the product of three independent inclusive single cross sections,

$$\sigma^{\mathrm{TPS}}_{\mathrm{hh}'\to a_1 a_2 a_3} = \frac{c}{3!}\,\frac{\sigma^{\mathrm{SPS}}_{\mathrm{hh}'\to a_1}\sigma^{\mathrm{SPS}}_{\mathrm{hh}'\to a_2}\sigma^{\mathrm{SPS}}_{\mathrm{hh}'\to a_3}}{\sigma^2_{\mathrm{eff,\,TPS}}}\,, \qquad (12)$$

normalized to the effective cross section of triple parton scattering (6) *squared*, which has in the factorization approach a simple geometrical representation [41]:

$$\sigma^2_{\mathrm{eff,\,TPS}} = \left[\int \mathrm{d}^2b\,T^3(\mathbf{b})\right]^{-1}. \qquad (13)$$

The effective cross sections of the double (10) and triple (13) parton scatterings may be estimated by using Eqn (7) for various forms of the parton profiles of colliding hadrons employed in the state-of-the-art generators of Monte Carlo events, such as Pythia-8 [49] or Herwig [50, 51]. The dependence of the proton overlap function on the impact parameter function is often parameterized in Pythia-8 as

$$T(\mathbf{b}) = \frac{m}{2\pi r_{\mathrm{p}}^2\,\Gamma(2/m)}\exp\left[\left(-\frac{b}{r_{\mathrm{p}}}\right)^m\right], \qquad (14)$$

where $T(\mathbf{b})$ is normalized to one, $\int T(\mathbf{b})\,\mathrm{d}^2b = 1$, r_{p} is the characteristic 'radius' of the proton, Γ is the gamma function, and the index of power m, which depends on generator 'settings', is determined by fitting to experimental data on pp-collisions [36]. This index of power ranges from $m = 2$ (purely Gaussian distribution) to $m = 0.7$, 1 (exponential-like distribution). The following formulas may be derived for the corresponding integrals of the second and third power of $T(\mathbf{b})$:

$$\sigma_{\mathrm{eff,\,DPS}} = \left(\int \mathrm{d}^2b\,T^2(\mathbf{b})\right)^{-1} = 2\pi r_{\mathrm{p}}^2\,\frac{2^{2/m}\Gamma(2/m)}{m}\,, \qquad (15)$$

$$\sigma_{\mathrm{eff,\,TPS}} = \left(\int \mathrm{d}^2b\,T^3(\mathbf{b})\right)^{-1/2} = 2\pi r_{\mathrm{p}}^2\,\frac{3^{1/m}\Gamma(2/m)}{m}\,. \qquad (16)$$

Equation (15) shows that, to reproduce the value $\sigma_{\mathrm{eff,\,DPS}} \simeq 15 \pm 5$ mb extracted from the measurements of double parton scattering in the Tevatron [27–30] and the LHC [32–40], the characteristic 'radius' of the proton should be set equal to $r_{\mathrm{p}} \simeq 0.11 \pm 0.02$, 0.24 ± 0.04, and 0.49 ± 0.08 fm for $m = 0.7$, 1, and 2, respectively. The values of the effective cross sections $\sigma_{\mathrm{eff,\,DPS}}$ and $\sigma_{\mathrm{eff,\,TPS}}$ (Eqns (15) and (16)) are not independent: they are related by the formula $\sigma_{\mathrm{eff,\,TPS}} = (3/4)^{1/m}\sigma_{\mathrm{eff,\,DPS}}$. This relation does not depend on the characteristic 'size' r_{p} of the proton but is sensitive to the overall form of the transverse parton profile that is characterized by the index of power m. Using Pythia-8 with the typical values of the index of power $m = 0.7$, 1, 2 determined by fitting the experimental data [36] yields $\sigma_{\mathrm{eff,\,TPS}} = [0.66,\ 0.75,\ 0.87]\sigma_{\mathrm{eff,\,DPS}}$, respectively.

The Herwig event generator uses another parameterization of the proton profile taken from the dipole fit to the two-gluon form factor in the momentum representation [52]:

$$F_{2\mathrm{g}}(\mathbf{q}) = \frac{1}{(q^2/m_{\mathrm{g}}^2+1)^2}\,, \qquad (17)$$

where the gluon mass m_{g} is a parameter that characterizes the distribution of partons over the transverse momentum q, while their distribution over transverse coordinates is determined using the Fourier transform

$$f(\mathbf{b}) = \int \exp\left(-\mathrm{i}\mathbf{b}\mathbf{q}\right)F_{2\mathrm{g}}(\mathbf{q})\,\frac{\mathrm{d}^2q}{(2\pi)^2}\,. \qquad (18)$$

The corresponding effective cross sections of the double (10) and triple (13) parton scattering have the form [50]

$$\sigma_{\mathrm{eff,\,DPS}} = \left[\int F_{2\mathrm{g}}^4(q)\,\frac{\mathrm{d}^2q}{(2\pi)^2}\right]^{-1} = \frac{28\pi}{m_{\mathrm{g}}^2} \qquad (19)$$

and [41]

$$\begin{aligned}\sigma^{-2}_{\mathrm{eff,\,TPS}} = \int (2\pi)^2\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)F_{2\mathrm{g}}(\mathbf{q}_1)F_{2\mathrm{g}}(\mathbf{q}_2)F_{2\mathrm{g}}(\mathbf{q}_3)\\ \times\,F_{2\mathrm{g}}(-\mathbf{q}_1)F_{2\mathrm{g}}(-\mathbf{q}_2)F_{2\mathrm{g}}(-\mathbf{q}_3)\,\frac{\mathrm{d}^2q_1}{(2\pi)^2}\,\frac{\mathrm{d}^2q_2}{(2\pi)^2}\,\frac{\mathrm{d}^2q_3}{(2\pi)^2}\,.\end{aligned}$$

A numerical calculation of the last integral and comparison of its value with result (19) yield the relation $\sigma_{\text{eff, TPS}} = 0.83\,\sigma_{\text{eff, DPS}}$, which is very close to that obtained earlier for the Gaussian overlap function used in Pythia-8. To reproduce the experimentally measured value $\sigma_{\text{eff, DPS}} \simeq 15 \pm 5$ mb, the characteristic 'size' of this parameterization should be $r_{\text{g}} = 1/m_{\text{g}} \simeq 0.13 \pm 0.02$ fm.

Thus, the study of a broad set of parameterizations of transverse parton densities used in the literature makes it possible to establish a very simple, reliable, and useful relationship [41] between the effective cross sections of double and triple parton scatterings:

$$\sigma_{\text{eff, TPS}} = k\sigma_{\text{eff, DPS}} \quad \text{at} \quad k = 0.82 \pm 0.11\,. \tag{20}$$

A typical value $\sigma_{\text{eff, DPS}} \simeq 15 \pm 5$ mb extracted from measurements in the Tevatron and LHC yields the following estimate for the effective cross section for triple parton scattering:

$$\sigma_{\text{eff, TPS}} = 12.5 \pm 4.5 \text{ mb}\,. \tag{21}$$

These are the values of the effective cross sections that are usually used [24, 47] in calculating the inclusive cross sections for those processes where particles are produced through the mechanisms of double and triple parton scattering.

It should be noted that, to describe the experimental data on double production of J/ψ, a much lower value of the effective double parton scattering cross section, at a level of 5 mb, should be used: the ATLAS, CMS, and D0 collaborations presented the values $\sigma_{\text{eff, DPS}} \simeq 6.3 \pm 1.9$ mb [33], $\sigma_{\text{eff, DPS}} \simeq 2.2-6.6$ mb [37], and $\sigma_{\text{eff, DPS}} \simeq 4.8 \pm 2.5$ mb [53], respectively. Such a noticeable disagreement with other measurements, which was interpreted in publications as the first indication of the possible dependence of $\sigma_{\text{eff, DPS}}$ on the type of scattering partons, triggered a number of studies to find a solution to this problem by taking into account both additional mechanisms of J/ψ production in single scattering and dynamic QCD correlations in double parton scattering. For example, it was shown in recent paper [54] in the k_{t}-factorization approach that the contributions from the color-octet mechanism, which take into account the combinatorial effects of multiple emission of gluons in the initial state, are of importance in the kinematic region of measurements of the CMS and ATLAS experiments, while the experimental data from the LHCb collaboration are well described by the $O(\alpha_{\text{s}}^4)$-color singlet contributions and the mechanism of double parton scattering. The authors of [54] obtained as a result of a comprehensive analysis $\sigma_{\text{eff, DPS}} \simeq 17.5 \pm 4.1$ mb, a value which is compatible with the estimate derived from other measurements and partly removes the significant disagreement between the estimated effective cross sections.

