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Generalization of the *k* coefficient method in relativity to an arbitrary angle between the velocity of an observer (source) and the direction of the light ray from (to) a faraway source (observer) at rest

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Contents

Abstract. **The *k* coefficient method proposed by Bondi is extended to the general case where the angle α between the velocity of a signal from a distant source at rest and the velocity of the observer does not coincide with 0 or π, as considered by Bondi, but takes an arbitrary value in the interval $0 \leqslant \alpha \leqslant \pi$, and to the opposite case where the source is moving and the observer is at rest, while the angle α between the source velocity and the direction of the signal to the observer takes any value between 0 and π. Functions $k_*(\beta, \alpha)$ and $k_+(\beta, \alpha)$ of the angle and relative velocity are introduced for the ratio ω/ω' of proper frequencies of the source and observer. Their explicit expressions are obtained without using Lorentz transformations, from the condition that the coherence of a bunch of rays is preserved in passing from the source frame to the observer frame. Owing to the analyticity of these functions in α, the ratio of frequencies in the cases mentioned is given by the formulas $\omega/\omega' = k_*(\beta, \alpha)$ and $\omega/\omega' = k_+(\beta, \pi - \alpha) \equiv 1/k_*(\beta, \alpha)$, which coincide with those for the Doppler effect, in which the angle α, the velocity β, and one of the frequencies are measured in the rest frame. A ray emitted by the source at an angle α to the observer's velocity in the source frame is directed at an angle α' to the same velocity in the observer frame. Owing to light aberration, the angles α and α' are functionally related through $k_*(\beta, \alpha) = k_+(\beta, \alpha')$. The functions $\alpha'(\alpha, \beta)$ and $\alpha(\alpha', \beta)$ are expressed as antiderivatives of $k_*(\beta, \alpha)$ and $k_*(\beta, \pi - \alpha')$. The analyticity of the functions $k_*(\beta, z)$ and $k_+(\beta, z)$ in $z \equiv \alpha$ in the interval $0 \leqslant z \leqslant \pi$ is extended to the entire plane of complex z, where k_* has poles at $z_n^{\pm} = 2\pi n \mp i \ln \cos \alpha_1$ (see (17)), and k_+ has zeros at the same points shifted by π. The spatiotemporal asymmetry of the Doppler and light aberration effects is explained by the closeness of these singularities to the real axis.**

Keywords: special relativity theory, invariance of coherence, invariance of phase, Doppler effect, aberration of light, analyticity in angle, conjugate poles and aberration scale

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1. Introduction

In the proposed generalized k coefficient method, the ratio ω/ω' of light ray frequencies in the proper frames of a source and an observer is equal to the coefficient $k_*(\beta, \alpha)$ when the source is at rest and the observer is moving, and to the coefficient $k_+(-\beta, \alpha) = k_+(\beta, \pi - \alpha) = 1/k_*(\beta, \alpha)$ when the source is moving and the observer is at rest. In both cases, $\beta = V/c$ is the relative velocity of the reference frames and α is the angle between the ray direction $\mathbf{n}_\gamma$ and the direction $\mathbf{n}_\beta$ of the relative velocity of the observer in the source frame or of the source in the observer frame.

Analytic expressions for the coefficients can be obtained from the condition that the coherence of a light beam is preserved in passing from the source frame to the observer frame. Using the fact that a monochromatic coherent light beam incident on a crystal lattice stays monochromatic and coherent under reflection at Bragg–Wulff angles, we show that in a frame M moving relative to the lattice opposite to its normal, the incident and reflected beams remain coherent and monochromatic, but their frequencies differ from those in the lattice frame by shifts to the red and blue sides as the incident ray 'runs after' M and the reflected one 'meets' M.

For $\alpha = \alpha_1$, $\cos\alpha_1 = \beta/(1+(1-\beta^2)^{1/2})$, the frequency shift of an incident ray vanishes because $k_*(\beta, \alpha_1) = 1$, and for $\alpha_1 < \alpha < \pi/2$, it turns blue, although in the lattice frame the ray is still 'running after' M (light aberration). As a result, for the ratio of frequencies in the frames of the source (lattice) and observer M we have the Doppler effect formula $\omega/\omega' = k_*(\beta, \alpha)$ valid when the source is at rest and the observer is moving.

A ray emitted by the source at an angle α to the velocity of the observer relative to the source in the observer frame is directed at an angle α' to that same velocity. The angles α and α' are functionally related: $\alpha'(\alpha, \beta)$ is the antiderivative of the coefficient $k_*(\beta, \alpha)$, and $\alpha(\alpha', \beta)$ is the antiderivative of the coefficient $k_*(\beta, \pi - \alpha') \equiv k_*(-\beta, \alpha')$. The coefficients $k_*(\beta, \alpha)$ and $k_+(\beta, \alpha)$, being analytic functions of the variable $z \equiv \alpha$ in the interval $0 \leqslant z \leqslant \pi$, are analytic in the entire plane of complex z, where $k_*(\beta, z)$ has poles at the points $z_n^{\pm} = 2\pi n \mp \mathrm{i} \ln \cos\alpha_1$ and $k_+(\beta, z)$ has zeros at these points shifted by π. Light aberration is related to the closeness of these poles and zeros to the real axis.

For the observer M moving with a velocity β in the environment of distant fixed sources sending rays to M at different angles α to its velocity with frequencies larger and smaller than ω_{L}, to detect them at angles $\alpha' = \alpha'(\alpha, \beta)$ with the same frequency ω_{L}, it is necessary that the ratios of the high ω_{G} and low ω_{S} frequencies to ω_{L} be mutually reciprocal and

$$\frac{\omega_{\mathrm{G}}}{\omega_{\mathrm{L}}} = k_*(\beta, \alpha) = \frac{\omega_{\mathrm{L}}}{\omega_{\mathrm{S}}} = \frac{1}{k_*(\beta, \pi - \alpha')} .$$

In this case, M coincides with a moving source of the frequency ω_{L}, whose rays are seen by distant observers at rest as making the angle α with its velocity and having the frequency $\omega = \omega_{\mathrm{L}} k_*(\beta, \alpha)$. This is the formula for the Doppler effect when the source is moving and the observer is at rest.

The method of the k coefficient was first proposed by Hermann Bondi, a professor at King's College, London, in the article "The space traveller's youth," published in *Discovery* [1] and then in his book *Relativity and Common Sense* [2]. Later, it appeared in books by D Bohm [3], L Marder [4], and others. I have to present it here because the goal of this article is an extension of this transparent method to the general case where the angle α between the velocity of a signal from a fixed distant source and the observer velocity is not equal to zero or π, as was assumed by Bondi, but can take an arbitrary value in the interval $0 \leqslant \alpha \leqslant \pi$, and to the opposite case where the source is moving and the observer is at rest, and the angle α between the source velocity and the direction of the signal to the observer takes any value between 0 and π.

2. The *k* coefficient method proposed by Bondi

Let there be two observers A and B at rest in the laboratory frame, separated by a distance d and keeping synchronized clocks. At the moments $t_n^{\mathrm{A}} = n t_1^{\mathrm{A}}$, $n = 0, 1, 2, \ldots$, which are multiples of the period t_1^{A}, A emits light signals in the direction of B. These signals reach B at the moments $t_n^{\mathrm{BA}} = t_n^{\mathrm{A}} + d/c$ with the same period t_1^{A} because $t_1^{\mathrm{B}} = t_{n+1}^{\mathrm{BA}} - t_n^{\mathrm{BA}} = t_1^{\mathrm{A}}$.

