Physics – Uspekhi **62** (5) 487 – 495 (2019)

METHODOLOGICAL NOTES PACS numbers: 03.50.De, 41.20.Jb

Material equations and Maxwell's equations for isotropic media; waves with negative group velocity and negative values of $\varepsilon(\omega)$ and $\mu(\omega)$

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DOI: https://doi.org/10.3367/UFNe.2019.01.038522

Contents

Abstract. The frequently used Maxwell's equations that contain E, B, D, and H fields are only substantiated in the framework of linear material equations and for isotropic media alone. We have shown that accounting for the deviation of magnetic permittivity $\mu(\omega)$ from unity in the usually employed dispersion equation implies a false precision. Therefore, if spatial dispersion is disregarded, transverse waves only exist in the energy region where $\varepsilon(\omega) > 0$ and have a positive group velocity.

Keywords: conductivity tensor, dielectric permittivity tensor, isotropic media, electric and magnetic permittivity, phase and group velocity of transverse electromagnetic waves

1. Introduction

The history of the problem discussed in this paper is quite completely, beginning from L I Mandelstam's work, presented in the review by V M Agranovich and Yu N Gartshtein [1]. L I Mandelstam [2] pointed out one of the possible selections of physically realized solutions of Maxwell's equations. According to [2], the field in an equilibrium medium far from a source should satisfy the condition at which the energy flux is directed from the source. Following [3], we will call this condition the Mandelstam emission principle. Using this principle, Mandelstam indicated an interesting feature of the refraction of a monochromatic plane wave by the interface of two isotropic media. If the waves in both media have a positive (or negative) group velocity,[1] then refraction is 'ordinary': the refracted and incident beams lie on different sides from the normal to the interface. If the group velocity of the wave in one of the media is positive and negative in another, refraction is 'extraordinary': the refracted and incident beams lie on one side of the normal to the interface. This refraction is now called negative refraction [1].

The question of the group velocity sign in a medium with dielectric permittivity $\varepsilon(\omega)$ and the magnetic permeability $\mu(\omega)$ was solved by D V Sivukhin [4] and V E Pafomov [5]. The dispersion equation for a transverse electromagnetic wave has the form [6, § 83]

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\,\mu(\omega). \tag{1.1}$$

If the field energy dissipation is ignored (i.e., in the case of real $\varepsilon(\omega)$ and $\mu(\omega)$), the wave does not decay (i.e., the wave vector $\mathbf{k}$ is real, $\mathbf{k}^2 > 0$) if the product $\varepsilon(\omega)\,\mu(\omega) > 0$, i.e., $\varepsilon(\omega)$ and $\mu(\omega) > 0$ or $\varepsilon(\omega)$ and $\mu(\omega) < 0$. Expression (1.1) gives the relation between the group and phase wave velocities:

$$\mathbf{v}_{\rm gr} = \frac{\partial\omega}{\partial\mathbf{k}} = \frac{2\omega\,\varepsilon(\omega)\,\mu(\omega)}{\partial(\omega^2\varepsilon\,\mu)/\partial\omega}\,\mathbf{v}_{\rm ph}\,,$$

$$\mathbf{v}_{\rm ph} = \frac{c}{\sqrt{\varepsilon(\omega)\,\mu(\omega)}}\,\frac{\mathbf{k}}{k}\,. \tag{1.2}$$

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Received 17 July 2017, revised 27 December 2018
Uspekhi Fizicheskikh Nauk **189** (5) 519 – 528 (2019)
DOI: https://doi.org/10.3367/UFNr.2019.01.038522
Translated by M Sapozhnikov

[1] In all cases, if not noted otherwise, an isotropic medium is considered. In an isotropic medium, the group and phase velocities of a wave are either parallel or antiparallel; it is assumed that in the first case the group velocity is positive and is negative in the second case.

