

METHODOLOGICAL NOTES

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Identity of the mechanisms of Weibel and Alfvén-cyclotron plasma instabilities

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Contents

1. **Introduction** 611
2. **Weibel instability** 612
3. **Alfvén ion-cyclotron instability** 613
   3.1 Laboratory reference frame K; 3.2. Rotating reference frame K′
4. **Conclusions** 615
5. **Appendix. Rigorous treatment of Alfvén ion-cyclotron waves in a rotating frame of reference** 615
   References 616

Abstract. **A plasma with an anisotropic velocity distribution of particles in a magnetic field is considered. It is shown that the Weibel instability arises in the reference frame rotating together with the particles, for example, ions. When considered in the immobile reference frame, this instability is known as the Alfvén cyclotron instability.**

Keywords: Weibel instability, Alfvén cyclotron instability, filamentation

1. Introduction

For definiteness, we consider the Alfvén ion-cyclotron (AIC) instability, which is one of the electromagnetic (nonpotential) plasma instabilities in a magnetic field.

Let ions with mass M, charge $q = Ze$, and volume density $n_{\rm i}$, which are charge-neutralized by electrons, have zero longitudinal (along the magnetic vector $\mathbf{B}_0$ in the plasma: hereinafter, this is the z-axis) and the same transverse velocities,

$$u \ll c\,. \tag{1}$$

Here, e and c are the elementary charge and the speed of light, respectively. According to theoretical [1–3] and experimental [4–8] results (see also reviews [9, 10]), the AIC instability plays the leading role in the case of a high value of the dimensionless parameter

$$\beta = \frac{4\pi n_{\rm i}\,(Mu^2/2)}{B_0^2}\,. \tag{2}$$

The excess energy of transverse motion results in an amplification of the AIC wave propagating along the magnetic field ($\mathbf{k}\,\|\,\mathbf{B}_0$). The AIC instability arises from random fluctuations, when the radiation of AIC waves by rotating ions prevails over their absorption [11]. A characteristic consequence of the AIC instability is the isotropization of the ion velocity distribution, which was determined in open trap experiments [4–8]. Suppressing the AIC instability also calls for taking special precautions in tokamaks [12–14].

The dispersion equation, which determines the dependence of the AIC wave frequency on the wave vector $\mathbf{k}$, is of the form

$$\tau\left[1+\frac{\beta}{(\Omega-1)^2}\right]+\frac{\Omega^2}{\Omega-1}=0\,, \tag{3}$$

where we introduced dimensionless quantities

$$\tau = \left(\frac{kc}{\omega_{\rm i}}\right)^2, \quad \Omega = \frac{\omega}{\omega_{B\rm i}}$$

and took into account the plasma quasineutrality condition. In accordance with this condition, the relation $n_{\rm e} = Zn_{\rm i}$ is satisfied, according to which the following relation holds:

$$\frac{\omega_{\rm e}^2}{\omega_{B\rm e}} = \frac{\omega_{\rm i}^2}{\omega_{B\rm i}}\,.$$

Here, $n_{\rm e}$ is the electron volume density, $\omega_{B\rm i} = qB_0/Mc$ and $\omega_{\rm i} = \sqrt{4\pi n_{\rm i}q^2/M}$ are the Larmor and plasma ion frequencies respectively, $\omega_{B\rm e} = eB_0/mc$ and $\omega_{\rm e} = \sqrt{4\pi n_{\rm e}e^2/m}$ are similar quantities for the electrons, and m is the electron mass.

For $\beta \to 0$, expression (3) corresponds to an AIC wave with a constant amplitude, which travels along the direction

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of the magnetic field. In particular, for $\omega \ll \omega_{Bi}$, we obtain the well-known dispersion law for low-frequency AIC waves: $\omega = u_A k$, where $u_A = B/\sqrt{4\pi\rho}$ and $\rho \approx M n_i$ is the density of the plasma substance. The last term in the square brackets in Eqn (3) imparts the instability property to this wave. As a result, its amplitude grows exponentially. In the limiting case

$$\beta \ll 1\,, \tag{4}$$

this term, which is responsible for the instability, becomes significant for $\omega \approx \omega_{Bi}$, which corresponds to

$$\Omega \approx 1\,. \tag{5}$$

Consequently, the AIC instability develops primarily in domain (5), and the solution is therefore sought in the form $\Omega = 1 + \Delta$, where $|\Delta| \ll 1$. As a result, for the branch of unstable oscillations, we obtain