We now assume that observer M is moving along the line connecting A and B, in the direction from A to B with a velocity V. Then the signals sent by A at moments t_n^{A} reach M at the moments t_n^* satisfying the requirement that the path of M and the nth signal are equal, $Vt_n^* = c(t_n^* - t_n^{\mathrm{A}})$ (Fig. 1). Hence, $t_n^* = t_n^{\mathrm{A}}/(1-\beta)$, $\beta = V/c$. This means that the period t_1^* between the signals arriving to M is, according to A's clock,

$$t_{n+1}^* - t_n^* = \frac{t_{n+1}^{\mathrm{A}} - t_n^{\mathrm{A}}}{1-\beta} = \frac{t_1^{\mathrm{A}}}{1-\beta} .$$

Because of the constant increase in the distance between the receiver and the source, this period is larger than that of emission.

But we are interested in the period of the signals coming to M according to M's clock. We let t_1^{M} denote it. Because it is also proportional to the emission period t_1^{A}, we introduce the proportionality coefficient k,

$$t_1^{\mathrm{M}} = k t_1^{\mathrm{A}} , \tag{1}$$

which depends on the velocity V and should exceed 1 because the detector of signals M is moving away from the signal source A.

We note that observer M can treat signals coming from A as being emitted by M itself toward B over intervals t_1^{M} (see Fig. 1). Because the interval between the signals received by observer B equals t_1^{A} and is related to t_1^{M} by (1), writing it in the form

$$t_1^{\mathrm{B}} = t_1^{\mathrm{A}} = \frac{1}{k}\, t_1^{\mathrm{M}} \tag{2}$$

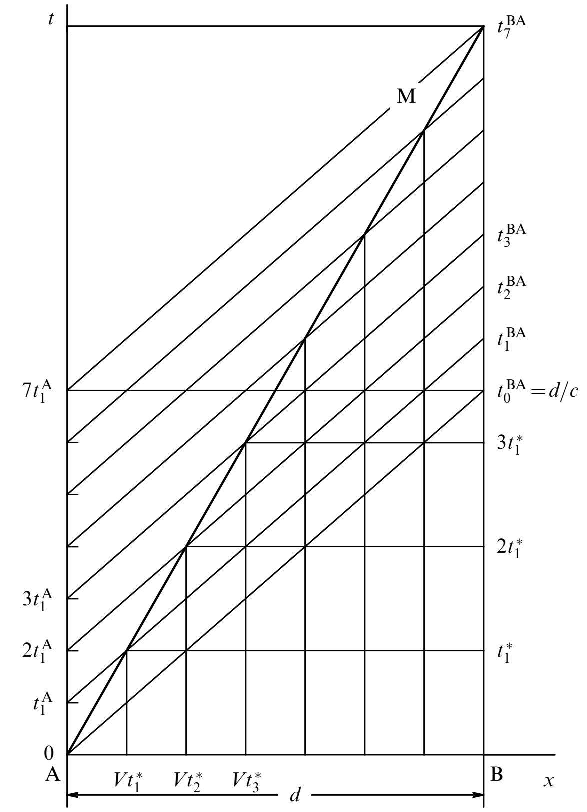

Figure 1. Computations of the relation between period-t_1^{A} signals emitted by the source A and their periods t_1^* and t_1^{M} received by a moving observer M by clocks of A and M, respectively.

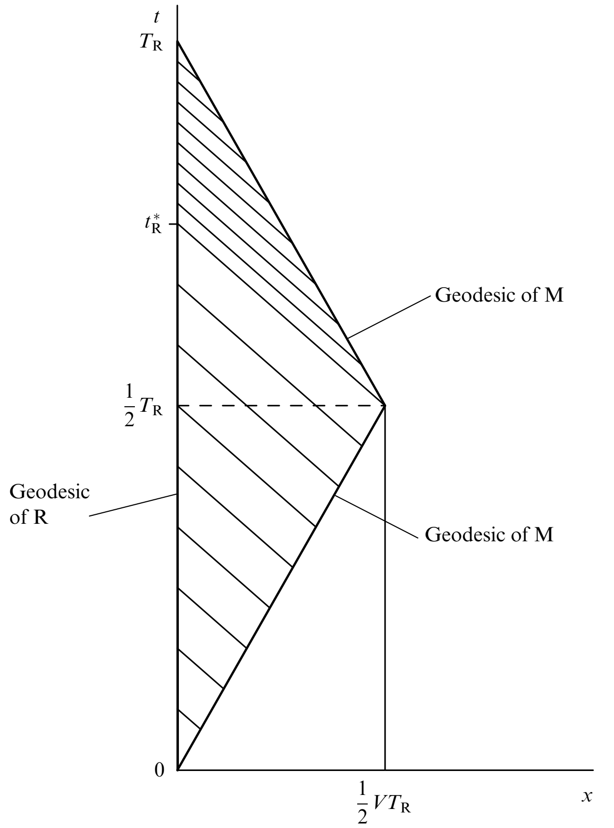

Figure 2. Computation of the relation between the proper time T_M and laboratory time T_R of the motion of M.

we can regard it as a relation between the emission period of the signals at the 'source' M and their detection period at B when the source is approaching the detector. Because of the constant increase in the distance between the source and the detector, the period at the detector is shorter than the emission period, which once again leads to $k > 1$.

We now consider the case where an observer at rest equipped with a detector R receives signals from a source M first moving away from R with a speed V to a maximum distance, and then approaching R at the same speed V (Fig. 2).

Let T_M be the total motion time of the source M measured by the clock of M, and T_R be this time measured by the clock of R. Then the signals emitted when M moves away reach R in the time

$$t_R^* = k\frac{1}{2}\,T_M\,, \tag{3}$$

and signals emitted by M when approaching R arrive in the time

$$T_R - t_R^* = \frac{1}{k}\frac{1}{2}T_M\,. \tag{4}$$

Eliminating t_R^* from these two equations, we find the important relation

$$T_R = \left(k + \frac{1}{k}\right)\frac{1}{2}\,T_M\,. \tag{5}$$

Because $k \neq 1$ (otherwise, $V = 0$), it follows that $k + 1/k > 2$ and hence

$$T_R > T_M\,.$$

Thus, the time T_R it takes M to return to R by the clock of R is always larger than the proper time T_M of M. This is the twin 'paradox' — a twin returning after a journey in space is younger than the one who stayed at home.

We now find the function $k(\beta)$. By the clock of the observer R, M was moving away from R for the time $T_R/2$, reaching the largest separation $VT_R/2$. A signal emitted by M at the moment of the largest separation takes the time $VT_R/2c$ to reach R, reaching R at the moment

$$t_R^* = \frac{T_R}{2} + \frac{VT_R}{2c} = \frac{1}{2}\,T_R(1+\beta)\,. \tag{6}$$

Eliminating the time ratios t_R^*/T_M, T_R/T_M, and t_R^*/T_R from the three equations (3), (5), and (6), we obtain the sought relation between k and β,

$$k(\beta) = \sqrt{\frac{1+\beta}{1-\beta}}\,. \tag{7}$$

Using this function in formula (5), we arrive at the known relation between the intervals of proper and laboratory motion time of M:

$$T_R = \frac{1}{\sqrt{1-\beta^2}}\,T_M\,. \tag{8}$$

The extension of the k coefficient method to the general case where the direction of rays from a source makes an arbitrary angle with the direction of the observer's velocity was prompted by an article submitted to *Physics–Uspekhi*, which claimed that the occurrence of maxima in reflected light intensity for particular incidence angles of a monochromatic beam on a crystal lattice (Bragg–Wulff angles) ceases to be Lorentz invariant, and constructive interference of reflected rays in the laboratory frame can become destructive if observed from another inertial frame. This is clearly not the case because it contradicts the phase invariance of a monochromatic light wave. The proposed extension of the k coefficient method offers a transparent and informal way to obtain general expressions for the Doppler effect and light aberration without resorting to Lorentz transformations and, in particular, to prove the Lorentz invariance of constructive interference of reflected rays in the Bragg–Wulff effect.