The transverse wave energy density is [6, § 83]

$$\overline{U} = \frac{1}{8\pi\,\omega\,\mu(\omega)}\,\frac{\partial\omega^2\varepsilon\,\mu}{\partial\omega}\,\overline{E^2}\,, \tag{1.3}$$

where $\overline{E^2}$ is the squared electric field strength averaged over the period $2\pi/\omega$. Because $\overline{U} > 0$ in an equilibrium medium [6, § 80], relations (1.2) and (1.3) give the Sivukhin conditions [4]:

$$\varepsilon(\omega),\ \mu(\omega) > 0 \Rightarrow \mathbf{v}_{\rm gr} \uparrow\uparrow \mathbf{v}_{\rm ph}\,,$$

$$\varepsilon(\omega),\ \mu(\omega) < 0 \Rightarrow \mathbf{v}_{\rm gr} \uparrow\downarrow \mathbf{v}_{\rm ph}\,. \tag{1.4}$$

Note that expressions (1.1)–(1.4) follow only from Maxwell's equations [6, § 77]

$$\operatorname{div}\mathbf{D} = 0\,,\ \operatorname{rot}\mathbf{E} = -\frac{1}{c}\,\frac{\partial\mathbf{B}}{\partial t}\,,\ \operatorname{div}\mathbf{B} = 0\,,\ \operatorname{rot}\mathbf{H} = \frac{1}{c}\,\frac{\partial\mathbf{D}}{\partial t} \tag{1.5}$$

and material equations, which for monochromatic fields have the form [6, § 77]

$$\mathbf{D} = \varepsilon(\omega)\,\mathbf{E},\quad \mathbf{B} = \mu(\omega)\,\mathbf{H}\,. \tag{1.6}$$

In the number of papers published after review [1], paper [7] by S G Rautian, which we already pointed out in [8], takes a special place. The most impressive result of paper [7] is the statement that waves with the negative group velocity in the optical spectral region do not exist in a homogeneous isotropic medium with dielectric permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$. The negative group velocity of waves in materials studied in experiments is caused, according to [7, p. 1023], not by the negative values of $\varepsilon(\omega)$ and $\mu(\omega)$, but by a periodic inhomogeneity of the medium.

We assumed that Rautian's paper [7] would be broadly discussed; however, it remained in fact unnoticed. For example, paper [7] was not even mentioned in reviews [9, 10] (although in [9], practical applications of waves with the negative group velocity were already discussed). This is all the more surprising because the proof of the fact that the group velocity of the wave can be only positive is absent in fact in [7]. The author of [7] recalls that D V Sivukhin himself wrote in paper [4]: "Media with $\varepsilon < 0$ and $\mu < 0$ are unknown. The question of the fundamental possibility of such media existing is open." Rautian continues [7, p. 1023]: "In the more than 50 years that have passed since those times, the situation has not changed, and I am sure it will never change: the existence of continuous homogeneous media with $\varepsilon < 0$ and $\mu < 0$ in the optical spectral region is impossible." This statement, of which the author is sure, was in no way discussed in [7], i.e., was formulated like a postulate. From this postulate, together with Sivukhin's conditions (1.4), the positive value of the wave group velocity is obtained as a direct consequence.

Note that, long before the publication of [7], Sivukhin discarded in fact his result (1.4) without any explanation. In the book *Optika* cited in [7], on another occasion, Sivukhin states without proof [11, § 64]: "It can be shown that in the case of electromagnetic waves in isotropic media, the propagation directions of the phase and energy coincide," and by deriving Fresnel formulas, he warns readers that "$\varepsilon(\omega)$ and $\mu(\omega)$ are substantially positive." [11, § 67]. In this connection, we note recent paper [12] in which Fresnel formulas are discussed for negative $\varepsilon(\omega)$ and $\mu(\omega)$.

The aim of this paper is, without using any postulates like the Rautian postulate [7], to answer the question: Can a transverse electromagnetic wave propagate in an isotropic medium if dielectric permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$ of the medium are negative at the wave frequency? As the first step, we present the solution to the problem of the propagation of an electromagnetic pulse from a quasi-monochromatic source in a medium with dielectric permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$.

