

METHODOLOGICAL NOTES

PACS numbers: **05.40.–a**, 05.60.Cd

Green's function method in the theory of Brownian motors

V M Rozenbaum, I V Shapochkina, L I Trakhtenberg

DOI: https://doi.org/10.3367/UFNe.2018.04.038347

Contents

Abstract. **We present the main results of the theory of Brownian motors obtained using the authors' approach, in which a Brownian particle moving in a slightly fluctuating potential profile is considered. By using the Green's function method, the perturbation theory in small fluctuations of potential energy is constructed. This approach allows obtaining an analytic expression for the mean particle velocity that is valid for two main types of Brownian motors (flashing and rocking ratchets) and any time dependence (stochastic or deterministic) of the fluctuations. The advantage of the proposed approach lies in the compactness of the description and, at the same time, in the variety of motor systems analyzed with its help: the overwhelming majority of known analytic results in the theory of Brownian motors follow from this expression. The mathematical derivations and analysis of those results are the main subject of these methodological notes.**

Keywords: Brownian motors, ratchets, driven diffusive systems, Green's functions

1. Introduction

Analytic approaches to solving many problems in theoretical physics arise when the model formulation of the system being studied allows proposing a suitable zeroth approximation and developing a perturbation series in small parameters to include factors that are absent in the zeroth approximation. The zeroth approximation is often selected such that it corresponds to the state of the system when its elements do not interact with each other or are in free motion. In this case, either the interactions between system elements or their potential energy in a time-dependent external field are regarded as perturbations.

Problems of this type are frequently solvable in the framework of an approximate approach based on differential equations, using iterations in small parameters describing perturbations. The solution procedure becomes more complicated if we attempt to include a larger number of factors into the zeroth approximation, for example, the stationary part of the potential energy, with its variable part considered small and treated as a perturbation. In this case, we have to first solve the unperturbed problem, i.e., find the response of the unperturbed system to a pointwise instantaneous forcing. The solution obtained, called the Green's function of the unperturbed problem, can then be used to analyze the full, 'perturbed' problem (see, e.g., Refs [1–3]).

The Green's function method is used in many areas of physics to solve both stationary and nonstationary problems. We mention boundary-value problems of electrostatics [4] as typical among stationary problems, and various problems of statistical physics and condensed-matter physics (irreversible processes, superconductivity, ferromagnetism, interaction of electrons with a lattice in metals and semiconductors) [5–7] as typical among nonstationary problems. Using Green's functions is especially fruitful when their closed analytic forms are available. For example, the Green's function for an ideal lattice of a solid body [8] was used to find the constant of tunnel atom transport in the solid phase [9], while the integral representation of the Coulomb Green's function [10, 11] has been used to compute rate constants for various processes occurring in the interaction of laser radiation with atoms (light scattering, photoexcitation, photoionization, and others) [10–13]. The Green's function method also becomes very fruitful in the analysis of perturbations in energy spectra induced by defects in the crystal lattice, the surface, or molecules with locally attached fragments [14–16].

In this paper, the Green's function method is applied to construct a theory of Brownian motors (called ratchets) — model systems in which temporal fluctuations of the potential

V M Rozenbaum (1, *), **I V Shapochkina** (2, †), **L I Trakhtenberg** (3, 4, ‡)
(1) Chuiko Institute of Surface Chemistry, National Academy of Sciences of Ukraine, ul. Generala Naumova 17, 03164 Kiev, Ukraine
(2) Belarusian State University, prosp. Nezavisimosti 4, 220030 Minsk, Belarus
(3) Semenov Federal Research Center for Chemical Physics, Russian Academy of Sciences, ul. Kosygina 4, 119991 Moscow, Russian Federation
(4) Lomonosov Moscow State University, Leninskie gory 1-3, 119991 Moscow, Russian Federation
E-mail: * vik-roz@mail.ru, † shapoch@mail.ru, ‡ litrakh@gmail.com

Received 7 November 2017, revised 19 April 2018
Uspekhi Fizicheskikh Nauk **189** (5) 529 – 543 (2019)
https://doi.org/10.3367/UFNr.2018.04.038347
Translated by S D Danilov; edited by A M Semikhatov

energy of a Brownian particle can lead to its directed motion [17–20]. The fluctuations can be different in nature: they can occur due to stochastic conformational transitions in molecules forming a Brownian motor, triggered by external nonequilibrium actions, can be an effect of an external alternating electromagnetic field applied to the system, etc. The first type of fluctuations (stochastic) is characteristic of biological molecular motors [21–27], while the second (periodic variations) are characteristic of artificially designed nanodevices [18, 28–34]. An important fact here is that the mean force applied to a particle through these fluctuations is zero (unbiased fluctuations), but system asymmetry and effects nonlinear in fluctuations lead to the directed motion. One of the first examples of the realization of directed motion arising in these conditions is a direct electric current excited under the action of a high-frequency electromagnetic field in media without a symmetry center (photogalvanic effect) [35].

Among the numerous approaches to describing the phenomenon of directed nanoparticle motion in nonequilibrium systems in the absence of mean forces and concentration gradients, a special place is occupied by theoretical analysis of the diffusive dynamics of an individual particle in a time-dependent potential field with a potential energy $U(x, t)$. Such approaches are the simplest because their main object is an individual particle or an ensemble of noninteracting particles, but at the same time are most instructive because they allow clarifying the basic condition for directed motion to occur. The basic equation describing inertialess motion of a Brownian particle is the Smoluchowski equation for the distribution function $\rho(x, t)$ that determines the probability of finding the particle at a point x at a time instant t. This equation is a particular case of the Fokker–Planck equation, the applications and solution methods of which are detailed in monograph [3] (see also its summary version [36] and monograph [37]). In many problems of Brownian motor theory, the Smoluchowski equation can be conveniently written as the continuity equation [17],

$$\frac{\partial}{\partial t}\,\rho(x,t) + \frac{\partial}{\partial x}\,J(x,t) = 0\,, \tag{1}$$

with the probability flux

$$\begin{aligned} J(x,t) &= -D\exp\left(-\beta U(x,t)\right)\frac{\partial}{\partial x}\exp\left(\beta U(x,t)\right)\rho(x,t) \\ &= -D\,\frac{\partial}{\partial x}\,\rho(x,t) - \beta D\,\frac{\partial U(x,t)}{\partial x}\,\rho(x,t)\,, \end{aligned} \tag{2}$$

where $D = (\beta\zeta)^{-1}$ is the diffusion coefficient, ζ is the friction coefficient depending on the medium viscosity and on the particle size and shape, and $\beta = (k_{\mathrm{B}}T)^{-1}$ is the inverse thermal energy (k_{B} is the Boltzmann constant and T is the absolute temperature). The mean force acting on the particle is assumed to be zero, and the averaging operation is understood as averaging over both space and time variables. For a spatially periodic potential profile $U(x, t)$, spatial averaging (over a period) always gives zero for the force $-\partial U(x, t)/\partial x$ that corresponds to this profile. If the form of the potential energy $U(x, t)$ is additive-multiplicative,

$$U(x,t) = u(x) + \sigma(t)w(x)\,, \tag{3}$$

where the function of time $\sigma(t)$ plays the role of a fluctuating variable, then the mean of $\sigma(t)$ must be equal to zero, $\langle\sigma(t)\rangle = 0$; the symbol $\langle\ldots\rangle$ denotes averaging over fluctuations, and its definition depends on the nature of $\sigma(t)$. We note that representation (3) is relevant for most theoretically and practically significant motor systems [17, 18, 38, 39] and that correlation functions of $\sigma(t)$ of different orders enable analyzing how the different natures of fluctuations are imprinted in the characteristics of motors.

The mean velocity for directed motion of a Brownian particle in the absence of mean forces can be different from zero due to the time dependence of the potential energy $U(x, t)$ under certain conditions on the system symmetry [17, 18, 20, 40–46]. Moreover, in a number of systems, the violation of nonobvious (so-called hidden) symmetries allows these systems to demonstrate motor behavior [45, 46]. We are typically interested in steady-state regimes of motion, when the system has already forgotten its initial condition. Then, by averaging continuity equation (1) with flux (2) over fluctuations, we can arrive at an important conclusion: the mean flux $J \equiv \langle J(x, t)\rangle$ is independent not only of time but also of the spatial coordinate, i.e., is a constant. This constant value multiplied with the spatial period L of the potential energy $U(x, t)$ defines the mean motion velocity $\langle v\rangle = LJ$, which is the main characteristic of Brownian motors.

To compute the mean velocity of motor motion, we need to find a solution of the Smoluchowski equation with periodic boundary conditions and then average flux (2) over potential energy fluctuations. An analytic solution of this complex problem is possible only under certain simplifying assumptions about the shape of the function $\sigma(t)$ that governs the temporal behavior of the particle potential energy, as well as about the shape of the potential profile and/or the presence of a small parameter that allows developing a perturbation theory. The function $\sigma(t)$ is often taken to be a dichotomous stochastic process in which $\sigma(t)$ takes two values , $+1$ and -1, with the given transition rates γ_+ and γ_- (particle transition rates between states with different potential energies (3)) [47]. A typical example of systems that exhibit such temporal behavior of potential energy is given by molecular (protein) motors [22–26, 48], with motion induced by the cyclic transition of a motor protein between conformational states with different potential profiles $U_+(x) = u(x) + w(x)$ and $U_-(x) = u(x) - w(x)$. For a dichotomous stochastic process, the correlation function $K(t) \equiv \langle\sigma(t_0 + t)\sigma(t_0)\rangle$ (where t_0 is an arbitrary initial instant) takes an exponential form $K(t) = \exp(-\Gamma|t|)$, where $\Gamma = \gamma_+ + \gamma_-$ is the inverse correlation time. In the class of processes where potential energy varies deterministically (the processes controlled through human-made techniques and considered in describing the mechanisms of the functioning of nanodevices), periodic processes are usually analyzed. In this case, $\sigma(t + \tau) = \sigma(t) = \sum_j \sigma_j \exp(-\mathrm{i}\omega_j t)$, where τ is the period, $\omega_j = 2\pi j/\tau$, $j = 0, \pm 1, \pm 2, \ldots$, and σ_j are the Fourier components of $\sigma(t)$. The averaging operation $\langle\ldots\rangle$ implies averaging over the period, whence $\langle\sigma(t)\rangle = \sigma_0 = 0$, and the correlation function is $K(t) = \sum_j |\sigma_j|^2 \exp(-\mathrm{i}\omega_j t)$.

The choice of the shape of the model potential profile is governed by the properties of the system considered. If the potential energy changes smoothly with the coordinate, we can limit its functional description to two (rarely, three) spatial harmonics [42, 50]. When dealing with potential energy profiles that have intervals of sharp behavior (in the limit, jumps), a saw-tooth profile is selected [51–53]. Describing motor systems with a potential in the form of two harmonics is simplified by resorting to the Fourier representation because summation of the occurring expressions over harmonics is then reduced to only the first terms in the series. For a saw-tooth potential, the coordinate representation can be quite useful because the piecewise linear form of the potential enables analytic integration. The shapes of the potential energy profiles mentioned above are clear leaders judging by the frequency of their use in problems of Brownian motor theory [17, 18, 38, 50].

The main parameters 'governing' the working regimes of a Brownian motor, the amplitude and sign of its mean velocity, and the energy characteristics are the frequency of fluctuations and the amplitude of spatial variations of the potential energy. The frequency can be chosen as the inverse correlation time Γ for a dichotomous stochastic process and as the inverse period τ^{-1} for a periodic deterministic process. Whether the frequencies are small or large is decided by comparing them to characteristic system frequencies, i.e., inverse characteristic times, for example, the particle diffusion time $\tau_{\rm D} = l^2/D$ over a characteristic length l or the time $\tau_{\rm sl} = \zeta l^2/\Delta V$ it rolls down a sloping part of the potential energy profile with the height ΔV and length l (we note that for inertial Brownian motors, an additional characteristic time related to the particle mass appears, together with nontrivial effects due to the competition between this time and the values of Γ^{-1} and τ [52–54]).