4. QCD evolution of multiparton distribution functions

4.1 Introduction. Momentum representation

As noted in Section 3, the effective cross section $\sigma_{\text{eff, DPS}}$ is either determined in phenomenological models or extracted from experimental data (due to the still unresolved confinement problem). However, much about the properties of multiparton momentum distribution functions may be learned in the QCD perturbation theory. Moreover, it is possible to establish in the collinear approach the status of factorization assumptions (4) and (5) and find the corrections caused by the QCD evolution of multiparton distributions.

In what follows, the momentum representation rather than the mixed (momentum-coordinate) representation turns out to be more convenient [52] for describing the inclusive double parton scattering cross section (8):

$$\begin{aligned}\sigma^{\text{DPS}}_{\text{hh}'\to a_1 a_2} &= \frac{c}{2}\sum_{i,j,k,l}\int \Gamma^{ij}_{\text{h}}(x_1,x_2;\mathbf{q};Q_1^2,Q_2^2)\\ &\times \hat{\sigma}^{ik}_{a_1}(x_1,x_1')\hat{\sigma}^{jl}_{a_2}(x_2,x_2')\Gamma^{kl}_{\text{h}'}(x_1',x_2';-\mathbf{q};Q_1^2,Q_2^2)\\ &\times \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_1'\,\mathrm{d}x_2'\,\frac{\mathrm{d}^2q}{(2\pi)^2}\,.\end{aligned} \tag{22}$$

The functions $\Gamma^{ij}_{\text{h}}(x_1,x_2;\mathbf{q};Q_1^2,Q_2^2)$ in the momentum representation depend on the transverse momentum $\mathbf{q}$, which is equal to the difference between the momenta of the partons in the amplitude of the hadron wave function and in the conjugate amplitude. This dependence arises because it is only the sum of the transverse momenta of the partons in a pair that is conserved. The momentum $\mathbf{q}$ is the Fourier-conjugate variable for the difference between the transverse coordinates of the partons used in the mixed representation.

One of the main problems is to find two-parton functions $\Gamma^{ij}_{\text{h}}(x_1,x_2;\mathbf{q};Q_1^2,Q_2^2)$ without simplifying assumptions (4) and (5). These functions were only known [55–57] for $\mathbf{q} = 0$ (in other words, integrated over the relative transverse distance between partons) in the collinear approach. In this approximation,

$$\Gamma^{ij}_{\text{h}}(x_1,x_2;\mathbf{q}=0;Q_1^2,Q_2^2) = D^{ij}_{\text{h}}(x_1,x_2;Q^2)\,,$$

provided that the scales of both processes are comparable ($Q_1^2 \simeq Q_2^2 = Q^2$), satisfying the generalized evolution equations first derived in [55–57]. A generalization for the case of two different scales is also available [58].

Similarly, in the case of triple parton scattering, we have in the momentum representation [48, 52], instead of (11),

$$\begin{aligned}&\sigma^{\text{TPS}}_{\text{hh}'\to a_1 a_2 a_3}\\ &= \frac{c}{2}\sum_{i,j,k,l,m,n}\int \Gamma^{ijk}_{\text{h}}(x_1,x_2,x_3;\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3;Q_1^2,Q_2^2,Q_3^2)\\ &\times (2\pi)^2\delta(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)\hat{\sigma}^{il}_{a_1}(x_1,x_1',Q_1^2)\\ &\times \hat{\sigma}^{jm}_{a_2}(x_2,x_2',Q_2^2)\hat{\sigma}^{kn}_{a_3}(x_3,x_3',Q_3^2)\\ &\times \Gamma^{lmn}_{\text{h}'}(x_1',x_2',x_3';-\mathbf{q}_1,-\mathbf{q}_2,-\mathbf{q}_3;Q_1^2,Q_2^2,Q_3^2)\\ &\times \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_1'\,\mathrm{d}x_2'\,\mathrm{d}x_3'\,\frac{\mathrm{d}^2q_1}{(2\pi)^2}\,\frac{\mathrm{d}^2q_2}{(2\pi)^2}\,\frac{\mathrm{d}^2q_3}{(2\pi)^2}\,.\end{aligned} \tag{23}$$

It is necessary to determine in this case the functions

$$\Gamma^{ijk}_{\text{h}}(x_1,x_2,x_3;\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3;Q_1^2,Q_2^2,Q_3^2)\,,$$

which are known for $\mathbf{q}_1 = \mathbf{q}_2 = \mathbf{q}_3 = 0$ [48, 55–57] (integrated over the relative transverse distances between partons) in the collinear approach, and in this approximation

$$\begin{aligned}&\Gamma^{ijk}_{\text{h}}(x_1,x_2,x_3;\mathbf{q}_1=\mathbf{q}_2=\mathbf{q}_3=0;Q^2,Q^2,Q^2)\\ &\quad = D^{ijk}_{\text{h}}(x_1,x_2,x_3;Q^2)\,.\end{aligned}$$

The properties of such multiparton distribution functions are discussed in Sections 4.2–4.5 in perturbative QCD.

4.2 Single-parton distribution functions in leading logarithmic approximation

An analysis of hard processes [59–61] (deep inelastic scattering of electrons by protons and electron–positron annihilation into hadrons) based on the QCD perturbation theory leads to a description of these processes in the leading logarithmic approximation in terms of the parton model with a variable cutoff parameter $\Lambda \sim Q$ in the transverse momenta of partons. The dependence of the multiparton distribution and fragmentation functions on the value of the cutoff parameter is determined by evolutionary equations, an elegant way of obtaining which was suggested by Lipatov [60] for any renormalizable field theory. Either the hardness of the process Q^2 (most often the transferred momentum squared) or its logarithm $\xi = \ln(Q^2/\mu^2)$ or the double logarithm (that explicitly takes into account the behavior of the effective coupling constant in the main logarithmic approximation) is used as an evolutionary variable:

$$t = \frac{1}{2\pi\beta} \ln\left[1 + \frac{g^2(\mu^2)}{4\pi}\,\beta \ln\left(\frac{Q^2}{\mu^2}\right)\right] = \frac{1}{2\pi\beta} \ln\left[\frac{\ln(Q^2/\Lambda^2_{\mathrm{QCD}})}{\ln(\mu^2/\Lambda^2_{\mathrm{QCD}})}\right],$$

where $\beta = (11N_{\mathrm{c}} - 2n_{\mathrm{f}})/(12\pi)$ in QCD, $g(\mu^2)$ is the coupling constant at a certain characteristic scale μ^2, starting from which the perturbation theory is applicable, n_{f} is the number of active flavors, Λ_{QCD} is the dimensional QCD parameter, and $N_{\mathrm{c}} = 3$ is the number of color degrees of freedom.

The evolution equations, most often known as the Dokshitzer–Gribov–Lipatov–Altarelli–Parisi (DGLAP) equations [59–62], acquire the simplest form if the natural dimensionless evolutionary variable t is used:

$$\frac{\mathrm{d}D_i^j(x,t)}{\mathrm{d}t} = \sum_{j'} \int_x^1 \frac{\mathrm{d}x'}{x'}\, D_i^{j'}(x',t) P_{j'\to j}\left(\frac{x}{x'}\right). \tag{24}$$

Equations (24) describe the change in single distribution functions of bare quarks and gluons ($j = \mathrm{q}, \bar{\mathrm{q}}, \mathrm{g}$) in dressed partons (quarks and gluons) with a change in the evolutionary variable t. The kernels P of these equations in the Lipatov method automatically include regularization at $x \to x'$, while in Ref. [62], the regularization was actually introduced 'by hand' based on the requirement that the momentum conservation law be satisfied.

The input in Lipatov's approach is a set of wave functions $\Psi_i^n(\beta_r, k_{\perp r})$, i.e., the amplitudes of the probabilities of finding a dressed i-type parton in a state consisting of n bare partons with fractions β_r of the longitudinal momentum and transverse momenta $k_{\perp r}$, $r = 1, \ldots, n$, that satisfy the normalization condition:

$$1 = Z_i + \sum_{n=2}^{\infty} \int \prod_{r=1}^{n} \frac{\mathrm{d}\beta_r}{\beta_r}\, \theta(\beta_r)\, \mathrm{d}^2 k_{\perp r} \times |\Psi_i^n|^2 \delta^2\left(\sum k_{\perp r}\right) \delta\left(\sum \beta_r - 1\right), \tag{25}$$

where Z_i is the renormalization constant of the wave function of an i-type parton. To determine some quantities of interest to us, there is no need to calculate Ψ_i^n in an explicit form: the Feynman integrals for Ψ_i^n and integrals over $k_\perp$ in the normalization condition (25) diverge logarithmically in renormalizable theories. They are regularized using a cutoff parameter Λ. Then, all constants Z_i, bare charges become functions of this parameter, just like quantities that can be expressed in terms of the wave functions Ψ_i^n. Closed equations determine their dependence on Λ. To derive these equations, we should straightforwardly differentiate the definitions of the quantities of interest to us and use the equations that are obtained by differentiating the normalization condition with respect to Λ.