$$\Omega \approx 1 + \mathrm{i}\sqrt{\beta - \frac{1}{4\tau^2}}\,. \tag{6}$$

Formula (6) holds when $\tau \gg 1$. It is seen from Eqn (6) that the AIC instability buildup increment reaches its maximum for $\tau \gg 1/\sqrt{\beta}$ and amounts to

$$\gamma = \sqrt{\beta}\,\omega_{Bi} = \frac{u\omega_i}{c\sqrt{2}}\,. \tag{7}$$

Thus, it follows that the AIC instability develops rapidly in comparison with the ion-ion collision time in the majority of cases of practical interest.

In the literature, the Weibel instability (WI) and the AIC instability are treated separately from each other and are considered to be different in nature. In particular, in books on plasma physics known to the authors, these two instabilities are considered in different sections unrelated to each other. We show that the WI and AIC instability mechanisms are identical.

2. Weibel instability

The WI is known to arise in a plasma with an anisotropic particle distribution in the absence of a magnetic field, i.e., for $\mathbf{B}_0 = 0$ [17]. A simple and lucid explanation for the mechanism of this instability is given in Refs [18, 19]. We briefly reproduce the conclusions given there, as we will need them below. For definiteness, we consider the ion motion, although the WI also develops on electrons, and does so much faster.

Let an electromagnetic wave with electric and magnetic fields (Fig. 1) related by Faraday's law of electromagnetic induction propagate along the z-axis in the plasma:

$$\begin{cases} \mathbf{E} = E(t)\,(\cos\eta,\, 0,\, 0)\,, \\ \mathbf{B} = \dfrac{kc}{\omega}\,E(t)\,(0,\, \cos\eta,\, 0)\,. \end{cases} \tag{8}$$

Here, $\eta = kz - \omega t$ and $E(t)$ is the wave amplitude varying slowly in comparison with the frequency ω.

A detailed description of many plasma instabilities may be obtained proceeding from the Vlasov equation for the particle distribution function $f(\mathbf{r}, \mathbf{V}_p, t)$, which defines the number dN of particles that are in a volume element d^3r and have velocities $\mathbf{V}_p$ in an interval d^3V_p, $dN = f(\mathbf{r}, \mathbf{V}_p, t)\,d^3r\,d^3V_p$. When it is known, the particle density $n(\mathbf{r}, t)$ and the electric current density $\mathbf{j}(\mathbf{r}, t)$ the particles produce at a given point r in

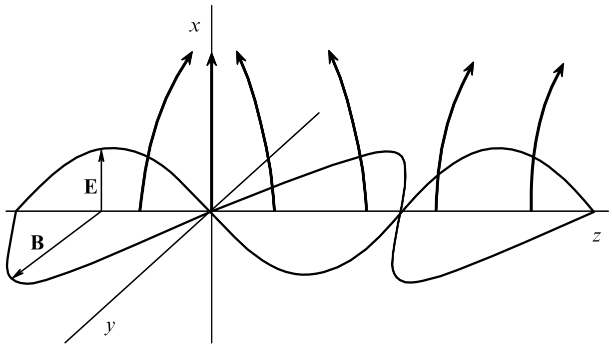

Figure 1. Weibel instability.

time t are expressed as

$$\begin{aligned} n(\mathbf{r},\, t) &= \int f(\mathbf{r}, \mathbf{V}_p, t)\,d^3V_p\,, \\ \mathbf{j}(\mathbf{r},\, t) &= q\int \mathbf{V}_p f(\mathbf{r}, \mathbf{V}_p,\, t)\,d^3V_p\,. \end{aligned} \tag{9}$$

Next, it would suffice to consider functions of the form

$$f(\mathbf{r}, \mathbf{V}_p, t) = n(\mathbf{r}, t)\,\delta\left[\mathbf{V}_p - \mathbf{V}(\mathbf{r}, t)\right].$$

In this case, the Vlasov equation is equivalent to the ordinary equation of motion (see, e.g., Refs [20, 21])

$$\dot{\mathbf{V}} = \frac{q}{M}\,\mathbf{E} + \frac{q}{Mc}\,(\mathbf{V}\times\mathbf{B})$$

(here and below, the ions are considered to be nonrelativistic).