To estimate characteristic values, l is often chosen as the spatial period L of potential energy variation. In this case, the low-frequency and high-frequency approximations are characterized by the respective small and large values of dimensionless frequencies $\Gamma\tau_{\rm D}$ and $\Gamma\tau_{\rm sl}$ (for dichotomous stochastic processes; for deterministic periodical processes, they are $\tau_{\rm D}/\tau$ and $\tau_{\rm sl}/\tau$). To use these dimensionless quantities in a general context, we introduce a common notation $\nu_{\rm ch}$ for them. The role of a dimensionless energy parameter is commonly played by the ratio of the energy barrier ΔU to the thermal energy $k_{\rm B}T$. Small and large values of this parameter respectively determine high-temperature (low-energy) and low-temperature (high-energy) approximations [41, 43, 48, 55, 56].

The main methods for obtaining an analytic description of the directed nanoparticle motion are based on the four approximations listed above. Figure 1 presents characteristic domains in the plane $(\Delta U/k_{\rm B}T, \nu_{\rm ch})$, determined by the choice of a small parameter that allows obtaining an analytic solution of the Smoluchowski equation. The low-frequency

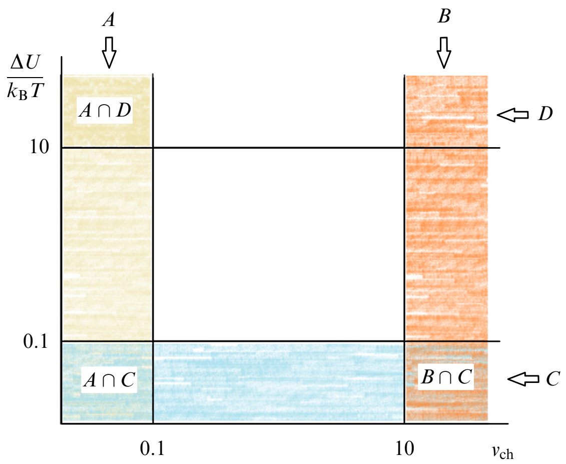

Figure 1. Validity domains of approximations allowing analytic solutions of the Smoluchowski equation (1), (2).

approximation (band A) or the adiabatic regime of motor operation allows deriving an explicit expression for the mean velocity of a Brownian particle for an arbitrary coordinate dependence of $U(x,t)$ [48, 52–54, 57–59]. Slow variations of the potential energy actually imply that the system is close to equilibrium. In this case, it suffices to seek corrections to the known equilibrium results, which significantly simplifies the solution process. The invariance of the low-frequency approximation to how the potential energy varies with time (stochastically or deterministically) [55] is another of its advantages, together with the high efficiency of energy conversion demonstrated by adiabatic motors [56, 59–61].

The high-frequency solutions (band B) are strongly dependent on the shape of the potential profile and the character of its changes with time [62, 63], but just this fact makes this approximation a kind of 'probe' enabling the problem to be 'tested' from different sides. A set of asymptotic behaviors is known for this regime [17, 64–66]. Motors of different classes — with a fluctuating periodic potential profile (flashing ratchets) and fluctuating inclining homogeneous force (rocking ratchets) — demonstrate different sensitivities to the presence of cusps and jumps in the potential profile in this regime [51].

The inequality $\Delta U/k_{\rm B}T \ll 1$ defines the low-energy range (band C), and solutions obtained in this range are called high-temperature approximations. The advantage of this approximation is that it can be used in the entire frequency range of potential energy fluctuations [41, 49, 55, 59]. However, this approximation cannot be applied to describe protein motors because the temperature of conformational changes in molecules exceeds $k_{\rm B}T$. The 'beneficiaries' of the high-temperature approximation are artificially designed devices, for example, dipole photomotors [66], in which the energy of interaction between a polar particle and a substrate are small because of the smallness of dipole interactions compared to $k_{\rm B}T$. Motor systems based on a semiconductor nanocluster [49] can also be assigned to this class. We note that abandoning the dipole approximation when considering photoinduced motion of stretched molecules along a substrate does not violate the validity of the high-temperature approximation [67].

The intersection of domains where approximations are valid ($A \cap C$ and $B \cap C$ in Fig. 1) enables comparing the results and offers additional criteria to check their validity. Additionally, the use of two approximations prompts conclusions that cannot be made in the framework of one of them. For example, the analysis of domain $B \cap C$ in Ref. [55] made it possible to establish that at $\Gamma \to \infty$ ($\tau \to 0$) the mean Brownian motor velocity is linear in the inverse frequency of potential energy fluctuations for a stochastic dichotomous process and quadratic for a deterministic one. This result is preserved even outside the high-temperature approximation [62–66], but obtaining it for arbitrary $\Delta U/k_{\rm B}T$ is much harder.

The analysis in the low-temperature domain (band D) does not offer an appropriate approximation on its own, due to the difficulty of working with differential equations having a small coefficient at the highest derivative [see Eqns (1) and (2), where the diffusion coefficient $D = k_{\rm B}T/\zeta$ is small for small T]. That is why the formulation of the low-temperature approximation imposes additional constraints on spatial and temporal characteristics of the potential profile. Briefly, their essence is that the original problem with a continuous description of Brownian particle motion is replaced by a description of its jumps over potential barriers separating potential wells. In this way, the low-temperature approximation can be introduced for potential profiles containing well-pronounced wells and barriers, and the kinetic approach serves as this approximation. It amounts to considering changes in potential well populations due to transitions between them, described in terms of transition rate constants. It is assumed that only the maxima and minima of the potential profile fluctuate, whereas their position and the profile curvature in the vicinity of the extrema stay without changes, that the barrier height ΔU exceeds the thermal energy $k_{\mathrm{B}}T$, and that the period of fluctuations $\tau\,(\Gamma^{-1})$ is much larger than τ_{D} (domain $A \cap D$).

These conditions ensure local thermodynamic equilibrium in each of the wells, which does not change under fluctuations of the potential and excludes energy losses related to the relaxation of the distribution function inside the wells [68]. The approach results in a suitable description of the main properties of Brownian motors for sufficiently low temperatures $k_{\mathrm{B}}T \ll \Delta U$ in a broad range of fluctuation frequencies Γ, $\tau^{-1} < \tau_{\mathrm{D}}^{-1}$ of potential wells and barriers [56]. The advantage of the kinetic approach lies in the simplicity of obtaining analytic results, facilitating comparisons with experimental observations for molecular motors, which are the main application of the low-temperature approximation. Its drawback is that it is impossible to pass to the high-frequency limit when just the intrawell motion provides the correct asymptotics [48, 50, 60, 61, 69].

In this paper, we present a theory of Brownian motors based on our approach, the construction of the perturbation theory in small potential energy fluctuations. According to this approach, the leading contribution to the additive–multiplicative form (3) of potential energy comes from the time-independent asymmetric periodic profile $u(x)$, whereas the product $\sigma(t)w(x)$ can be considered a small correction. We assume that values taken by the function $\sigma(t)$ are of the order of unity, and therefore the order of $\sigma(t)w(x)$ is defined by the order of the function $w(x)$, which is denoted as $O(w)$. As mentioned above, representation (3) embraces the majority of practically and theoretically important variants of potential energy variations [17, 18, 38, 39, 49]. Two main classes of Brownian motors — with a fluctuating periodic potential (flashing ratchet) and with a fluctuating homogeneous force (rocking ratchet) — are particular cases of representation (3). The functions $u(x)$ and $w'(x) \equiv \mathrm{d}w(x)/\mathrm{d}x$ must then be periodic. For motors of the flashing ratchet class, $w(x+L) = w(x)$, and of the rocking ratchet class, $w(x) = Fx$, $F = \mathrm{const}$.

2. Perturbation theory in small fluctuations: general consideration

In this section, assuming that fluctuations of the potential energy $U(x,t)$ described by the term $\sigma(t)w(x)$ in expression (3) are small, we develop a perturbation theory in these fluctuations. With this aim, we insert Eqn (3) into representation (2) for the flux and average it over the fluctuations:

$$J \equiv \big\langle J(x,t)\big\rangle = \hat{J}(x)\big\langle \rho(x,t)\big\rangle - \beta D w'(x)\big\langle \sigma(t)\rho(x,t)\big\rangle\,, \quad (4)$$

where

$$\hat{J}(x) = -D\exp\big(-\beta u(x)\big)\,\frac{\partial}{\partial x}\exp\big(\beta u(x)\big) \quad (5)$$

is the flux operator in the stationary potential profile $u(x)$. If Eqn (4) is considered as a differential equation for the function of coordinate $\langle \rho(x,t)\rangle$, its general solution can be readily written as

$$\begin{aligned}\big\langle \rho(x,t)\big\rangle &= C\exp\big(-\beta u(x)\big) - \beta\exp\big(-\beta u(x)\big)\\ &\quad\times\int_0^x \mathrm{d}x'\,w'(x')\exp\big(\beta u(x')\big)\big\langle\sigma(t)\rho(x',t)\big\rangle\\ &\quad-\frac{J}{D}\exp\big(-\beta u(x)\big)\int_0^x \mathrm{d}x'\exp\big(\beta u(x')\big)\,. \end{aligned}\quad (6)$$

Representation (6) contains two arbitrary variables, J and C, to be determined from additional conditions. The normalization condition $\int_0^L \mathrm{d}x\langle \rho(x,t)\rangle = 1$ fixes the constant C. Then, requiring that for periodic functions $u(x)$ and $w'(x)$ the solution $\langle \rho(x,t)\rangle$ be also periodic for an steady-state process, $\langle \rho(x+L,t)\rangle = \langle \rho(x,t)\rangle$, the flux J can be written as

$$\begin{aligned}J &= -\beta D\int_0^L \mathrm{d}x\,q(x)w'(x)\big\langle\sigma(t)\rho(x,t)\big\rangle\,,\\ q(x) &= \exp\big(\beta u(x)\big)\left(\int_0^L \mathrm{d}x\,\exp\big(\beta u(x)\big)\right)^{-1}.\end{aligned}\quad (7)$$

We assume that the function $w(x)$ is of the order $O(w)$ and find an approximate expression for $\langle\sigma(t)\rho(x,t)\rangle$ up to $O(w)$. As the first step, we develop a perturbation theory with respect to the small function $w(x)$, relying on Eqns (1)–(3) [3, 70]:

$$\begin{aligned}\rho(x,t) &= \rho^{(0)}(x) - \beta D\int_0^L \mathrm{d}x'\,\frac{\partial}{\partial x'}\big[w'(x')\rho^{(0)}(x')\big]\\ &\quad\times\int_{-\infty}^{\infty}\mathrm{d}t'\sigma(t')g(x,x',t-t') + O(w^2)\,.\end{aligned}\quad (8)$$

Here, the unperturbed distribution function $\rho^{(0)}(x)$ satisfies the equation $\hat{J}(x)\rho^{(0)}(x) = 0$, which is the condition that the unperturbed flux is zero. A normalized solution of this equation is the Boltzmann distribution

$$\rho^{(0)}(x) = \exp\big(-\beta u(x)\big)\left(\int_0^L \mathrm{d}x\,\exp\big(-\beta u(x)\big)\right)^{-1}.$$

The retarded Green's function $g(x,x',t)$ ($g(x,x',t) = 0$ for $t<0$), present in (8), satisfies the equation

$$\frac{\partial}{\partial t}\,g(x,x',t) + \frac{\partial}{\partial x}\,\hat{J}(x)g(x,x',t) = -\delta(x-x')\delta(t)\,. \quad (9)$$

As the second step, we multiply all terms in Eqn (8) by $\sigma(t)$ and average the result over fluctuations using the equality $\langle\sigma(t)\rangle = 0$, which leads to the sought expression for $\langle\sigma(t)\rho(x,t)\rangle$:

$$\begin{aligned}\big\langle\sigma(t)\rho(x,t)\big\rangle &= -\beta D\int_0^L \mathrm{d}x'\,\frac{\partial}{\partial x'}\big[w'(x')\rho^{(0)}(x')\big]\\ &\quad\times\int_{-\infty}^{\infty}\mathrm{d}t'\,\big\langle\sigma(t)\sigma(t')\big\rangle g(x,x',t-t') + O(w^2)\,.\end{aligned}\quad (10)$$

We let the function $S(x,x')$ of two spatial variables denote the inner integral in the right-hand side of (10). Taking the definition of the correlation function $K(t-t') \equiv \langle\sigma(t)\sigma(t')\rangle$ and the fact that $g(x,x',t-t') = 0$ for $t' > t$ into account, the function $S(x, x')$ can be written as

$$S(x, x') = \int_0^\infty \mathrm{d}t\, g(x, x', t) K(t)\,. \tag{11}$$

Finally, substituting expression (10) in Eqn (7) with due regard for (11) leads to the result

$$J = (\beta D)^2 \int_0^L \mathrm{d}x\, q(x) w'(x) \times \int_0^L \mathrm{d}x' S(x, x') \frac{\partial}{\partial x'} w'(x') \rho^{(0)}(x') + O(w^3)\,. \tag{12}$$

This expression, obtained for the first time in Ref. [71], is the most general result of the approximation of small fluctuations. Its analysis is of high importance because it leads to a wide spectrum of analytic results, including those known in the theory of Brownian motors in particular cases. Analysis of expression (12) is carried out in the next sections of this paper.