3. Coefficients $k_*(\beta, \alpha)$ and $k_+(\beta, \alpha)$ in the problem of invariance of the coherence of a bunch of monochromatic rays

We consider light reflection by two layers A and B of a crystal lattice (Fig. 3). Laser beams incident on surfaces A and B are coherent. This means that constant phases of wave front surfaces that are normal to the rays and separated by the wavelength λ differ by 2π. The surfaces with definite phases travel with the speed of light. For neighboring rays 1 and 2 reflected at the neighboring (nearest) points of neighboring layers A and B, the optical path difference is

$$D = \frac{2d}{\sin\theta} - 2d\cot\theta\cos\theta = 2d\sin\theta\,,$$

where d is the distance between layers A and B, and $\theta = \pi/2 - \alpha$, with α being the incidence angle, equal to the reflection angle. If this difference is equal to an integer number of wavelengths,

$$2d\sin\theta = m\lambda\,, \qquad 0 < \sin\theta_m = \frac{m\lambda}{2d} \leqslant 1\,, \tag{9}$$

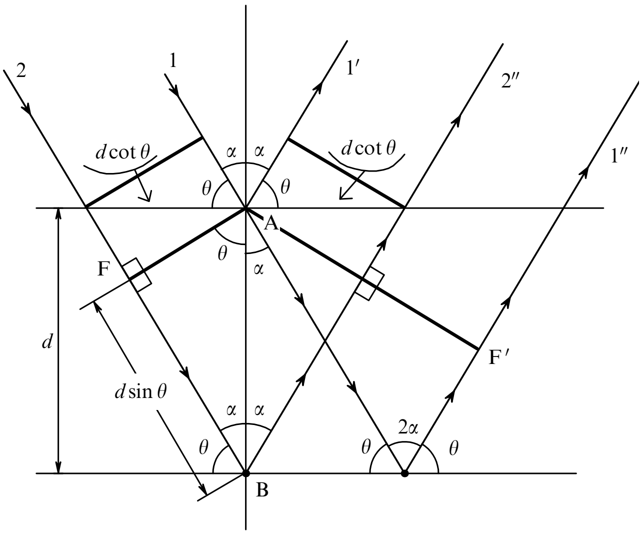

Figure 3. For a coherent bunch of rays, the optical path between the front AF that has just passed the layer A and the front AF′ passing through it equals $2d\sin\theta$.

the rays reflected from layers A and B at such angles remain coherent. This is the Bragg–Wulff condition for the direction of maximum reflected light intensity.

We define a signal to be the part of the coherent light rays between two fronts with the phase difference of 2π, i.e., with the length λ and duration $T = \lambda/c$ of one period.

We assume that $t = t_0^{\mathrm{A}} = 0$ is the moment incident ray 1 is reflected from layer A; it coincides with the moment when the front of rays 1 and 2 crosses layer A (see Fig. 3). Then the front of rays 1 and 2 reflected from layer B starts crossing layer A at the moment

$$t = t_0^{\mathrm{ABA}} = \frac{2d\sin\theta}{c} = \frac{m\lambda}{c} = mT$$

(see Fig. 3). The next front of rays 1 and 2 crosses layer A at the moment $t = t_1^{\mathrm{A}} = \lambda/c = T$ and after reflection from layer B starts crossing layer A at the moment

$$t = t_1^{\mathrm{ABA}} = mT + T\,.$$

The signals incident on layer A at the moments $t_n^{\mathrm{A}} = nt_1^{\mathrm{A}} = nT$, $n = 0, 1, 2, 3, \ldots$, spend the same time $d\sin\theta/c$ on the path from A to B and arrive at B at the moments

$$t_n^{\mathrm{BA}} = t_n^{\mathrm{A}} + \frac{d\sin\theta}{c}\,.$$

Hence, the period the subsequent signals from A arrive at B equals the period they appear in A:

$$t_{n+1}^{\mathrm{BA}} - t_n^{\mathrm{BA}} = t_{n+1}^{\mathrm{A}} - t_n^{\mathrm{A}} = t_1^{\mathrm{A}}\,.$$

In turn, the signals reflected at the moments t_n^{BA} arrive at A at the moments

$$t_n^{\mathrm{ABA}} = t_n^{\mathrm{BA}} + \frac{d\sin\theta}{c} = t_n^{\mathrm{A}} + \frac{2d\sin\theta}{c}\,,$$

and hence the period at which the signals reflected from B arrive at A coincides with the period they originally appeared at A,

$$t_{n+1}^{\mathrm{ABA}} - t_n^{\mathrm{ABA}} = t_{n+1}^{\mathrm{BA}} - t_n^{\mathrm{BA}} = t_1^{\mathrm{A}}\,.$$

Signals reflected from layer A at the moments t_n^{A} and signals reflected by layer B and starting to cross A at the moments t_n^{ABA} propagate in the same direction and are coherent if the interval

$$t_n^{\mathrm{ABA}} - t_n^{\mathrm{A}} = \frac{2d\sin\theta}{c}$$

is a multiple of the light period T, i.e, if the Bragg–Wulff condition is satisfied,

$$\frac{2d\sin\theta}{c} = mT\,.$$

Let observer M move away from A and approach B at a constant velocity V normal to the layers. Then the arrival of the signal sequence from A to M is uniform according to the clocks of both A and M, but their arrival period t_1^{MA} measured by the clock of M is larger than their emission period at A. We let

$$k_* = \frac{t_1^{\mathrm{MA}}}{t_1^{\mathrm{A}}} > 1 \tag{10}$$

denote the ratio of these two periods. On the other hand, because M approaches B with the same velocity V, signals reflected from B also arrive at M uniformly as measured by the clock of both A and M, but their arrival period t_1^{MBA} measured by the clock of M is k_+ times shorter than t_1^{A},

$$t_1^{\mathrm{MBA}} = \frac{1}{k_+}\,t_1^{\mathrm{A}}\,. \tag{11}$$

Indeed, because M, on receiving signals reflected from layer B, could have considered them to be its own signals emitted in the direction of A, they would have reached layer A with the period t_1^{A}, which should be larger than the emission period t_1^{MBA} by k_+ times, because M is moving away from A with the velocity V.

We find the functions k_* and k_+. The observer M is at the distance Vt_n^* from A when at the moment t_n^* it receives the front of the signal emitted at A at the moment t_n^{A}. But the path that the front should pass during the time $t_n^* - t_n^{\mathrm{A}}$ is $Vt_n^*\cos\alpha$ (Fig. 4). Therefore, $Vt_n^*\cos\alpha = c(t_n^* - t_n^{\mathrm{A}})$, whence

$$t_n^* = \frac{t_n^{\mathrm{A}}}{1 - \beta\cos\alpha}\,, \qquad \beta = \frac{V}{c}\,, \qquad \alpha = \frac{\pi}{2} - \theta\,, \qquad \cos\alpha = \sin\theta\,.$$

The signals from A arrive to M with the period

$$t_{n+1}^* - t_n^* = \frac{t_{n+1}^{\mathrm{A}} - t_n^{\mathrm{A}}}{1 - \beta\cos\alpha} = \frac{t_1^{\mathrm{A}}}{1 - \beta\cos\alpha}\,,$$

and the respective period of the proper time of M is

$$t_1^{\mathrm{MA}} = \sqrt{1-\beta^2}\,(t_{n+1}^* - t_n^*) = \frac{\sqrt{1-\beta^2}\,t_1^{\mathrm{A}}}{1 - \beta\cos\alpha}\,,$$
$$k_*(\beta, \alpha) = \frac{\sqrt{1-\beta^2}}{1 - \beta\cos\alpha}\,. \tag{12}$$

We note that $k_* < 1$ if

$$\cos\alpha < \cos\alpha_1 = \frac{\beta}{1 + \sqrt{1-\beta^2}}\,.$$

This phenomenon, unexpected for angles $\alpha_1 < \alpha < \pi/2$, is due to aberration.