Following Refs [18, 19], we consider the case when the ions, prior to the appearance of the wave, possess the same velocity $\mathbf{u}$ directed along the x-axis. On appearance of the wave, the ion velocities change slightly: $\mathbf{V} = \mathbf{u} + \delta\mathbf{V}$, where $\delta\mathbf{V}$ satisfies the equation linearized in the wave amplitude:

$$\delta\dot{\mathbf{V}} = \frac{q}{M}\,\mathbf{E} + \frac{q}{Mc}\,(\mathbf{u}\times\mathbf{B})\,. \tag{10}$$

When projected on the axes, Eqn (10) takes the form

$$\begin{cases} \delta\dot{V}_x = \dfrac{q}{M}\,E\cos\eta\,, \\ \delta\dot{V}_y = 0\,, \\ \delta\dot{V}_z = \dfrac{qku}{M\omega}\,E\cos\eta\,. \end{cases} \tag{11}$$

We write the ion number conservation law directly in the form linearized in the wave amplitude:

$$\frac{\partial\delta n_i}{\partial t} + n_i\,\frac{\partial\delta V_z}{\partial z} = 0\,.$$

We differentiate this equation with respect to time to obtain, in view of the last equation of system (11),

$$\frac{\partial^2\delta n_i}{\partial t^2} - \frac{q n_i k^2 u}{M\omega}\,E\,(t)\sin\eta = 0\,.$$

We seek the solution in the form $E(t) \propto \exp(-\mathrm{i}\Omega t)$. For the AIC wave induced ion density variation, this yields a value

$$\delta n_i = -\frac{q n_i k^2 u}{M\omega\,\Omega^2}\,E\,(t)\sin\eta\,. \tag{12}$$

Therefore, in agreement with Fig. 1, layers with an excess and deficiency of ions appear, and the WI is sometimes referred to as the filamentation instability.

An additional electric current $\delta \mathbf{j} = (\delta j_x,\ 0,\ 0)$ arises along the x-axis,

$$\delta j_x = qu\,\delta n_{\mathrm{i}} = -\frac{q^2 n_{\mathrm{i}} k^2 u^2}{M\omega\,\Omega^2}\,E(t)\sin\eta\,. \tag{13}$$

In the nonrelativistic ion case (1), the displacement current may be disregarded, and therefore

$$\nabla\times\mathbf{B}\approx\frac{4\pi}{c}\,\delta\mathbf{j}\,. \tag{14}$$

Hence, in view of expressions (8) and (13), we obtain the relation

$$\frac{k^2 c}{\omega}E(t)\sin\eta\left(1+\frac{\gamma^2}{\Omega^2}\right)=0\,.$$

Consequently, an instability emerges with increment (7).

One can see from the above treatment that the main role in the onset of the WI is played by the presence of the magnetic field rather than the electric field, which, in addition, may be screened by electrons. An anisotropic particle velocity distribution is the necessary condition for the onset of the WI. That is why the WI also results from the field fluctuation of the form $\mathbf{E}=0$, $\mathbf{B}\neq 0$, which corresponds to the $\omega=0$ case in the example discussed above. In Refs [17–19] a field of this type was considered with $\mathbf{B}=B(t)(0,\cos(kz),0)$. In this case, instead of the last equation in system (11), we obtain

$$\delta\dot{V}_z=\frac{qu}{Mc}\,B(t)\cos(kz)\,, \tag{15}$$

and instead of formula (12), we obtain the equation

$$\delta n_{\mathrm{i}}=-\frac{qn_{\mathrm{i}}ku}{Mc\Omega^2}\,B(t)\sin(kz)\,, \tag{16}$$

whence there follows a conclusion about the WI development with increment (7) once again.