We note that in the particular case of a stochastic dichotomous process, for which $K(t) = \exp(-\Gamma|t|)$, where Γ is the inverse correlation time, the function $S(x, x')$ is the Laplace transform of the Green's function $g(x, x', t)$ describing diffusion in the potential profile $u(x)$ and satisfies the equation

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} \hat{J}(x) + \Gamma\right] S(x, x') = -\delta(x - x')\,, \tag{13}$$

which can be easily obtained from Eqn (9) if it is multiplied by $\exp(-\Gamma t)$ and integrated over t from $-\varepsilon$ to ∞, where ε is an infinitesimal positive value. We must take into account that the contribution of $\exp(\Gamma\varepsilon) g(x, x', -\varepsilon)$ vanishes because $g(x, x', t)$ is the retarded Green's function. In Section 3, an alternative derivation of Eqn (12) is proposed, with the function $S(x, x')$ satisfying Eqn (13); this derivation follows directly from the Smoluchowski equation with sources and sinks (master equation), which describes the stochastic dichotomous process and underlies stochastic models of Brownian motors [17, 37, 47, 72]. We also discuss the physical meaning of the function $S(x, x')$ and give its explicit expression for a saw-tooth potential profile.

3. Perturbation theory in small fluctuations: dichotomous stochastic process

We consider the motion of a Brownian particle with the potential energy given by a random variable described by a stochastic dichotomous process. Two states of the dichotomous process, denoted by indices '+' and '–' in what follows, are characterized by different potential profiles $U_+(x)$ and $U_-(x)$, which are described by periodic functions of the coordinate with the period L: $U_\pm(x + L) = U_\pm(x)$. Such a behavior of the potential energy can be characterized by additive-multiplicative dependence (3) when the function $\sigma(t)$ takes two values $+1$ and -1 with given transition rates between these values, γ_+ and γ_-. We assume in what follows that the dichotomous process is symmetric (i.e., $\gamma_+ = \gamma_-$) and the inverse correlation time Γ is $2\gamma_+$. This simplification does not modify the resulting expression because in the approximation of small fluctuations, the characteristics of both symmetric and asymmetric processes depend only on the sum of γ_+ and γ_-. The correctness of this assertion can be seen, for example, by analyzing the high-temperature result in Ref. [55]: up to terms of the order $O(w^3)$, the mean velocity at $\gamma_+ = \gamma_-$ depends only on $\Gamma = \gamma_+ + \gamma_-$ [see also Eqn (68) below].

The probability densities $\rho_+(x, t)$ and $\rho_-(x, t)$ of finding a Brownian particle in states '+' and '–' satisfy the Smoluchowski equation with an additional term $r(x, t)$ describing the rate of particle transitions between these states [17, 37, 38, 55]:

$$\begin{aligned}
&\frac{\partial}{\partial t} \rho_\pm(x, t) = -\frac{\partial}{\partial x} J_\pm(x, t) \mp r(x, t)\,,\\
&r(x, t) = \frac{1}{2}\Gamma\left[\rho_+(x, t) - \rho_-(x, t)\right],\\
&J_\pm(x, t) = -D\,\frac{\partial \rho_\pm(x, t)}{\partial x} - \beta D\,\frac{\partial U_\pm(x)}{\partial x}\,\rho_\pm(x, t)\,.
\end{aligned} \tag{14}$$

The probability densities are subject to the periodicity condition and are normalized:

$$\rho_\pm(x + L, t) = \rho_\pm(x, t)\,, \quad \int_0^L \left[\rho_+(x, t) + \rho_-(x, t)\right] \mathrm{d}x = 1\,. \tag{15}$$

For stationary processes $(\partial \rho_\pm(x, t)/\partial t = 0)$, the total flux of particles through an arbitrary cross section x is given by the sum of fluxes $J_\pm(x)$, $J(x) = J_+(x) + J_-(x)$ and does not depend on x. Precisely this flux $J \equiv J(x)$ is the basic quantity of the Brownian motor theory, because the mean motor velocity is expressed in terms of it as $\langle v \rangle = LJ$.

We introduce new probability densities $\xi(x)$ and $\eta(x)$, such that

$$\begin{cases} \xi(x) = \rho_+(x) + \rho_-(x)\,,\\ \eta(x) = \rho_+(x) - \rho_-(x)\,, \end{cases} \tag{16}$$

$$\rho_\pm(x) = \frac{1}{2}\left[\xi(x) \pm \eta(x)\right]$$

and, following (3) with $\sigma(t) = \pm 1$, expand the potential profiles $U_\pm(x)$ as some mean profile $u(x)$ and the fluctuating contribution $w(x)$,

$$U_\pm(x) = u(x) \pm w(x)\,. \tag{17}$$

We relate the flux operator defined by expression (5) to the mean profile. The total flux can then be written as

$$J = \hat{J}(x)\xi(x) - \beta D w'(x) \eta(x)\,. \tag{18}$$

Relation (18), taking (5) into account, is treated as a differential equation for the function $\xi(x)$. Its general solution

$$\begin{aligned}
\xi(x) = {} & C \exp\left(-\beta u(x)\right) - \beta \exp\left(-\beta u(x)\right)\\
& \times \int_0^x \mathrm{d}x'\, w'(x') \eta(x') \exp\left(\beta u(x')\right)\\
& - \frac{J}{D} \exp\left(-\beta u(x)\right) \int_0^x \mathrm{d}x' \exp\left(\beta u(x')\right)
\end{aligned} \tag{19}$$

contains an arbitrary constant C and the required constant quantity (flux J) that is independent of C and can be found from the periodicity condition $\xi(x + L) = \xi(x)$. The flux J then becomes

$$\begin{aligned}
&J = -\beta D \int_0^L \mathrm{d}x\, q(x) w'(x) \eta(x)\,,\\
&q(x) = \exp\left(\beta u(x)\right) \left(\int_0^L \mathrm{d}x \exp\left(\beta u(x)\right)\right)^{-1}.
\end{aligned} \tag{20}$$

The equation for $\eta(x)$ can be obtained from system of equations (14) written for the stationary case,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[J_+(x) - J_-(x)\right] = -\Gamma\eta(x)\,. \tag{21}$$

Inserting the expression for the flux difference $J_+(x) - J_-(x) = \hat{J}(x)\eta(x) - \beta D w'(x)\xi(x)$ into Eqn (21), we obtain a differential equation for $\eta(x)$:

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\,\hat{J}(x) + \Gamma\right]\eta(x) = \beta D\,\frac{\mathrm{d}}{\mathrm{d}x}\,w'(x)\xi(x) \tag{22}$$

(the function $\xi(x)$ is assumed to be given).

The solution of this equation can be written using the notation $S(x,x')$ introduced above for the function (the solution of Eqn (13)) representing the Laplace transform of the Green's function $g(x,x',t)$:

$$\eta(x) = -\beta D\int_0^L \mathrm{d}y\, S(x,y)\,\frac{\mathrm{d}}{\mathrm{d}y}\,w'(y)\xi(y)\,. \tag{23}$$

Substituting (23) in Eqn (20) allows the sought stationary flux to be expressed in terms of the function $\xi(x)$:

$$J = (\beta D)^2\int_0^L \mathrm{d}x\; q(x)w'(x)\int_0^L \mathrm{d}y\; S(x,y)\,\frac{\mathrm{d}}{\mathrm{d}y}\,w'(y)\xi(y)\,. \tag{24}$$

Equation (24) defines the total flux J precisely if the function $\xi(x)$ — the solution of the system of differential equations (18), (22) (or integral equations (19), (23)) — is found, satisfying periodic boundary conditions and normalization conditions

$$\int_0^L \xi(x)\,\mathrm{d}x = 1\,, \qquad \int_0^L \eta(x)\,\mathrm{d}x = 0\,. \tag{25}$$

We now consider small fluctuations in the potential energy of a nanoparticle (small compared to the thermal energy), i.e., consider the approximation $|w(x)| \ll k_{\mathrm{B}}T$. The structure of expression (24) is such that in this approximation the flux is a small quantity of the order $O(w^2)$. If we limit ourselves to considering only contributions quadratic in $w(x)$ in the flux, the function $\xi(x)$ should be computed in the zeroth approximation in $w(x)$. This means that the second and third terms can be ignored in general solution (19) (the third term is zero, as follows from relation (20), which gives $J = 0$ for $w(x) \equiv 0$). The remaining first term contains the integration constant C, which can be easily found from normalization condition (25) for $\xi(x)$. As a result, in the zeroth order in fluctuations, the function $\xi(x)$, denoted by $\xi_0(x)$ in what follows, is

$$\xi_0(x) = \exp\left(-\beta u(x)\right)\left(\int_0^L \mathrm{d}x\,\exp\left(-\beta u(x)\right)\right)^{-1}, \tag{26}$$

i.e., is the Boltzmann distribution normalized to unity for the unperturbed potential profile $u(x)$. In other words, we obtain expression (12) with $\rho^{(0)}(x) = \xi_0(x)$ in an alternative way, which is valid under the assumption that stochastic dichotomous fluctuations of potential energy are small [72].

We further consider some important properties of the function $S(x,y)$ that follow from Eqn (13) in some particular cases. In the high-frequency limit $\Gamma \to \infty$ and under the condition that the potential profile $u(x)$ (defining the operator $\hat{J}(x)$ in accordance with (5)) is a smooth function, Eqn (13) implies that, approximately, $S(x,y) = -\Gamma^{-1}\delta(x-y)$ (up to the exchange $x' \to y$), which when inserted into Eqn (12) gives

$$J = \frac{D^2\beta^3}{\Gamma}\,\frac{\int_0^L \mathrm{d}x\,u'(x)\left[w'(x)\right]^2}{\int_0^L \mathrm{d}x\exp\left(\beta u(x)\right)\int_0^L \mathrm{d}x\exp\left(-\beta u(x)\right)}\,. \tag{27}$$

This result coincides with expressions known from [64, 66] (see also review [17]). The low-frequency ($\Gamma \to 0$) representation for the flux (12) for a dichotomous process (stochastic or deterministic) is obtained in Section 4.