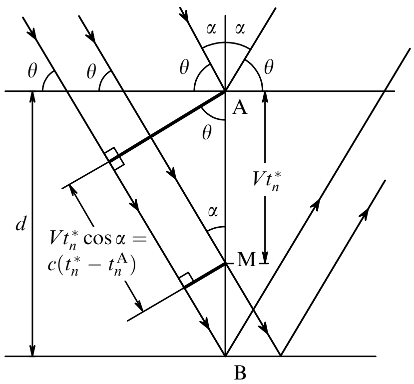

Figure 4. Computation of the moment t_n^* when observer M is taken over by the front of the nth signal, which passes layer A at the moment t_n^{A}.

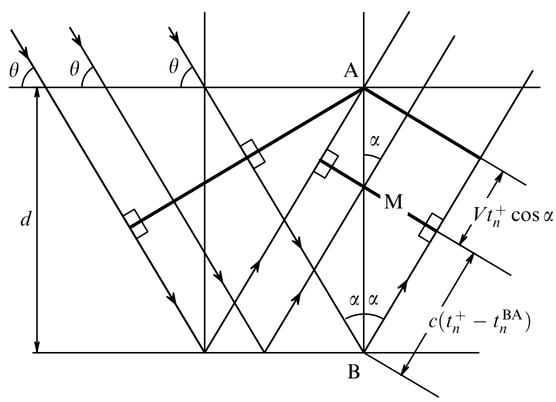

Figure 5. Computation of the moment t_n^+ when observer M meets the front of the nth signal reflected from layer B.

At the moment t_n^+, having passed the distance Vt_n^+ from A, observer M receives the signal that was reflected from B at the moment $t_n^{\mathrm{BA}} = t_n^{\mathrm{A}} + d\sin\theta/c$. The optical path of the front of the signal reflected from layer B to layer A can be expressed as the sum of the path $c(t_n^+ - t_n^{\mathrm{BA}})$ from layer B to the encounter with M and the remainder $Vt_n^+ \cos\alpha$ from the encounter with M to layer A (Fig. 5). Because this sum equals $d\sin\theta$, from the equation

$$c(t_n^+ - t_n^{\mathrm{BA}}) + Vt_n^+ \cos\alpha = d\sin\theta$$

it follows that

$$t_n^+ = \frac{t_n^{\mathrm{A}}}{1+\beta\cos\alpha} + \frac{2d\sin\theta}{c(1+\beta\cos\alpha)}\,.$$

Thus, by the clock at A, the period at which M sees the signals reflected from B is

$$t_{n+1}^+ - t_n^+ = \frac{t_1^{\mathrm{A}}}{1+\beta\cos\alpha}\,,$$

while by the clock of M it is

$$t_1^{\mathrm{MBA}} = \sqrt{1-\beta^2}\,(t_{n+1}^+ - t_n^+) = \frac{\sqrt{1-\beta^2}\,t_1^{\mathrm{A}}}{1+\beta\cos\alpha}\,,$$
$$k_+(\beta,\alpha) = \frac{1+\beta\cos\alpha}{\sqrt{1-\beta^2}}\,. \tag{13}$$

Formulas (12), (13) relate the frequency ω of signals from a distant source in the laboratory frame to their frequency ω' measured by an observer M moving either from or to the source; the signs are $-$ or $+$, respectively,

$$\omega = \omega' \, \frac{\sqrt{1-\beta^2}}{1 \mp \beta\cos\alpha}\,, \qquad \cos\alpha = \sin\theta \tag{14}$$

(see formula (15) in [5]). This formula was first obtained by Einstein in his first paper [6] on the theory of relativity.

Thus, signals reflected from layers A and B are also coherent for observer M if they are coherent for A (i.e., if $\theta = \theta_{\mathrm{m}}$; see (9)), and their periods (or frequencies) are related by the usual formulas (12)–(14) for the Doppler effect. This is a consequence of the invariance of the light wave phase. Here, this is proved with the help of the coefficients k_* and k_+.

4. Analyticity of the coefficients $k_*(\beta,\alpha)$ and $k_+(\beta,\alpha)$ in the angle α and the relation between them

We note that the angle α that we use is confined within $0 \leqslant \alpha \leqslant \pi/2$ and, according to Figs 3–5, defines the arrangement of the velocities of the signal and observer with respect to each other and relative to layers A and B in the laboratory frame. The angle α_{P} used by Pauli in formula (15) has the physical sense of the angle between the velocities of the observer and the signal from a remote source that is at rest in the laboratory frame. It can range the interval $0 \leqslant \alpha_{\mathrm{P}} \leqslant \pi$. When the signal runs after the observer (see Fig. 4), our angle $\alpha = \alpha_{\mathrm{P}}$, and when the signal and observer move toward each other (see Fig. 5), α is related to α_{P} as $\alpha = \pi - \alpha_{\mathrm{P}}$. For this reason, the passage from formula (14) to formula (15) by Pauli amounts to replacing α with α_{P} for the upper sign and α with $\pi - \alpha_{\mathrm{P}}$ for the lower sign, such that the term $\mp\beta\cos\alpha$ is replaced with $-\beta\cos\alpha_{\mathrm{P}}$.

This is nothing but an analytic continuation of the functions $k_*(\beta,\alpha)$ and $1/k_+(\beta,\alpha)$ in the angle α to the entire interval $0 \leqslant \alpha \leqslant \pi$, in which, both by value and by physical sense, $\alpha \equiv \alpha_{\mathrm{P}}$ for $k_*(\beta,\alpha)$ and $\alpha \equiv \pi - \alpha_{\mathrm{P}}$ for $1/k_+(\beta,\alpha)$. In other words, these functions coincide as functions of α_{P}:

$$\frac{1}{k_+(\beta,\pi-\alpha_{\mathrm{P}})} = k_*(\beta,\alpha_{\mathrm{P}}) = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\alpha_{\mathrm{P}}}\,, \qquad 0 \leqslant \alpha_{\mathrm{P}} \leqslant \pi\,. \tag{15}$$

The coincidence of these two analytic functions of the variable $z \equiv \alpha_{\mathrm{P}}$ in the interval $0 \leqslant z \leqslant \pi$ means that they coincide in the whole plane of the complex variable z, having simple poles at the points $z_n^{\pm} = 2\pi n \mp \mathrm{i}\ln\cos\alpha_1$, with n an integer, $\cos\alpha_1 = \beta/(1+(1-\beta^2)^{1/2})$, with the residues $\mp\mathrm{i}$.

The angle α_1 plays an important role in the generalized k coefficient method.

5. Derivation of relativistic Einstein formulas for the Doppler effect and light aberration using the generalized k coefficient method

5.1 Relation between the angles α and α' made by ray directions and the velocity of the detector relative to the source in the source and detector frames

With the help of functions (15), the Einstein formula for the ratio $\omega_{\mathrm{L}}/\omega'_{\mathrm{M}}$ of proper frequencies of a distant source L resting in the laboratory frame and a moving observer M can be written as

$$\frac{\omega_{\mathrm{L}}}{\omega'_{\mathrm{M}}}=\frac{1}{k_{+}(\beta,\pi-\alpha_{\mathrm{P}})}=k_{*}(\beta,\alpha_{\mathrm{P}})=\frac{\sqrt{1-\beta^{2}}}{1-\beta\cos\alpha_{\mathrm{P}}}\,, \tag{16}$$

where α_{P} is the angle between the direction $\mathbf{n}_\gamma$ of the signal velocity and the direction $\mathbf{n}_\beta$ of the velocity of observer M in the laboratory frame. For $0<\alpha_{\mathrm{P}}<\pi/2$, formula (16) describes the situation depicted in Fig. 4, and for $\pi/2<\alpha_{\mathrm{P}}<\pi$, the situation in Fig. 5, because as α_{P} increases, the source L is displaced to a different position relative to M along an arc lying far from M.