3. Alfvén ion-cyclotron instability

3.1 Laboratory reference frame K

We consider case (4). Bearing in mind relation (5), for ease of calculations we consider the resonance case

$$\omega=\omega_{B\mathrm{i}}\,. \tag{17}$$

Now,

$$\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1\,, \tag{18}$$

where $\mathbf{E}$ and $\mathbf{B}_1$ are the low (in comparison with $\mathbf{B}_0=(0,\,0,\,B_0)$) electric and magnetic fields of the AIC wave with projections on the X-, Y-, and Z-axes of frame K:

$$E_X=E(t)\cos\eta\,,\quad E_Y=E(t)\sin\eta\,,\quad E_Z=0\,, \tag{19}$$

$$\mathbf{B}_1=\frac{kc}{\omega}\,E(t)(-\sin\eta\,,\,\cos\eta\,,\,0)\,. \tag{20}$$

Here, as in Section 2, $\eta=kz-\omega t$ and $E(t)$ is the AIC wave amplitude varying slowly in comparison with the frequency ω. The corresponding condition is of the form

$$\frac{\omega E}{\dot{E}}\sim\frac{\omega}{\gamma}\gg 1\,.$$

It is satisfied in the case of inequality (4).

With condition (17), fields (19) and (20) rotate together with the ions with the same angular velocity. In the absence of an AIC wave, the projections of the velocity of an individual ion on the X-, Y-, and Z-axes are

$$\mathbf{u}(t,\,\theta)=u(\sin\chi\,,\,-\cos\chi\,,\,0)\,, \tag{21}$$

where $\chi=-\omega t+\theta$, θ is the angle between the x-axis and the ion velocity at the time $t=0$, $u=\omega\rho_0$, and ρ_0 is the radius of ion orbits. We first assume that angle θ is also the same for all ions. This ion motion is described by the distribution function

$$f(\mathbf{r},\mathbf{V}_{\mathrm{p}},t)=n_{\mathrm{i}}\delta\left(V_{\mathrm{p}x}-u\sin\chi\right)\delta\left(V_{\mathrm{p}y}+u\cos\chi\right)\delta\left(V_{\mathrm{p}z}\right).$$

The ion and electron charges are mutually compensated, and so the net charge density is equal to zero,

$$\rho_{\mathrm{tot}}=0\,. \tag{22}$$

The Larmor orbit dimensions for electrons are negligible compared to ρ_0, and therefore they hardly move in the XY plane. Consequently, the density of the ion current perpendicular to external magnetic field $\mathbf{B}_0$ is not compensated by electrons. In the frame K, according to expression (9), the ion current density $\mathbf{j}=qn_{\mathrm{i}}\,\mathbf{u}(t,\,\theta)$. Accordingly, the current density 4-vector in the plasma in the frame K is given by the expression

$$j^{\mathrm{i}}=(\rho_{\mathrm{tot}}c,\mathbf{j})=qn_{\mathrm{i}}u\,(0,\,\sin\chi,\,-\cos\chi,\,0)\,. \tag{23}$$

With the appearance of an AIC wave, the ion velocities change slightly: $\mathbf{V}=\mathbf{u}+\delta\mathbf{V}$, where the ion velocity variations $\delta\mathbf{V}$ are quantities of the first-order in the amplitude of the AIC wave. Since $u_Z=0$, the same is true for the V_Z velocity component itself. In the general case, the ion density also changes: $n_{\mathrm{i}}(\mathbf{r},t)=n_{\mathrm{i}}+\delta n_{\mathrm{i}}(\mathbf{r},\,t)$. The electrons move freely along the magnetic lines of force and compensate the ion charge density variations in a time $\tau_{\mathrm{e}}\sim 1/\omega_{\mathrm{e}}$. The characteristic times for the processes under consideration, $\sim 1/\omega_{B\mathrm{i}}$, exceed τ_{e} by many orders of magnitude, and so the quasineutrality condition (22) is safely satisfied not only in the absence of the AIC wave but also in its presence.

3.2 Rotating reference frame K′

From laboratory reference frame K we go to frame K′, which rotates about the Z-axis together with the ions (Fig. 2), i.e., with the angular velocity defined by vector $\omega=-\omega\hat{\mathbf{Z}}$, where $\hat{\mathbf{Z}}$ is the unit vector aligned with the Z-axis. This transition is described in detail in the Appendix. Here, we keep the terms $\sim 1/c$ and ignore the relativistic quantities of higher order in $1/c$. Account should also be taken of the centrifugal and Coriolis forces of inertia, which emerge in K′. As a result, within the above accuracy order, we obtain the equation of motion in the rotating reference frame:

$$M\,\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}=q\left(\mathbf{E}'+\frac{1}{c}\,\mathbf{v}\times\mathbf{B}'\right)+2M\left(\mathbf{v}\times\boldsymbol{\omega}\right)+M\omega^2\boldsymbol{\rho}\,. \tag{24}$$

