

METHODOLOGICAL NOTES

PACS numbers: 02.30.Em, 41.20.Cv, 41.20.Gz, 47.15.Hg, 47.15.km

Irrotational flow (of a magnetic field or incompressible fluid) around a screen with a slot

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DOI: https://doi.org/10.3367/UFNe.2020.03.038733

Contents

Abstract. **Problems concerning irrotational flows past obstacles may have physical applications in magneto- and electrostatics, as well as in the description of hydrodynamical flows of incompressible fluids. It is shown that for a magnetic field flow (or a flow of incompressible fluid in the hydrodynamical case) around an ideally conducting (impermeable) screen of width D with a narrow slot of width Δ a substantial flux passes trough the slot, so that, for example, a magnetic field (the velocity in the hydrodynamical problem) averaged over a slot with the width $\Delta = 0.01D$ will be 26 times greater than its far upstream value. The hydrodynamical problem is also formulated for an axisymmetric case for a circular screen and orifice. In this case, if the orifice is small enough, the flux of fluid proves to be proportional to the orifice diameter Δ, whereas fluid speed in the orifice increases as $1/\Delta$ if Δ is decreased, i.e., even faster than in plane geometry.**

Keywords: magnetic field, plane problem, axisymmetric problem, screen with a slot

1. Introduction

In many problems regarding magneto- and electrostatics and in hydrodynamical flows of incompressible fluid, the distributions of fields and velocities are described with the help of the Laplace equation and, as a consequence, may coincide for many problems. This is why the results obtained for one magnetostatic configuration or another might be carried over to the distribution of the electric field or velocity in a hydrodynamical flow. In this study, we will be mainly interested in patterns of field lines around screens with slots or orifices and in the field line concentration there. Because of the equivalence in the problem statements in these cases, we will begin with magnetostatic problems, and then discuss how they and their results will look for electrostatics and hydrodynamics. Since in an axisymmetric geometry the flow past a screen with an orifice is of interest only for hydrodynamics, we will be directly referring to a flow of fluid in this case.

When considering magnetostatic configurations, we draw attention to the fact that, if a magnetic field flows around an ideally conducting screen with a narrow slot (Fig. 1), its field lines become concentrated in the slot, which, in principle, can be used to generate strong magnetic fields [1–3] and possibly to carry out research utilizing such magnetic fields [4]. The field lines are concentrated because, in this case, the screen is subdivided into two conductors, and owing to no net current in each of them, a substantial magnetic flux should pass through the slot. The results obtained for the magnetic field configuration in plane geometry are easily transferred to an irrotational problem for electric field, in which the magnetic field is replaced by the electric one directed perpendicularly to the magnetic field, depicted in Fig. 1b, i.e., in the plane of Fig. 1b, parallel to the screen, and to a hydrodynamical problem for a flow of incompressible fluid. The electric field in the electrostatic problem and fluid velocity in the hydrodynamical one are amplified in the slot according to the same law as for the magnetic field in the magnetic case.

The hydrodynamical problem for a flow past a screen with a slot can be formulated in an axisymmetric geometry (Fig. 2), in the coordinates r and z, when the screen and the orifice are circles, so that the condition that the flow be irrotational (zero circulation along a contour around the ring in the flow) also results in an enhanced velocity $\mathbf{v}$ in the slot. In an axisymmetric magnetic problem, magnetic field lines will not

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Received 31 January 2020, revised 28 February 2020
Uspekhi Fizicheskikh Nauk **190** (10) 1109 – 1114 (2020)
Translated by D S Danilov