As the angle α_{P} increases, the ratio $\omega_{\mathrm{L}}/\omega'_{\mathrm{M}}$ decreases and, according to the note after formula (12), becomes less than 1 when α_{P} crosses the value

$$\alpha_1=\arccos\frac{\beta}{1+\sqrt{1-\beta^{2}}}<\frac{\pi}{2}\,. \tag{17}$$

For angles α_{P} in the interval $\alpha_1<\alpha_{\mathrm{P}}<2\pi-\alpha_1$, the function $k_*(\beta,\alpha_{\mathrm{P}})$ and the ratio $\omega_{\mathrm{L}}/\omega'_{\mathrm{M}}$ become smaller than 1. This means that the frequency ω'_{M} of signals received by the moving observer M is higher than the proper frequency ω_{L} of signals emitted by the source L, owing to aberration of these signals or to M approaching L.

If the source L, ω_{L} in the laboratory frame, which sends signals to M at the angle $\alpha=\pi-\alpha'_{\mathrm{P}}$ lying in the interval $\alpha_1<\alpha=\pi-\alpha'_{\mathrm{P}}<\pi$, $0<\alpha'_{\mathrm{P}}<\pi-\alpha_1$, is replaced with a source S, ω_{S} of signals with a reduced frequency $\omega_{\mathrm{S}}<\omega_{\mathrm{L}}$ such that observer M receives these signals in his proper frame with the frequency ω_{M} equal to ω_{L},

$$\omega_{\mathrm{M}}=\omega_{\mathrm{L}}=\omega'_{\mathrm{M}}k_*(\beta,\alpha)=\omega'_{\mathrm{M}}\frac{\sqrt{1-\beta^{2}}}{1-\beta\cos\alpha}\,,\qquad \alpha_1<\alpha<\pi\,, \tag{18}$$

then the ratio of the frequency of the emitted signal to the frequency of the signal received by M,

$$\frac{\omega_{\mathrm{S}}}{\omega_{\mathrm{M}}}\equiv\frac{\omega_{\mathrm{S}}}{\omega_{\mathrm{L}}}=\frac{\omega_{\mathrm{L}}}{\omega'_{\mathrm{M}}}=k_*(\beta,\pi-\alpha'_{\mathrm{P}})=\frac{\sqrt{1-\beta^{2}}}{1+\beta\cos\alpha'_{\mathrm{P}}}\,,\qquad 0<\alpha'_{\mathrm{P}}<\pi-\alpha_1\,, \tag{19}$$

must remain the same as before the replacement of L with S, because for $\omega_{\mathrm{M}}=\omega_{\mathrm{L}}$ it is also defined by the proportionality coefficient $k_*(\beta,\alpha)=k_*(\beta,\pi-\alpha'_{\mathrm{P}})$, which depends only on the velocity β and the angle $\alpha=\pi-\alpha'_{\mathrm{P}}$, $\alpha_1<\alpha<\pi$, between the velocity direction $\mathbf{n}_\beta$ and the direction $\mathbf{n}_\gamma$ of the signal coming from the source (Fig. 6).

On the other hand, if we require that observer M receiving signals from the source L, ω_{L} *at an acute angle* α_{P}, $0<\alpha_{\mathrm{P}}<\alpha_1$, between the directions of velocities of the observer and the signal, $\cos\alpha_{\mathrm{P}}=\mathbf{n}_\beta\mathbf{n}_\gamma$, detect them not with a frequency ω'_{M} lower than ω_{L} but with the frequency $\omega_{\mathrm{M}}=\omega_{\mathrm{L}}$, i.e., the same frequency as from the source S, ω_{S}, then this source L, ω_{L} has to be replaced by a source G, ω_{G} whose frequency exceeds ω_{L} by $\omega_{\mathrm{L}}/\omega_{\mathrm{S}}$ times, $\omega_{\mathrm{G}}=\omega_{\mathrm{L}}(\omega_{\mathrm{L}}/\omega_{\mathrm{S}})$. The boundary conditions $\omega_{\mathrm{S}}=\omega_{\mathrm{L}}$ and $\omega_{\mathrm{G}}=\omega_{\mathrm{L}}$ are automatically satisfied in this case and are consequences of one another, and $\omega_{\mathrm{L}}=\sqrt{\omega_{\mathrm{G}}\omega_{\mathrm{S}}}$ (see Fig. 6).

In this and only in this case,

$$\frac{\omega_{\mathrm{L}}}{\omega'_{\mathrm{M}}}=k_*(\beta,\alpha_{\mathrm{P}})=\frac{\omega_{\mathrm{G}}}{\omega_{\mathrm{L}}}=\frac{\omega_{\mathrm{L}}}{\omega_{\mathrm{S}}}=\frac{1}{k_*(\beta,\pi-\alpha'_{\mathrm{P}})}\equiv k_+(\beta,\alpha'_{\mathrm{P}})\,, \tag{20}$$

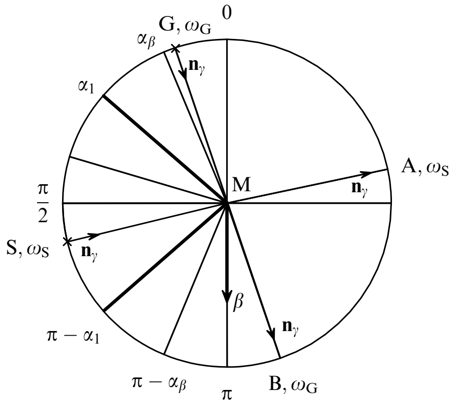

Figure 6. Signals of sources G and S with frequencies $\omega_{\mathrm{G}}>\omega_{\mathrm{L}}$ and $\omega_{\mathrm{S}}<\omega_{\mathrm{L}}$ arrive to M with the frequency $\omega_{\mathrm{L}}=\sqrt{\omega_{\mathrm{G}}\omega_{\mathrm{S}}}$ and are sent by M with the frequency ω_{L} to the observers B and A, who detect them with frequencies ω_{G} and ω_{S}. In the plot, $\beta=0.9$, $\alpha_{\mathrm{G}}=20^\circ$, and $\alpha_{\mathrm{S}}=105^\circ$.

which is equivalent to the angle α'_{P} being related to the angle α_{P} as

$$\frac{\sqrt{1-\beta^{2}}}{1-\beta\cos\alpha_{\mathrm{P}}}=\frac{1+\beta\cos\alpha'_{\mathrm{P}}}{\sqrt{1-\beta^{2}}}\,. \tag{21}$$

Pauli gives this formula under the number (16c) and obtains it, just as Einstein did in [7], from the invariance of the monochromatic light wave phase under Lorentz transformations. Here, we obtained it with the generalized k coefficient method, without resorting to Lorentz transformations.

We also note that the intervals $0\leqslant\alpha_{\mathrm{P}}\leqslant\alpha_1$, $0\leqslant\alpha'_{\mathrm{P}}\leqslant\pi-\alpha_1$ of the angles α_{P} and α'_{P} used in the derivation of relation (21) can be extended owing to the analyticity of all functions that depend on these angles. This is why relation (21) between α_{P} and α'_{P} is automatically preserved in the extended interval $0\leqslant\alpha_{\mathrm{P}},\alpha'_{\mathrm{P}}\leqslant\pi$ of these angles. We consider this relation, viewing α_{P} as an argument and α'_{P} as a function of α_{P} and omitting the index P, $\alpha_{\mathrm{P}}=\alpha$, $\alpha'_{\mathrm{P}}=\alpha'(\alpha)$. The function $\alpha'(\alpha)$ also depends, certainly, on the parameter β, $\alpha'(\alpha)\equiv\alpha'(\alpha,\beta)$, which we keep in mind in what follows.