Hereinafter, the vectors represented by the projections on the x-, y-, and z-axes of frame K′ are denoted by lowercase letters,

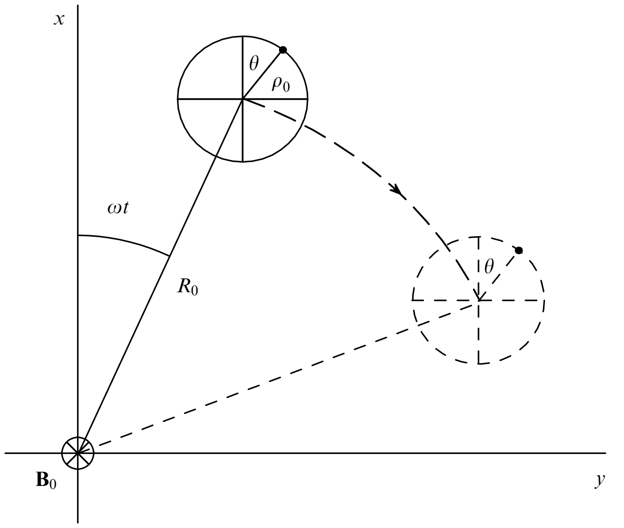

Figure 2. Motion of one of the ions in the rotating reference frame K$'$. Position of its Larmor orbit at the next point in time is shown by dashed lines. Angle θ remains time-independent. Also shown is the Larmor orbit of radius ρ_0, along which the ion moves in the frame K.

e.g., $\mathbf{r} = (x, y, z)$ is the radius vector of an ion, $\mathbf{v} = (v_x, v_y, v_z)$ is its velocity, $\boldsymbol{\rho} = (x, y, 0)$, and the fields, as usual, are denoted by uppercase letters:

$$\mathbf{E}' \approx \mathbf{E} + \frac{1}{c}\,\mathbf{V}_0 \times \mathbf{B}\,, \quad \mathbf{B}' \approx \mathbf{B} - \frac{1}{c}\,\mathbf{V}_0 \times \mathbf{E}\,, \tag{25}$$

where $\mathbf{V}_0 = \boldsymbol{\omega} \times \mathbf{r} = -\omega\hat{\mathbf{Z}} \times \boldsymbol{\rho}$ is the linear velocity of motion of frame K$'$ relative to K at the point $\mathbf{r}$ where the ion is located.

As in Section 2, hereafter we take into account only the terms linear in the amplitude of the AIC wave and ignore the higher-order terms in smallness. Then, from expressions (24) and (25) we obtain

$$\begin{cases} \dfrac{\mathrm{d}v_x}{\mathrm{d}t} = \dfrac{q}{M}\,E_x - \omega v_y\,, \\ \dfrac{\mathrm{d}v_y}{\mathrm{d}t} = \dfrac{q}{M}\,E_y + \omega v_x\,, \\ \dfrac{\mathrm{d}v_z}{\mathrm{d}t} = \dfrac{q}{Mc}\,\big[B_{1y}(v_x + \omega y) - B_{1x}(v_y - \omega x)\big]\,. \end{cases} \tag{26}$$

Here the $\mathbf{E}$ and $\mathbf{B}_1$ field projections on the x-, y-, and z-axes of frame K$'$ are expressed as

$$\begin{cases} E_x = E_X \cos\varphi + E_Y \sin\varphi = E(t)\cos(kz)\,, \\ E_y = -E_X \sin\varphi + E_Y \cos\varphi = E(t)\sin(kz)\,, \\ E_z = E_Z = 0\,, \end{cases} \tag{27}$$

$$\mathbf{B}_1 = \frac{kcE}{\omega}\,\big(-\sin(kz),\, \cos(kz),\, 0\big)\,, \tag{28}$$

where $\varphi = -\omega t$ is the angle of rotation of frame K$'$ relative to K.