2. Emission of a quasi-monochromatic source in an isotropic medium

Consider a field source producing charge density $\rho_{\rm ext}(\mathbf{r}, t)$ and current density $\mathbf{j}_{\rm ext}(\mathbf{r}, t)$ in the region under study. Instead of equations (1.5), we now have equations [6, § 75]

$$\operatorname{div}\mathbf{D} = 4\pi\rho_{\rm ext}\,,\quad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\,\frac{\partial\mathbf{B}}{\partial t}\,,$$

$$\operatorname{div}\mathbf{B} = 0\,,\quad \operatorname{rot}\mathbf{H} = \frac{1}{c}\,\frac{\partial\mathbf{D}}{\partial t} + \frac{4\pi}{c}\,\mathbf{j}_{\rm ext}\,. \tag{2.1}$$

The law of conservation of foreign (with respect to the medium) charges being related to the source, the continuity equation

$$\frac{\partial\rho_{\rm ext}}{\partial t} + \operatorname{div}\mathbf{j}_{\rm ext} = 0 \tag{2.2}$$

is obtained from the first and fourth equations in (2.1).

We assume that the source is quasi-monochromatic,

$$\rho_{\rm ext}(\mathbf{r}, t) = \operatorname{Re}\rho^{(0)}_{\rm ext}(\mathbf{r}, t)\exp(-\mathrm{i}\,\omega t)\,,$$

$$\mathbf{j}_{\rm ext}(\mathbf{r}, t) = \operatorname{Re}\mathbf{j}^{(0)}_{\rm ext}(\mathbf{r}, t)\exp(-\mathrm{i}\,\omega t)\,, \tag{2.3}$$

where $\rho^{(0)}_{\rm ext}$ and $\mathbf{j}^{(0)}_{\rm ext}$ are slow (compared to $\exp(-\mathrm{i}\,\omega t)$) functions of time: if these functions are characterized by some time τ, then

$$\frac{1}{\omega\tau} \ll 1\,. \tag{2.4}$$

Equations (2.1) and material equations are linear. Therefore, fields **E**, **B**, **D**, and **H** have a form similar to (2.3). For example, $\mathbf{E}(\mathbf{r}, t) = \operatorname{Re}\mathbf{E}^{(0)}(\mathbf{r}, t)\exp(-\mathrm{i}\,\omega t)$.

Assuming that the field energy dissipation in the medium is negligible (the dielectric permittivity and the magnetic permeability are real), we write the material equations in the zero-order approximation over parameter (2.4) in the form (1.6). Then, equations (2.1) are reduced to equations for the fields **E** and **H**:

$$\operatorname{div}\mathbf{E} = \frac{4\pi}{\varepsilon(\omega)}\,\rho_{\rm ext}\,,\quad \operatorname{rot}\mathbf{E} = -\frac{\mu(\omega)}{c}\,\frac{\partial\mathbf{H}}{\partial t}\,,$$

$$\operatorname{div}\mathbf{H} = 0\,,\quad \operatorname{rot}\mathbf{H} = \frac{\varepsilon(\omega)}{c}\,\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\,\mathbf{j}_{\rm ext}\,. \tag{2.5}$$

Similarly to [13, § 17], we introduce the vector potential $\mathbf{A}(\mathbf{r}, t)$ and the scalar potential $\varphi(\mathbf{r}, t)$ in order to satisfy the second and third equations in (2.5):

$$\mathbf{H}(\mathbf{r}, t) = \operatorname{rot}\mathbf{A}\,,\quad \mathbf{E}(\mathbf{r}, t) = -\frac{\mu(\omega)}{c}\,\frac{\partial\mathbf{A}}{\partial t} - \frac{1}{\varepsilon(\omega)}\nabla\varphi\,. \tag{2.6}$$

In this case, the first and fourth equations in (2.5) are reduced to equations for φ and $\mathbf{A}$:

$$\nabla^2\varphi + \frac{\varepsilon(\omega)\,\mu(\omega)}{c^2}\,\frac{\partial}{\partial t}\,\mathrm{div}\,\mathbf{A} = -4\pi\rho_{\mathrm{ext}}(\mathbf{r},t)\,,$$
$$\nabla^2\mathbf{A} - \frac{\varepsilon(\omega)\,\mu(\omega)}{c^2}\,\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left(\frac{1}{c}\,\frac{\partial\varphi}{\partial t} + \mathrm{div}\,\mathbf{A}\right) = -\frac{4\pi}{c}\,\mathbf{j}_{\mathrm{ext}}(\mathbf{r},t)\,. \tag{2.7}$$