5.2 Function $\alpha'(\alpha,\beta)$ as an antiderivative of the coefficient $k_*(\beta,\alpha)$

Differentiating both terms in (21) with respect to α, we obtain an expression for the derivative that relates it directly to $k_*(\beta,\alpha)$:

$$\frac{\partial\alpha'(\alpha)}{\partial\alpha}\equiv\dot{\alpha}'(\alpha)=\frac{\sqrt{1-\beta^{2}}}{1-\beta\cos\alpha}\equiv k_*(\beta,\alpha)\,. \tag{22}$$

Now, integrating this expression over α, we find the sought function, the antiderivative of $k_*(\beta,\alpha)$:

$$\alpha'(\alpha,\beta)=\sqrt{1-\beta^{2}}\int_0^{\alpha}\frac{\mathrm{d}\alpha}{1-\beta\cos\alpha}=2\arctan\left(\sqrt{\frac{1+\beta}{1-\beta}}\tan\frac{\alpha}{2}\right). \tag{23}$$

At the ends of the interval $0\leqslant\alpha\leqslant\pi$, it takes the values

$$\alpha'(0)=0\quad\text{and}\quad\alpha'(\pi)=\pi\,,$$

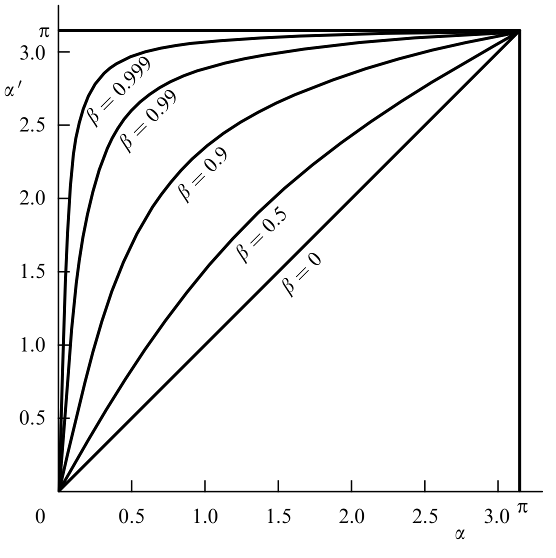

Figure 7. Function $\alpha'(\alpha, \beta)$ for five values of the parameter $\beta = 0, 0.5, 0.9, 0.99$, and 0.999. This function increases monotonically with the angle α, $0 \leqslant \alpha \leqslant \pi$, as well as with the parameter β, $0 \leqslant \beta \leqslant 1$.

and at intermediate points, characterized by a given β, $\alpha = \alpha_\beta = \arccos \beta$, $\alpha = \alpha_1$, $\alpha = \pi/2$, it takes the values

$$\alpha'(\alpha_\beta) = \frac{\pi}{2}\,, \quad \alpha'(\alpha_1) = \pi - \alpha_1\,, \quad \alpha'\left(\frac{\pi}{2}\right) = \pi - \alpha_\beta\,.$$

The derivative $\dot{\alpha}'(\alpha)$ attains its maximum and minimum values at the ends of the interval,

$$\dot{\alpha}'(0) = k_*(\beta, 0) = k_+(\beta, 0) = \sqrt{\frac{1+\beta}{1-\beta}}\,,$$
$$\dot{\alpha}'(\pi) = k_*(\beta, \pi) = k_+(\beta, \pi) = \sqrt{\frac{1-\beta}{1+\beta}}\,,$$

which coincide with $k(\beta)$ and $1/k(\beta)$ in Bondi's method, and at the intermediate points mentioned above, it takes the values

$$\frac{1}{\sqrt{1-\beta^2}}\,, \quad 1\,, \quad \sqrt{1-\beta^2}\,.$$

The function $\alpha'(\alpha, \beta)$ is plotted in Fig. 7. Its curvature reaches an extremum at $\alpha = \alpha_1$, equal to $-\cot\alpha_1/\sqrt{2}$.

We note that the inverse function $\alpha(\alpha')$ follows from (23) by the sign change at the parameter β,

$$\alpha(\alpha') = 2\arctan\left(\sqrt{\frac{1-\beta}{1+\beta}}\tan\frac{\alpha'}{2}\right). \tag{24}$$

5.3 Derivation of the Einstein formula for a moving source and a resting observer

We assume that on the arc $0 \leqslant \alpha_{\rm P} \leqslant \alpha_1$ there are sources G, $\omega_{\rm G}$ whose frequency $\omega_{\rm G}(\alpha_{\rm P})$ *monotonically decreases as* $\alpha_{\rm P}$ *increases* from $\omega_{\rm G}(0) = \omega_{\rm L}\sqrt{(1+\beta)/(1-\beta)}$ for $\alpha_{\rm P} = 0$ to $\omega_{\rm G}(\alpha_1) = \omega_{\rm L}$ for $\alpha_{\rm P} = \alpha_1$.

We assume that on the arc $\alpha_1 \leqslant \alpha_{\rm P} \leqslant \pi$ there are sources S, $\omega_{\rm S}$, whose frequency $\omega_{\rm S}(\alpha_{\rm P})$ *monotonically decreases as* $\alpha_{\rm P}$ *increases* from $\omega_{\rm S}(\alpha_1) = \omega_{\rm L}$ for $\alpha_{\rm P} = \alpha_1$ to $\omega_{\rm S}(\pi) = \omega_{\rm L}\sqrt{(1-\beta)/(1+\beta)}$ for $\alpha_{\rm P} = \pi$.

The moving observer M sees all these signals *with the same frequency* $\omega_{\rm L}$ *in his proper frame* and sends them to observers B and A in the laboratory frame. They reach observer B with the frequency $\omega_{\rm G}(\alpha_{\rm P}) \geqslant \omega_{\rm L}$ if $0 \leqslant \alpha_{\rm P} \leqslant \alpha_1$ and with the frequency $\omega_{\rm S}(\alpha_{\rm P}) \leqslant \omega_{\rm L}$ if $\alpha_1 \leqslant \alpha_{\rm P} \leqslant \pi/2$. We draw attention to the fact that because of light aberration, observer B detects the frequency $\omega_{\rm S} < \omega_{\rm L}$ for the angle $\alpha_{\rm P}$ in the range $\alpha_1 < \alpha_{\rm P} < \pi/2$, even though the source M is approaching.

The signals reach observer A with the frequency $\omega_{\rm S}(\alpha_{\rm P}) < \omega_{\rm L}$ because for $\pi/2 \leqslant \alpha_{\rm P} \leqslant \pi$, the source M moves away from A. Here, as previously, $\alpha_{\rm P}$ is the angle between the direction $\mathbf{n}_\beta$ of the velocity of the source M and the direction $\mathbf{n}_\gamma$ of the velocity of signals arriving to B or A, $\cos\alpha_{\rm P} = \mathbf{n}_\gamma\mathbf{n}_\beta$.

The requirement that the frequency ratio $\omega_{\rm G}/\omega_{\rm L}$ be equal to the ratio $\omega_{\rm L}/\omega_{\rm S}$ is equivalent to the transfer of the source with the frequency $\omega_{\rm L}$ from the laboratory frame to the proper frame of observer M. As a result, M becomes a source moving with the velocity β emitting at the frequency $\omega_{\rm M} \equiv \omega_{\rm L}$, and B and A become detectors of signals of the frequency ω from the source M, $\omega_{\rm L}$, approaching B and moving away from A,

$$\frac{\omega}{\omega_{\rm L}} = k_*(\beta, \alpha_{\rm P}) = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\alpha_{\rm P}}\,. \tag{25}$$

Observer B detects a frequency $\omega \geqslant \omega_{\rm L}$ for the interval $0 \leqslant \alpha_{\rm P} \leqslant \alpha_1$, and $\omega \leqslant \omega_{\rm L}$ for the interval $\alpha_1 \leqslant \alpha_{\rm P} \leqslant \pi/2$. Observer A always detects a frequency $\omega \leqslant \omega_{\rm L}$ because in this case $\pi/2 \leqslant \alpha_{\rm P} \leqslant \pi$.

