

METHODOLOGICAL NOTES

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From Lord Rayleigh to Professor A A Vlasov (from 1906 to 1945)

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Abstract. **The authors' view on the development of theoretical concepts of collisionless plasma is presented. The review begins with studies by Lord Rayleigh, who established the basics of multi-electron system theory in 1906 as a development of J J Thompson's atomic theory. His study was based on the concept that electrons move in a self-consistent electric field produced by the electrons themselves. Rayleigh then used this concept to predict the oscillatory motion of electrons and obtained a formula for the frequency of a system's collective oscillations. Those studies established a foundation for Rayleigh's model of a multi-electron system, and his formula for the frequency of collective oscillations may be considered the first analytical result of the science pioneered by Lord Rayleigh.**

Keywords: Rayleigh model, Langmuir – Tonks model, self-consistent field, Boltzmann kinetic equation, Landau collision integral, Vlasov equation

In memory of
Leonid Veniaminovich Keldysh

1. Introduction

Twenty years after Lord Rayleigh had published his paper [1], the experimental results of Langmuir [2, 3], Dittmer [4], and Penning [5, 6] appeared. The subject of their studies was gas discharges, in which they observed high-frequency oscillations of an electric field that were hardly understood at that time. The frequency of those oscillations coincided with that predicted by Rayleigh [1]: Rayleigh himself had long ago explained the field nature of those oscillations whose frequency depends on the density of electrons. Langmuir [7] introduced a concept of plasma as a generally neutral gas that consists of electrons, ions, and neutral particles, atoms and molecules. Langmuir and Tonks [8, 9], developing Langmuir's concept and using Rayleigh's model of a self-consistent field [1], suggested a model to interpret the experimental results that now looks simple and generalizes the Rayleigh model. It was assumed in that new model that charged particles move in a self-consistent, but this time electromagnetic, Maxwell field, owing to which the model becomes electromagnetic, rather than merely electric one. The model was quite successful in explaining experimental data on plasma oscillations in regions of not only high (electron) but also low (ion) frequencies. Studies [8, 9] thus showed the efficiency of using the concept of a self-consistent Maxwell field for describing plasma.

Development of the kinetic theory of plasma was, however, slowed down due to the belief that the Boltzmann equation is not applicable to a gas with Coulomb interaction, i.e., plasma. This opinion stemmed from the long-range character of Coulomb interaction, as a result of which the collision integral in the Boltzmann equation diverges for a pair of charged particles provided that they interact in a vacuum. As early as in his paper [7], Langmuir noted, however, that the Coulomb potential is screened in a plasma, unlike a vacuum, at distances larger than the Debye radius [10, 11] and showed that account of the Debye screening eliminates the divergence in cross sections of pair collisions of charged plasma particles.

In his approach [12], Landau while not abandoning pairwise collisions, took into account Debye screening in the Coulomb potential to derive the Boltzmann equation with a finite collision integral, thus introducing the collective effect into the kinetic theory of plasma.

A crucial step was made by A A Vlasov [13–15]: following [1–9], he added to the kinetic equation, which describes plasma particles, self-consistent electromagnetic fields, while

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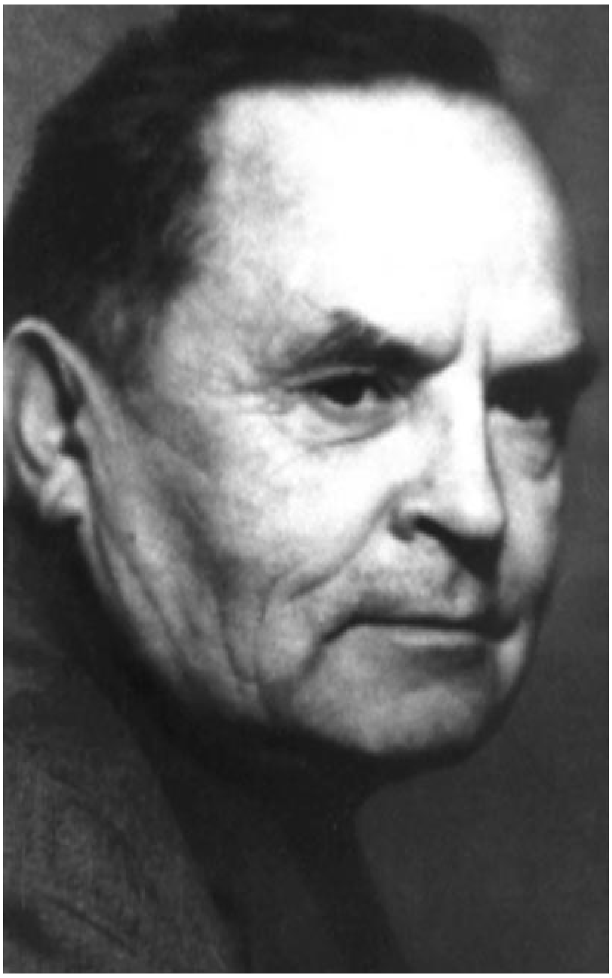

Anatolii Aleksandrovich Vlasov
(20.08.1908 – 22.12.1975)

fully ignoring pair particle collisions. It is this equation that was named after Vlasov, whose star rose.