The choice of potentials $\mathbf{A}$ and φ in (2.6) is ambiguous (see [13, § 18]): potentials

$$\tilde{\mathbf{A}} = \mathbf{A} + \nabla f\,, \quad \tilde{\varphi} = \varphi - \frac{\varepsilon(\omega)\,\mu(\omega)}{c}\,\frac{\partial f}{\partial t}\,, \tag{2.8}$$

where $f(\mathbf{r},t)$ is an arbitrary function, give the same expressions for fields $\mathbf{H}$ and $\mathbf{E}$ as potentials $\mathbf{A}$ and φ. We will choose the function $f(\mathbf{r},\mathbf{t})$ in (2.8) so as to satisfy the Lorentz condition [13, § 46],

$$\mathrm{div}\,\mathbf{A} + \frac{1}{c}\,\frac{\partial\varphi}{\partial t} = 0\,. \tag{2.9}$$

In this case, equations (2.7) take the form

$$\nabla^2\varphi - \frac{\varepsilon(\omega)\,\mu(\omega)}{c^2}\,\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho_{\mathrm{ext}}(\mathbf{r},t)\,,$$
$$\nabla^2\mathbf{A} - \frac{\varepsilon(\omega)\,\mu(\omega)}{c^2}\,\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\,\mathbf{j}_{\mathrm{ext}}(\mathbf{r},t)\,. \tag{2.10}$$

In the zero-order approximation in parameter (2.4), we have $\partial^2\mathbf{A}/\partial t^2 = -\omega^2\mathbf{A}$, $\partial^2\varphi/\partial t^2 = -\omega^2\varphi$, and equations (2.10) outside the source are equations for a monochromatic wave, which can be treated as a plane wave in small spatial regions at a large enough distance from the source [13, § 66]. In this case, the wave vector and frequency are related by dispersion equation (1.1). The field will not decay if the inequality $\varepsilon(\omega)\,\mu(\omega) > 0$ is fulfilled.

The solution of equations (2.10) is similar to that of the same problem for radiation in a vacuum [13, § 62] in the form of retarded or advanced potentials

$$\varphi(\mathbf{r},t) = \int\frac{\mathrm{d}V'}{|\mathbf{r}-\mathbf{r}'|}\,\rho_{\mathrm{ext}}\left(\mathbf{r}',t\mp\frac{|\mathbf{r}-\mathbf{r}'|}{v_{\mathrm{ph}}}\right),$$
$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int\frac{\mathrm{d}V'}{|\mathbf{r}-\mathbf{r}'|}\,\mathbf{j}_{\mathrm{ext}}\left(\mathbf{r}',t\mp\frac{|\mathbf{r}-\mathbf{r}'|}{v_{\mathrm{ph}}}\right), \tag{2.11}$$

where $\mathrm{d}V' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$, and $v_{\mathrm{ph}} = \omega\sqrt{\varepsilon(\omega)\,\mu(\omega)}/c$ is the phase velocity of the wave. By using the continuity equation (2.2), we can verify that the Lorentz condition with φ and $\mathbf{A}$ from (2.11) is satisfied for any (but the same for φ and $\mathbf{A}$) sign at $|\mathbf{r}-\mathbf{r}'|/v_{\mathrm{ph}}$.