In differentiating the limits of integration with respect to $k_\perp$, it should be kept in mind that the main contribution to Ψ_i^n in the leading logarithmic approximation is made by tree skeleton diagrams that do not interfere with each other in (25) and suchlike relations. This leads to the classical probabilistic interpretation of individual terms in the normalization condition: the probability of decay into n partons is determined by a product of the probability of decay into $n-1$ partons and the probability of subsequent decay into two partons. Moreover, the transverse momenta are strictly ordered: they increase along the skeleton diagram in the case of deep inelastic scattering and decrease in $\mathrm{e}^+\mathrm{e}^-$ annihilation.

The Lipatov method naturally enables deriving an evolution equation also for multiparton distribution (and fragmentation) functions by differentiating their definitions in terms of the introduced sets of wave functions, as is discussed in Sections 4.3–4.5.

Equations (24), after the Mellin transform (introducing the moments of functions)

$$M_i^j(n,t) = \int_0^1 \mathrm{d}x\, x^n D_i^j(x,t), \tag{26}$$

are reduced to a system of ordinary linear differential equations of the first order with constant coefficients:

$$\frac{\mathrm{d}M_i^j(n,t)}{\mathrm{d}t} = \sum_{j'} M_i^{j'}(n,t) P_{j'\to j}(n), \tag{27}$$

where

$$P_{j'\to j}(n) = \int_0^1 x^n P_{j'\to j}(x)\, \mathrm{d}x. \tag{28}$$

The solutions of this system may be found explicitly by diagonalization [59–62]. Then, the inverse Mellin transform

$$xD_i^j(x,t) = \int \frac{\mathrm{d}n}{2\pi\mathrm{i}}\, x^{-n} M_i^j(n,t) \tag{29}$$

enables the distribution functions to be determined in the x-representation. The integration is carried out in this case along the imaginary axis to the right of all singularities in n, and in the general case is not possible in an explicit form. However, the asymptotic behavior can be assessed in some interesting and useful limits of the kinematic variable x. For example, it is possible to explicitly calculate in the double logarithmic approximation the Green's function, solutions of Eqns (24) with singular initial conditions $D_i^j(x, t=0) = \delta(x-1)\delta_{ij}$. For instance, the distribution of gluons in gluons may be obtained (see, e.g., [61, 63]):

$$xD_{\mathrm{g}}^{\mathrm{g}}(x,t) = 4N_{\mathrm{c}} t \exp(-at) \frac{I_1(v)}{v} \simeq 4N_{\mathrm{c}} t v^{-3/2} \frac{\exp(v - at)}{\sqrt{2\pi}}, \tag{30}$$

where

$$v = \sqrt{8N_c t \ln \frac{1}{x}}, \quad a = \frac{11}{6} N_c + \frac{1}{3} \frac{n_f}{N_c^2}, \tag{31}$$

and I_1 is the standard modified Bessel function. Result (30) shows that the unitarity condition is violated at small x due to too rapid a growth of gluon densities. It should be noted as well that the average number of bare partons of type j in the dressed parton i,

$$\langle n^j \rangle_i = M_i^j(0,t) = \left[\exp\left(P(0)t\right)\right]_i^j, \tag{32}$$

cannot be correctly determined in the collinear approach, since the kernels $P_{g\to g}(0)$ and $P_{q\to g}(0)$ diverge at small x, and it is necessary to go beyond this approach.

4.3 Two- and three-parton distribution functions in the leading logarithmic approach

The evolution equations for two-parton distribution functions have been derived in [55–57]:

$$\begin{aligned}
&\frac{\mathrm{d}D_i^{j_1 j_2}(x_1,x_2,t)}{\mathrm{d}t} \\
&= \sum_{j_1'} \int_{x_1}^{1-x_2} \frac{\mathrm{d}x_1'}{x_1'} D_i^{j_1' j_2}(x_1',x_2,t) P_{j_1'\to j_1}\left(\frac{x_1}{x_1'}\right) \\
&+ \sum_{j_2'} \int_{x_2}^{1-x_1} \frac{\mathrm{d}x_2'}{x_2'} D_i^{j_1 j_2'}(x_1,x_2',t) P_{j_2'\to j_2}\left(\frac{x_2}{x_2'}\right) \\
&+ \sum_{j'} D_i^{j'}(x_1+x_2,t) \frac{1}{x_1+x_2} P_{j'\to j_1 j_2}\left(\frac{x_1}{x_1+x_2}\right),
\end{aligned} \tag{33}$$

and the kernel

$$\frac{1}{x_1+x_2} P_{j'\to j_1 j_2}\left(\frac{x_1}{x_1+x_2}\right), \tag{34}$$

included in the inhomogeneous part of the equations, does not contain a δ-like term that is present in the $P_{j_1'\to j_1}(x_1/x_1')$ kernels. The equations describe the behavior in the double distribution functions of bare quarks and gluons in dressed partons (quarks and gluons) as the evolution variable t changes, i.e., when the scales of both hard processes are comparable, $Q_1^2 \simeq Q_2^2$, and there is no other large logarithm $|\ln(Q_1^2/Q_2^2)|$, which would require going beyond the leading logarithmic approximation, in which, it should be recalled, another more frequently used evolution variable ξ is defined up to a constant due to the nonunique choice of the normalization scale μ^2.

All terms on the right-hand side of Eqns (33) may be interpreted in the parton model in simple physical terms. Consider the inclusive probability $D_i^{j_1 j_2}(x_1,x_2,t)\delta x_1 \delta x_2$ to find in a dressed parton i a pair of bare partons like j_1 and j_2 with fractions of longitudinal momentum ranging from x_1 to $x_1+\delta x_1$ and from x_2 to $x_2+\delta x_2$, respectively, on a scale t. Apparently, as t increases to $t+\Delta t$, this probability may be altered by two types of processes. The decay of partons with large momenta, which increases the number of $j_1 j_2$ pairs with the sought momenta, leads to an increase in the probability, while the decay from the $j_1 j_2$ pair into partons with smaller momenta decreases its value.

Three types of decay processes are possible, which increase the number of pairs of partons $j_1 j_2$ with longitudinal momenta in the range $x_1 \to x_1+\delta x_1$, $x_2 \to x_2+\delta x_2$. In the first process, we begin our consideration with a pair of partons $j_1' j_2$ with longitudinal momenta in the range $x_1' \to x_1'+\delta x_1'$, $x_2 \to x_2+\delta x_2$. The quantities x_1' must satisfy the condition $x_1 < x_1' < 1-x_2$, i.e., be sufficiently large so that parton j_1' can decay into parton j_1, and sufficiently small so that the momentum of the initial pair of partons does not exceed that of the dressed parton i. The parton j_1' decays then in such a way that one of the newly produced partons is of type j_1 with longitudinal momentum in the range $x_1 \to x_1+\delta x_1$. This process is described by the first term on the right-hand side of Eqn (33), in which the kernel $P_{j_1'\to j_1}(x_1/x_1')$ does not contain terms proportional to the δ-function, i.e., it corresponds to the actual decay of the parton. The second process (the second term on the right-hand side of Eqn (33)) is similar to the first one and corresponds to the decay of the second parton. The interpretation of these two contributions just corresponds to the interpretation of a similar contribution from the actual decay of partons in the case of evolution (24) of one-parton distribution functions. In the third process (the third term on the right-hand side of Eqn (33)), one parton j' with the 'correct' longitudinal momentum $x_1+x_2 \to x_1+x_2+\delta x_2$ decays in such a way that a $j_1 j_2$ pair is produced with the required longitudinal momenta in the required range. The number of $j_1 j_2$ pairs decreases when either parton j_1 or parton j_2 decays into partons with smaller (than the sought) longitudinal momenta. These are just the contributions from the first two terms on the right-hand side of the equations generated by the presence of δ-functions in the kernels ('virtual corrections').