The physical meaning of the function $S(x,y)$ is that it describes the probabilistic process of particle motion from point y to point x and contains all important information on diffusion in the potential $u(x)$, with the fluctuation variable $\sigma(t)$ characterized by the correlation function $K(t)$. For a sawtooth potential specified by a piecewise linear periodic function with two linear intervals $[0,l]$ and $[l,L]$ within the period L and the energy barrier u_0 ($u(x) = u_0x/l$ for $x \in [0,l]$ and $u(x) = u_0(L-x)/(L-l)$ for $x \in [l,L]$), Eqn (13), which corresponds to a stochastic dichotomous process with $K(t) = \exp(-\Gamma|t|)$, complemented by the boundary conditions that the function $S(x,y)$ and its partial derivative over x are periodic and that $S(x,y)$ is continuous at $x = y$ and the flux jumps at this point,

$$\hat{J}(x)S(x,y)\Big|_{x=y+\varepsilon} - \hat{J}(x)S(x,y)\Big|_{x=y-\varepsilon}\;\underset{\varepsilon\to 0}{\to}\; -1\,, \tag{28}$$

can be solved analytically [71, 73]:

$$\begin{aligned}
-\Gamma S(x,y) \underset{0<x<y<l<L}{=} \lambda^2\,&\frac{\exp\left(f_l(y-x)\right)}{\Delta\Lambda_l}\Big\{-(f_l - f_{L-l})^2\\
&\times \sinh\Lambda_{L-l}(L-l)\sinh\Lambda_l(l-y)\sinh\Lambda_l x\\
&-(f_l - f_{L-l})\Lambda_l\sinh\Lambda_{L-l}(L-l)\sinh\Lambda_l(x+y-l)\\
&+\Lambda_l\Lambda_{L-l}\big[\cosh\Lambda_{L-l}(L-l)\sinh\Lambda_l(x-y+l)\\
&-\sinh\Lambda_l(x-y)\big] + \sinh\Lambda_{L-l}(L-l)\big[\Lambda_l^2\cosh\Lambda_l(l-y)\\
&\times\cosh\Lambda_l x + \Lambda_{L-l}^2\sinh\Lambda_l(l-y)\sinh\Lambda_l x\big]\Big\}\,,
\end{aligned} \tag{29}$$

$$\begin{aligned}
-\Gamma S(x,y) \underset{0<y<x<l<L}{=} \lambda^2\,&\frac{\exp\left(f_l(y-x)\right)}{\Delta\Lambda_l}\Big\{-(f_l - f_{L-l})^2\\
&\times \sinh\Lambda_{L-l}(L-l)\sinh\Lambda_l y\sinh\Lambda_l(l-x)\\
&-(f_l - f_{L-l})\Lambda_l\sinh\Lambda_{L-l}(L-l)\sinh\Lambda_l(x+y-l)\\
&+\Lambda_l\Lambda_{L-l}\big[-\cosh\Lambda_{L-l}(L-l)\sinh\Lambda_l(x-y-l)\\
&+\sinh\Lambda_l(x-y)\big] + \sinh\Lambda_{L-l}(L-l)\big[\Lambda_l^2\cosh\Lambda_l y\\
&\times\cosh\Lambda_l(x-l) + \Lambda_{L-l}^2\sinh\Lambda_l y\sinh\Lambda_l(l-x)\big]\Big\}\,,
\end{aligned} \tag{30}$$

$$\begin{aligned}
-\Gamma S(x,y) \underset{0<y<l<x<L}{=} \lambda^2\,&\frac{\exp\left(f_l y + f_{L-l}(L-x)\right)}{\Delta}\\
&\times\Big\{(f_l - f_{L-l})\big[\sinh\Lambda_l(l-y)\sinh\Lambda_{L-l}(x-l)\\
&-\sinh\Lambda_l y\sinh\Lambda_{L-l}(L-x)\big]\\
&+\Lambda_{L-l}\big[\sinh\Lambda_l(l-y)\cosh\Lambda_{L-l}(x-l)\\
&+\sinh\Lambda_l y\cosh\Lambda_{L-l}(L-x)\big]\\
&+\Lambda_l\big[\cosh\Lambda_l(l-y)\sinh\Lambda_{L-l}(x-l)\\
&+\cosh\Lambda_l y\sinh\Lambda_{L-l}(L-x)\big]\Big\}\,,
\end{aligned} \tag{31}$$

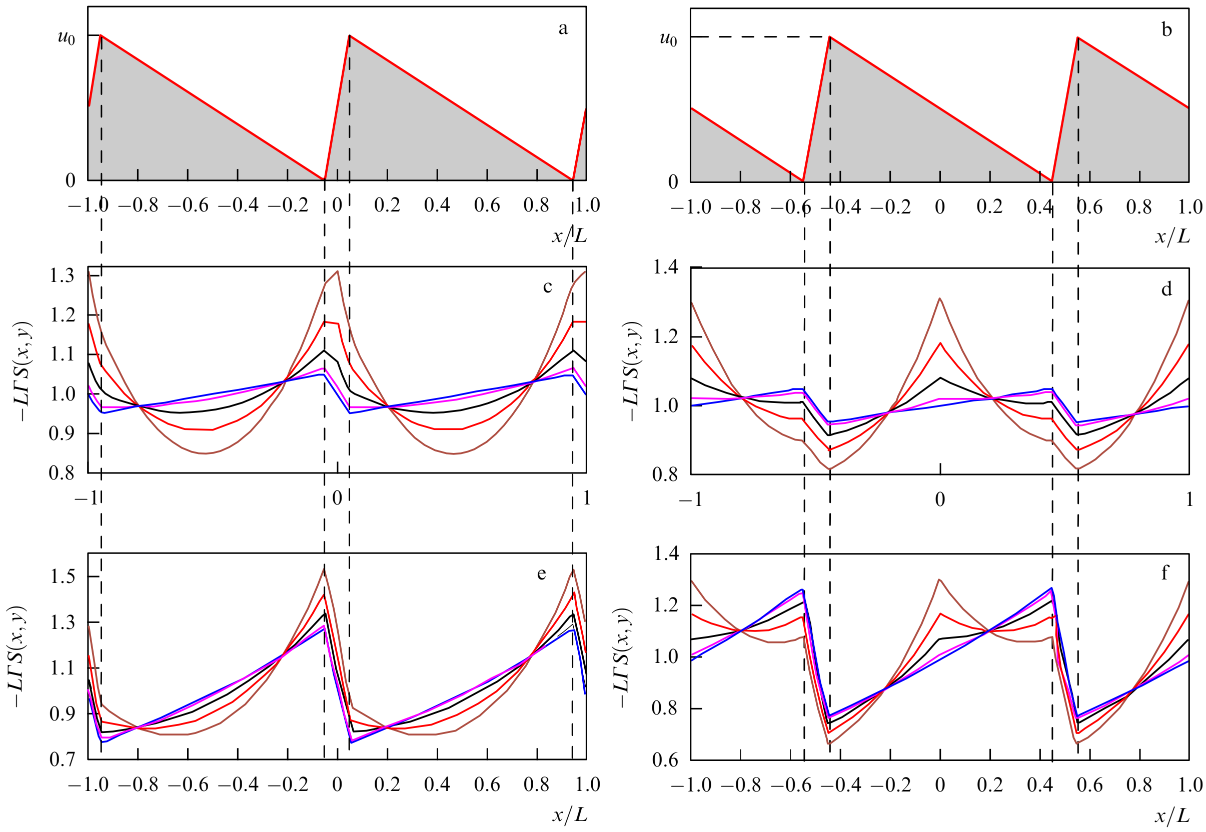

Figure 2. Saw-tooth potentials $u(x)$ for normalized lengths of 'tooth' links equal to $l/L = 0.1$ and $1 - l/L = 0.9$, where L is the spatial period; the initial particle position $y = 0$ is respectively taken in the middle of the base of (a) steep and (b) gentle slopes of the potential, and associated probability densities $-\Gamma S(x, y)$ (in units of L^{-1}) of finding the particle at a point x in the saw-tooth potential with the lifetime Γ^{-1} (panels c, e and d, f) computed for different values of the amplitude u_0 of $u(x)$: (c, d) $\beta u_0 = 0.1$ and (e, f) $\beta u_0 = 0.5$. The curves in panels (c–f), in the vicinity of $x = 0$ from top down are in the order of decreasing the parameter $\Gamma L^2/D = 2, 1.5, 1, 0.5, 0$ (the last value corresponds to an infinite lifetime, when $-\Gamma S(x, y)$ does not depend on y and coincides with the Boltzmann distribution).

where

$$\Delta = 2(\lambda^2 + f_l f_{L-l}) \sinh \Lambda_l l \sinh \Lambda_{L-l}(L - l)$$
$$+ 2\Lambda_l \Lambda_{L-l}\big[\cosh \Lambda_l l \cosh \Lambda_{L-l}(L - l) - 1\big] ,$$
$$\Lambda_l = \sqrt{f_l^2 + \lambda^2} , \quad \Lambda_{L-l} = \sqrt{f_{L-l}^2 + \lambda^2} , \tag{32}$$
$$f_l = \frac{\beta u_0}{2l} , \quad f_{L-l} = -\frac{\beta u_0}{2(L - l)} , \quad \lambda^2 = \frac{\Gamma}{D} .$$

To obtain analogous formulas in the interval $l < y < L$, it suffices to make the following changes in expressions (29)–(32):

$$l \leftrightarrow L - l , \quad x \leftrightarrow L - x , \quad y \leftrightarrow L - y , \quad f_l \leftrightarrow -f_{L-l} . \tag{33}$$

The quantity $-\Gamma S(x, y)$ gives the probability density of finding the particle at a point x in a saw-tooth potential with a lifetime Γ^{-1} under the condition that it was initially placed at a point y. For long-lived potentials ($\Gamma \to 0$), the quantity $-\Gamma S(x, y)$ tends to the Boltzmann distribution $\rho^{(0)}(x)$. For short-lived potentials ($\Gamma \to \infty$), it is close to the initial delta-shaped distribution $\delta(x - y)$.

The functions $-\Gamma S(x, y)$ (29)–(33) are plotted in Fig. 2 [71]. The shape of the curves is highly dependent on where the initial particle position $y = 0$ was chosen: in the narrow or wide parts of the profile base (cf. Fig. 2c, e and d, f). The value of the ratio of the amplitude u_0 of $u(x)$ to the thermal energy $k_B T$ also influences the shape of the curves. The positions of the maxima of curves $-\Gamma S(x, y)$ correspond to the positions of potential wells and the initial particle position. It can be seen that if the particle 'starts' at the wide link of the potential, then the lower the frequency Γ (the longer the lifetime of the state with this potential), the larger is the contribution of potential minima and the smaller the contribution of the initial position y in this link to the probability of finding the particle at the point x (Fig. 2d, f). If at the initial instant the particle is at the narrow link of the potential, such a tendency is absent: the values of the probability density for those x that coincide with the positions of potential wells and for $x = y$ decrease together with the reduction in the frequency (Fig. 2c, e). Additionally, in this case, the highest point in the graph of the function $-\Gamma S(x, y)$ corresponds to the well position; this means that if the slope of the profile is sufficiently steep, the particle has enough time to roll down the well, even for a short-lived potential profile (the curve with $\Gamma L^2/D = 2$ in Fig. 2c is the only exception: the amplitude of the potential is taken such that it is still insufficiently large, and the lifetime is sufficiently small). To summarize, we stress that just the details of the behavior of $-\Gamma S(x, y)$ define, by virtue of relation (12), the main features in the dependences of the Brownian ratchet velocity on the model parameters.