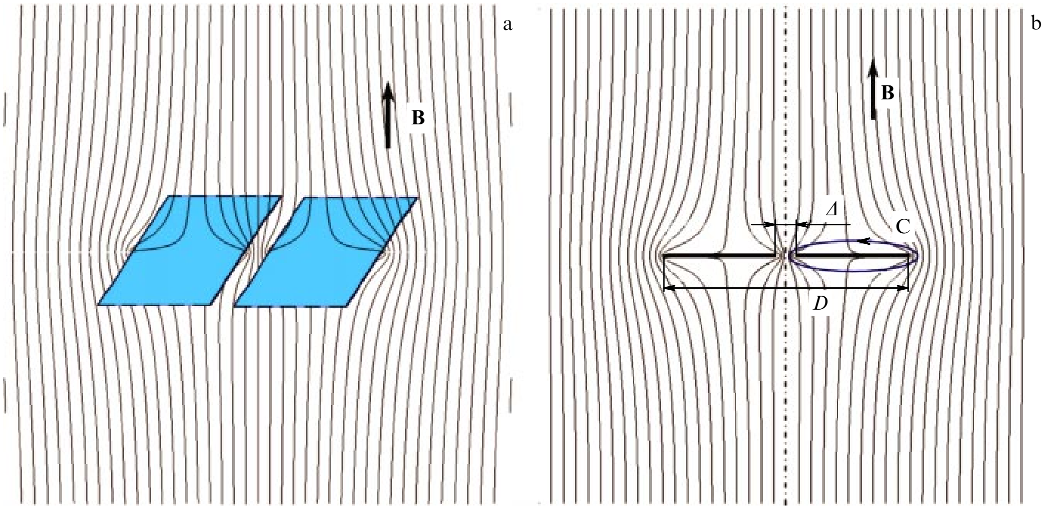

Figure 1. Pattern of field lines for a screen with a slot. (a) Three-dimensional isometric view, (b) two-dimensional view. C denotes a contour circulating around the conductor.

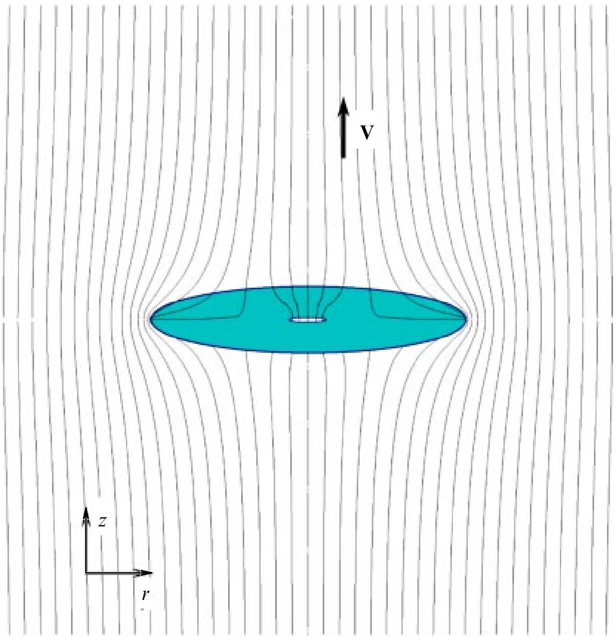

Figure 2. Pattern of field lines in the axisymmetric geometry.

concentrate inside the orifice since, in this case, a current will circulate in the closed ring and the integral along a contour around the ring will be different from zero. The electrostatic problem cannot be posed in an axisymmetric case at all.

2. Problem statement

We assume that in the magnetostatic problem the field outside the conductors obeys the standard equations

$$\operatorname{div} \mathbf{B} = 0\,,$$

$$\operatorname{rot} \mathbf{B} = 0\,,$$

and hence can be described with the help of potential $\mathbf{B} = \operatorname{grad}\varphi$. The magnetic field wrapping around ideal conductors is assumed to take a constant value $\mathbf{B}_0$ at infinity. The normal field component is zero on the surfaces of the conductors. The conductors are not connected to current sources (as depicted in Fig. 1a), so that currents on their surface are determined by the external magnetic field, and the net current through any section equals zero. As a result, the integral of the magnetic field along any contour circulating around the conductors is equal to zero:

$$\oint \mathbf{B}\,\mathrm{d}\mathbf{l} = 0\,. \tag{1}$$

Note that, in the absence of an external magnetic field for any single-connected ideal conductor that does not contain a magnetic field inside (for example, for a superconductor), all its surface currents, and hence the magnetic field outside the conductor, should be equal to zero. Indeed, in this case, the boundary conditions for the potential are that the normal magnetic field component is zero, $(\operatorname{grad}\varphi)_{\mathrm{n}} = 0$, both on the conductor surface and at infinity, and the only solution in this case $\varphi = \mathrm{const}$ corresponds to a zero magnetic field. It is different if a conductor is doubly connected (for example, has the shape of a bagel). The magnetic field can then cross the orifice in the conductor, and a current may circulate in its closed contour. In this case, generally speaking, a solution with the potential in the entire space cannot be constructed, and the problem will have a nontrivial solution without an external magnetic field, depending on a single parameter (the net current in the contour or net magnetic flux through the orifice). If there is an external field, the full problem solution will be a sum of the solution with nonzero current in the contour and the solution with zero current, but an external field at infinity. For the problem we are interested in, we will consider single-connected conductors, in agreement with Fig. 1.