Let a be the scale of the linear size of the source. At a large distance r from the source

$$\frac{a}{r} \ll 1\,. \tag{2.12}$$

Functions φ and $\mathbf{A}$ (2.11) in the zero-order approximation over parameter (2.12) can be written in the form [13, § 66]

$$\varphi(\mathbf{r},t) = \frac{1}{r}\int\mathrm{d}V'\rho_{\mathrm{ext}}\left(\mathbf{r}',t\mp\frac{|\mathbf{r}-\mathbf{r}'|}{v_{\mathrm{ph}}}\right),$$
$$\mathbf{A}(\mathbf{r},t) = \frac{1}{cr}\int\mathrm{d}V'\,\mathbf{j}_{\mathrm{ext}}\left(\mathbf{r}',t\mp\frac{|\mathbf{r}-\mathbf{r}'|}{v_{\mathrm{ph}}}\right). \tag{2.13}$$

The time $\sim a/v_{\mathrm{ph}}$ in $t\mp|\mathbf{r}-\mathbf{r}'|/v_{\mathrm{ph}} \approx t\mp r/v_{\mathrm{ph}} \pm \mathbf{r}\,\mathbf{r}'/(r\,v_{\mathrm{ph}})$ can be ignored if the charge distribution in the source does not change noticeably, i.e., if $a/v_{\mathrm{ph}} \ll 1/\omega$. This condition is equivalent to the condition [13, § 66]

$$\frac{a}{\lambda} \ll 1\,, \tag{2.14}$$

where $\lambda = 2\pi v_{\mathrm{ph}}/\omega$ is the wavelength in the medium. From (2.13) in the zero-order approximation over small parameter (2.14), we find [13, § 66]

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{cr}\int\mathrm{d}V'\,\mathbf{j}_{\mathrm{ext}}\left(\mathbf{r}',t\mp\frac{r}{v_{\mathrm{ph}}}\right) = \frac{1}{cr}\,\dot{\mathbf{d}}\left(t\mp\frac{r}{v_{\mathrm{ph}}}\right), \tag{2.15}$$

where the dipole moment of the source is

$$\mathbf{d}\,(t) = \int\mathrm{d}V\,\rho_{\mathrm{ext}}(\mathbf{r},t)\,\mathbf{r}\,. \tag{2.16}$$

In the expression for $\varphi(\mathbf{r},t)$ (2.13), it is also necessary to retain the first-order term in parameter (2.14); assuming that the total charge in the source is zero, we find

$$\varphi(\mathbf{r},t) = \pm\,\mathbf{r}\,\dot{\mathbf{d}}\left(t\mp\frac{r}{v_{\mathrm{ph}}}\right)\frac{1}{r^2}\,. \tag{2.17}$$

By substituting potentials (2.15) and (2.17) into expressions (2.6), we obtain expressions for fields at distances from the source $r \gg \lambda \gg a$ [see (2.12) and (2.14)]:

$$\mathbf{H}(\mathbf{r},t) = \pm\,\frac{1}{cr v_{\mathrm{ph}}}\,\ddot{\mathbf{d}}\left(t\mp\frac{r}{v_{\mathrm{ph}}}\right)\times\frac{\mathbf{r}}{r}\,,$$
$$\mathbf{E}(\mathbf{r},t) = \pm\,\frac{\mu(\omega)\,v_{\mathrm{ph}}}{c}\,\mathbf{H}(\mathbf{r},t)\times\frac{\mathbf{r}}{r}\,. \tag{2.18}$$

By using these expressions, we find the energy flux density vector (the Umov–Poynting vector) [6, § 80]:

$$\mathbf{S} = \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H} = \pm\,\mu(\omega)\,v_{\mathrm{ph}}H^2(\mathbf{r},t)\,\frac{\mathbf{r}}{4\pi r}\,. \tag{2.19}$$

According to the Mandelstam principle (see Section 1), the energy flux at a large distance from the source is directed from it: $\mathbf{S}\uparrow\uparrow\mathbf{r}$. Therefore, according to (2.19), the upper sign should be taken in all formulas if $\mu(\omega) > 0$, and the lower sign should be taken if $\mu(\omega) < 0$. The final expressions for the fields and the energy flux density have the form