Solutions to Eqns (33) can be represented as a convolution of single distribution functions and kernels of equations [55–57]:

$$\begin{aligned}
&D_i^{j_1 j_2}(x_1,x_2,t) \\
&= \sum_{j' j_1' j_2'} \int_0^t \mathrm{d}t' \int_{x_1}^{1-x_2} \frac{\mathrm{d}z_1}{z_1} \int_{x_2}^{1-z_1} \frac{\mathrm{d}z_2}{z_2} D_i^{j'}(z_1+z_2,t') \\
&\times \frac{1}{z_1+z_2} P_{j'\to j_1' j_2'}\left(\frac{z_1}{z_1+z_2}\right) D_{j_1'}^{j_1}\left(\frac{x_1}{z_1}, t-t'\right) D_{j_2'}^{j_2}\left(\frac{x_2}{z_2}, t-t'\right).
\end{aligned} \tag{35}$$

This convolution, which coincides with the jet calculus rules [64, 65] formulated for multiparton fragmentation functions, is a generalization of the well-known Gribov–Lipatov relation derived for single functions [59–61, 66]: the distribution of bare partons in dressed ones is identical to the fragmentation of bare partons into dressed ones in the leading logarithmic approximation. Solutions (35) displayed above show that the two-parton functions are strongly correlated in the leading logarithmic approximation:

$$D_i^{j_1 j_2}(x_1,x_2,t) \neq D_i^{j_1}(x_1,t) D_i^{j_2}(x_2,t). \tag{36}$$

In the case of triple parton distributions, we have [48, 55, 67]

$$\begin{aligned}
&\frac{\mathrm{d}D_i^{j_1 j_2 j_3}(x_1,x_2,x_3,t)}{\mathrm{d}t} \\
&= \sum_{j_1'} \int_{x_1}^{1-x_2-x_3} \frac{\mathrm{d}x_1'}{x_1'} D_i^{j_1' j_2 j_3}(x_1',x_2,x_3,t) P_{j_1'\to j_1}\left(\frac{x_1}{x_1'}\right) \\
&+ \sum_{j_2'} \int_{x_2}^{1-x_1-x_3} \frac{\mathrm{d}x_2'}{x_2'} D_i^{j_1 j_2' j_3}(x_1,x_2',x_3,t) P_{j_2'\to j_2}\left(\frac{x_2}{x_2'}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j_3'}\int_{x_3}^{1-x_1-x_2}\frac{\mathrm{d}x_3'}{x_3'}\,D_i^{j_1j_2j_3'}(x_1,x_2,x_3',t)P_{j_3'\to j_3}\left(\frac{x_2}{x_2'}\right)\\
&+\sum_{j'}D_i^{j'j_3}(x_1+x_2,x_3,t)\frac{1}{x_1+x_2}\,P_{j'\to j_1j_2}\left(\frac{x_1}{x_1+x_2}\right)\\
&+\sum_{j'}D_i^{j'j_2}(x_1+x_3,x_2,t)\frac{1}{x_1+x_3}P_{j'\to j_1j_3}\left(\frac{x_1}{x_1+x_3}\right)\\
&+\sum_{j'}D_i^{j_1j'}(x_1,x_2+x_3,t)\frac{1}{x_2+x_3}P_{j'\to j_2j_3}\left(\frac{x_2}{x_2+x_3}\right).\qquad(37)
\end{aligned}
$$

The two-parton distribution functions $D_i^{j'j_3}(x_1+x_2,x_3,t)$ determined above, which are contained in the inhomogeneous part of evolution equations (37), satisfy equations (33) with known solution (35). The solutions of Eqns (37) may be represented as a convolution of single and double distribution functions with the kernels of equations [48]:

$$
\begin{aligned}
&D_i^{j_1j_2j_3}(x_1,x_2,x_3,t)\\
&\quad=\sum_{j'j_1'j_2'j_3'}\int_0^t\mathrm{d}t'\int_{x_1}^1\frac{\mathrm{d}z_1}{z_1}\int_{x_2}^1\frac{\mathrm{d}z_2}{z_2}\int_{x_3}^1\frac{\mathrm{d}z_3}{z_3}\,\theta(1-z_1-z_2-z_3)\\
&\quad\times\Bigg[D_i^{j'j_3'}(z_1+z_2,z_3,t')\frac{1}{z_1+z_2}\,P_{j'\to j_1'j_2'}\left(\frac{z_1}{z_1+z_2}\right)\\
&\quad+D_i^{j'j_2'}(z_1+z_3,z_2,t')\frac{1}{z_1+z_3}\,P_{j'\to j_1'j_3'}\left(\frac{z_1}{z_1+z_3}\right)\\
&\quad+D_i^{j_1'j'}(z_1,z_2+z_3,t')\frac{1}{z_2+z_3}\,P_{j'\to j_2'j_3'}\left(\frac{z_2}{z_2+z_3}\right)\Bigg]\\
&\quad\times D_{j_1'}^{j_1}\left(\frac{x_1}{z_1},t-t'\right)D_{j_2'}^{j_2}\left(\frac{x_2}{z_2},t-t'\right)D_{j_3'}^{j_3}\left(\frac{x_3}{z_3},t-t'\right).\qquad(38)
\end{aligned}
$$

Equations for n-parton distribution functions and their solutions expressed in terms of a convolution of $(n-1)$-parton functions and kernels of the equations have also been derived and obtained in [55, 67] using the Lipatov method; however, they are very cumbersome and not displayed here.

4.4 Multiparton distribution functions in hadrons

More interesting in phenomenological applications are the distribution functions of bare quarks and gluons in hadrons. Equations for these functions can be obtained using the widely employed and phenomenologically justified hypothesis of factorization [68] of the physics of small and large distances. The evolution of the distribution functions of bare quarks and gluons in hadrons is described then by the same equations that were discussed in Sections 4.1–4.3, where the index of the dressed parton i is replaced by the index of the hadron under consideration h. We only consider here two-parton distributions, as the most interesting in phenomenology, the equations for which have the form [56, 57]

$$
\begin{aligned}
&\frac{\mathrm{d}D_{\mathrm{h}}^{j_1j_2}(x_1,x_2,t)}{\mathrm{d}t}\\
&\quad=\sum_{j_1'}\int_{x_1}^{1-x_2}\frac{\mathrm{d}x_1'}{x_1'}\,D_{\mathrm{h}}^{j_1'j_2}(x_1',x_2,t)P_{j_1'\to j_1}\left(\frac{x_1}{x_1'}\right)\\
&\quad+\sum_{j_2'}\int_{x_2}^{1-x_1}\frac{\mathrm{d}x_2'}{x_2'}\,D_{\mathrm{h}}^{j_1j_2'}(x_1,x_2',t)P_{j_2'\to j_2}\left(\frac{x_2}{x_2'}\right)\\
&\quad+\sum_{j'}D_{\mathrm{h}}^{j'}(x_1+x_2,t)\frac{1}{x_1+x_2}\,P_{j'\to j_1j_2}\left(\frac{x_1}{x_1+x_2}\right).\qquad(39)
\end{aligned}
$$

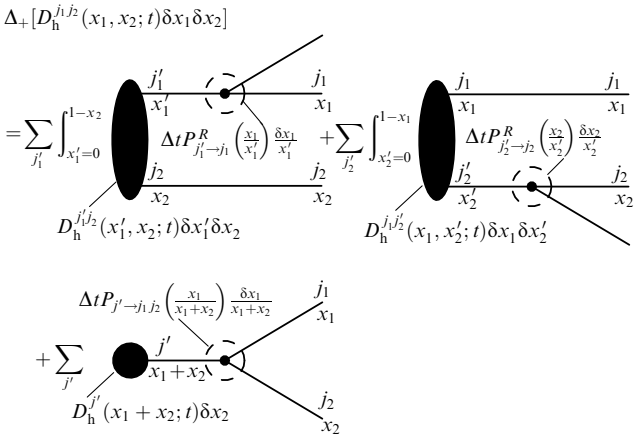

Figure 1. Decays of partons in which the number of j_1j_2-pairs with the fractions of longitudinal momentum in the range $x_1\to x_1+\delta x_1$, $x_2\to x_2+\delta x_2$ increases by the value $\Delta_+[D_{\mathrm{h}}^{j_1j_2}(x_1,x_2;t)\delta x_1\delta x_2]$. $P^R_{j'\to j}(x)$ is the part of the $P_{j'\to j}(x)$ kernel that corresponds to the actual decay of the partons, i.e., does not contain the terms proportional to $\delta(1-x)$.

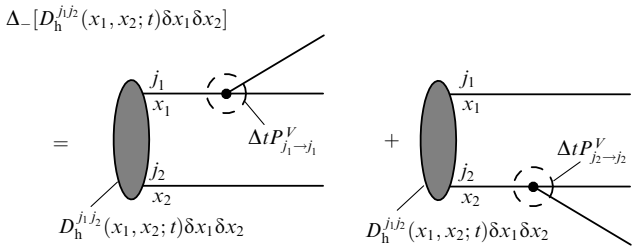

Figure 2. Decays of partons that decrease the number of j_1j_2-pairs with the fractions of the longitudinal momentum in the range $x_1\to x_1+\delta x_1$, $x_2\to x_2+\delta x_2$ by the value $\Delta_-[D_{\mathrm{h}}^{j_1j_2}(x_1,x_2;t)\delta x_1\delta x_2]$. $P^V_{j'\to j}(x)$ is the part of the $P_{j'\to j}(x)$ kernel that contains the terms proportional to $\delta(1-x)$.

In addition to Section 4.3, in which a probabilistic interpretation of all contributions on the right-hand side of the evolution equations is given, they are displayed for clarity in a graphical form in Figs 1 and 2.