(1) We begin our brief and largely personal presentation by describing Rayleigh's role in the development of physical concepts of plasma, including those still little known. Rayleigh redesigned in 1906 [1] the Thompson model of the atom into what is now called the plasma atom model. The key point of that transformation was the concept of the self-consistent field introduced by Rayleigh. It is therefore quite natural that already Vlasov's first publication [13] contains a reference to Rayleigh's study [1].

In 1925, Langmuir started in [2, 3] to methodically explore in his experiments the properties of plasma, including spectra of oscillations of plasma charge density using a high-frequency probe he designed (Langmuir probe). He discovered that there is a limiting frequency of long waves that had been predicted by Rayleigh as early as the early 20th century [1]. That frequency was shown earlier by Rayleigh and afterwards by Langmuir to depend on plasma electron density.

The dependence of the frequency on electron density enabled Langmuir to make an important conclusion regarding the collective nature of those oscillations that were later named plasma (or Langmuir) oscillations. In developing this approach, Langmuir and Tonks applied a simple mechanical model, in which electrons moved in self-consistent electromagnetic fields described by Maxwell equations, to explain in qualitative terms the spectrum of longitudinal (potential) waves observed by Langmuir. Moreover, the model also enabled them to explain in qualitative terms both the dispersion of Langmuir waves they observed and the presence in plasma of a low-frequency (involving ions) ion-acoustic oscillation branch, as well as possible propagation of transverse electromagnetic waves in plasma.

It should be noted that Langmuir used in his study [7] dated 1928 an important provision that the Coulomb potential in plasma is screened at distances larger than the Debye radius to show that the cross section of scattering of a particle on such a potential is finite. This conclusion allowed him to see how transport coefficients and relaxation times may be determined in the description of phenomena in non-uniform and non-stationary plasma based on the kinetic equation.

(2) The work by Landau [12] may be considered the next important step in the development of the basic science of plasma. Landau made the Boltzmann kinetic equation applicable to a description of the gas with Coulomb interaction between particles where the total cross section of elastic scattering of the pair of charged particles in a vacuum is infinite. The Boltzmann collision integral is in this case naturally senseless. Landau used the known result of Debye and Huckel [10] and took into account the Debye screening of the Coulomb potential of interaction between charged particles at large distances, owing to which the cross section of Coulomb scattering in a plasma becomes finite.

Landau thus took into account the static self-consistent field that involves many particles to obtain a meaningful integral of pair Coulomb collisions in the Boltzmann equation. Although this problem seems at first glance to be beyond the topic of collisionless plasma that we are discussing, its solution was a significant achievement with implications for practice. Namely, taking into account the effect of collective phenomena of pair Coulomb collisions according to Debye enabled Landau to describe many important phenomena in plasma similar to those that had been studied and explained by that time in the physics of gases. This result extending the physical kinetics of plasma beyond Boltzmann's concept of pair collisions was emphasized by Vlasov in his lecture course delivered to students in the Physics Department of Moscow State University.

(3) An important step that essentially determined the development of the basic theory of collisionless plasma as the physics of collective phenomena was made by Vlasov in [13–15]. He asserted in [13, p. 292] that "the kinetic equation method that only takes into account pair interaction, i.e., interaction by means of impact, is for the system of charged particles an approximation that is, strictly speaking, unsatisfactory; an essential role in the theory of such phenomena should be played as well by interaction forces operating at large distances, and the system of charged particles is therefore not a gas but a kind of system drawn together by long-range forces." We continue the direct quotation from Vlasov: "The account of 'long-range forces' naturally results in the properties that are missing in a normal gas medium whose properties comply well with the standard scheme of the kinetic equation. They include unusual oscillation properties of electron plasma, the existence of which was noted by Rayleigh in 1906 in a special problem of the behavior of an array of electrons in the old Thomson model of the atom and, independently in 1929, by Langmuir and Tonks, who used a similar approach to gas plasma."

In accordance with his assertion made in [13], Vlasov proposed a kinetic equation to describe a fully ionized plasma in which pair collisions of particles are fully ignored, and interaction between the particles is considered, following Langmuir and Tonks, to be due to self-consistent electromagnetic fields. To describe the self-consistency, the Vlasov equation was complemented, similarly to Langmuir's and Tonks's model in [8, 9], by a system of Maxwell equations in which the field is generated by the plasma particles themselves that move at the same time in the electromagnetic fields they generate. This plasma model became to be referred to as self-consistent field approximation for collisionless plasma or Vlasov plasma. The model not only confirmed the concepts on which the Rayleigh–Tonks–Langmuir theory is based but also provided a quantitatively correct description of dispersion of Langmuir oscillations, which was discussed in qualitative terms by Langmuir and Tonks [8, 9].