$$\mathbf{H}(\mathbf{r},t) = \pm\,\frac{1}{cv_{\mathrm{ph}}r}\,\ddot{\mathbf{d}}\left(t\mp\frac{r}{v_{\mathrm{ph}}}\right)\times\frac{\mathbf{r}}{r}\,,$$
$$\mathbf{E}(\mathbf{r},t) = \sqrt{\frac{\mu(\omega)}{\varepsilon(\omega)}}\,\mathbf{H}(\mathbf{r},t)\times\frac{\mathbf{r}}{r}\,, \tag{2.20}$$
$$\mathbf{S}(\mathbf{r},t) = \frac{c}{4\pi}\,\sqrt{\frac{\mu(\omega)}{\varepsilon(\omega)}}\,H^2(\mathbf{r},t)\,\frac{\mathbf{r}}{r}\,,$$

where the upper (lower) sign corresponds to the case $\varepsilon(\omega)$, $\mu(\omega) > 0$ ($\varepsilon(\omega)$, $\mu(\omega) < 0$). Recall that $\mathbf{d}(t) = \mathrm{Re}\,\mathbf{d}^{(0)}(t)\exp(-\mathrm{i}\omega t)$, where $\mathbf{d}^{(0)}(t)$ is a slow function (compared to $\exp(-\mathrm{i}\omega t)$).

We see from (2.20) that the energy flux from the source in a medium with $\varepsilon(\omega) < 0$ and $\mu(\omega) < 0$ received at an instant t is determined by the motion of charges in the source that will occur only in the future at the moment $t + r/v_{\mathrm{ph}}$. Thus, a

consequence proves to be ahead of the cause, which is inconsistent with the principle of causality. [2]

Thus, dispersion equation (1.1) is invalid for all media in the frequency region where the dielectric permittivity and magnetic permeability are negative. However, as mentioned in Section 1, dispersion equation (1.1) directly follows from Maxwell's equations (1.5) and material equations (1.6). To find the reason for the incorrectness of dispersion equation (1.1) and to obtain the correct dispersion equation, it is necessary to elucidate how material equations (1.6) appear in the electromagnetic field theory. We will discuss this question in what follows [6, 16–18].

3. Maxwell's equations and material equations for an isotropic medium

Consider the Maxwell–Lorentz equations for an electromagnetic field [16, § 27; 17, § 1; 18, § 1],

$$\begin{aligned} &\operatorname{div}\mathbf{E}=4\pi\rho\,, \quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\,,\\ &\operatorname{div}\mathbf{B}=0\,, \quad \operatorname{rot}\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}+\frac{4\pi}{c}\,\mathbf{j}\,, \end{aligned} \tag{3.1}$$

where (in the absence of foreign charges) ρ and $\mathbf{j}$ are the mean charge density and the mean charge current density of the medium (mean values in the sense of disregarding fluctuations [18, § 1]). Electric field strength $\mathbf{E}$ and magnetic induction $\mathbf{B}$ are determined by the force acting of the point charge q moving at velocity $\mathbf{v}$ (the Lorentz force),

$$\mathbf{F}=q\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right). \tag{3.2}$$

The first and fourth equations (3.1) contain the law of conservation of charges in a medium:

$$\frac{\partial\rho}{\partial t}+\operatorname{div}\mathbf{j}=0\,. \tag{3.3}$$

Equations (3.1) can be represented in an equivalent form, which is "more convenient for macroscopic electrodynamics" [16, § 28], if we introduce into the right-hand side of the fourth equation the vector $\tilde{\mathbf{D}}(\mathbf{r},t)$ called, like $\mathbf{D}(\mathbf{r},t)$, the electric induction [6, § 103; 17, § 1]:

$$\frac{\partial\tilde{\mathbf{D}}}{\partial t}=\frac{\partial\mathbf{E}}{\partial t}+4\pi\,\mathbf{j}\,. \tag{3.4}$$