However, the initial conditions for the new distribution functions in hadrons on the initial scale $t=0$ ($Q^2=\mu^2$) are now a priori unknown, in contrast to the initial conditions for the parton-level functions (for which $D_i^j(x,t=0)=\delta_{ij}\,\delta(x-1)$; $D_i^{j_1j_2}(x_1,x_2,t=0)=0$), and they must be introduced in a phenomenological manner or determined from experiment or models that claim to have the confinement problem solved. Nevertheless, solutions of equations with given initial conditions may be represented in terms of convolutions of parton-level functions in the form [56, 57]

$$
D_{\mathrm{h}}^{j_1j_2}(x_1,x_2,t)=D_{\mathrm{h(fact)}}^{j_1j_2}(x_1,x_2,t)+D_{\mathrm{h(QCD)}}^{j_1j_2}(x_1,x_2,t)\,,\qquad(40)
$$

where $D_{\mathrm{h(fact)}}^{j_1j_2}(x_1,x_2,t)$ and $D_{\mathrm{h(QCD)}}^{j_1j_2}(x_1,x_2,t)$ are the perturbation-theory dynamic correlations induced by the evolution of the two-parton functions (compare (35) and (42)),

$$
\begin{aligned}
D_{\mathrm{h(fact)}}^{j_1j_2}(x_1,x_2,t)&=\sum_{j_1'j_2'}\int_{x_1}^{1-x_2}\frac{\mathrm{d}z_1}{z_1}\int_{x_2}^{1-z_1}\frac{\mathrm{d}z_2}{z_2}\,D_{\mathrm{h}}^{j_1'j_2'}(z_1,z_2,0)\\
&\quad\times D_{j_1'}^{j_1}\left(\frac{x_1}{z_1},t\right)D_{j_2'}^{j_2}\left(\frac{x_2}{z_2},t\right),\qquad(41)
\end{aligned}
$$

$$D^{j_1 j_2}_{\mathrm{h(QCD)}}(x_1, x_2, t) = \sum_{j' j'_1 j'_2} \int_0^t \mathrm{d}t' \int_{x_1}^{1-x_2} \frac{\mathrm{d}z_1}{z_1} \int_{x_2}^{1-z_1} \frac{\mathrm{d}z_2}{z_2} D^{j'}_{\mathrm{h}}(z_1 + z_2, t') \frac{1}{z_1 + z_2} \times P_{j' \to j'_1 j'_2}\left(\frac{z_1}{z_1 + z_2}\right) D^{j_1}_{j'_1}\left(\frac{x_1}{z_1}, t - t'\right) D^{j_2}_{j'_2}\left(\frac{x_2}{z_2}, t - t'\right). \tag{42}$$

The first term on the right-hand side of Eqn (40) is a solution of homogeneous QCD equations (independent evolution of two branches of a parton cascade), in which the two-parton distribution functions $D^{j'_1 j'_2}_{\mathrm{h}}(z_1, z_2, 0)$ at the initial scale μ^2 are, generally speaking, unknown. Factorization may be suggested for these nonperturbative two-parton functions at small z_1 and z_2,

$$D^{j'_1 j'_2}_{\mathrm{h}}(z_1, z_2, 0) \simeq D^{j'_1}_{\mathrm{h}}(z_1, 0) D^{j'_2}_{\mathrm{h}}(z_2, 0)\,, \tag{43}$$

if the restrictions $(z_1 + z_2 < 1)$ imposed by the law of conservation of longitudinal momentum are disregarded. This approach results in

$$D^{ij}_{\mathrm{h(fact)}}(x_1, x_2, t) \simeq D^{i}_{\mathrm{h}}(x_1, t) D^{j}_{\mathrm{h}}(x_2, t) \tag{44}$$

and provides a partial justification of the factorization hypothesis for the two-parton distribution functions, which is widely employed in the literature in the form

$$D^{j_1 j_2}_{\mathrm{h(fact)}}(x_1, x_2, t) = D^{j_1}_{\mathrm{h}}(x_1, t) D^{j_2}_{\mathrm{h}}(x_2, t)(1 - x_1 - x_2)\,. \tag{45}$$

The longitudinal momentum factor $(1 - x_1 - x_2)$ was actually introduced 'by hand' to smoothly zero the two-parton distribution functions at $x_1 + x_2 \to 1$ as required by the law of conservation of momentum.

Equations (40) and (42) show that, even if the two-parton distribution functions are factorized (43) on a certain scale, this factorization is inevitably broken as a result of evolution, and additional dynamic correlations emerge, which was first noticed in [69–71]. A similar conclusion can be reached in the case of triple parton distributions in hadrons, which were studied in detail in [48], but they are very cumbersome and, therefore, not presented here.

4.5 Estimation of correlations

It is of interest for practical applications to determine the magnitude of induced correlations (42) in comparison with the factorization component (41). This problem was partially investigated in a theoretical approach in Ref. [72]. If moments of the distribution functions are introduced,

$$M^{j_1 j_2}_{\mathrm{h}}(n_1, n_2, t) = \int_0^1 \mathrm{d}x_1\, \mathrm{d}x_2\, \theta(1 - x_1 - x_2) \times x_1^{n_1} x_2^{n_2} D^{j_1 j_2}_{\mathrm{h}}(x_1, x_2, t)\,, \tag{46}$$

the evolution integrodifferential equations are transformed into systems of ordinary differential equations of the first order with constant coefficients, whose solutions are found in explicit form. Then, the inverse Mellin transform

$$x_1 x_2 D^{j_1 j_2}_{\mathrm{h}}(x_1, x_2, t) = \int \frac{\mathrm{d}n_1}{2\pi \mathrm{i}}\, x_1^{-n_1} \int \frac{\mathrm{d}n_2}{2\pi \mathrm{i}}\, x_2^{-n_2} M^{j_1 j_2}_{\mathrm{h}}(n_1, n_2, t) \tag{47}$$

enables the distribution functions to be determined in the x-representation. Integration is carried out in this case along the imaginary axis to the right of all singularities in n, and in the general case is not possible in an explicit form. However, the asymptotic behavior can be assessed: as t increases, the second term in solution (40) for finite x_1 and x_2 becomes dominant [72], and thus the asymptotic behavior of two-parton distribution functions does not depend on the a priori unknown two-parton initial data $D^{j_1 j_2}_{\mathrm{h}}(x_1, x_2, 0)$.

The asymptotic behavior only indicates a trend, but does not provide any information about the values of x_1, x_2, and $t(Q)$, starting from which the correlations caused by the QCD evolution become significant (especially since the asymptotic behavior occurs in terms of the double-logarithmic variable t). The answer to this specific question can be obtained as well. The contribution of evolution-induced correlations in comparison with the factorization component was first estimated numerically in Ref. [70]. The initial data required in these calculations of single parton distribution functions $D^{j}_{\mathrm{h}}(x, 0)$ were specified at the scale $Q_0 = \mu = 1.3$ GeV in accordance with the CTEQ parameterization (Coordinated Theoretical Experimental project on QCD) [73]. Results for the ratio of gg-correlations in the proton, which emerge as a result of evolution, and the factorization component

$$R(x, t) = \left.\frac{D^{\mathrm{gg}}_{\mathrm{p(QCD,\,corr)}}(x_1, x_2, t)}{D^{\mathrm{g}}_{\mathrm{p}}(x_1, t) D^{\mathrm{g}}_{\mathrm{p}}(x_2, t)(1 - x_1 - x_2)^2}\right|_{x_1 = x_2 = x} \tag{48}$$

are shown in Fig. 3. It can be seen that, at the scale $Q \sim 5$ GeV, which is typical of measurements of the CDF collaboration, ratio (48) is about 10% and increases to 30% at the scale $Q \sim 100$ GeV, accessible for measurements at the LHC, for fractions of the longitudinal momentum partons $x \leqslant 0.1$. For fractions of longitudinal momentum $x \sim 0.2 - 0.4$, correlations are large, up to 90%. They become noticeable with increasing $t(Q)$ for an ever larger range of x in accordance with the asymptotic behavior obtained in [72].

It should be noted that, instead of the longitudinal momentum factor $(1 - x_1 - x_2)$, which 'zeroes out' in a smooth manner the product of two single parton distribution functions at $x_1 + x_2 \to 1$ in factorization ansatz (45), in calculating ratio (48), the factor $(1 - x_1 - x_2)^2$ was introduced with a higher power of zero at $x_1 + x_2 \to 1$, as follows from the evolution equations. Integration over the phase volume alone leads in (41) and (42) to the second power of zero:

$$\int_{x_1}^{1-x_2} \mathrm{d}z_1 \int_{x_2}^{1-z_1} \mathrm{d}z_2 = \frac{(1 - x_1 - x_2)^2}{2}\,.$$

This power should actually be greater than two and depend on t, increasing with the growing hardness of the process, as is the case for single distribution functions in the regime $x \to 1$ [61, 74]. A similar result was obtained for two-particle fragmentation functions in the asymptotic regime $x_1 + x_2 \to 1$ in [75]. Numerical results confirm that the power of zero at $x_1 + x_2 \to 1$ is greater than two and increases with increasing $t(Q)$, as is shown in Fig. 3. However, the introduced factor $(1 - x_1 - x_2)^2$ barely affects relation (48) in the region of relatively small fractions of longitudinal momenta x_1, x_1, which is the most interesting from the experimental perspective, since it is from this region that the contribution to the cross section from multiple interactions may be experimentally extracted.