Thus, the function $k_*(\beta, \alpha_{\rm P})$ in formula (16) with the same sense of the angle $\alpha_{\rm P}$ in the range $0 < \alpha_{\rm P} < \pi$, $\cos\alpha_{\rm P} = \mathbf{n}_\beta\mathbf{n}_\gamma$, describes the relation of the proper frequency $\omega_{\rm L}$ of the laboratory source L to the frequency $\omega'_{\rm M}$ of signals reaching observer M moving away ($0 < \alpha_{\rm P} < \pi/2$) or approaching ($\pi/2 < \alpha_{\rm P} < \pi$) the source.

In formula (25), the same function $k_*(\beta, \alpha_{\rm P})$ with the same sense of the angle $\alpha_{\rm P}$ relates the proper frequency $\omega_{\rm L}$ of the moving source M to the frequency ω of signals arriving to a laboratory observer, if the source M is approaching ($\omega > \omega_{\rm L}$ for $0 < \alpha_{\rm P} < \alpha_1$ and $\omega < \omega_{\rm L}$ for $\alpha_1 < \alpha_{\rm P} < \pi/2$) or moving away from ($\omega < \omega_{\rm L}$ for $\pi/2 < \alpha_{\rm P} < \pi$) this observer.

Formula (25) coincides with formula (48.12) in [8] and with the solution of problem 1.21 in [9], found by the traditional methods of relativity theory. Both formulas, (16) and (25), were first obtained by Einstein in [6, 7] relying on the phase preservation of a monochromatic light wave under Lorentz transformations.

In this respect, Einstein's explanations of these formulas are very enlightening. We present them using Pauli's notation for the angle and dimensionless velocity as adopted here and removing the indices M and P. Einstein writes [7]:

"1. If an observer moves with the velocity v with respect to an infinitely distant light source of frequency ω so that the line 'light source–observer' makes an angle α with the observer velocity with respect to a reference frame that is at rest relative to the light source, the frequency ω' of light detected by the observer is given by the relationship

$$\omega' = \omega\,\frac{1-\beta\cos\alpha}{\sqrt{1-\beta^2}}\,.$$

2. If a source emitting light of frequency ω_0 in the reference frame moving together with it moves so that the line 'light source–observer' makes an angle α with the velocity

of light source in the frame resting with respect to the observer, the frequency ω detected by the observer is given by the relationship

$$\omega = \omega_0 \, \frac{\sqrt{1-\beta^2}}{1-\beta\cos\alpha} \, .\text{''}$$

We stress that irrespective of the object of motion — be it the observer or the source — the velocity β of this object M and the angle α are defined by Einstein relative to a rest frame, which we call the laboratory one. The magnitude β and the direction $\mathbf{n}_\beta$ of velocity, the frequency ω, and the direction $\mathbf{n}_\gamma$ of light emitted by the laboratory source or detected by the laboratory observer are measured by laboratory instruments, whereas frequencies ω' and ω_0 are predicted or considered to be known in advance.

We also note that the use of an 'infinitely distant' light source is equivalent to the condition that the angle α remains constant as M moves.

6. Aberration of light from stars and the function $\alpha'(\alpha, \beta)$ describing a shift in the angular position and color of stars

Formulas (16) and (25) obtained for the Doppler effect (which coincide with Einstein's formulas) are the general form of the dependence of the ratio of proper frequencies of light emitted by a source and detected by an observer at the relative velocity β and the angle α measured in the laboratory frame.

At the same time, the description of light aberration mentioned above requires the knowledge of how the angle α' depends on α. We recall that α is the angle between the ray direction $\mathbf{n}_\gamma$ in the laboratory frame and the velocity direction $\mathbf{n}_\beta$ of the moving frame relative to the laboratory one (i.e., the direction of the x axis), and α' is the angle between the ray direction $\mathbf{n}'_\gamma$ in the moving frame and the direction $\mathbf{n}_\beta$ of the velocity of this frame relative to the laboratory one (i.e., the direction of axes x, x').

The function $\alpha'(\alpha)$ and its derivative $\dot\alpha'(\alpha)$ by α are already obtained above and are given by formulas (23) and (22). Because they also depend on the parameter β in addition to the dependence on α, we let them be denoted as $\alpha'(\alpha, \beta)$ and $\dot\alpha'(\alpha, \beta)$ in what follows.

These functions were given in the preceding section for five characteristic values of α in the interval $0 \leqslant \alpha \leqslant \pi$. These values are sufficient to qualitatively describe a picture of the sky seen by the crew on a spacecraft M moving with the velocity β in the center of a giant spherical laboratory with stars at its periphery.

Stars on the celestial sphere can be to a very good approximation considered fixed light sources that are almost 'infinitely far,' according to Einstein, from the region where spacecraft M is traveling. The position of a star in the sky can be characterized by the angles θ and φ of the spherical coordinate system with the polar axis (the x axis) in the direction $\mathbf{n}_\beta$ of the velocity of spacecraft M. Because of azimuthal symmetry, the Doppler and aberration effects depend only on the magnitude of β and the polar angle θ, replaced here by the angle $\alpha = \pi - \theta$, which was also selected by both Einstein and Pauli.

If for $\beta = 0$ the position of a star in the sky is determined by the angle α between the ray direction $\mathbf{n}_\gamma$ from this star to spacecraft M and the direction $\mathbf{n}_\beta$ of its future velocity (the direction of the x axis), then, upon acquiring the velocity β, the position of this star in the proper frame of spacecraft M is described by the angle $\alpha'(\alpha, \beta)$, which is larger than α,

$$\alpha'(\alpha, \beta) = 2\arctan\left(\sqrt{\frac{1+\beta}{1-\beta}}\,\tan\frac{\alpha}{2}\right) > \alpha \, , \tag{26}$$

except for the cases $\alpha = 0$ and $\alpha = \pi$, when $\alpha' = \alpha$. Thus, the stars in the + and − directions of the x axis stay without displacement in the spacecraft M proper frame, whereas the stars off the x axis, acquiring a positive shift, are displaced in the positive direction of the x axis such that their concentration in the rear hemisphere decreases, increasing in the forward hemisphere.

Stars with the characteristic angular coordinates for a given parameter β

$$\alpha = \alpha_\beta = \arccos\beta \, , \qquad \alpha = \alpha_1 \, , \qquad \alpha = \frac{\pi}{2} \tag{27}$$

in the moving frame of M are situated at angles α' equal to

$$\alpha'(\alpha_\beta) = \frac{\pi}{2} \, , \qquad \alpha'(\alpha_1) = \pi - \alpha_1 \, , \qquad \alpha'\left(\frac{\pi}{2}\right) = \pi - \alpha_\beta \, , \tag{28}$$

having acquired the displacements $\alpha'(\alpha, \beta) - \alpha$ equal to

$$\frac{\pi}{2} - \alpha_\beta \, , \qquad \pi - 2\alpha_1 \, , \qquad \frac{\pi}{2} - \alpha_\beta \, . \tag{29}$$

Stars with the coordinate $\alpha = \alpha_1$ undergo the largest shift. Stars from the rear sky hemisphere in the interval $\alpha_\beta < \alpha < \pi/2$ are to be seen in the forward hemisphere of spacecraft M inside the interval $\pi/2 < \alpha' < \pi - \alpha_\beta$ neighboring the spherical sector $\pi - \alpha_\beta < \alpha' \leqslant \pi$ that hosts stars from the forward hemisphere for $\beta = 0$ (Fig. 8).

We note that for a small β and an arbitrary α, the shift can be expressed as a series in powers of β,

$$\alpha'(\alpha, \beta) - \alpha$$
$$= \beta\sin\alpha + \frac{1}{4}\,\beta^2\sin 2\alpha + \frac{1}{4}\,\beta^3\left(\sin\alpha + \frac{1}{3}\sin 3\alpha\right) + \ldots \, . \tag{30}$$

It can be readily obtained from integral (23).