By integrating (3.4), we obtain (see 17, § 1])

$$\tilde{\mathbf{D}}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)+4\pi\int_{-\infty}^{t}\mathbf{j}(\mathbf{r},t')\,\mathrm{d}t'+4\pi\,\mathbf{P}^{(-)}(\mathbf{r})\,, \tag{3.5}$$

where $\mathbf{P}^{(-)}(\mathbf{r})$ is a so far arbitrary time-independent vector. Taking into account the first equation in (3.1) and continuity equation (3.3) and requiring the fulfilment of the condition

$$\operatorname{div}\mathbf{P}^{(-)}=-\rho^{(-)}(\mathbf{r})\,, \tag{3.6}$$

where $\rho^{(-)}(\mathbf{r})\equiv\rho(\mathbf{r},t=-\infty)$, we find that $\operatorname{div}\tilde{\mathbf{D}}=0$. Thus, along with system of equations (3.1), the equivalent system of equations [3] [6, § 103; 16, § 28; 17, § 1; 18, § 1],

$$\begin{aligned} &\operatorname{div}\tilde{\mathbf{D}}=0\,, \quad \operatorname{rot}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}\,,\\ &\operatorname{div}\mathbf{B}=0\,, \quad \operatorname{rot}\mathbf{B}=\frac{1}{c}\frac{\partial\tilde{\mathbf{D}}}{\partial t}\,, \end{aligned} \tag{3.7}$$

is valid, in which $\tilde{\mathbf{D}}(\mathbf{r},t)$ is determined from (3.5) and (3.6). Taking into account the electric neutrality of the entire medium, equation (3.6) should be supplemented by the condition that $\mathbf{P}^{(-)}(\mathbf{r})=0$ outside the medium (in a vacuum, where $\rho^{(-)}(\mathbf{r})=0$) [6, §§ 6, 77]. Note that in this case the vector $\mathbf{P}^{(-)}(\mathbf{r})$ is not yet defined unambiguously, and any vector of the form $\operatorname{rot}\mathbf{f}$ can be added to it in the region inside the substance [6, § 6].

By expanding all the quantities in (3.7) to Fourier integrals in time and space, we can obtain relations between corresponding Fourier components:

$$\begin{aligned} &\mathbf{k}\,\tilde{\mathbf{D}}_{\omega\mathbf{k}}=0\,, \quad \mathbf{k}\times\mathbf{E}_{\omega\mathbf{k}}=\frac{\omega}{c}\,\mathbf{B}_{\omega\mathbf{k}}\,,\\ &\mathbf{k}\,\mathbf{B}_{\omega\mathbf{k}}=0\,, \quad \mathbf{k}\times\mathbf{B}_{\omega\mathbf{k}}=-\frac{\omega}{c}\,\tilde{\mathbf{D}}_{\omega\mathbf{k}}\,. \end{aligned} \tag{3.8}$$

Similarly, we find from (3.5) (see [17, § 2])

$$\tilde{\mathbf{D}}_{\omega\mathbf{k}}=\mathbf{E}_{\omega\mathbf{k}}+8\pi^2\,\mathbf{j}_{\omega\mathbf{k}}\delta_+(\omega)+4\pi\delta(\omega)\,\mathbf{P}^{(-)}_{\mathbf{k}}\,, \tag{3.9}$$

where [19, § 5]

$$\delta_+(\omega)=\frac{1}{2\pi}\int_0^\infty\exp(\mathrm{i}\omega t)\,\mathrm{d}t=\frac{1}{2}\delta(\omega)+\frac{\mathrm{i}}{2\pi\omega}\,; \tag{3.10}$$

the singularity at $\omega=0$ in the last term should be treated in the sense of the principal value. Considering time-dependent fields, we will omit terms with $\delta(\omega)$ in (3.9) [6, § 96] and assume that

$$\tilde{\mathbf{D}}_{\omega\mathbf{k}}=\mathbf{E}_{\omega\mathbf{k}}+\frac{4\pi\,\mathrm{i}}{\omega}\,\mathbf{j}_{\omega\mathbf{k}}\,. \tag{3.11}$$