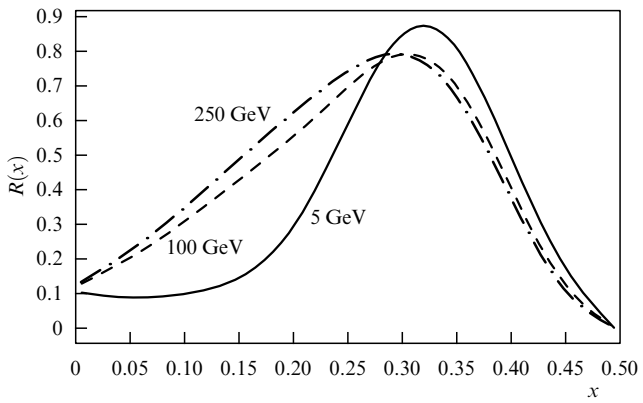

Figure 3. Ratio of the evolution-induced correlations to the factorization component for double gluon–gluon distributions in the proton as a function of $x = x_1 = x_2$ for $Q = 5$ GeV (solid curve), $Q = 100$ GeV (dashed curve), and $Q = 250$ GeV (dashed-dotted curve).

The properties of double parton distributions in hadrons were investigated in more detail later using direct integration of evolution equations (39) [76, 77] (in [77], only homogeneous equations). This method turned out to be more efficient in numerical calculations than using solutions (40) of these equations in the form of convolutions of single distribution functions, since in this case there is no need to deal with singular Green's functions (single parton-level functions with singular δ-like initial conditions). Available programs allow tabulating double parton distribution functions in a wide range of variables: $10^{-6} < x_1 < 1$; $10^{-6} < x_2 < 1$; $1 < Q^2 < 10^9$ GeV^2. The initial data for double distributions are specified in this case in an improved 'factorized' form, taking into consideration additional constraints that follow from the momentum and quark sum rules:

$$D_{\mathrm{h}}^{j_1 j_2}(x_1, x_2, 0) = D_{\mathrm{h}}^{j_1}(x_1, 0) D_{\mathrm{h}}^{j_2}(x_2, 0)(1 - x_1 - x_2)^2 \times (1 - x_1)^{-2-\alpha(j_1)}(1 - x_2)^{-2-\alpha(j_2)}\,, \tag{49}$$

where $\alpha(j) = 0$ for sea partons and $\alpha(j) = 0.5$ for valence partons. The point is that evolution equations (39), due to the properties of their kernels, ensure conservation of both the total momentum of partons and the number of valence quarks (as is the case for single distributions):

$$\sum_{j_1} \int_0^{1-x_2} \mathrm{d}x_1\, x_1 D_{\mathrm{h}}^{j_1 j_2}(x_1, x_2, t) = (1 - x_2) D_{\mathrm{h}}^{j_2}(x_2, t)\,, \tag{50}$$

$$\int_0^{1-x_2} \mathrm{d}x_1\, D_{\mathrm{h}}^{j_1 j_2}(x_1, x_2, t) = N D_{\mathrm{h}}^{j_2}(x_2, t)\,, \tag{51}$$

where $N = N_{j_{1\mathrm{v}}}$ for j_2 not equal to j_1 or $\bar{j}_1$; $N = N_{j_{1\mathrm{v}}} - 1$ for $j_2 = j_1$; $N = N_{j_{1\mathrm{v}}} + 1$ for $j_2 = \bar{j}_1$; $j_{1\mathrm{v}} \equiv j_1 - \bar{j}_1$ (j_1 is not equal to g); and $N_{j_{1\mathrm{v}}}$ is the number of valence quarks of type j_1 in the proton (hadron). It should be noted that without additional factors $(1 - x_1)^{-2-\alpha(j_1)}(1 - x_2)^{-2-\alpha(j_2)}$ in the improved 'factorization' ansatz (49) neither the momentum (50) no quark (51) sum rules may be satisfied for two-parton initial data.

It is also worth noting here that the particular solution (42) of inhomogeneous equations contributes to the inclusive cross section for two-parton scattering with a weight larger (with a different effective cross section [78–84]) than the solution of homogeneous equations (41). This issue is discussed in more detail in Section 5. Solutions to homogeneous equations, as noted, are usually taken in factorization form, considered to be a good approximation to the exact solution if there are no nonperturbative correlations. These initial correlation conditions are not known *a priori*, but they are not entirely arbitrary either, since they must satisfy nontrivial sum rules [76, 85, 86], which are preserved in the course of evolution. The problem of setting 'correct' initial data for evolution equations with the correct behavior near kinematic boundaries has also been actively studied and discussed [76, 85, 87–92].

5. Cross sections for double and triple parton scattering taking into consideration QCD evolution

Numerous successful attempts have been made over the last decade [24, 78–83, 93–98] to take into account in hadron-hadron collisions the evolution-related dynamic QCD correlations discussed in Section 4. Solutions to evolution equations for double distribution functions are the sum of solutions for homogeneous and inhomogeneous equations. The former describes the independent (concurrent) evolution of two branches of the parton cascade: one branch contains a parton with a fraction of the longitudinal momentum x_1 in the final state, while the other, a parton with x_2. Solutions of inhomogeneous equations support the feasibility of splitting one-parton evolution (one branch j splits into two different branches: j_1 and j_2) with the standard splitting kernel $P_{j \to j_1 j_2}(z)$. These two different structures generate three different contributions to the inclusive cross section of double parton scattering [78]:

$$\sigma_{\mathrm{hh}' \to a_1 a_2}^{\mathrm{DPS}} = \sigma_{\mathrm{hh}' \to a_1 a_2}^{\mathrm{DPS},\,1\times1} + \sigma_{\mathrm{hh}' \to a_1 a_2}^{\mathrm{DPS},\,1\times2+2\times1} + \sigma_{\mathrm{hh}' \to a_1 a_2}^{\mathrm{DPS},\,2\times2}\,, \tag{52}$$

where

$$\sigma_{\mathrm{hh}' \to a_1 a_2}^{\mathrm{DPS},1\times1} = \frac{c}{2} \sum_{i,j,k,l} \int \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_1'\, \mathrm{d}x_2' \int F_{2\mathrm{g}}^4(q)\, \frac{\mathrm{d}^2 q}{(2\pi)^2} \times D_{\mathrm{h}}^i(x_1; \mu^2, Q_1^2) D_{\mathrm{h}}^j(x_2; \mu^2, Q_2^2) \hat{\sigma}_{a_1}^{ik}(x_1, x_1') \times \hat{\sigma}_{a_2}^{jl}(x_2, x_2') D_{\mathrm{h}'}^k(x_1'; \mu^2, Q_1^2) D_{\mathrm{h}'}^l(x_2'; \mu^2, Q_2^2)\,, \tag{53}$$

$$\begin{aligned}\sigma_{\mathrm{hh}' \to a_1 a_2}^{\mathrm{DPS},\,1\times2+2\times1} &= \frac{c}{2} \sum_{i,j,k,l} \int \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_1'\, \mathrm{d}x_2' \int^{\min(Q_1^2, Q_2^2)} F_{2\mathrm{g}}^2(q)\, \frac{\mathrm{d}^2 q}{(2\pi)^2} \\ &\times \big[D_{\mathrm{h}}^i(x_1; \mu^2, Q_1^2) D_{\mathrm{h}}^j(x_2; \mu^2, Q_2^2) \hat{\sigma}_{a_1}^{ik}(x_1, x_1') \\ &\times \hat{\sigma}_{a_2}^{jl}(x_2, x_2') D_{\mathrm{h}'2}^{kl}(x_1', x_2'; q^2, Q_1^2, Q_2^2) \\ &+ D_{\mathrm{h}2}^{ij}(x_1, x_2; q^2, Q_1^2, Q_2^2) \hat{\sigma}_{a_1}^{ik}(x_1, x_1') \\ &\times \hat{\sigma}_{a_2}^{jl}(x_2, x_2') D_{\mathrm{h}'}^k(x_1'; \mu^2, Q_1^2) D_{\mathrm{h}'}^l(x_2'; \mu^2, Q_2^2) \big]\,,\end{aligned} \tag{54}$$