It should be noted that Vlasov and Landau in the mid-1940s virtually independently of each other theoretically predicted (with the point of a pen) a phenomenon that was quite new in the physics of plasma: collisionless damping of plasma waves (CDPW). The simplicity of the prediction made by Vlasov was due to the smart choice of the model function for electron velocity distribution. Vlasov also made the first step to understanding the physical nature of the phenomenon he predicted, indicating that collisionless damping is due to absorption of waves by plasma electrons. The Vlasov plasma model later became a basis for a broad range of explorations of new plasma phenomena and applied research and development that generated a plethora of publications. Landau published his prediction of CDPW in 1946 virtually simultaneously with Vlasov. His elegant method for predicting CDPW rapidly became generally accepted; the Vlasov vs. Landau priority problem has, however, emerged, distressing romantics of Russian science. This is, however, an issue that is 'beyond science'.

2. The formation of first theoretical concepts of plasma

The first theoretical concepts of plasma as a gas of charged particles were formulated in [1, 8, 9]. They were based on Rayleigh's simple arguments and Langmuir's experiments [2, 3]. Those experiments explored spectra of plasma oscillations in the glow discharge in mercury vapors and air under a pressure of $P_0 \approx 10^{-4} - 10^{-3}$ Torr at distances from electrodes that are much larger than the Debye radius of field screening. This choice of experimental conditions ensured a high uniformity of the plasma and its quasineutrality. Two experimental techniques were used: the method of plasma oscillation eigenfrequencies that manifest themselves in the spectra of resonance absorption of external radiation and the probe method, in which eigenfrequencies, the temperature of electrons, and plasma density are measured. This Langmuir probe later gained popularity in scientific laboratories.

The first phenomenon experimentally discovered by Langmuir was that plasma exhibits high-frequency oscillations whose frequency spectrum depends on plasma electron density in the way predicted by Rayleigh [1] in 1906. The oscillation frequency varied with changes in discharge current, i.e., depended on electron density. Langmuir suggested for that frequency an approximate formula that describes the main dependences of the observed frequency ν on experimentally measured physical parameters. According to [3], this qualitative formula has the following form:

$$\nu^2 \approx \frac{e^2 n_e}{m} \,. \tag{1}$$

The dependence of the oscillation frequency on charge e and mass m of the electron that follows from Eqn (1) enabled Langmuir to conclude that the oscillations he observed were electron ones; he also considered the dependence on the electron density n_e (number of electrons per cubic centimeter) a manifestation of collective phenomena.

It was noted above that the formula for the electron oscillation frequency was derived by Rayleigh [1] as early as 1906. Rayleigh predicted in that paper collective oscillations of electrons in the model of the multi-electron Thomson atom, in which positive charge is assumed to be uniformly distributed over a small spherical region while electrons are embedded in that charge (so-called plum pudding model). Each electron may oscillate around its equilibrium position. The atom as a whole is electrically neutral.

Rayleigh analytically described oscillations of the electron in the multi-electron Thomson atom in the following way (modern notations are used [16]): if a minor displacement of the electron from the equilibrium position is denoted as S, he presented the uncompensated electric charge that emerged as a result of that displacement in the form

$$\rho = 4\pi \operatorname{div}(e n_e \mathbf{S}) \,. \tag{2}$$

The electric field created by that charge, $\mathbf{E} = 4\pi e n \mathbf{S}$, acts on each electron with the force $\mathbf{F} = -e\mathbf{E} = 4\pi e^2 n \mathbf{S}$ that tends to bring the electron back to the equilibrium position. As a result, the electron oscillates, and Rayleigh described that oscillation motion using the equation

$$m \frac{d^2 \mathbf{S}}{dt^2} + 4\pi e^2 n_e \mathbf{S} = 0 \,. \tag{3}$$

This equation describes oscillations of electrons and the field they create with a frequency of

$$\omega = \omega_{Le} = \sqrt{\frac{4\pi e^2 n_e}{m}} \approx \sqrt{3 \times 10^9 n_e}\ [\mathrm{s}^{-1}] \,. \tag{4}$$

It was in the studies by Langmuir and Tonks [8, 9] that Eqn (1) found a rigorous justification, was named Langmuir frequency, and yielded an estimate of its value.

The dependence of frequency (4) on plasma electron density prompted Langmuir and Tonks [8, 9], who operated with the concept of elasticity as the driver of oscillations, to hypothesize that the elasticity of plasma is due to the 'large number' of electrons. It is, however, in [8, 9] that they started to methodically abandon such primitive concepts. They demonstrated that they had switched to understanding plasma waves as the plasma's electromagnetic field and matter waves by suggesting a new plasma model (that may be referred to as an electromagnetic model) of their own. The starting point in the model is the Maxwell equations:

$$\begin{gathered} \operatorname{rot} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \,, \qquad \operatorname{rot} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j} \,, \\ \operatorname{div} \mathbf{E} = 4\pi n_e \rho \,, \qquad \operatorname{div} \mathbf{H} = 0 \,, \end{gathered} \tag{5}$$

and the required constitutive relations:

$$m \frac{\partial \mathbf{V}}{\partial t} = e\left(\mathbf{E} + \frac{1}{c}\mathbf{V} \times \mathbf{H}\right) , \quad \mathbf{j} = e n_e \mathbf{V} \,, \quad \rho = e n_e \,. \tag{6}$$