We now consider how the color of stars changes if viewed from a spacecraft moving with a velocity β. According to formulas (16) and (22), the frequency ω'_{M} of star light seen from the spacecraft is lower or higher than its proper frequency ω_{L} if

$$k_*(\beta, \alpha) \equiv \dot\alpha'(\alpha, \beta) > 1 \qquad \text{or} \quad < 1 \, .$$

The functions take the value 1 for $\alpha = \alpha_1$ (see (17)). Stars with this angular coordinate in the laboratory frame have a shifted coordinate $\alpha' = \pi - \alpha_1$ in the frame of spacecraft M, but the proper frequency of their light is preserved. Consequently, as seen from the spacecraft, the frequency ω'_{M} of stars with the coordinate α' in the range $0 \leqslant \alpha' < \pi - \alpha_1$ is lower than their proper frequency ω_{L} (the red shift), and is higher than the proper frequency (the blue shift) for the interval $\pi - \alpha_1 < \alpha' \leqslant \pi$. If the spacecraft velocity increases, the angle α_1 decreases, approaching zero. As a result, the region of the sky where the red shift is observed steadily grows, whereas the region where the blue shift is observed steadily decreases.

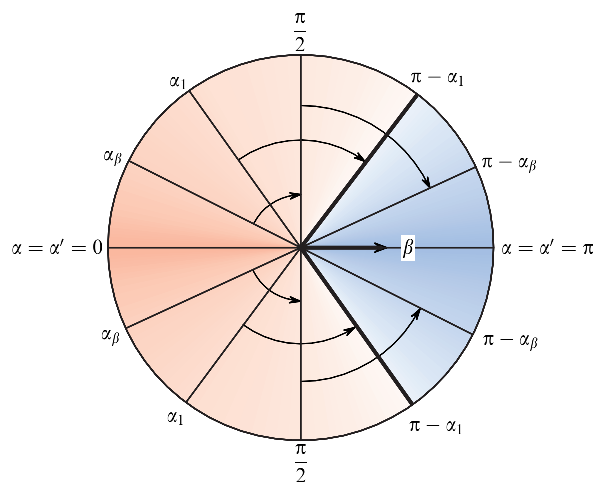

Figure 8. Stars at angles $\alpha = 0$, α_β, α_1, $\pi/2$, and π in the laboratory frame, are seen in the frame of spacecraft M at angles $\alpha' = 0, \pi/2, \pi - \alpha_1, \pi - \alpha_\beta$, and π with a red shift if $\alpha' < \pi - \alpha_1$, and with a blue shift if $\alpha' > \pi - \alpha_1$. In the plot, the parameters are $\beta = 0.9$, $\alpha_\beta = 26°$, and $\alpha_1 = 51°$.

In conclusion, we note that the aberration of light from stars was discovered by the English astronomer James Bradley in 1725–1728 as a fairly small change (with a semi-annual period) in angles between directions to various stars [10]. He then found the speed of light, which was correct to within 2%.

Because the scale of aberration is governed by the difference of the angle $2\alpha_1$ from π or the angle α_β from $\pi/2$ (see (29)), for Earth, with its orbital velocity of 30 km s^{-1}, $\beta = 10^{-4}$, we find $\pi - 2\alpha_1 = \beta = 20{,}5''$. This observation confirmed Copernicus's idea that Earth orbits the Sun.

7. Conclusions

Alternative ways of describing complex, enigmatic, or paradoxical phenomena missing in our everyday life are always helpful for their deeper understanding.

The Lorentz transformations connect the descriptions or measurement results of the same physical quantity at the same point by two observers moving relative to each other at a constant speed. Examples of such quantities primarily are the 4-coordinates in proper inertial frames of observers and units of length and time used in them.

Another important quantity in this work is the wave 4-vector k^μ of a plane monochromatic wave. Its scalar product $k^\mu x_\mu$ with the coordinate 4-vector forms the wave phase, which is invariant under Lorentz transformations. Wave 4-vectors $k^\mu = (\mathbf{k}, \omega/c)$ and $k'^\mu = (\mathbf{k}', \omega'/c)$ of a ray in a given reference frame and a frame moving relative to it are related by the direct and inverse Lorentz transformations (see Eqn (6.1) in [8]).

For frequencies ω, ω' and angles α, α' ($k_1 = (\omega/c)\cos\alpha$, $k_1' = (\omega'/c)\cos\alpha'$), these relations take the form

$$\omega = \omega' \frac{1 + \beta\cos\alpha'}{\sqrt{1-\beta^2}} \equiv \omega' k_+(\beta, \alpha')\,,$$
$$\omega' = \omega\,\frac{1 - \beta\cos\alpha}{\sqrt{1-\beta^2}} \equiv \frac{\omega}{k_*(\beta,\alpha)}\,, \tag{31}$$

whence

$$\frac{\sqrt{1-\beta^2}}{1-\beta\cos\alpha} = \frac{1+\beta\cos\alpha'}{\sqrt{1-\beta^2}} \quad \text{or} \quad k_*(\beta,\alpha) = k_+(\beta,\alpha')\,. \tag{32}$$

These formulas for the coefficients k_*, k_+ and the relation between them coincide with formulas (16), (20), and (21) obtained with the generalized k coefficient method.

Measurements indicate that in agreement with Lorentz transformations, moving rulers are shorter than their counterparts at rest and that moving clocks run slower than those at rest. And yet, to explain the twin paradox, the phenomenon of aberration, and the relativistic Doppler effect, one has to compare the rate of identical clocks separated by large distances (measuring time at a distance) or even resort to noninertial frames.

In the k coefficient method proposed by Bondi and in the generalized k coefficient method proposed here, determining time at a distance is intrinsically a necessary condition. Furthermore, in the generalized method the coefficients $k_*(\beta,\alpha)$ and $k_+(\beta,\pi-\alpha) \equiv 1/k_*(\beta,\alpha)$ are mutually reciprocal analytic functions of the complex variable $z \equiv \alpha$ in the interval $0 \leqslant z = \alpha \leqslant \pi$ and in the entire plane of complex z, which presents a natural domain where these functions exist as a single entity. Preserving a periodic dependence on z under its change parallel to the real axis, these functions have singularities outside the real axis at the points

$$z_n^{\pm} = 2\pi n \mp \mathrm{i}\,\ln\cos\alpha_1\,, \qquad \cos\alpha_1 = \frac{\beta}{1+\sqrt{1-\beta^2}}\,, \tag{33}$$

which are the poles for $k_*(\beta,z)$ and zeros for $k_+(\beta,\pi-z)$. Just these poles and zeros *are responsible for the essential spatiotemporal asymmetry of the relativistic Doppler and light aberration effects*, which consist of a shift of the angular position and color of light rays from distant sources in the direction of the observer velocity. If z changes parallel to the imaginary axis, $k_*(\beta,z)$ decays exponentially, and $k_+(\beta,\pi-z)$ exponentially increases.

The mathematical elegance of the derivation of these functions from Lorentz transformations competes with the common sense of their appearance in the k coefficient method. The Lorentz transformations emphasize the local equivalence of inertial frames (with and without prime), whereas the generalized method collects and integrates the information on distant events with the help of periodic signals to and from them. The coherence of such signals does not change if their phase is incremented by $2\pi n$. The periodicity of the functions $k_*(\beta,z)$ and $k_+(\beta,\pi-z)$ when z runs parallel to the real axis is apparently related to this.

In this respect we cannot resist recalling the words of Dirac from his article "The relation between mathematics and physics" [11] as regards the principle of mathematical beauty and the theory of functions of a complex variable. "This branch of mathematics is of exceptional beauty, and further, the group of transformations in the complex plane, is the same as the Lorentz group governing the space-time of restricted relativity. One is thus led to suspect the existence of some deep-lying connection between the theory of functions of a complex variable and the space-time of restricted relativity, the working out of which will be a difficult task for the future."

The use of analytic functions $k_*(\beta,z)$ and $k_+(\beta,\pi-z)$ with poles and zeros (33) is in line with this connection.

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