$$\sigma_{\mathrm{hh}' \to a_1 a_2}^{\mathrm{DPS},\,2\times2} = \frac{c}{2} \sum_{i,j,k,l} \int \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_1'\, \mathrm{d}x_2' \int^{\min(Q_1^2, Q_2^2)} \frac{\mathrm{d}^2 q}{(2\pi)^2} \times D_{\mathrm{h}2}^{ij}(x_1, x_2; q^2, Q_1^2, Q_2^2) \hat{\sigma}_{a_1}^{ik}(x_1, x_1') \times \hat{\sigma}_{a_2}^{jl}(x_2, x_2') D_{\mathrm{h}'2}^{kl}(x_1', x_2'; q^2, Q_1^2, Q_2^2)\,, \tag{55}$$

$$D_{\mathrm{h}2}^{ij}(x_1, x_2; q^2, Q_1^2, Q_2^2) = \sum_{j'j_1'j_2'} \int_{q^2}^{\min(Q_1^2, Q_2^2)} \mathrm{d}k^2 \, \frac{\alpha_{\mathrm{s}}(k^2)}{2\pi k^2} \int_{x_1}^{1-x_2} \frac{\mathrm{d}z_1}{z_1} \int_{x_2}^{1-z_1} \frac{\mathrm{d}z_2}{z_2} \times D_{\mathrm{h}}^{j'}(z_1 + z_2; \mu^2, k^2) \frac{1}{z_1 + z_2} P_{j' \to j_1' j_2'}\left(\frac{z_1}{z_1 + z_2}\right) \times D_{j_1'}^{i}\left(\frac{x_1}{z_1}; k^2, Q_1^2\right) D_{j_2'}^{j}\left(\frac{x_2}{z_2}; k^2, Q_2^2\right), \quad (56)$$

where $\alpha_{\mathrm{s}}(k^2) = g^2(k^2)/(4\pi)$ is the running coupling constant. The single distribution functions $D_{\mathrm{h}}^{i}(x_1; \mu^2, Q_1^2)$, presented here explicitly as functions of both scales, are solutions of DGLAP equations with given initial conditions $D_{\mathrm{h}}^{i}(x_1; \mu^2)$ on the initial scale μ^2. They can be defined through Green's functions $D_{i'}^{i}(z; \mu^2, Q^2)$:

$$D_{\mathrm{h}}^{i}(x; \mu^2, Q^2) = \sum_{i'} \int_x^1 \frac{\mathrm{d}z}{z} \, D_{\mathrm{h}}^{i'}(z; \mu^2) \, D_{i'}^{i}\left(\frac{x}{z}; \mu^2, Q^2\right). \quad (57)$$

It should be noted from the very beginning that the three terms (53)–(55) contribute to the inclusive cross section with different dimensional (geometric) weights, which are referred to as effective cross sections. For example, the first (1×1) component describes the production of two hard particles, a_1 and a_2, as a result of interaction between two parton pairs from two independent branches of parton cascades. The probability of such a double parton interaction depends on the spatial distribution (in the transverse plane of the impact parameter) of the parton-cascade branches. This spatial distribution is governed in the momentum representation by a two-parton form factor (primarily two-gluon at small x) $F_{2\mathrm{g}}$ [52]. Having integrated over q, we obtain

$$\int F_{2\mathrm{g}}^4(q) \, \frac{\mathrm{d}^2 q}{(2\pi)^2} = \frac{1}{\sigma_{\mathrm{eff,\,DPS}}}. \quad (58)$$

The quantity $\sigma_{\mathrm{eff,\,DPS}}$, usually called the effective cross section, characterizes the transverse area occupied by the partons that are involved in the hard collision.

Thus, the inclusive cross section of double parton scattering depends on the spatial correlations between two partons in the wave function of the colliding proton (hadron). Due to the strong ordering of transverse momenta in the course of DGLAP evolution, the position of partons in the transverse plane of the impact parameter b_t is actually 'frozen', and the form factor $F_{2\mathrm{g}}$ describes the initial b_t-distribution formed in the nonperturbative region at a certain scale smaller than the scale μ^2 from which DGLAP evolution begins.

However, there is another type of correlation that is due to the splitting of one branch of the parton cascade into two branches. The splitting at the scale k^2 results in the emergence of two branches with a relatively small transverse spatial separation, $\delta b_t^2 \sim 1/k^2$. This effect of a purely perturbative origin can significantly increase the inclusive cross section for double parton scattering and result in a significant decrease in the effective cross section $\sigma_{\mathrm{eff,\,DPS}}$ (extracted from relation (9)) in comparison with the value estimated in the geometric representation (10).

Depending on the kinematics of the process under consideration and experimental cutoffs, the study may be focused on exploring either two-parton correlations that are due to the nonperturbative region, i.e., the (1×1) component, or correlations of a purely perturbative origin, i.e., those that are due to the (2×2) component.

The contribution of the mixed $(1 \times 2 + 2 \times 1)$ component is governed by the form factor $F_{2\mathrm{g}}(q)$ from the side of one incident proton and perturbative splitting from the other. Since the form factor $F_{2\mathrm{g}}(q)$ alone ensures the rapid convergence of the integral over q in the region of small $q^2 < \mu^2$, here we are actually dealing with primarily long-range correlations from the nonperturbative region.

The (2×2) component has been actively discussed [79–82, 93–97]; however, in our opinion [82], this discussion is primarily of a terminological nature. This contribution includes two splitting kernels, and the integration over q does not contain the strong suppressing factor $F_{2\mathrm{g}}(q)$. Formally, this contribution in the collinear approach in the region of not too small x should be considered a result of the interaction of a *single* pair of partons that includes a $2 \to 4$ hard subprocess [79–82, 93, 94], since the dominant contribution to the phase volume is due to the region of large $q^2 \sim \min(Q_1^2, Q_2^2)$. However, as shown in [78, 82], configurations are possible with a relatively large interval between the splitting point $(k^2, z_1 + z_2)$ and the (momentum) coordinates (Q_i^2, x_i) of the hard subprocess. Should this be the case, an evolution (described either by DGLAP, if $k^2 \ll Q_i^2$, or Balitskii–Fadin–Kuraev–Lipatov (BFKL) [63, 99–101] if $z_1 + z_2 \gg x_i$) with the production of many secondary particles becomes possible. The corresponding process cannot be described by a $2 \to 4$-hard matrix element, and all contributions should be taken into account in Eqn (52), especially in the case of configurations with completely different scales (for example, $Q_1^2 \ll Q_2^2$).

The structure of the additional contributions $(1 \times 2 + 2 \times 1)$ and (2×2) that contain one and two splitting kernels, respectively, was investigated in the double logarithmic approximation in Ref. [82], where the kinematic region is specified in which the (2×2) component should be attributed to double parton scattering. The problem of so-called double counting was studied in detail later with a prescription for its solution [96, 97] and confirmation of the need to consider the (2×2) component as double parton scattering in the region of sufficiently small longitudinal momenta.

The formulas for calculating the inclusive cross section for triple parton scattering taking into consideration the QCD evolution were also obtained in [48] with an estimate of geometric factors for various contributions generated by solutions of the corresponding homogeneous and inhomogeneous equations, and a discussion of the double counting problem; however, the corresponding expressions are cumbersome and not displayed here.

6. Phenomenological impacts of QCD evolution

As noted, the double parton distribution functions as a result of evolution become strongly correlated at large Q^2 and finite fractions of the longitudinal momenta x_1 and x_2, even if they were factorized on the initial scale $Q_0^2 = \mu^2$. However, experimental data do not yet support the presence of strong correlations in the longitudinal momentum fractions, which can be explained by the fact that the kinematic region of variables (relatively small x and Q^2) accessible for investigation is still far from the asymptotic region where the dynamic QCD correlations dominate. With an increase in luminosity in the LHC, it will be possible to study double parton

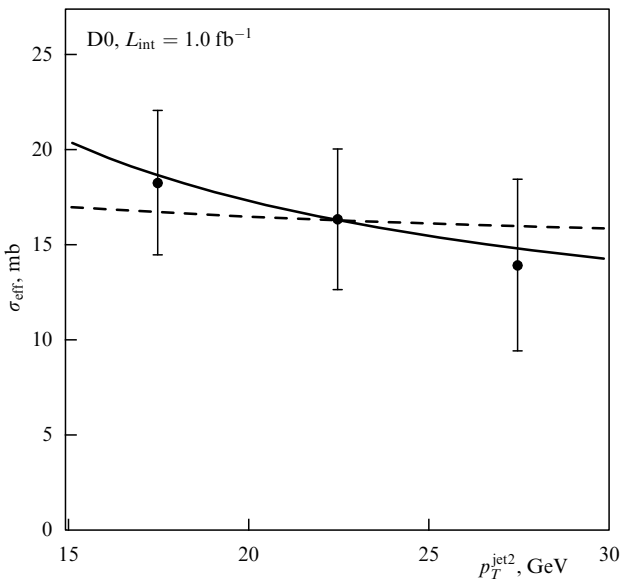

Figure 4. Effective cross section $\sigma^{\rm exp}_{\rm eff,\,DPS}$ measured for three values of the transverse momentum of the second jet $p_T^{\rm jet2}$ (taken from [29]). The solid curve ($k = 0.5$) and dashed curve ($k = 0.1$) present calculations made using Eqn (60) with $p_{T0}^{\rm jet2} = 22.5$ GeV and $\sigma^0_{\rm eff} = 16.3$ mb.

scattering in the regime of large x and Q^2, in which correlations can manifest themselves in a direct way.