In the hydrodynamical case, the flow is considered irrotational, so that its velocity is expressed as $\mathbf{v} = \operatorname{grad}\varphi$. The obstacles in the flow are assumed to be rigid and allowing an ideal slip, so that the normal component of velocity on their surface is zero. Since the flow is irrotational, the circulation around any closed contour equals zero, just as for the magnetic field,

$$\oint \mathbf{v}\,\mathrm{d}\mathbf{l} = 0\,.$$

3. Concentration of field lines in an orifice for the Laplace equation (for plane and axisymmetric irrotational flows)

3.1 Plane geometry

Let us derive an approximate law describing the field line concentration in a slot using the condition that the integral of the magnetic field along a contour C circulating around the conductor in the plane of Fig. 1b is zero. We will use the circumstance that, for a solution in the case without a slot, the integral along this contour, i.e., the nonclosed contour (for half of the screen) in this case, differs from zero and can be calculated, and for the solution when there is a slot with a flux passing through it, but the screen is infinite, the integral for this contour, i.e., the nonclosed contour in this case too (for half of the passing flux), is also different from zero and can also be calculated. The contributions of these integrals in (1) will have different signs, and since their sum equals zero, the flux passing through the slot can be calculated. In this reasoning, we essentially use the assumption that the slot is small with respect to the screen and its effect on the magnetic field configuration outside the slot is small too.

In the plane geometry, the estimate can be done with a logarithmic accuracy when it is supposed that not only the ratio D/Δ but also its logarithm are large, and the accuracy is $1/\ln(D/\Delta)$ in order of magnitude. For the estimate, one can assume that the magnetic field is composed of two constituents: the magnetic field around the screen in the absence of the slot, making a nonzero contribution in (1), and the magnetic field defined by the flux passing through the slot, which contributes in (1) with the opposite sign.

The problem of finding the magnetic field around the screen when the slot is ignored can be solved exactly [5], giving for the complex-valued magnetic field potential

$$w = B_0\sqrt{-z^2 + R^2}\,, \tag{2}$$

where B_0 is the magnetic field at infinity and $R = D/2$ is the screen half-width. The contribution from (2) in (1) is equal to the difference between potentials (2), i.e.,

$$2B_0R\,. \tag{3}$$

At distances r that are large compared to the slot size Δ, the magnetic flux f passing through the slot creates the magnetic field

$$B = \frac{f}{\pi r}$$

and hence contributes in (1) as

$$-\frac{2f}{\pi}\ln\left(\frac{2R}{\Delta}\right). \tag{4}$$

Equating the sum of (3) and (4) to zero, we find, with logarithmic accuracy, the magnitude of flux

$$f = \frac{\pi}{\ln(2R/\Delta)}B_0R = \frac{\pi}{2\ln(D/\Delta)}B_0D\,. \tag{5}$$

The flux magnitude estimation can be somewhat improved by estimating the coefficient in the argument of the logarithm in (5). With this aim, we can use the analytical solution [5] for the complex potential of a flow with a finite flux (flux f) close to the ends of the screen,

$$w \sim \ln\left(z + \sqrt{z^2 - (x_i)^2}\right),$$

where x_i is the coordinate of each of two screen edges, $|x_1| = \Delta/2$, $|x_2| = R$. Then, each edge gives the factor 2 in the sign of the logarithm, and the improved flux estimate will take the form

$$f = \frac{\pi}{2\ln(4D/\Delta)}B_0D\,. \tag{6}$$

The mean magnetic field will increase with the decrease in the slot width as

$$\bar{B} = \frac{\pi}{2}\frac{D}{\Delta}\frac{1}{\ln(4D/\Delta)}B_0\,. \tag{7}$$

The distribution of the magnetic field within the slot for small Δ/D is described by the analytical solution [5] for the field in the slot according to the formula

$$B = \frac{B(0)}{\sqrt{1-(2x/\Delta)^2}}\,, \tag{8}$$

where x is the coordinate counted from the slot center and $B(0)$ is the field at the center. According to (8), the slot-mean magnetic field magnitude is $\bar{B} = (\pi/2)B(0)$.

Note that the problem above on the concentration of magnetic field lines in a slot is close to the problem on current distribution over the surface of two flat bars separated by a narrow gap, considered in [6]. The solution methods are also similar (matching a solution ignoring the gap with the other one treating the width of the bars as infinite), as are the solution results, which take the form of (6), (7) with a logarithmic accuracy.