Using results (29)–(33) in formula (12), we can compute the mean velocity of a stochastic Brownian motor function-

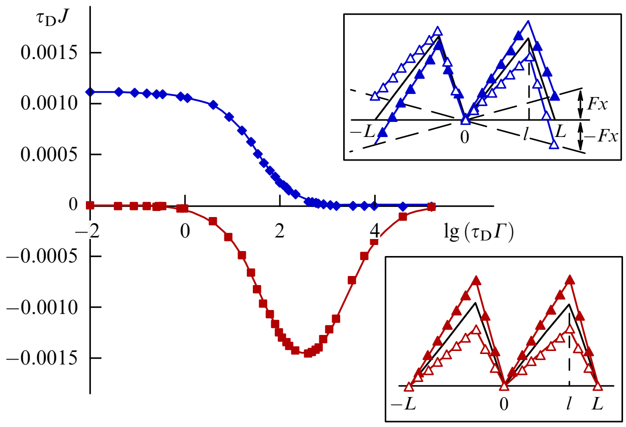

Figure 3. Frequency dependence of the flux defining the mean velocity of a ratchet with a fluctuating force (upper curve, $w(x) = Fx$, $F = w_0$) and with a fluctuating potential (lower curve, $w(x) = (w_0/u_0)u(x)$); $\tau_D \equiv L^2/D$. The insets to the right of the curves plot the profiles $u(x)$ (no marker), $u(x) + w(x)$ and $u(x) - w(x)$ (filled and unfilled triangles).

ing due to fluctuations $\pm w(x)$ of a saw-tooth profile $u(x)$ with the amplitude u_0 and the characteristic lengths l and L. Figure 3 presents the frequency dependences of the flux computed for those ratchet models with a fluctuating potential for which $w(x) = (w_0/u_0)u(x)$, and a fluctuating force where $w(x) = w_0 x$. It can be seen that for the same asymmetry of the potential profile $u(x)$, the directions of motion for these motors are opposite. One more distinction of these motors lies in the zero low-frequency asymptotic behavior of the mean velocity for a motor with fluctuating potential energy, whereas the mean velocity of a motor with a fluctuating force differs from zero at small frequencies. We note that, for $w_0/u_0 = 0.1$ and $l/L = 0.9$, computations with analytic formulas (12), (29)–(33) fully reproduce the results in Ref. [38], whose authors did not use the Green's function technique, exploring the same system based on a certain numerical procedure. Such agreement serves as an additional test of the fidelity of the results obtained and confirmation of the possibility of using them for studies of motor characteristics with changing geometrical parameters of potential profiles with arbitrarily shaped fluctuations $w(x)$.

To conclude this section, we note that the mean velocity of a particle driven by small fluctuations in the potential energy is written here in a rather simple analytic form. Furthermore, all known analytic expressions for the mean motor velocity expanded in a series in small $w(x)$, obtained in the framework of various other approximations, can be directly derived from Eqn (12). Indeed, in the case of small frequencies of potential energy fluctuations, additionally assuming that the potential profile changes instantaneously, we arrive at a generalization of the celebrated lemma by Parrondo [57]. For the high-frequency variant of a dichotomous stochastic process, Eqn (12) leads to the relations in Ref. [17] obtained for smooth potential profiles. The applicability of Eqn (2) to the description of ratchets with a fluctuating force and fluctuating periodic potential profile adds to its universality.

4. Low-frequency approximation

The approximation of low frequencies of potential energy fluctuations, or the adiabatic approximation, occupies a special place in the Brownian motor theory. It allows obtaining closed analytic expressions for the mean velocity of motors, whose subsequent analysis reveals many common features in their functioning: a similarity in the behavior of stochastic and deterministic motor systems and motors controlled by extremely fast or extremely slow changes in potential energy, symmetry properties, conditions concerning the high efficiency of energy transformation, and others [41, 50, 52–55, 59, 61]. The analysis is simplified for small fluctuations because of the adiabaticity of processes. The absence of heat exchange between the subsystem being considered and a heat bath can be realized in two ways [59], which delineates so-called 'adiabatically slow' and 'adiabatically fast' motors that function due to fluctuations in a periodic potential profile (flashing ratchets). The first way implies that the variation in potential energy is so slow that a quasiequilibrium is reached at each instant [57]. The second way corresponds to instantaneous transitions of the system between two different potential profiles (a fast change in potential energy) under the condition that the lifetime of each of the states is large enough for the equilibrium distribution to set in. The flux determining the mean velocity of such Brownian motors with a fluctuating periodic potential is [57, 59]

$$J = \tau^{-1} \int_0^L \mathrm{d}x \left[q_+(x) - q_-(x) \right] \int_0^x \mathrm{d}y \left[\rho_+^{(0)}(y) - \rho_-^{(0)}(y) \right], \tag{34}$$

where the notation

$$\rho_\pm^{(0)}(x) \equiv \exp\left(-\beta U_\pm(x)\right) \left(\int_0^L \mathrm{d}x \exp\left(-\beta U_\pm(x)\right) \right)^{-1}$$

is used for equilibrium distributions in states of a dichotomous process with potential energies $U_\pm(x)$ (17),

$$q_\pm(x) \equiv \exp\left(\beta U_\pm(x)\right) \left(\int_0^L \mathrm{d}x \exp\left(\beta U_\pm(x)\right) \right)^{-1};$$

τ is the sum of lifetimes of the states, equal to the period of potential energy change if it varies periodically, or to the sum of inverse transition rates γ_+^{-1} and γ_-^{-1} between the states if these transitions are random. We stress that result (34) is independent of whether the dichotomous process is deterministic or stochastic [55]. In the approximation of small potential energy fluctuations, which is considered in this study, the expansion of expression (34) in small $w(x)$ gives

$$J = -\Gamma \beta^2 \int_0^L \mathrm{d}x \, q(x) \left[w(x) - \int_0^L \mathrm{d}z \, q(z) w(z) \right] \times \int_0^x \mathrm{d}y \, \rho^{(0)}(y) \left[w(y) - \int_0^L \mathrm{d}z \, \rho^{(0)}(z) w(z) \right]. \tag{35}$$

Here, we took into account that for a symmetric stochastic dichotomous process, the inverse correlation time Γ is related to τ as $\Gamma = 4/\tau$.

For Brownian motors with a homogeneous force fluctuating in sign between two symmetric states $\pm F$ (a rocking ratchet) with F not necessarily small, the expression for the flux in the adiabatic limit takes the form [39, 74]

$$J = \frac{1}{2}\left[J(F) + J(-F)\right],$$
$$J(F) = D\left(1 - \exp(-\beta FL)\right)\left[\int_0^L \mathrm{d}x \exp\left(-\beta U(x)\right)\right.$$
$$\times \int_0^L \mathrm{d}x \exp\left(\beta U(x)\right) - \left(1 - \exp(-\beta FL)\right) \tag{36}$$
$$\left.\times \int_0^L \mathrm{d}x \exp\left(-\beta U(x)\right) \int_0^x \mathrm{d}y \exp\left(\beta U(y)\right)\right]^{-1},$$
$$U(x) = u(x) - Fx.$$

We note that the dependence $J(F)$ is the known analytic solution of the problem of particle motion in a viscous medium in a periodic potential $u(x)$ under the action of a uniform stationary force F [3, 75]. For this type of motor, the approximation of small fluctuations discussed here corresponds to small amplitudes of the force F. In this case, the expansion of expressions (36) in F gives [53]

$$J = -\beta F^2 \mu \Phi_{\mathrm{on-off}},$$
$$\Phi_{\mathrm{on-off}} = \int_0^L \mathrm{d}x \left[q(x) - L^{-1}\right] \int_0^x \mathrm{d}y \left[\rho^{(0)}(y) - L^{-1}\right], \tag{37}$$
$$\mu = \frac{L^2}{\zeta \int_0^L \mathrm{d}x \exp\left(-\beta u(x)\right) \int_0^L \mathrm{d}x \exp\left(\beta u(x)\right)},$$

where μ is the particle mobility in the potential $u(x)$; it is the proportionality coefficient between the particle velocity and the small applied force F: $LJ(F) = \mu F$; the function $q(x)$ is the same as $q_\pm(x)$ above up to the replacement of $U_\pm(x)$ with $u(x)$. The quantity $\Phi_{\mathrm{on-off}}$ represents the integral flux equal to the fraction of particles crossing some given cross section in one cycle of switching the periodic potential $u(x)$ on and off [57]. This same quantity determines the velocity of a Brownian motor with a flashing potential, or the so-called on–off ratchet for which $w(x) = u(x)$ and $\sigma(t) = \pm 1$ [see (3), (17)]: $LJ = (L/\tau)\Phi_{\mathrm{on-off}}$; in other words, up to τ^{-1}, the quantity $\Phi_{\mathrm{on-off}}$ coincides with the flux J (34) for $U_+(x) = u(x)$ and $U_-(x) = 0$. As shown in Ref. [53], the mean velocities of adiabatic Brownian motors with a fluctuating force and flashing potential are determined by the same quantity $\Phi_{\mathrm{on-off}}$, even if small inertial contributions are taken into account.

To conclude this consideration, we solve the inverse problem, i.e., we demonstrate that low-frequency expressions (35) and (37) can be obtained from general relation (12) by writing its adiabatic limit. For definiteness, we consider a dichotomous process with the function $S(x, x')$ satisfying Eqn (13). We relabel the variables as $x \to x'$, $x' \to y$ and integrate all terms of this equation over x' from 0 to x,

$$\Gamma \int_0^x \mathrm{d}x' S(x', y) + \hat{J}(x) S(x, y) + C_1(y)$$
$$= -\int_0^x \mathrm{d}x' \delta(x' - y). \tag{38}$$

The integral in the right-hand side is the Heaviside theta function $\theta(x - y)$. Using explicit expression (5) for the flux operator $\hat{J}(x)$ in a stationary potential $u(x)$ and multiplying both sides of Eqn (38) by $D^{-1} \exp\left(\beta u(x)\right)$, after the integration over x, we obtain

$$\exp\left(\beta u(x)\right) S(x, y) = C_2(y) + D^{-1} C_1(y) \int_0^x \mathrm{d}x' \exp\left(\beta u(x')\right)$$
$$+ D^{-1} \int_0^x \mathrm{d}x' \exp\left(\beta u(x')\right) \theta(x' - y)$$
$$+ D^{-1} \Gamma \int_0^x \mathrm{d}x' \exp\left(\beta u(x')\right) \int_0^{x'} \mathrm{d}x'' S(x'', y). \tag{39}$$

Two integration constants $C_1(y)$ and $C_2(y)$ can be found from the conditions

$$S(0, y) = S(L, y), \qquad \int_0^L \mathrm{d}x\, S(x, y) = -\Gamma^{-1}, \tag{40}$$

which follow from the periodicity of the functions $u(x)$ and $\hat{J}(x) S(x, y)$ (the second equality in Eqns (40) is obtained after integration of Eqn (13) over x from zero to L).

Integrating by part in the inner integral in the expression for flux (12) and using periodic boundary conditions, we rewrite Eqn (12) in a form convenient for computations:

$$J = -(\beta D)^2 \int_0^L \mathrm{d}x\, q(x) w'(x)$$
$$\times \int_0^L \mathrm{d}y\, \rho^{(0)}(y) w'(y) \frac{\partial}{\partial y} S(x, y) + O(w^3). \tag{41}$$

The partial derivative $\partial S(x, y)/\partial y$ entering (41) can be written, using formula (39) and the found integration constants $C_1(y)$ and $C_2(y)$, in the form

$$\frac{\partial}{\partial y} S(x, y) = \left(\frac{\partial S(x, y)}{\partial y}\right)_{\Gamma = 0} + \Gamma R(x, y) + O(\Gamma^2), \tag{42}$$

where

$$\left(\frac{\partial S(x, y)}{\partial y}\right)_{\Gamma = 0} = D^{-1} \rho^{(0)}(x) q(y) \int_0^L \mathrm{d}\xi \exp\left(-\beta u(\xi)\right)$$
$$\times \int_0^L \mathrm{d}\eta \exp\left(\beta u(\eta)\right) \left[-\int_0^L \mathrm{d}x' \rho^{(0)}(x') \int_0^{x'} \mathrm{d}x'' q(x'')\right.$$
$$\left.+ \int_y^L \mathrm{d}x' \rho^{(0)}(x') + \int_0^x \mathrm{d}x' q(x') - \theta(x - y)\right], \tag{43}$$

$$R(x, y) = D^{-1} \rho^{(0)}(x) \int_0^L \mathrm{d}\xi \exp\left(-\beta u(\xi)\right) \int_0^L \mathrm{d}\eta \exp\left(\beta u(\eta)\right)$$
$$\times \left\{\left[\int_0^L \mathrm{d}x' \rho^{(0)}(x') \int_0^{x'} \mathrm{d}x'' q(x'') - \int_0^x \mathrm{d}x' q(x')\right] A(L, y)\right.$$
$$\left.+ A(x, y) - \int_0^L \mathrm{d}x' \rho^{(0)}(x') A(x', y)\right\}, \tag{44}$$

$$A(x, y) \equiv \int_0^x \mathrm{d}x' q(x') \int_0^{x'} \mathrm{d}x'' \left(\frac{\partial S(x'', y)}{\partial y}\right)_{\Gamma = 0}. \tag{45}$$