The ion velocity perturbations $\delta\mathbf{v} = (\mathrm{d}\xi_x/\mathrm{d}t,\ \mathrm{d}\xi_y/\mathrm{d}t,\ \mathrm{d}z/\mathrm{d}t)$ caused by the electromagnetic field of the AIC wave itself are responsible for the AIC instability. Based on Fig. 2, we write

$$x = R_0 \cos(\omega t) + \rho_0 \cos\theta + \xi_x\,,$$

$$y = R_0 \sin(\omega t) + \rho_0 \sin\theta + \xi_y\,.$$

Hence, we obtain

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = -\omega R_0 \sin(\omega t) + \delta v_x\,,$$

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \omega R_0 \cos(\omega t) + \delta v_y\,,$$

where $\delta v_x = \mathrm{d}\xi_x/\mathrm{d}t$ and $\delta v_y = \mathrm{d}\xi_y/\mathrm{d}t$. Then, we disregard the second-order terms in the amplitude of the AIC wave to bring Eqns (26) to the form

$$\begin{cases} \dfrac{\mathrm{d}\delta v_x}{\mathrm{d}t} = \dfrac{q}{M}\,E\cos(kz) - \omega\delta v_y\,, \\ \dfrac{\mathrm{d}\delta v_y}{\mathrm{d}t} = \dfrac{q}{M}\,E\sin(kz) + \omega\delta v_x\,, \\ \dfrac{\mathrm{d}v_z}{\mathrm{d}t} = -\dfrac{qku}{M\omega}\,E\sin(kz - \theta)\,. \end{cases} \tag{29}$$

As in Section 2, we obtain, in view of Eqns (29), the following expression for the ion density variation:

$$\delta n_{\mathrm{i}} = -\frac{q n_{\mathrm{i}} k^2 u}{M\omega\Omega^2}\,E\cos(kz - \theta)\,. \tag{30}$$

In the planes $z = \text{const}$, the plasma remains uniform in the presence of an AIC wave. This is also evident from the first two equations (29), according to which δv_x and δv_y are independent of the x and y coordinates. Consequently, the motion of particles along these coordinates does not make a contribution to their density variation.

In going from K$'$ to K, the thicknesses Δz of plasma layers perpendicular to the rotation axis remain invariable, even with the inclusion of relativistic effects. That is why formula (30) is valid in the laboratory frame K as well. In the frame K at the time t, the ions travel with the velocity defined by expression (21). In view of the plasma quasineutrality discussed above, as in the case (23), we conclude that the current density 4-vector in frame K is of the form

$$\delta j^{\mathrm{i}} = q\delta n_{\mathrm{i}} u\,(0,\, \sin\chi,\, -\cos\chi,\, 0)\,.$$

Hence, we find its components in frame K$'$ (see Appendix):

$$(\delta j')^{\mathrm{i}} = \sum_{k=0}^{3} \frac{\partial x^{\mathrm{i}}}{\partial X^k}\,\delta j^k = q\delta n_{\mathrm{i}} u\,(0,\, \sin\theta,\, -\cos\theta,\, 0)\,.$$

Therefore, the projections of the electric current density vector on the spatial unit vectors of frame K$'$ are expressed as

$$\delta\mathbf{j} = q\delta n_{\mathrm{i}} u\,(\sin\theta,\, -\cos\theta,\, 0)\,. \tag{31}$$

We note that, due to quasineutrality in our case, $E_z = E_Z = 0$.

Now, let us compare the AIC instability and the WI. Their kinship becomes evident even on comparing formulas (15) and (16) with the last equation (29) and formula (30). Hence, it is clear that the AICI development is attended by filamentation, i.e., by the formation of bunches and rarefactions in the plasma, as well as the isotropization of particle velocities.

In a quantitative comparison of these instabilities, the particles should be assumed to be uniformly distributed in angle θ. Indeed, in the limiting case $B_0 \to 0$ the particle beam becomes uniform in density, as in the example discussed in Section 2, and the radius of curvature ρ_0 of particle trajectories becomes infinite, i.e., the particle trajectories straighten. We average the current density (31) over angle θ to obtain

$$\delta j_x = -\frac{\omega_{\mathrm{i}}^2 k^2 u}{8\pi\omega\Omega^2}\, E\sin(kz)\,, \quad \delta j_y = \frac{\omega_{\mathrm{i}}^2 k^2 u}{8\pi\omega\Omega^2}\, E\cos(kz)\,. \qquad (32)$$