External electric and magnetic fields are disregarded in the proposed model; plasma is considered to be an isotropic medium in which equations for the longitudinal field ($\mathrm{rot}\,\mathbf{E}=0$) and transverse field ($\mathrm{div}\,\mathbf{E}=0$) decouple, and the system of Eqns (5) and (6) splits into two independent equations:

$$\frac{\partial^2 \mathbf{E}^{\mathrm{l}}}{\partial t^2}+\frac{4\pi e^2 n_{\mathrm{e}}}{m}\,\mathbf{E}^{\mathrm{l}}=0\,,\qquad \frac{\partial^2 \mathbf{E}^{\mathrm{tr}}}{\partial t^2}+\frac{4\pi e^2 n_{\mathrm{e}}}{m}\,\mathbf{E}^{\mathrm{tr}}=c^2\Delta\mathbf{E}^{\mathrm{tr}}\,, \tag{7}$$

while the conditions under which solutions of those equations have the form of plane monochromatic waves, $\sim\exp(-\mathrm{i}\omega t+\mathrm{i}\mathbf{kr})$, are reduced to the following dispersion relations for the longitudinal and transverse waves:

$$\omega_{\parallel}^2=\omega_{\mathrm{Le}}^2\,,\qquad \omega_{\perp}^2=\omega_{\mathrm{Le}}^2+k^2c^2\,, \tag{8}$$

which are the same as the dispersion relations for longitudinal and transverse waves in a modern model of cold electron plasma [16]. Experimentally determined absorption spectra and coefficients of reflection and refraction of electromagnetic waves in mercury vapor plasma [7–9] for the measured electron density $n_{\mathrm{e}}\approx 10^{10}\ \mathrm{cm}^{-3}$ also agree with the values obtained with Eqns (7) taken into account.

Langmuir's and Tonks's simple electromagnetic model thus explained quite satisfactorily the experimental results and showed that a self-consistent field may be effectively used to describe plasma.

3. Boltzmann kinetic equation and Landau collision integral

The results and ideas presented in Section 2 clearly demonstrated that the kinetic model should be developed to describe the oscillatory properties of plasma. Researchers with a conservative physical mentality could hardly apply to plasma the then-standard physical kinetics of gases based on a kinetic equation with the Boltzmann collision integral. Indeed, the core of that integral was, for example, the cross section of the elastic scattering of two colliding particles. If two particles collide in a vacuum, that cross section is infinite due to a slow decrease of the Coulomb potential with distance. In other words, the corresponding collision integral is meaningless.

On the other hand, the Debye–Huckel effect was known in a medium containing immobile charges, where the Coulomb potential exponentially diminishes at distances that are larger than the Debye radius due to the screening caused by the particles. Landau showed in [12] that such a multi-particle effect as Debye screening may be introduced in the Boltzmann equation, where the interaction of a pair of colliding particles alone was earlier taken into account. Landau reproduced to this end the derivation of the collision integral based on Boltzmann's initial assumptions and extended it by an assumption that the momentum transferred in the process of collision is small; this assumption corresponds to the scattering of particles at small angles or collisions at large impact parameters, which is the same thing. The result obtained coincides with that found directly from the Boltzmann collision integral (see [17]). It proves to be proportional to the following logarithmically diverging integral over the transferred momentum Δp:

$$\Lambda=\int\frac{\mathrm{d}\Delta p}{\Delta p}=\ln\frac{r_{\mathrm{D}}}{r_{\min}}=\ln\frac{k_{\mathrm{B}}T\sqrt{\mu}}{\hbar\sqrt{4\pi e^2 n}}\,. \tag{9}$$

Several comments regarding Eqn (9) are relevant. First, in deriving this equation, the inverse proportionality between the momentum transferred in the collision and the corresponding impact parameter is taken into account. Second, the emergence of the minimum impact parameter is not due to the defectiveness of the Boltzmann collision integral that does not exist at small impact parameters but reflects the limited applicability of Landau's assumption regarding the smallness of the momentum transferred in collisions. We used here the well-known estimate that follows from the Heisenberg relation. Equation (9) is valid at high temperatures when $k_{\mathrm{B}}T>e^4\mu/\hbar^2$. If the effective mass μ is set to be the electron mass, the last inequality shows that quantum cutoff at small impact parameters becomes of importance at a temperature over 27 eV.

Third, the most important point for our presentation is that Landau used the Debye screening radius r_{D} as the maximum impact parameter. Owing to this, the Coulomb logarithm L becomes dependent on electron density.