Next, in view of the smallness of Γ (the adiabatic limit), it suffices only to substitute the first term $(\partial S(x, y)/\partial y)_{\Gamma = 0}$, independent of Γ, from expansion (42) in Eqn (41) in order to show that for Brownian motors with a fluctuating homogeneous force ($w'(x) = F$), the expression for the flux takes the form given in (37). But a similar substitution for motors with a fluctuating periodic potential, when the function $w(x)$ is periodic, $w(x + L) = w(x)$, leads to the vanishing flux J. This implies that for this class of motors, a nonzero contribution can be provided by the second term $\Gamma R(x, y)$ in representation (42) for $\partial S(x,y)/\partial y$ (the term linear in Γ), and therefore

$$\begin{aligned} J &\approx -\Gamma(\beta D)^2 \int_0^L \mathrm{d}x\, q(x) w'(x) \int_0^L \mathrm{d}y\, \rho^{(0)}(y) w'(y) R(x,y) \\ &= -\Gamma\beta^2 D \int_0^L \mathrm{d}x\, w'(x) \int_0^L \mathrm{d}y\, \rho^{(0)}(y) w'(y) \\ &\times \left[-A(L,y) \int_0^x \mathrm{d}x' q(x') + A(x,y) \right] \\ &= -\Gamma\beta^2 D \int_0^L \mathrm{d}x\, w'(x) \int_0^x \mathrm{d}x' q(x') \\ &\times \left[\int_0^{x'} \mathrm{d}x'' \varphi(x'') - \int_0^L \mathrm{d}x'' q(x'') \int_0^{x''} \mathrm{d}x''' \varphi(x''') \right] \\ &= \Gamma\beta^2 D \int_0^L \mathrm{d}x\, q(x) \left[w(x) - \int_0^L \mathrm{d}z\, q(z) w(z) \right] \int_0^x \mathrm{d}x' \varphi(x') \,. \end{aligned} \tag{46}$$

Here, the function $\varphi(x)$ is the result of integration over y:

$$\begin{aligned} \varphi(x) &\equiv \int_0^L \mathrm{d}y\, \rho^{(0)}(y) w'(y) \left(\frac{\partial S(x,y)}{\partial y} \right)_{\Gamma=0} \\ &= -D^{-1} \rho^{(0)}(x) \left[w(x) - \int_0^L \mathrm{d}x' \rho^{(0)}(x') w(x') \right] . \end{aligned} \tag{47}$$

Inserting (47) into Eqn (46), we arrive at expression (35), valid for Brownian motors with a fluctuating periodic potential.

As we already mentioned in Section 3, a major distinction of Brownian motors with a fluctuating homogeneous force from motors with a fluctuating periodic potential is that in the adiabatic limit the velocity is different from zero for the first and tends to zero for the second. Because the high-frequency limit gives zero velocity in both cases [see, e.g., expression (27)], it is obvious that the frequency dependence of velocity is a monotonically decreasing function for motors with a fluctuating force and is nonmonotonic for motors with a fluctuating potential (see Fig. 3). In the latter case, for a small Γ, the law of velocity increase with the frequency of fluctuations is sensitive to the degree to which the process of potential switching can be considered a purely dichotomous one. Namely, the presence of short time intervals in which the function $\sigma(t)$ varies fast (step-wise in the limit) leads to linear low-frequency asymptotics [76]. The proportionality of the Brownian motor velocity to Γ squared is characteristic of smooth $\sigma(t)$, i.e., the existence of transient processes (making the step-wise potential changes continuous) leads to the replacement of the linear law of velocity increase with frequency by a quadratic one [76]. We mention that in an approximation linear in frequency, the velocity depends additively on the contributions from each time interval with fast changes of the function $\sigma(t)$: each jump makes a contribution linear in Γ, and each deviation from the junp-like behavior gives a correction quadratic in the duration of the jump. The form of intervals where $\sigma(t)$ varies smoothly does not contribute to the velocity in this approximation [76].

Reference [71] shows that for Brownian photomotors in which directed motion is induced by resonance laser radiation cyclically acting on a particle [49], accounting for fast relaxation processes with a duration τ_{rel} (described by a supersymmetric periodic function $\sigma(t) = -\sigma(t+\tau/2)$; $\sigma(t) = 1 - 2\exp(-t/\tau_{\mathrm{rel}})$, $t \in [0, \tau/2]$, $\tau \gg \tau_{\mathrm{rel}}$) also produces a nonadiabatic correction to the velocity. The character of the dependence of the mean velocity on $\tau_{\mathrm{rel}}/\tau_{\mathrm{D}}$ is determined by the presence or absence of jumps in the spatial dependence of the potential energy. For a saw-tooth potential profile, the velocity decreases with increasing $\tau_{\mathrm{rel}}/\tau_{\mathrm{D}}$, and for small $\tau_{\mathrm{rel}}/\tau_{\mathrm{D}}$ the decrease is linear for extremely asymmetric potential profiles and quadratic for profiles without jumps. The character of the dependence of the mean motor velocity on $\tau_{\mathrm{rel}}/\tau_{\mathrm{D}}$ and spatial asymmetry of the potential differs for small and large τ_{rel} compared with $\tau_l \equiv l^2/D$, which is the characteristic diffusion time over the characteristic small length l of the potential. Interestingly, the nonadiabatic contribution in the mean velocity coming from taking relaxation processes in the periodic temporal dependence of the potential energy into account coincides with the expression obtained in Ref. [51] directly for the velocity of a motor moving owing to stochastic dichotomous (instantaneous) changes in the potential energy with the correlation function $K(t) = \exp(-\Gamma|t|)$ (up to a formal replacement of τ_{rel} with Γ^{-1}). Thus, in the last case, the parameter Γ is 'responsible' for the occurrence of the motor effect proper, whereas the parameter τ_{rel} in Ref. [71] governs only the nonadiabatic correction. Such a nontrivial coincidence of results invites a thought on a deep similarity between the two different mechanisms of ratchet effect (stochastic with the inverse correlation time Γ^{-1} and deterministic periodic with transient processes whose duration is determined by τ_{rel}): the presence of exponential behavior in the 'decay' of the potential profile [71].

5. High-temperature approximation

The smallness of fluctuations $\sigma(t)w(x)$ of the potential energy $U(x,t)$ assumed in this paper implies that their magnitude is small compared to the thermal energy $k_{\mathrm{B}}T$, whereas the contribution of $u(x)$ to $U(x,t)$ can be arbitrary. If additionally the ratio $u(x)/k_{\mathrm{B}}T$ is assumed to be small, we arrive at the well-known high-temperature [55] or low-energy [41] approximations used in some applications [49, 66, 67].

We derive the relations of the high-temperature approximation from the formulas obtained in the preceding sections for small fluctuations by using the operator form of transformations like

$$\hat{S}(x) f(x) = \int_0^L \mathrm{d}y\, S(x,y) f(y) \,. \tag{48}$$

Here, the action of an integral operator $\hat{S}(x)$ on an arbitrary periodic function $f(x)$ is determined by the integral, with the kernel given by the function $S(x,y)$ introduced in Eqn (11). The use of integral operators makes the expressions of the required transformations compact. Indeed, with the use of integral operators, relations (12), (11), and (9) take the succinct form

$$J = (\beta D)^2 \int_0^L \mathrm{d}x\, q(x) w'(x) \hat{S}(x) \frac{\partial}{\partial x} w'(x) \rho^{(0)}(x) + O(w^3) \,, \tag{49}$$

where

$$\hat{S}(x) = \int_0^\infty \mathrm{d}t\, \hat{g}(x,t) K(t) \,, \tag{50}$$

and the Green's operator $\hat{g}(x,t)$ satisfies the equation

$$\frac{\partial}{\partial t} \hat{g}(x,t) + \frac{\partial}{\partial x} \hat{J}(x) \hat{g}(x,t) = -\hat{1}\delta(t) \,, \tag{51}$$

($\hat{1}$ is the unit operator). If we introduce the Green's operator $\hat{g}_0(x,t)$ for the unperturbed Eqn (51) with $u(x)=0$ and $\hat{J}(x) = -D\,\partial/\partial x$, then the original operator $\hat{g}(x,t)$ satisfies the Dyson equation [70]

$$\hat{g}(x,t) = \hat{g}_0(x,t) - \beta D \int_0^t \mathrm{d}t'\, \hat{g}_0(x,t-t')\, \frac{\partial}{\partial x}\, u'(x)\hat{g}(x,t')\,. \tag{52}$$

We note that the integral over time in this expression can be extended to the entire range of t' because the Green's operators are retarded, i.e., $\hat{g}_0(x,t-t') = 0$ for $t' > t$ and $\hat{g}(x,t') = 0$ for $t' < 0$. Therefore, in the frequency Fourier representation where

$$\begin{aligned} \hat{g}(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \exp(-\mathrm{i}\omega t)\hat{\tilde{g}}(x,\omega)\,, \\ \hat{\tilde{g}}(x,\omega) &= \int_{-\infty}^{\infty} \mathrm{d}t\, \exp(\mathrm{i}\omega t)\hat{g}(x,t)\,, \end{aligned} \tag{53}$$

Eqn (52) can be written in the form

$$\hat{\tilde{g}}(x,\omega) = \hat{\tilde{g}}_0(x,\omega) - \beta D \hat{\tilde{g}}_0(x,\omega)\, \frac{\partial}{\partial x}\, u'(x)\hat{\tilde{g}}(x,\omega)\,. \tag{54}$$

Then the operator $\hat{S}(x)$ defined in Eqn (50) can be expressed in terms of $\hat{\tilde{g}}(x,\omega)$ as

$$\begin{aligned} \hat{S}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{K}(-\omega)\hat{\tilde{g}}(x,\omega) \\ &= \hat{S}_0(x) - \frac{\beta D}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{K}(-\omega)\hat{\tilde{g}}_0(x,\omega)\, \frac{\partial}{\partial x}\, u'(x)\hat{\tilde{g}}(x,\omega)\,, \end{aligned} \tag{55}$$

where

$$\hat{S}_0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{K}(-\omega)\hat{\tilde{g}}_0(x,\omega) \tag{56}$$

(the second equality in Eqns (55) is obtained using Eqn (54)).