Formula (7) shows that for narrow slots the magnetic field is strongly enhanced in the slot: for example, for a slot with the width $\Delta = 0.01D$, the magnitude $\bar{B} = 26B_0$ and even the minimum field in the slot is $B(0) = 17B_0$. Such an increase in magnetic field (and also in electric field in the electrostatic case and fluid velocity in the hydrodynamical case) can be used in various applications.

An interpolation formula for the flux can be proposed, passing in the limit when the screen disappears ($D/\Delta = 1$) in $f = B_0D$,

$$f = \frac{B_0D}{(2/\pi)\ln(D/\Delta) + A - (A-1)(\Delta/D)}\,, \tag{9}$$

where A is a constant. For a screen shaped like a narrow band ($\Delta/D \simeq 1$), the flux is approximately equal:

$$f = \frac{B_0(D+\Delta)}{2}\,. \tag{10}$$

Imposing the condition that (9) pass into (10), in this case, we find for the constant

$$A = \frac{3}{2} - \frac{2}{\pi} \simeq 0.86\,.$$

Interestingly, for this value of A the improved coefficient under the logarithm sign becomes 3.9 for $\Delta/D \ll 1$, in excellent agreement with the coefficient 4 in formula (6). Thus, one can hope that formula (9) with $A = 0.86$ will well describe the magnitude of flux through the slot for any value of Δ/D.

3.2 Axisymmetric case

An axisymmetric screen with an orifice is a doubly connected domain. For the magnetic case, the ideal conductivity will prevent magnetic flux from penetrating through the orifice, and hence the external magnetic field will generate a current in the contour such that the net magnetic flux through the orifice will be zero, excluding a field increase in the orifice. This is why we consider here a hydrodynamical problem: the increase in fluid velocity will occur only in it.

In the axisymmetric case, the condition that the integral along a contour around the screen from the orifice to the boundary be zero enables calculating the fluid volume flux through the orifice with the accuracy of $\sim a/R$ instead of the logarithmic accuracy previously, where a is the orifice radius and R is that of the screen. In this case, the flow of fluid in the absence of an orifice is described by formulas presented in [7] which give for the potential distribution at the screen surface

$$\varphi = \frac{2}{\pi} v_0 \sqrt{R^2 - \rho^2}\,, \tag{11}$$

where v_0 is the fluid velocity at infinity and $\rho = \sqrt{x^2 + y^2}$ is the radius in cylindrical coordinates. The contribution from this flow in contour integral (1) is equal to twice the difference of the potential between $\rho = 0$ and $\rho = R$,

$$\frac{4}{\pi} v_0 R\,.$$

On the other hand, the flow of the fluid with flux f through the orifice is described by formulas for an electric field of a charged conducting cylindrical disk, the potential of which [8]

$$\varphi = \frac{\pi}{2}\frac{e}{a} = \frac{1}{4}\frac{f}{a}\,,$$

where e is the disk charge, which contributes to integral (1) with the opposite sign as

$$\frac{1}{2}\frac{f}{a}\,.$$

Equating these contributions, we obtain

$$f = \frac{8}{\pi} a R v_0\,.$$

We find that in the axisymmetric case the mean velocity in the orifice,

$$\bar{v} = \frac{8}{\pi^2}\frac{R}{a} v_0\,,$$

will increase for a decreasing even faster than in the plane geometry.

Noteworthy is the distribution of velocity in a small orifice as a function of radius is described, according [8], by a formula which is analogous to (8) of the plane problem,

$$v = \frac{v(0)}{\sqrt{1 - (r/a)^2}}\,, \tag{12}$$

so that the mean velocity through the orifice is related to the velocity at the center by $\bar{v} = 2v(0)$.

Just as in the plane geometry, an interpolation formula can be proposed to express the dependence of the volume flux on a/R, which has the correct asymptotic behavior for small a/R,

$$f = \left(\frac{8}{\pi^2}\xi + C_1\xi^2 + C_2\xi^3\right)\pi R^2 v_0\,, \tag{13}$$

where $\xi = a/R$, and C_1 and C_2 are some constants. Imposing on (13) the condition that, in the limit of vanishing screen ($a/R \simeq 1$), it gives a correct dependence for a thin ring $f = \pi R^2 v_0 \xi$, we find for the constants