The most important conclusion is that, as a result of truncating the divergence of Coulomb logarithm at large impact parameters due to Debye screening according to Landau, the Coulomb logarithm itself proves to be dependent on electron density. Following the way of thinking that was promoted by Langmuir, this dependence is a manifestation of the collective, albeit weak, nature of the Landau Coulomb logarithm. It is for this reason that Vlasov was saying to students in the Physics Department of Moscow State University that Landau had made a breakthrough from the paradigm of pair collisions, a cornerstone of Botzmann's early kinetics, that impeded analysis of the physical picture. Apart from that event, which is central to our narration, one should not miss the following useful Focker–Planck form of the collision integral derived by Landau in [12]:

$$\frac{\partial f_\alpha(\mathbf{r},\mathbf{V}_\alpha,t)}{\partial t}+\mathbf{V}_\alpha\frac{\partial f_\alpha(\mathbf{r},\mathbf{V}_\alpha,t)}{\partial\mathbf{r}}+e_\alpha\left\{\mathbf{E}+\frac{1}{c}[\mathbf{V}_\alpha\mathbf{B}]\right\}\frac{\partial f_\alpha(\mathbf{r},\mathbf{V}_\alpha,t)}{\partial\mathbf{p}_\alpha}=\frac{\partial}{\partial p_{\alpha i}}\left(D_{ij}^{\alpha}\frac{\mathrm{d}f_\alpha(\mathbf{r},\mathbf{V}_\alpha,t)}{\mathrm{d}p_{\alpha j}}-A_i^{\alpha}f_\alpha(\mathbf{r},\mathbf{V}_\alpha,t)\right), \tag{10}$$

where

$$D_{ij}^{\alpha}=\sum_\beta\int\mathrm{d}\mathbf{p}_\beta\,I_{ij}^{\alpha\beta}(\mathbf{p}_\alpha,\mathbf{p}_\beta)f_\beta(\mathbf{p}_\beta)\,,\qquad A_i^{\alpha}=\sum_\beta\int\mathrm{d}\mathbf{p}_\beta\,I_{ij}^{\alpha\beta}(\mathbf{p}_\alpha,\mathbf{p}_\beta)\frac{\partial f_\beta}{\partial p_{\beta j}}\,.$$

Here, $I_{ij}^{\alpha\beta}=2\pi e_\alpha^2 e_\beta^2 L(u^2\delta_{ij}-u_iu_j)u^3$, where $\mathbf{u}=\mathbf{V}_\alpha-\mathbf{V}_\beta$, ${}_\alpha=\mathrm{d}\mathbf{r}_\alpha/\mathrm{d}t$, and $f_\alpha(\mathbf{p}_\alpha,\mathbf{r}_\alpha,t)$ is the one-particle distribution function of type α particles ($\alpha=\mathrm{e},\mathrm{i}$).

Finally, Landau made the diverging Coulomb logarithm L (Landau logarithm) finite by truncating the interaction at large distances as a result of Debye screening and, at small distances, at $r_{\min}=e^2/T$, the distance where the approximation of small-angle scattering that he used fails. As a result, the Coulomb logarithm proves to be

$$L = \ln \frac{r_{\mathrm{D}}}{r_{\min}} = \ln \frac{r_{\mathrm{D}} T}{e^2} \,.$$

It is this logarithm that Langmuir obtained in 1928 [7].

Landau thus made the Boltzmann kinetic equation applicable to the description of plasma and significantly simplified in that way the calculation of transport coefficients, relaxation times, and other parameters that characterize the collisional kinetics of plasma. It should be stressed once again that, following Rayleigh, Langmuir, and Tonks, Landau introduced into the kinetic theory of plasma a collective effect, the manifestation of a self-consistent field, by replacing pair collisions of particles with collisions of collective Debye–Huckel–Langmuir–Landau clouds.

It would be relevant here to assess the importance of Landau's derivation of the collision integral (which was later named after him). Landau led physical kinetics in his systematic derivation of the collision integral of charged particles away from Boltzmann's paradigm of pair collisions to a new paradigm emerging in plasma theory that takes into account the collective effects related to the Maxwell electromagnetic field. If formulated in the language of the earlier concepts of Boltzmann pair collisions, the collective effect that Landau takes into account actually removes the error of his predecessors, who overestimated the role of large impact parameters: it is of importance if pairs of charged particles collide in a vacuum but proves to be suppressed as a result of Debye screening in a plasma.

Unlike Landau, Vlasov did not pursue the path of cleaning Augean stables. He followed Rayleigh, Langmuir, and Tonks up to introducing into the physical kinetics the Maxwell field electrodynamics that identifies, as is clear up to now, the field as a mechanism of long-range interaction between plasma particles already within the picture of the Debye–Huckel screening.

4. Vlasov kinetic equation with a self-consistent field

Shortly after that, the ideas of Rayleigh, Langmuir, and Tonks about the self-consistent field were revived by introducing the Maxwell electrodynamics into the physical kinetics. The bold and significant step that essentially guided the development of plasma physics as the physics of collective phenomena was made in Vlasov's breakthrough work [13–15].