We insert result (55) into Eqn (49) and introduce the notation $J_0\{u\}$ for the first term,

$$J_0\{u\} \equiv (\beta D)^2 \int_0^L \mathrm{d}x\, q(x)w'(x)\hat{S}_0(x)\, \frac{\partial}{\partial x}\, w'(x)\rho^{(0)}(x)\,, \tag{57}$$

which reflects the fact that the quantity introduced is a functional of $u(x)$. For further transformations, we need the property of oddness of this functional. We prove it. For this, we need first to prove that the operator $\hat{S}_0(x)$ is Hermitian and commutes with the operator $\partial/\partial x$. Owing to equality (56), it suffices to prove that the unperturbed Green's operator $\hat{g}_0(x,t)$ has these properties. Its Hermiticity property follows from the equation for $\hat{g}_0(x,t)$, and the commutativity with $\partial/\partial x$ follows from spatial homogeneity, which holds for free diffusion in the absence of a potential and results in $g_0(x,y,t)$ being dependent only on the argument $x-y$:

$$\begin{aligned} \frac{\partial}{\partial x}\hat{g}_0(x,t)f(x) &= \int_0^L \mathrm{d}y\, \frac{\partial g_0(x-y,t)}{\partial x}\, f(y) \\ &= -\int_0^L \mathrm{d}y\, \frac{\partial g_0(x-y,t)}{\partial y}\, f(y) \\ &= \int_0^L \mathrm{d}y\, g_0(x-y,t)\, \frac{\partial}{\partial y}\, f(y) = \hat{g}_0(x,t)\, \frac{\partial}{\partial x}\, f(x)\,. \end{aligned} \tag{58}$$

Then the oddness of the functional $J_0\{u\}$ is proved by a chain of transformations

$$\begin{aligned} J_0\{u\} &= (\beta D)^2 \int_0^L \mathrm{d}x\, q(x)w'(x)\, \frac{\partial}{\partial x}\, \hat{S}_0(x)w'(x)\rho^{(0)}(x) \\ &= -(\beta D)^2 \int_0^L \mathrm{d}x \left[\frac{\partial}{\partial x}\, q(x)w'(x)\right] \hat{S}_0(x)w'(x)\rho^{(0)}(x) \\ &= -(\beta D)^2 \int_0^L \mathrm{d}x\, \rho^{(0)}(x)w'(x)\hat{S}_0(x)\, \frac{\partial}{\partial x} w'(x)q(x) = -J_0\{-u\}\,, \end{aligned} \tag{59}$$

where we use the fact (see the last equality) that the function $q(x)$ differs from $\rho^{(0)}(x)$ only by the replacement of $u(x)$ with $-u(x)$. From the equality $J_0\{-u\} = -J_0\{u\}$, it follows that $J_0\{0\} = 0$; thus, in the linear approximation in $u(x)$, by expanding functions $q(x)$ and $\rho^{(0)}(x)$ up to $u(x)$, the functional $J_0\{u\}$ can be written as

$$J_0\{u\} \approx -2\beta^3 D^2 \int_0^L \mathrm{d}x \left[\frac{\partial}{\partial x}\, u(x)w'(x)\right] \hat{S}_0(x)w'(x)\,. \tag{60}$$

Returning to formula (49) with the operator $\hat{S}(x)$ defined by relation (55) and using equality (60), we present the flux J in the approximation linear in $u(x)$ as

$$\begin{aligned} J &\approx J_0\{u\} - \frac{\beta^3 D^2}{L^2}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \tilde{K}(-\omega)W(\omega)\,, \\ W(\omega) &= \frac{D}{2\pi}\int_0^L \mathrm{d}x\, w'(x)\hat{\tilde{g}}_0(x,\omega)\, \frac{\partial}{\partial x}\, u'(x)\hat{\tilde{g}}_0(x,\omega)\, \frac{\partial}{\partial x}\, w'(x)\,. \end{aligned} \tag{61}$$

The expression for $W(\omega)$ can be simplified using the commutativity of the operators $\hat{\tilde{g}}_0(x,\omega)$ and $\partial/\partial x$, as well as the identities

$$\begin{aligned} \frac{\partial}{\partial x}\, u'(x)\, \frac{\partial}{\partial x} &= \frac{\partial^2}{\partial x^2}\, u(x)\, \frac{\partial}{\partial x} - \frac{\partial}{\partial x}\, u(x)\, \frac{\partial^2}{\partial x^2}\,, \\ D\frac{\partial^2}{\partial x^2} &= \hat{\tilde{g}}_0^{-1}(x,\omega) - \mathrm{i}\omega\hat{1}\,. \end{aligned} \tag{62}$$

Hence, because the operator $\hat{\tilde{g}}_0(x,\omega)$ is Hermitian, the expression for $W(\omega)$ takes the form

$$\begin{aligned} W(\omega) = &-\frac{1}{\pi}\int_0^L \mathrm{d}x \left[\frac{\partial}{\partial x}\, u(x)w'(x)\right] \hat{\tilde{g}}_0(x,\omega)w'(x) \\ &+ \frac{\mathrm{i}\omega}{2\pi}\int_0^L \mathrm{d}x\, u'(x)\left[\hat{\tilde{g}}_0(x,\omega)w'(x)\right]^2. \end{aligned} \tag{63}$$

Inserting (63) into approximate equality (61) and taking (56) into account, we can readily see that the first term in Eqn (63) is compensated by $J_0\{u\}$, and therefore the flux in the high-temperature approximation takes the final form

$$J = -\frac{\mathrm{i}\beta^3 D^2}{2\pi L^2}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \omega\tilde{K}(-\omega)\int_0^L \mathrm{d}x\, u'(x)\left[\tilde{w}'(x,\omega)\right]^2, \tag{64}$$

which includes the so-called fluctuating effective potential [51]

$$\tilde{w}(x,\omega) = \hat{\tilde{g}}_0(x,\omega)w(x) = \int_0^L \mathrm{d}y\, \tilde{g}_0(x-y,\omega)w(y)\,. \tag{65}$$

Relation (64) is the most general expression for the flux determining the velocity of a high-temperature Brownian motor with a fluctuating periodic potential profile [51, 71], which is valid for any law of the profile variation with time. In the case of a stochastic dichotomous process with the inverse correlation time Γ, we have $\tilde{K}(-\omega) = 2\Gamma/(\omega^2 + \Gamma^2)$ and

$$J = \frac{\beta^3 D^2 \Gamma}{L^2} \int_0^L \mathrm{d}x\, u'(x) \left[\tilde{w}'(x, \mathrm{i}\Gamma)\right]^2 , \tag{66}$$

where the function $\tilde{w}'(x, \mathrm{i}\Gamma)$ is defined by formula (65) where the Green's function $\tilde{g}_0(x, \mathrm{i}\Gamma)$ is written in the explicit form

$$\tilde{g}_0(x, \mathrm{i}\Gamma) = -\sum_q \frac{\exp\left[\mathrm{i}k_q(x-y)\right]}{\Gamma + Dk_q^2} = -\frac{z\cosh\left[z(1-2|x|/L)\right]}{\Gamma \sinh z} , \tag{67}$$

$$z = \frac{1}{2}\sqrt{\Gamma \tau_{\mathrm{D}}} , \qquad \tau_{\mathrm{D}} = \frac{L^2}{D} .$$

Here and hereafter, we use wave vectors $k_q = 2\pi q/L$, which are functions of an integer argument q.

The relations presented here are convenient for the analysis of potential profiles with large gradients (see, for example, the analysis of Brownian motors with saw-tooth potentials in Refs [51, 71]). In the case of smooth potential profiles, in contrast, the representation in terms of Fourier harmonics is more convenient because it allows one to keep only a few first harmonics. For example, for a nonsymmetric dichotomous process, such a high-temperature Fourier representation is obtained in Ref. [55]:

$$J = \frac{\mathrm{i}\beta^3 D\Gamma}{L} \sum_{qq'} \frac{Dk_q k_{q+q'} u_{q'} + (\gamma_+ - \gamma_-) w_{q'}}{(\Gamma + Dk_q^2)(\Gamma + Dk_{q+q'}^2)}\, k_{q'} w_q w_{-q-q'} . \tag{68}$$

In the symmetric case ($\gamma_+ = \gamma_-$), formula (68) corresponds to representation (66) written in terms of Fourier components.

For a periodic process,

$$\tilde{K}(-\omega) = 2\pi \sum_j |\sigma_j|^2 \delta(\omega - \omega_j) ,$$

and therefore formula (64) takes the form

$$J = -\frac{\mathrm{i}\beta^3 D^2}{L^2} \sum_j |\sigma_j|^2 \omega_j \int_0^L \mathrm{d}x\, u'(x) \left[\tilde{w}'(x, \omega_j)\right]^2 . \tag{69}$$

Similarly to the situation with a stochastic process, expression (69) is convenient for the analysis of potential profiles with large gradients, whereas for smooth coordinate dependences of the potential energy, the representation in terms of Fourier components becomes more productive [55],

$$J = \frac{\mathrm{i}\beta^3 D^3}{L} \sum_{qj,\, q'j'} \frac{k_q^2 k_{q+q'}^2 k_{q'} U_{qj} U_{q'j'} U_{-q-q',-j-j'}}{(\mathrm{i}\omega_j + Dk_q^2)(\mathrm{i}\omega_{j+j'} + Dk_{q+q'}^2)} . \tag{70}$$

Formula (70) is valid for an arbitrary function $U(x,t)$, not only the one reducible to the additive-multiplicative form (3). However, if we resort to formula (3) and the approximation of small fluctuations up to terms of the order $O(w^2)$, expression (69), written in Fourier components can be represented in the form [49]

$$J = \frac{2\mathrm{i}\beta^3 D^3}{L} \sum_{\substack{qq'(\neq 0)\\ (q+q'\neq 0)}} k_q k_{q'} k_{q+q'} (k_q^2 + k_{q'}^2)$$

$$\times\, w_q w_{q'} u_{-q-q'} \psi(Dk_q^2, Dk_{q'}^2) ,$$

$$\psi(a,b) = \sum_{j=1}^{\infty} \frac{\omega_j^2 |\sigma_j|^2}{(\omega_j^2 + a^2)(\omega_j^2 + b^2)} , \tag{71}$$

which follows from Eqn (70) with an accuracy of $O(w^2)$.

6. Conclusions

Brownian motors (ratchets), which are models of controlled diffusive systems with broken mirror symmetry [17–20] and are being actively explored at present, function under the action of external fluctuating perturbation in the absence of macroscopic driving forces. The material of this methodological paper covers the most typical examples of theoretical descriptions of systems in which the external perturbation $w(x)$ is created artificially, and commonly turns out to be substantially smaller than stationary interactions $u(x)$ of motors with the environment. Such (nonbiological) systems can include, for example, particles moving in solutions under the action of a periodic asymmetric potential [77, 78], vortices in superconductors [79], atoms in dissipative optical lattices [80], and electrons in organic semiconductors [81]. Brownian motors with a fluctuating force or fluctuating periodic potential energy can also function because of the electrophoretic or dielectrophoretic effect [78, 82], which seems to promise to widen the variety of methods of radioelectric delivery and segregation of biological drugs [83, 84]. As a controlling perturbation in this case, one may use spatial harmonic signals with different time dependences [72, 85]. A situation inverse to the small-fluctuation approximation, assuming large perturbations of potential profiles, is characteristic of molecular motors of biological origin. For this reason, the description of motor proteins such as proton ATP-synthase [24], kinesin, myosin V [25], and biological microtubules [26] is outside the scope of this paper.

The smallness of perturbations suggests that the well-developed and effective theoretical technique of Green's functions can be used to derive analytic expressions describing the dependence of the mean velocity of a Brownian particle (Brownian motor) on the problem parameters (temperature, fluctuation frequency, geometrical parameters of the potential profile, etc.). All the variety of possible dependences is embraced by a single compact expression (12), derived in detail in Sections 2 and 3 in the framework of a general description (assuming an arbitrary temporal behavior of perturbations), as well as in the case of a dichotomous stochastic process.

The compact character of representation (12) is achieved through the introduction of the function of two variables $S(x, x')$ defined by the Green's function $g(x, x', t)$ for diffusion in a stationary potential profile $u(x)$; finding the function $S(x, x')$ for arbitrary $u(x)$ is far from a trivial task. Analytic expressions for $S(x, x')$ can be obtained, for example, for a saw-tooth potential exhibiting stochastic dichotomous fluctuations [see expressions (29)–(32) and Fig. 2]. Computations based on these expressions give characteristic frequency dependences of the mean velocity, which differ substantially for the main ratchet types (see Fig. 3), those with a fluctuating force and fluctuating periodic potential energy. Explicit expressions for $S(x, x')$ can also be obtained within certain approximations (regarding the frequencies of fluctuations and amplitudes of potential profiles), which are actively used in the theory of Brownian motors. The proposed approach allows the results of these approximate analyses to be reproduced starting from the same basic expression, i.e., view them from a common standpoint. All this is realized in this article.