$$C_1 = 2 - \frac{16}{\pi^2}\,, \quad C_2 = \frac{8}{\pi^2} - 1\,.$$

4. Results of computations in plane geometry

We carried out two-dimensional computations for the distribution of the magnetic field (fluid velocity) for the values of Δ/D in the range from 0.004 to 0.1. The computations were performed using a code based on the finite-element method. A mesh was constructed with a refinement around the slot using mostly second-order rectangular elements, the maximum number of elements being $\sim 5 \times 10^5$, which proves to be sufficient for computations with slots with $D/\Delta \sim 1000$. As an example, Fig. 3 plots the computed field lines (lines wrapped around the screen) and potential (lines that are parallel to the screen at large distances) for $D/\Delta = 5$. For the electrostatic problem with an external field directed along the screen, the field lines of the magnetic problem will correspond to equipotential lines of the electrostatic problem, and equipotential lines of the magnetic problem will correspond to the electric field lines.

Flux computations were carried out for the cross section of the slot and some section covering the slot (lines *1–2* in Fig. 3). The values of fluxes through these sections should be equal, because the flux is preserved for an irrotational field, but in the computational technique the sections distant from the slot are preferable, since the velocity does not diverge there. The dependence of the flux f through the slot on Δ/D is

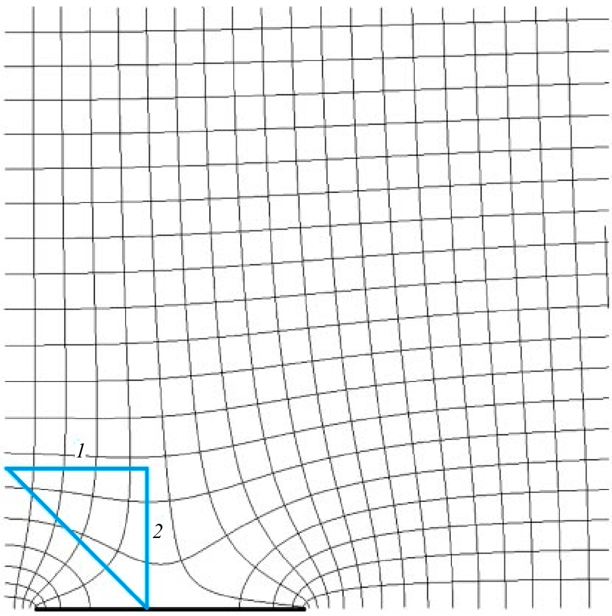

Figure 3. Distribution of field lines and potential for $D/\Delta = 10$.

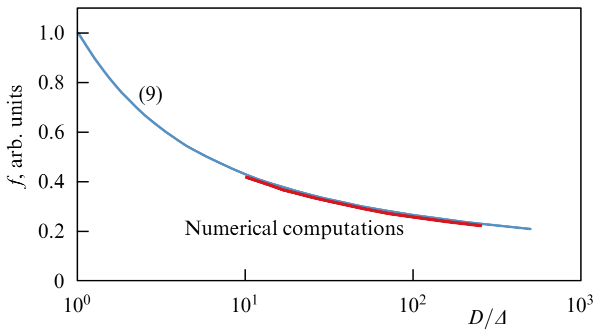

Figure 4. (Color online.) Dependence of flux through the slot, normalized by $f_0 = B_0 D$, on D/Δ obtained numerically and analytically using formula (9).

depicted in Fig. 4. For comparison, the value of f computed by formula (9) with the value of constant $A = 0.86$ is also plotted. It can be seen that both dependences agree with each other remarkably well. Computations indicated that the relationship $\bar{B} = (\pi/2)B(0)$ connecting the slot-mean magnetic field with its value at the center is observed with an accuracy of several percent even for a slot with $\Delta/D = 0.1$. Accordingly, the distribution of the magnetic field over the slot width will also be described by formula (8) with an accuracy of the same order.

5. Results of computations for an irrotational flow in the axisymmetric case

We carried out numerical computations of velocity distributions for various values of a/R. Since the computations used a three-dimensional code, to reduce the number of elements, a segment of a cylinder was selected with the opening angle of several degrees. The mesh was constructed with a refinement close to the orifice. These measures enabled us to perform computations for the ratio $R/a \sim 100$. To compute the volume flux, a surface was used in the orifice section, and also a cylindrical surface around the slot (by analogy with surface *1–2* in Fig. 3). As an example of the distribution obtained, Fig. 5 plots field lines and equipotential lines for $R/a = 5$ in comparison with the solution for the plane geometry.