Vlasov formulated in 1938 [13] his own definition of plasma: "Plasma is a medium with a long-range interaction in which account for only pair interaction between particles is apparently insufficient; the interaction should be taken into account by means of the electromagnetic fields generated by the particles." This assertion made by Vlasov corresponds to using in the kinetic equation the electromagnetic fields that are determined as self-consistent ones via the Maxwell equations. In accordance with this definition, Vlasov fully disregarded pair collisions of particles in deriving his kinetic equation to represent it in the following form:

$$\frac{\partial f_a(\mathbf{r}, \mathbf{V}_a, t)}{\partial t} + \mathbf{V}_a \frac{\partial f_\alpha(\mathbf{r}, \mathbf{V}_a, t)}{\partial \mathbf{r}} + e_\alpha \left\{ \mathbf{E} + \frac{1}{c} \left[\mathbf{V}_\alpha \mathbf{B} \right] \right\} \frac{\partial f_\alpha(\mathbf{r}, \mathbf{V}_\alpha, t)}{\partial \mathbf{p}_\alpha} = 0 \,. \tag{11}$$

Vlasov complemented Eqn (11) with the system of Maxwell equations:

$$\begin{aligned} &\operatorname{rot} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \,, \qquad \operatorname{div} \mathbf{E} = 4\pi \sum_\alpha e_\alpha \int f_\alpha \, \mathrm{d}\mathbf{p} \,, \\ &\operatorname{rot} \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \sum_\alpha e_\alpha \int \mathbf{V}_\alpha f_\alpha \, \mathrm{d}\mathbf{p} \,, \qquad \operatorname{div} \mathbf{B} = 0 \,. \end{aligned} \tag{12}$$

In 1938, Vlasov used Eqn (11) combined with Eqns (12) as a foundation for his plasma model. They were named the system of equations with self-consistent interaction, and Eqn (11) was named after Vlasov. The plasma model is referred to as collisionless plasma or, in other words, Vlasov plasma. In addition to providing a quantitative interpretation of Langmuir's experiments where plasma waves were observed, this model became a basis for a broad range of explorations of new plasma phenomena and applied developments.

5. Stationary longitudinal plasma waves. 'The future' is for Vlasov

One of the results of Vlasov's study [13] is the dispersion equation (Vlasov law) that he published in 1938. This equation describes stationary longitudinal oscillations of plasma (with real-valued ω and $\mathbf{k}$) in terms of the theory of Cauchy-type singular integrals that are directly related, as is clearly seen now, to the mathematical apparatus for the theoretical solution of collisionless plasma problems in the following form:

$$1 - \frac{4\pi e^2}{k^2} \, \text{v.p.} \int \frac{\mathbf{k} \, \partial f_0 / \partial \mathbf{p}}{\omega - \mathbf{k}\mathbf{v}} = 0 \,. \tag{13}$$

Here, v.p. means that the integral in Eqn (13) should be understood as the Cauchy principal value that corresponds, according to Vlasov, to the requirement of stationary oscillations. This choice of the Cauchy principal value made by Vlasov helps one to understand the Sokhotskii relations that had already been known by that time. A single glance at these relations is sufficient to understand that Vlasov's choice for describing stationary oscillations is quite natural.

Not everyone, however, was enthusiastic about that option. There were opponents who failed to see 'sufficient justifications' for the choice made. Things moved on, however. Equation (13) was used to obtain the spectrum of plasma oscillation and dispersion when the electron distribution function is maxwellian, and the wave phase velocity is much larger than the thermal velocity of electrons: $\omega \gg k v_{\mathrm{Te}}$,

$$\omega = \pm \sqrt{\omega_{\mathrm{Le}}^2 + k^2 v_{\mathrm{Te}}^2} \,. \tag{14}$$

Expression (14) correctly describes in qualitative terms the volume resonances that Tonks discovered as early as 1931 in his early studies [18, 19], i.e., long before Vlasov. The volume plasma resonances were explored after Tonks by a number of authors. A detailed description of Tonks's studies may be found in the review published by Golant and Piliya [20] (it also contains references on the issue). Dattner's study [21] published in 1957 (see also [22]) is the most comprehensive one. The term 'Tonks–Dattner resonances' was coined. A list of publications by authors from various countries who use Eqn (14) enables a judgment to be made of how popular this formula is. We list for illustration some of the publications listed in review [20]: Gil'denburg [23], Parker, Nickel, and Gould [24], and Vandenplas [25]. These are only some examples of studies, including experimental ones, in which stationary plasma waves have been explored.

The future (or more accurately, a careful reading of the past) has shown that Vlasov's dispersion equation (13) is also justified by the experiment, thus refuting the allegations regarding 'insufficient justifications'. The discussion that emerged later in relation to stationary plasma waves has, however, become a kind of PR campaign for some academics and a topic to be hushed up for the others, but this is an issue that is 'beyond science'.

As a conclusion: *an important discovery by our protagonists regarding nonstationary waves.*

Vlasov made another rather important discovery in 1945 that significantly enriched plasma physics. Namely, he published that year [14] a solution of the Cauchy problem for the relaxation of small potential plasma oscillations on the basis of linearized Vlasov equation (11) to predict, with the point of his pen, collisionless damping of plasma waves.