For example, the mean velocity of a motor with the potential energy varying in a high-frequency dichotomous stochastic process, given by formula (27) is easily derived from general expression (12). Section 4 is devoted to a much more involved derivation of formulas for the low-frequency (adiabatic) approximation. The main result of this approximation is the expression for a contribution, linear in the fluctuation frequency, to the mean velocity of Brownian motors with a fluctuating potential profile. Section 5 addresses one more important approximation, that of high temperatures, which allows the motor velocity to be analyzed for an arbitrary frequency of potential energy fluctuations. The relations presented in that section are convenient for the analysis of both smooth potential profiles [41, 55] and profiles containing intervals with large potential gradients [51, 71]. The opposite approximation of low temperatures can also be obtained directly from expression (12) using the saddle point method; however, this case is not touched on in this paper. The reason is that, on the one hand, such a method to derive low-temperature expressions is fairly obvious (see, e.g., [68]), while, on the other hand, it would require the presentation of cumbersome expressions, the simplified variants of which can be easily obtained for a jumping motion description [56].

This study was carried out in the framework of State Assignment 0082-2018-0003 (registration number AAAA-A18-118012390045-2) and was supported by the Russian Foundation for Basic Research (projects 18-57-00003 and 18-29-02012-mk) and the Foundation for Fundamental Research of Belarus (project F18R-022). I V Sh expresses her gratitude for the support of her work through a grant from the President of the Republic of Belarus in the area of education.

References

1. Vladimirov V S *Equations of Mathematical Physics* (Moscow: Mir, 1984); Translated from Russian: *Uravneniya Matematicheskoi Fiziki* (Moscow: Nauka, 1981)
2. Lifshitz E M, Pitaevskii L P *Statistical Physics* Pt. 2 *Theory of the Condensed State* (Oxford: Pegramon Press, 1980); Translated from Russian: *Statisticheskaya Fizika* Pt. 2 *Teoriya Kondensirovannogo Sostoyaniya* (Moscow: Nauka, 1978)
3. Riskin H *The Fokker – Plank Equation. Methods of Solution and Applications* (Berlin: Springer-Verlag, 1989)
4. Eyges L *The Classical Electromagnetic Field* (New York: Dover Publ., 1972)
5. Zubarev D N *Sov. Phys. Usp.* **3** 320 (1960); *Usp. Fiz. Nauk* **71** 71 (1960)
6. Bonch-Bruevich V L, Tyablikov S V *The Green Function Method in Statistical Mechanics* (Amsterdam: Interscience Publ., 1962); Translated from Russian: *Metod Funktsii Grina v Statisticheskoi Mekhanike* (Moscow: Fizmatlit, 1961)
7. Abrikosov A A, Gorkov L P, Dzyaloshinski I E *Methods of Quantum Field Theory in Statistical Physics* (Englewood Cliffs, N.J.: Prentice-Hall, 1963); Translated from Russian: *Metody Kvantovoi Teorii Polya v Statisticheskoi Fizike* (Moscow: Fizmatgiz, 1962)
8. Ludwig W *Dynamics of Crystal Lattice with Defects* (Prague: Academics, 1966)
9. Goldanskii V I, Trakhtenberg L I, Fleurov V N *Tunneling Phenomena in Chemical Physics* (New York: Gordon and Breach, 1989); Translated from Russian: *Tunnel'nye Yavleniya v Khimicheskoi Fizike* (Moscow: Nauka, 1986)
10. Vetchinkin S I, Kchristenko S V *Chem. Phys. Lett.* **1** 437 (1967)
11. Zon B A, Rapoport L I *JETP Lett.* **7** 52 (1968); *Pis'ma Zh. Eksp. Teor. Fiz.* **7** 70 (1968)
12. Zon B A, Manakov N L, Rapoport L I *Sov. Phys. JETP* **28** 480 (1969); *Zh. Eksp. Teor. Fiz.* **55** 924 (1968)
13. Rapoport L I, Zon B A, Manakov N L *Sov. Phys. JETP* **29** 220 (1969); *Zh. Eksp. Teor. Fiz.* **56** 400 (1969)
14. Lifshitz I M *Zh. Eksp. Teor. Fiz.* **17** 1017 (1947)
15. Lifshitz I M *Zh. Eksp. Teor. Fiz.* **17** 1075 (1947)
16. Dyadyusha G G, Rozenbaum V M, Dekhtyar M L *Sov. Phys. JETP* **73** 581 (1991); *Zh. Eksp. Teor. Fiz.* **100** 1051 (1991)
17. Reimann P *Phys. Rep.* **361** 57 (2002)
18. Hänggi P, Marchesoni F *Rev. Mod. Phys.* **81** 387 (2009)
19. Schadschneider A, Chowdhury D, Nishinari K *Stochastic Transport in Complex Systems: From Molecules to Vehicles* (Amsterdam: Elsevier, 2010)
20. Cubero D, Renzoni F *Brownian Ratchets: From Statistical Physics to Bio and Nano-motors* (Cambridge: Cambridge Univ. Press, 2016)
21. Bressloff P C, Newby J M *Rev. Mod. Phys.* **85** 135 (2013)
22. Kolomeisky A B *Motor Proteins and Molecular Motors* (Boca Raton, FL: CRS Press, 2015)
23. Howard J *Mechanics of Motor Proteins and the Cytoskeleton* (Sunderland, MA: Sinauer Associates, 2001)
24. Romanovsky Yu M, Tikhonov A N *Phys. Usp.* **53** 893 (2010); *Usp. Fiz. Nauk* **180** 931 (2010)
25. Romanovsky Yu M, Tikhonov A N *Phys. Usp.* **59** 121 (2016); *Usp. Fiz. Nauk* **186** 125 (2016)
26. Zakharov P N et al. *Phys. Usp.* **59** 773 (2016); *Usp. Fiz. Nauk* **186** 853 (2016)
27. Chowdhury D *Phys. Rep.* **529** 1 (2013)
28. Goychuk I *Beilstein J. Nanotechnol.* **7** 328 (2016)
29. Browne W R, Feringa B L *Nature Nanotechnol.* **1** 25 (2006)
30. Cheng C, Stoddart J F *Chem. Phys. Chem.* **17** 1780 (2016)
31. Kay E R, Leigh D A, Zerbetto F *Angew. Chem. Int. Ed.* **46** 72 (2007)
32. Saha S, Stoddart J F *Chem. Soc. Rev.* **36** 77 (2007)
33. Zheng Y B et al. *J. Nanophoton.* **4** 042501 (2010)
34. Michl J, Sykes E C H *ACS Nano* **3** 1042 (2009)
35. Belinicher V I, Sturman B I *Sov. Phys. Usp.* **23** 199 (1980); *Usp. Fiz. Nauk* **130** 415 (1980)
36. Olemskoi A I *Phys. Usp.* **41** 411 (1998); *Usp. Fiz. Nauk* **168** 475 (1998)
37. Gardiner C W *Stochastic Methods. Handbook for the Natural and Social Sciences* 4th ed. (Berlin: Springer-Verlag, 2009); Translated into Russian: *Stokhasticheskie Metody v Estestvennykh Naukakh* (Moscow: Mir, 1986)
38. Astumian R D, Bier M *Phys. Rev. Lett.* **72** 1766 (1994)
39. Sokolov I M *Phys. Rev. E* **63** 021107 (2001)
40. Reimann P *Phys. Rev. Lett.* **86** 4992 (2001)
41. Rozenbaum V M *JETP* **110** 653 (2010); *Zh. Eksp. Teor. Fiz.* **137** 740 (2010)
42. Shved N Yu, Shapochkina I V, Rozenbaum V M *Vestn. Belorussk. Gos. Univ. Ser. 1* (2) 27 (2014)
43. Rozenbaum V M *Low Temp. Phys.* **40** 469 (2014); *Fiz. Nizk. Temp.* **40** 604 (2014)
44. Denisov S, Flach S, Hänggi P *Phys. Rep.* **538** 77 (2014)
45. Cubero D, Renzoni F *Phys. Rev. Lett.* **116** 010602 (2016)
46. Rozenbaum V M et al. *JETP Lett.* **107** 506 (2018); *Pis'ma Zh. Eksp. Teor. Fiz.* **107** 525 (2018)
47. Van den Broeck C, Hänggi P *Phys. Rev. A* **30** 2730 (1984)
48. Astumian R D *J. Phys. Chem.* **100** 19075 (1996)
49. Rozenbaum V M et al. *J. Chem. Phys.* **145** 064110 (2016)
50. Rozenbaum V M et al. *Phys. Rev. E* **83** 051120 (2011)
51. Rozenbaum V M et al. *Phys. Rev. E* **94** 052140 (2016)
52. Rozenbaum V M et al. *Phys. Rev. E* **92** 062132 (2015)
53. Rozenbaum V M et al. *Phys. Rev. E* **89** 052131 (2014)
54. Rozenbaum V M, Shapochkina I V *JETP Lett.* **102** 248 (2015); *Pis'ma Zh. Eksp. Teor. Fiz.* **102** 275 (2015)

55. Rozenbaum V M *JETP Lett*. **88** 342 (2008); *Pis'ma Zh. Eksp. Teor. Fiz*. **88** 391 (2008)
56. Rozenbaum V M, Shapochkina I *Phys. Rev. E* **84** 051101 (2011)
57. Parrondo J M R *Phys. Rev. E* **57** 7297 (1998)
58. Parrondo J M R et al. *Europhys. Lett*. **43** 248 (1998)
59. Rozenbaum V M et al. *Phys. Rev. E* **85** 041116 (2012)
60. Rozenbaum V M et al. *Phys. Rev. E* **71** 041102 (2005)
61. Rozenbaum V M *JETP Lett*. **79** 388 (2004); *Pis'ma Zh. Eksp. Teor. Fiz*. **79** 475 (2004)
62. Doering C R, Dontcheva L A, Klosek M M *Chaos* **8** 643 (1998)
63. Reimann P, in *Stochastic Processes in Physics, Chemistry, and Biology* (Eds J A Freund, T Poeschel) (Berlin: Springer-Verlag, 2000) p. 50
64. Mielke A *Ann. Physik* **4** 721 (1995)
65. Elston T C, Doering C R *J. Stat. Phys*. **83** 359 (1996)
66. Dekhtyar M L, Ishchenko A A, Rozenbaum V M *J. Phys. Chem. B* **110** 20111 (2006)
67. Dekhtyar M L, Rozenbaum V M *J. Chem. Phys*. **134** 044136 (2011)
68. Rozenbaum V M et al. *Physica A* **363** 211 (2006)
69. Ambaye H, Kehr K W *Physica A* **267** 111 (1999)
70. Dyson F J *Phys. Rev*. **75** 486 (1949)
71. Rozenbaum V M et al. *JETP Lett*. **105** 542 (2017); *Pis'ma Zh. Eksp. Teor. Fiz*. **105** 521 (2017)
72. Vysotskaya V A et al. *Khim. Fiz. Tekhnol. Poverkhnosti* **8** 299 (2017)
73. Vysotskaya V A et al. *Zh. Belorussk. Gos. Univ. Fiz*. (3) 33 (2017)
74. Magnasco M O *Phys. Rev. Lett*. **71** 1477 (1993)
75. Lifson S *J. Chem. Phys*. **36** 2410 (1962)
76. Rozenbaum V M, Shapochkina I V *JETP Lett*. **92** 120 (2010); *Pis'ma Zh. Eksp. Teor. Fiz*. **92** 124 (2010)
77. Rousselet J et al. *Nature* **370** 446 (1994)
78. Gorre-Talini L, Spatz J P, Silberzan P *Chaos* **8** 650 (1998)
79. De Souza Silva C C et al. *Nature* **440** 651 (2006)
80. Gommers R, Bergamini S, Renzoni F *Phys. Rev. Lett*. **95** 073003 (2005)
81. Kedem O, Lau B, Weiss E A *Nano Lett*. **17** 5848 (2017)
82. Pethig R *Dielectrophoresis. Theory, Methodology and Biological Applications* (Hoboken, NJ: John Wiley and Sons, Inc. 2017)
83. Pommer M S *Electrophoresis* **29** 1213 (2008)
84. Mahabadi S, Labeed F H, Hughes M P *Electrophoresis* **36** 1493 (2015)
85. Rozenbaum V M et al. *Phys. Rev. E* **99** 012103 (2019)