Taking into consideration expression (32), we write the fourth Maxwell equation in frame K$'$. In frame K, in this equation the displacement current may be ignored according to inequality (1). To be more precise, this may be done provided [17]

$$kc \gg \omega_{\mathrm{i}}\,. \qquad (33)$$

Note that the smallest spatial scale characteristic of Alfvén waves is the ion cyclotron orbit radius $r_{B\mathrm{i}}$, and so

$$k \lesssim k_{\max} = \frac{1}{r_{B\mathrm{i}}}\,.$$

Hence, and from inequality (33), we conclude that the displacement current may be disregarded provided that

$$\frac{c\omega_{B\mathrm{i}}}{u\omega_{\mathrm{i}}} \gg 1\,,$$

which is almost always fulfilled. Actually, this condition need not be fulfilled. Our analysis is performed in frame K$'$, in which the forces of inertia operate, and so the Maxwell equations should be written in the framework of general relativity. In frame K$'$, the space-time metric is stationary (see Appendix). Furthermore, according to expressions (27), the electric field is time-independent in frame K$'$. Therefore, according to Ref. [23, § 90], the displacement current is exactly equal to zero, and the fourth equation takes the form (14). From expression (28), it follows that

$$\nabla \times \mathbf{B}_1 = \frac{ck^2}{\omega}\, E\big(\sin(kz)\,,\ -\cos(kz)\,,\ 0\big)\,.$$

We write the x component of Eqn (14) to obtain the relation

$$\left(1 + \frac{\gamma^2}{\Omega}\right) E\sin(kz) = 0\,.$$

Therefore, $\omega = \mathrm{i}\gamma$ in accordance with formula (7). The same conclusion follows when we consider the y component of Eqn (14).

4. Conclusions

The physical meaning of the Alfvén ion-cyclotron plasma instability and the identity of the AIC instability and Weibel instability mechanisms become clear when the AIC instability is considered in a rotating reference frame. It is clear from our analysis that the AIC instability is the Weibel filamentation instability, which takes place in the rotating reference frame. The sequence of events is as follows. The thermal fluctuation of plasma currents gives rise to an electromagnetic field. The magnetic component of this field bends the trajectories of charged particles in plasma, which results in a redistribution of their density (see formulas (29) and (30)). Due to the anisotropy of the particle velocity distribution, this density redistribution gives rise to currents, which enhance the field fluctuation. Therefore, the indicated anisotropy is the cause of the appearance of positive feedback in the plasma. The WI is the special case of the AICI in the limit of zero external magnetic field $\mathbf{B}_0$.

5. Appendix. Rigorous treatment of Alfvén ion-cyclotron waves in a rotating frame of reference

Let $X^i = (X^0, X^1, X^2, X^3)$ denote the four-dimensional coordinates of an event (for instance, a point-like flash of light) in the laboratory inertial frame K. Here, $i = 0, 1, 2, 3$, $X^0 = ct$, t is the time synchronized in K, and $X^\alpha = (X, Y, Z)$ are the Cartesian coordinates of the event. The magnetic field $\mathbf{B}_0$ is aligned with the Z-axis.

We construct the frame K$'$ rotating about the Z-axis of frame K with an angular velocity $-\omega$. To this end, we consider a system of numbered clocks describing circumferences about the Z-axis according to the law

$$\begin{cases} \rho = \sqrt{X^2 + Y^2} = \mathrm{const}\,, \\ Z = \mathrm{const}\,, \\ \varphi = \psi - \omega t\,. \end{cases}$$

Here, $\varphi = \arctan(Y/X)$, $X = \rho\cos\varphi$, $Y = \rho\sin\varphi$, and ψ is the value of angle φ at the point in time $t = 0$. The number that labels the clock is a set of three numbers: ρ, Z, ψ.

According to general relativity, systems K and K$'$ are equivalent. From the equivalence of K and K$'$, there follows the invariance of the interval between two close events, which gives the following expression for the space-time metric in frame K$'$ [22, 23]:

$$\begin{aligned} \mathrm{d}s_{\mathrm{K}'}^2 &= \mathrm{d}s_{\mathrm{K}}^2 = (\mathrm{d}X^0)^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2 \\ &= f(\mathrm{d}x^0)^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 + \frac{2\omega}{c}\,(x\,\mathrm{d}y - y\,\mathrm{d}x)\,\mathrm{d}x^0\,, \end{aligned} \qquad (34)$$

where $x^0 = ct = X^0$, $f = 1 - (\omega\rho/c)^2$,

$$\begin{cases} x = \rho\cos\psi = X\cos(\omega t) - Y\sin(\omega t)\,, \\ y = \rho\sin\psi = X\sin(\omega t) + Y\cos(\omega t)\,, \\ z = Z\,. \end{cases}$$

The clocks on the axis of rotation are at rest both in K and in K$'$, and they are therefore suited for use in these two frames of reference. The remaining clocks are synchronized to those located on the axis of rotation. In other words, the figures on their clock faces are assigned in accordance with the procedure performed, for instance, by exchanging radially traveling light signals between the neighboring clocks with the coinciding values of numbers ψ and Z [22, 23].