Landau virtually simultaneously with Vlasov published in *JETP* a solution to the Cauchy problem containing his prediction of CDPW [26]. Owing to the result being published in a renowned journal and the author's scientific prestige, the damping phenomenon was named after Landau. The prediction of plasma wave damping now belongs to the treasury of Russian science.

Because the articles by Vlasov [14] (the manuscript sent to the printer's on May 26, 1945 and signed for publication on November 13, 1945) and Landau [26] (submitted to the journal on June 2, 1945) were published at about the same time, some academics try to reduce the scientific assessments to discussions typical of a sports contest: who was the first?

A matter of concern is that the very formulation of the question is reminiscent of the situation of a bull in a china shop. It may be typical of sports but is it relevant to science? The answer we suggest is: both were the first (this may occur in sports as well). The main reason is that, although the old building of the Physics Department of MSU was crammed with people, due to which contacts between them were very close, and it was very difficult to conceal discoveries prior to publishing, the articles by Vlasov [14] and Landau [26] exhibit high and instructive standards of scientific creativity typical of classics. Moreover, today as well, any indelicacy in assessing scientific activities, especially those of classics, should be subject to ostracism. This attitude is apparently of special importance in times like ours, when attempts are being made to reform science. We would like to hope that ostracism of that kind will someday become an effective tool of the scientific journals published by the Russian Academy of Sciences.

As to the Vlasov vs Landau question, L Keldysh, former editor-in-chief of *Physics–Uspekhi*, having met shortly before his death Silin and learned that he and Rukhadze were struggling with that problem, said firmly: "It's time to end it." Our suggestion regarding the final resolution of the problem is: definitely both, but this only refers to plasma and the times of the mid-20th century. In our opinion, studying the events of the rapidly passed time that we witnessed will educate the successors exploring those events. Interest in the experience of predecessors in both science and everyday life is growing increasingly strong, as the example of Yu A Sokhotskii, whose work [27] was ignored by predecessors, shows.

To understand the atmosphere in which discoveries were made at that time, we briefly describe the history of Sokhotskii formulas. One can see in the oblivion and later 'independent' derivation of the formulas by our compatriot Sokhotskii, who discovered them in 1873, an omen of that hard and heroic time, recurrences of which are still alive in our age of consumerism. (That happened to Landau when his successors inadvertently used one of the Sokhotskii formulas, which is now known to everyone, and some of them declared it the Landau formula with half-residue, a statement that may be correct according to [28].) This blunder being eventually identified allows one to hope that the professional skills of our reading successors will be enhanced to become the certainty that respect to ancestors and predecessors will get stronger and firmer. This is, well, so much needed for Russia and its science to get revived. We are grateful to those who left us precious scientific heritage as a memory of them.

Hoping to compose a continuation,

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One of the authors, Anri Amvrosievich Rukhadze, died unexpectedly on March 8, 2018. Both authors, however, had a chance to sign the manuscript. [1]

Appendix [2]

Difficult-to-access materials related to Vlasov's creative activities [14–16].

The two 3D distributions that are competitively used in applications have the form

$$f_0(v) = \frac{n_{\rm e}}{(2\pi m T)^3} \exp\left(-\frac{mv^2}{2T_{\rm e}}\right),$$
$$f_0(v) = \frac{n_{\rm e} v_{\rm Te}^3/\pi^3}{v_z^2 + v_{\rm Te}^2} \frac{1}{v_x^2 + v_{\rm Te}^2} \frac{1}{v_y^2 + v_{\rm Te}^2}\,. \qquad (15)$$

The latter formula contains a product of three functions of independent arguments, three projections of the electron velocity.

Vlasov used Eqns (11) and (15) to obtain an equation for the function $\varphi(\mathbf{v}, t)$:

$$\frac{\partial \varphi(v_z, t)}{\partial t} + {\rm i}kv_z\varphi(v_z, t) - {\rm i}eE_z\,\frac{\partial f_0(v_z)}{\partial v_z} = 0\,. \qquad (16)$$

Here, $E_z = -\partial\Phi/\partial z$ is the small strength of the potential electric field of plasma oscillations. Vlasov represents the initial condition as

$$\varphi(v, 0) = \varphi_0(v)\,. \qquad (17)$$

[1] The second author of the article, Viktor Pavlovich Silin, died on January 12, 2019 (an obituary of Silin was published in *Usp. Fiz. Nauk* **189** 559 (2019) [*Physics–Uspekhi* **62** 524 (2019)]). (*Editor's note.*)

[2] The appendix was written by A A Rukhadze.