The dependence of the mean velocity through the orifice $\bar{v}$ on R/a found in these computations is given in Fig. 6. For comparison, we also plot the dependence of $\bar{v}$ computed with the help of interpolation (13),

$$\frac{\bar{v}}{v_0} = \left(\frac{8}{\pi^2}\xi + C_1\xi^2 + C_2\xi^3\right)\frac{1}{\xi^2}\,. \tag{14}$$

It is seen that both dependences agree remarkably well. Computations showed that the relationship $\bar{v} = 2v(0)$ linking the slot-mean velocity with its value at the center holds with an accuracy of several percent even for the orifice with $a/R = 0.2$. Accordingly, the distribution of velocity over the width of the slot is described by formula (12) with an accuracy of the same order.

6. Conclusions

In plane-geometry problems of magnetostatics (hydrodynamics of an incompressible fluid), for a magnetic field

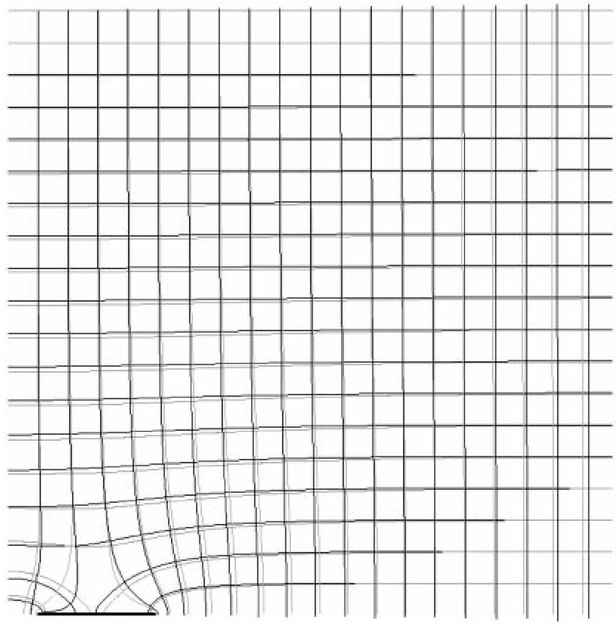

Figure 5. Distribution of field lines and potential for $R/a = 5$ in axisymmetric geometry (black lines) and for $D/\Delta = 5$ in plane geometry (gray lines).

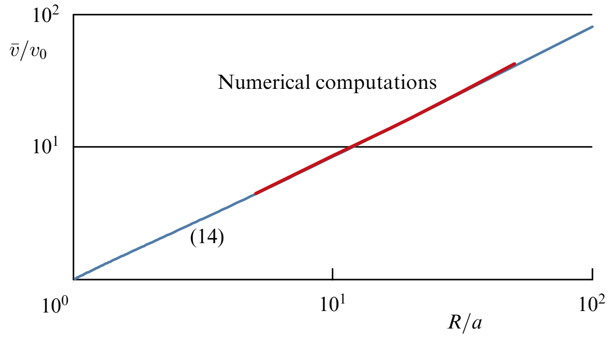

Figure 6. (Color online.) Dependence of the slot-mean velocity in the orifice on R/a obtained numerically and using formula (14).

around an ideally conducting screen (for an irrotational flow of fluid around an obstacle), a small slot in the screen (obstacle) modifies the configuration of the magnetic field (the flow character) rather dramatically, leading to increased values of the magnetic field (fluid velocity) in the slot. For a thin screen, the magnetic flux passing through the slot decreases only logarithmically with the slot width, equaling 26% of the flux impinging on the screen for a slot with the width $0.01D$.

In the axisymmetric case, the volume flux through the orifice decreases proportionally to its radius if the orifice size is decreased and the mean velocity through it increases even more strongly than in the plane geometry. So, for a flow past a flat disk, for an orifice with a radius 100 times smaller than the screen radius, the mean velocity through the orifice increases by a factor of 81.

The effect of field (flow velocity) amplification in the presence of small orifices is common for other geometries, not only for those considered in this paper. The difference for other geometries will only be in the amplification coefficients. The effect can be used to concentrate magnetic field (flow velocity) in various setups.

In everyday life, the phenomenon of enhanced heat leakage through small slits is very common, and it is generally known that one must remove even the smallest of them to preserve heat. In this study, we show that the magnitude of flux through narrow slots is weakly dependent on their size.

Summarizing, we reiterate that a method to estimate magnetic and hydrodynamical fluxes through narrow slots in obstacles has been presented, together with quantitative results for some important special cases of obstacle and slot configurations.

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