It was shown in Ref. [102] that measurements by the D0 collaboration [29] of the effective cross section $\sigma^{\rm exp}_{\rm eff,\,DPS}$, presented as a function of the momentum of a second jet $p_T^{\rm jet2}$ (ordered in the transverse momentum), can be regarded as the first indirect indication of the evolution of double parton distributions. Figure 4 shows that this cross section tends to decrease with an increase in the hardness of the process, which can be characterized in this case by the transverse momentum of the jet. The dimensional geometric factor $\sigma_{\rm eff,\,DPS}$ contains at first glance information about the nonperturbative structure of the proton and reflects the distribution of parton matter in the overlap region of two colliding hadrons; it therefore should not depend a priori on the hardness of the parton subprocess. However, in both CDF and D0 experiments, the effective cross section $\sigma^{\rm exp}_{\rm eff,\,DPS}$ was not measured directly, but was calculated (extracted) using the normalization to the product of two single cross sections (as in Eqn (9)):

$$\frac{\sigma^{\gamma+3j}_{\rm DPS}}{\sigma^{\gamma j}\sigma^{jj}} = (\sigma^{\rm exp}_{\rm eff,\,DPS})^{-1}\,, \tag{59}$$

where $\sigma^{\gamma+3j}_{\rm DPS}$ is the inclusive cross section of events with a photon and three jets in the final state, which are produced in double parton scattering, $\sigma^{\gamma j}$ is the inclusive cross section of the process with a photon and a jet in the final state, and σ^{jj} is the inclusive cross section of two-jet events. It should be noted that neither collaboration has used any theoretical calculations (predictions) for inclusive cross sections but compared the number of double parton events with a photon and three jets in the final state observed in one hard $p\bar{p}$ collision with the number of events with a photon and three jets in the interactions that occur in two different $p\bar{p}$ collisions.

With this normalization (59) and the presence of additional correlations in double parton distributions, the experimentally 'extracted' $\sigma^{\rm exp}_{\rm eff,\,DPS}$ will differ from $\sigma_{\rm eff,\,DPS}$ determined from Eqn (10) in terms of the parton matter density $f(b)$ in the plane of the impact parameter, and will depend on the hardness of the process. A functional form of the dependence of $\sigma^{\rm exp}_{\rm eff,\,DPS}$ on $p_T^{\rm jet2}$, inspired by the explicit form of the evolutionary variable t and the solution of evolutionary equations, was proposed in [102],

$$\sigma^{\rm exp}_{\rm eff,\,DPS} = \sigma^0_{\rm eff}\left(1 + k\ln\frac{p_T^{\rm jet2}}{p_{T0}^{\rm jet2}}\right)^{-1}\,, \tag{60}$$

and is shown in Fig. 4 for $k = 0.1$ (dashed line) and $k = 0.5$ (solid line). These curves illustrate two possible behaviors of $\sigma^{\rm exp}_{\rm eff,\,DPS}$: a barely noticeable decrease (virtually constant value) and a distinct decrease in the measurement area. The normalization point $p_{T0}^{\rm jet2} = 22.5$ GeV with $\sigma^0_{\rm eff} = 16.3$ mb was fixed to reproduce the measured value in the central bin $p_T^{\rm jet2}$. The measurements carried out in three $p_T^{\rm jet2}$ ranges do not yet enable making assertions about a serious check of the 'prediction' (60) with two fitting parameters, $\sigma^0_{\rm eff}$ and k.

Thus, in contrast to naïve expectations, the existence of additional dynamic QCD correlations implies the dependence of the experimentally extracted effective cross section $\sigma^{\rm exp}_{\rm eff,\,DPS}$ on the energy scale of the resolution, an observation that has been confirmed in subsequent theoretical studies [78, 83, 98, 103]. Moreover, the relative contribution of the mixed $(1 \times 2 + 2 \times 1)$ component (54) to the inclusive cross section of double parton scattering proves to be numerically significant [83, 98] at LHC energies and grows as the energy scale increases. The first estimates of these additional contributions to the inclusive cross section of actual processes were made in Ref. [84] in a simplified phenomenological approach with an encouraging conclusion regarding their experimental observability. A close estimate for the observability frequency of so-called joint interactions [19] was also obtained using the Pythia generator [49]. It was shown in [83] that taking into account additional dynamic QCD correlations (three-parton interactions in the author's terminology) provides a resolution to a long-standing problem: why the observed inclusive cross section of double parton scattering is twice as large as its estimate made in the approximation of independent parton interactions (factorization (1×1) component (53) alone with a 'standard' geometric transverse profile and proton radius).

As noted in the introduction, the contribution from double parton scattering has been by now reliably measured and picked out [24] in a number of processes that contain in the final state heavy quarks (c, b), quarkonia ($J/\psi, \Upsilon$), jets, and gauge bosons (γ, W, Z). There are also many theoretical estimates of cross sections for rarer processes, the observation and study of which will become possible with an increase in LHC luminosity and at higher energies of the colliders under design. Most of these estimates have been made in the approximation of independent parton interactions (the factorization (1×1) component alone (53), but with a typical value $\sigma_{\rm eff,\,DPS} \simeq 15 \pm 5$ mb rather than with the 'standard' geometric transverse profile and radius of the proton, to effectively take into account additional nonfactorization contributions). However, more detailed calculations of the cross sections are now possible in the recently developed Monte Carlo model [104] of double parton scattering, which explicitly takes into account evolution effects.

In concluding this section, attention should also be paid to processes with the production of two W-bosons with the same-sign electric charge and the production of a W-boson accompanied by heavy mesons with the same-sign electric charge: despite the low probability of such events, these

processes were proposed in [105–108] as the most suitable and 'purest' for studying the mechanism of double parton scattering owing to a significantly lower relative background from single scattering. Moreover, if two same-sign W bosons are produced, the dynamic QCD correlations can manifest themselves in nontrivial kinematic correlations [106] of these bosons by their rapidity, which are absent in the approximation of independent parton interactions. Processes with the production of two heavy quarkonia in various combinations are also very promising candidates [24, 109–111] for studying double parton scatterings.

7. Conclusions

Two-parton distribution functions in the leading logarithmic approximation of QCD perturbation theory satisfy certain evolutionary equations. It follows from these equations that the factorization hypothesis, which is often used for two-parton distribution functions in the analysis of experimental data on double parton scattering, may be taken as a first approximation in a limited range of variation in the hardness of the process. The momentum and quark sum rules, which are preserved during evolution due to the properties of the kernels of the equations, yield additional constraints on the form of the 'factorization' ansatz for the initial data. As the hardness of the process grows, the contribution of the emerging dynamic QCD correlations to the two-parton distribution functions and the inclusive double parton scattering cross section increases in comparison with the factorization component, behavior that gives hope for their direct observation.

The effective cross section $\sigma_{\text{eff, DPS}}^{\text{exp}}$ measured by the D0 collaboration [29], if represented as a function of the momentum of the second jet p_T^{jet2}, may be regarded as the first indirect indication of the evolution of double parton distributions. Measurements in a wider range of variation of the hardness of the process with lower experimental uncertainties are needed to observe this effect more clearly and compare it in detail with calculations [83], which explicitly take into account dynamic correlations (three-parton interactions).

It should be noted that double parton scattering is also actively discussed in proton–nucleus (pA) collisions [42, 112–126], since their relative contribution increases in comparison with the naïve scaling expectation, and new unique options for the further study and measurement of QCD momentum correlations emerge. The latest achievements in and prospects for these studies may be found in review [47]. Nucleus–nucleus (AA) collisions are less interesting [47, 124] for studying multiparton interactions, since in this case multiple nucleon–nucleon scatterings that reflect the well-known distribution of nucleons in the nucleus rather than the parton distribution in the nucleon of interest to us dominate.

As the collision energy increases, the region of increasingly smaller fractions of longitudinal parton momenta in which parton densities strongly increase becomes accessible for exploration, and it is necessary to go beyond the standard approaches [63, 127–129]. In relation to this, the issue of the relationship between competing effects — an increase in parton densities due to the splitting of the parton cascade branches vs. their decrease as a result of parton diffusion by rapidity [129] — becomes especially interesting.

The results covered in this review emphasize the importance of a profound understanding of the dynamics of multiparton interactions in hadronic collisions in operating and projected colliders. This observation refers to both investigating basic QCD phenomena and the background for the search for new physics in rare events in which many heavy particles and particles with large transverse momenta are produced.

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