Using the Laplace transformation (direct and inverse),

$$\varphi_p(\mathbf{v}) = \int_0^\infty \mathrm{d}t\, \varphi(\mathbf{v}, t) \exp(-pt)\,,$$
$$\varphi(\mathbf{v}, t) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+\sigma}^{+\mathrm{i}\infty+\sigma} \mathrm{d}t\, \varphi_p(\mathbf{v}) \exp(pt)\,, \tag{18}$$

where $\sigma > 0$ (formulas for the transformation of $\Phi(z, t)$ have a similar form), Vlasov derives from Eqn (16) with initial condition (17) at the Poisson equation

$$(p + \mathrm{i}kv)\varphi_p = \varphi_0 \tilde{f}_0(v) + \frac{\mathrm{i}k\Phi_p}{m} \frac{\partial \tilde{f}_0(v)}{\partial v_z}\,,$$
$$k^2 \Phi_p = -4\pi e \int_0^\infty \mathrm{d}v\, \varphi_p(v)\,. \tag{19}$$

Vlasov employs these equations and the inverse Laplace transformation to derive the relation

$$\varphi_p = \frac{\varphi_0 \tilde{f}_0(v) + \mathrm{i}k\Phi_p\, \partial \tilde{f}_0(v)/\partial p_z}{p + \mathrm{i}kv}\,, \tag{20}$$

which is further used to find a solution to the Cauchy problem for the electric field potential:

$$\Phi(k, t) = \frac{\mathrm{i}}{2\pi} \int_{-\mathrm{i}\infty+\sigma}^{+\mathrm{i}\infty+\sigma} \mathrm{d}p\, \exp(pt)\Phi_p$$
$$= \frac{2\mathrm{i}e}{k^2} \int_{-\mathrm{i}\infty+\sigma}^{+\mathrm{i}\infty+\sigma} \mathrm{d}p\, \exp(pt) \int_{-\infty}^{+\infty} \frac{\varphi_0 \tilde{f}_0(v)}{p + \mathrm{i}kv}\, \mathrm{d}v$$
$$\times \left(1 - \frac{4\pi e^2}{km} \int_{-\infty}^{+\infty} \frac{\mathrm{d}v}{p + \mathrm{i}kv} \frac{\partial \tilde{f}_0(v)}{\partial v}\right)^{-1}. \tag{21}$$

To make the demonstration of the phenomenon he predicted simpler, Vlasov uses, instead of the one-dimensional model Maxwell distribution

$$\tilde{f}_0(v) = \frac{n_\mathrm{e}}{\sqrt{2\pi T_\mathrm{e}/m}} \exp\left(-\frac{mv^2}{2T_\mathrm{e}}\right), \tag{22}$$

a one-dimensional dispersion Lorentz-type distribution, a substitution that does not affect the Cherenkov nature of CDPW:

$$\tilde{f}_0(v) = \frac{n_\mathrm{e} v_\mathrm{Te}/\pi}{v^2 + v_\mathrm{Te}^2}\,. \tag{23}$$

Vlasov used function (23) to handle both integrals in Eqn (23) in a standard way prescribed by the theory of functions of a complex variable and obtained as a result the following formula that describes the time evolution of the perturbation potential of the equilibrium distribution function [15]:

$$\Phi(k, t) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty+\sigma}^{+\mathrm{i}\infty+\sigma} \exp(pt)\Phi_p\, \mathrm{d}p$$
$$= \frac{2e}{k^2}\, \varphi_0 \int_{-\mathrm{i}\infty+\sigma}^{+\mathrm{i}\infty+\sigma} \exp(pt)\, \mathrm{d}p\, \frac{p + kv_\mathrm{Te}}{(p + kv_\mathrm{Te})^2 + \omega_\mathrm{Le}^2}$$
$$= \Phi_0(k, 0) n_\mathrm{e} \exp(-kv_\mathrm{Te} t) \cos(\omega_\mathrm{Le} t)\,, \tag{24}$$

where $\Phi(k, 0)$ and $\varphi_0(k, 0)$ are the initial values of the potential proportional to the initial distribution function. As follows from Eqn (24), the Langmuir wave is damping with time; its frequency has a negative imaginary-valued part, $\mathrm{Im}\,\omega = -kv_\mathrm{Te}$. Vlasov relates the collisionless damping he found with the absorption of waves by plasma electrons. However, he only learns about relation of the phenomenon he discovered to the Vavilov–Cherenkov effect from the Nobel lecture of his teacher, I E Tamm. Indeed, the ways of Lord are inscrutable.

Owing to distribution (23) being chosen in a smart way, Vlasov succeeded in his pioneering work to theoretically show 'with the point of his pen' the existence of collisionless damping of plasma waves. It is not without reason that distribution (23), albeit a model one, is named after Lorentz.

At present, Vlasov's approach that implements analytical simulation of new predicted phenomena may cause objections if compared with other similar predictions. It yielded, however, the result (24) that is of importance for plasma physics: a prediction of collisionless damping of plasma waves. The fact that Vlasov predicted this phenomenon virtually simultaneously with Landau has been under a veil of silence until recently. Meanwhile, it ignited an information war against Vlasov and his disciples that was waged for a long time. Delivering a speech at the scientific council of the Physics Department of MSU, Dmitrii Blokhintsev as early as the 1940s described the situation at the department with the following words: "Kings go mad, and the people suffer for it." It is only recently that Vlasov's results have become a matter of public discussion [29]. If fate is benevolent, although this is not obvious, we will also write about it.

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