In frame K, the clocks may not travel at a velocity exceeding the speed of light. For this reason, the indicated choice of the rotating frame of reference is feasible in a limited spatial domain $f > 0$, which is, however, sufficiently broad for our consideration.

In frame K$'$, the ion motion is described by the equation [23]

$$\frac{Du^i}{\mathrm{d}s} = \frac{q}{Mc^2}\, F^{ik} u_k\,. \qquad (35)$$

Hereinafter, we perform summation over repeated pairs of indices, and $u^i = \mathrm{d}x^i/\mathrm{d}s$ and u_{i} are the contravariant and covariant components of the ion velocity 4-vector. Next, F^{ik} and $F^{ik}(\mathrm{K})$ are the respective electromagnetic field tensors in frames K′ and K taken at the point of ion location and related by the expression

$$F^{ik} = \frac{\partial x^i}{\partial X^l}\frac{\partial x^k}{\partial X^m}F^{lm}(\mathrm{K}). \tag{36}$$

Last, $\mathrm{D}u^i/\mathrm{d}s$ is the covariant derivative of the velocity 4-vector,

$$\frac{\mathrm{D}u^i}{\mathrm{d}s} = \frac{\mathrm{d}u^i}{\mathrm{d}s} + \Gamma^i_{kl}u^k u^l.$$

The nonzero Christoffel symbols for metric (34) are of the form

$$\Gamma^1_{00} = -\frac{\omega^2}{c^2}x, \quad \Gamma^1_{02} = \frac{\omega}{c},$$
$$\Gamma^2_{00} = -\frac{\omega^2}{c^2}y, \quad \Gamma^2_{02} = -\frac{\omega}{c}.$$

Then, equation of motion (35) written for the velocity components $v_x = \mathrm{d}x/\mathrm{d}t$, $v_y = \mathrm{d}y/\mathrm{d}t$, and $v_z = \mathrm{d}z/\mathrm{d}t$ assumes the form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\gamma v_x) - \omega^2 x + 2\omega\gamma v_y = \dfrac{1}{M}F_x, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\gamma v_y) - \omega^2 y - 2\omega\gamma v_y = \dfrac{1}{M}F_y, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\gamma v_z) = \dfrac{q}{M}F_z, \end{cases} \tag{37}$$

$$F_x = q\left[E_x Q + \frac{B_z}{c}(v_y - \omega x) - \frac{B_y}{c}v_z\right],$$

$$F_y = q\left[E_y Q - \frac{B_z}{c}(v_x + \omega y) + \frac{B_x}{c}v_z\right],$$

$$F_z = q\left[E_z Q + \frac{B_y}{c}(v_x + \omega y) - \frac{B_x}{c}(v_y - \omega x)\right],$$

$$\gamma = \left[1 - \left(\frac{\omega\rho}{c}\right)^2 + \frac{2\omega}{c^2}(xv_y - yv_x) - \frac{\mathbf{v}^2}{c^2}\right]^{-1/2},$$

$$Q = 1 - \left(\frac{\omega\rho}{c}\right)^2 + \frac{\omega}{c^2}(xv_y - yv_x).$$

Here, E_α and B_α are the field components on K′, $\alpha = 1, 2, 3 \equiv x, y, z$. Tensor F^{ik} from expression (36) is expressed through them in the usual way: $F^{0\alpha} = -E_\alpha$, $F^{\alpha\beta} = -\varepsilon_{\alpha\beta\gamma}B_\gamma$ [23].

In our case, $E_z = 0$, and the remaining field components are defined by formulas (18), (27), and (28) with $\mathbf{B}_0 = (0, 0, B_0)$.

We ignore the terms $\sim V^2/c^2$ and $\sim \omega^2\rho^2/c^2$ to obtain Eqns (26).

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* In the print version the last name of the first author was misspelled as "Velikov". (*Editor's note to English proof.*)