The passage from Maxwell's equations (3.7) for three fields, $\mathbf{E}$, $\mathbf{B}$, and $\tilde{\mathbf{D}}$, to Maxwell's equations (1.5) for four fields, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, and $\mathbf{H}$, is performed by dividing the current $\mathbf{j}$ into a part related to the polarization $\mathbf{P}(\mathbf{r},t)$ and a part related to the magnetization $\mathbf{M}(\mathbf{r},t)$ [6, § 79; 17, § 2; 18, § 1],

$$\mathbf{j}(\mathbf{r},t)=\frac{1}{4\pi}\,\frac{\partial(\tilde{\mathbf{D}}-\mathbf{E})}{\partial t}=\frac{\partial\mathbf{P}}{\partial t}+c\operatorname{rot}\mathbf{M}\,; \tag{3.12}$$

then, the fields

$$\begin{aligned} &\mathbf{D}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)+4\pi\,\mathbf{P}(\mathbf{r},t)\,,\\ &\mathbf{H}(\mathbf{r},t)=\mathbf{B}(\mathbf{r},t)-4\pi\,\mathbf{M}(\mathbf{r},t) \end{aligned} \tag{3.13}$$

are introduced to obtain equations (1.5).

Vectors $\mathbf{P}(\mathbf{r},t)$ and $\mathbf{M}(\mathbf{r},t)$ are defined as in the theory of static fields. The polarization $\mathbf{P}$ is defined so that [6, § 6]

$$\operatorname{div}\mathbf{P}=-\rho(\mathbf{r}) \tag{3.14}$$

and $\mathbf{P}(\mathbf{r})=0$ outside the body. It is proved that the polarization itself is the dipole moment of the unit volume in

[2] Note that this problem was solved in [14, 15]. According to [14, 15], radiation in a medium with negative dielectric permittivity and negative magnetic permeability is directed to the source!

[3] Maxwell's equations in this form are presented already in the first edition (1957) [6, § 83]).

the sense that the integral taken over the entire body volume is

$$\int \mathbf{r}\rho(\mathbf{r})\,\mathrm{d}V = -\int \mathbf{r}\,\mathrm{div}\,\mathbf{P}\,\mathrm{d}V = \int \mathbf{P}(\mathbf{r})\,\mathrm{d}V. \tag{3.15}$$

Definition (3.14) in alternate fields does not contradict equality (3.12) or continuity equation (3.3) [6, § 77]. The magnetization $\mathbf{M}$ is defined so that [6, § 29]

$$c\,\mathrm{rot}\,\mathbf{M} = \mathbf{j} \tag{3.16}$$

and $M = 0$ outside the body. It is proved that the magnetization itself is the magnetic moment of the unit volume in the sense that the integral taken over the entire body volume is

$$\frac{1}{2c}\int \mathbf{r}\times\mathbf{j}\,\mathrm{d}V = \frac{1}{2}\int \mathbf{r}\times\mathrm{rot}\,\mathbf{M}\,\mathrm{d}V = \int \mathbf{M}\,\mathrm{d}V. \tag{3.17}$$

One can see from (3.12) that definition (3.16) introduced in the static limit is invalid in an alternate field, and $\mathbf{M}(\mathbf{r}, t)$ cannot be treated as the magnetic moment of the unit volume [6, § 79]. Thus, the question of the correctness of equality (3.12) (and, therefore, equations (1.5)) is reduced to the sense of the vector $\mathbf{M}(\mathbf{r}, t)$. We will show below how this question can be solved within the framework of the linear electrodynamics of a homogeneous isotropic medium.

The material equation for a homogeneous medium under stationary conditions in the linear approximation has the form [4] [6, § 103; 17, § 2]

$$j_i(\mathbf{r}, t) = \int \mathrm{d}t' \int \mathrm{d}V' g_{ij}(t - t', \mathbf{r} - \mathbf{r}')\, E_j(\mathbf{r}', t'). \tag{3.18}$$

For the corresponding Fourier components, we obtain the relation

$$j_{i\omega\mathbf{k}} = \sigma_{ij}(\omega, \mathbf{k})\, E_{j\omega\mathbf{k}}, \tag{3.19}$$

where the conductivity tensor is

$$\sigma_{ij}(\omega, \mathbf{k}) = \int \mathrm{d}V \exp(-\mathrm{i}\mathbf{k}\mathbf{r}) \int \mathrm{d}t \exp(\mathrm{i}\omega t)\, g_{ij}(t, \mathbf{r}). \tag{3.20}$$

It follows from (3.11) and (3.19) that

$$\tilde{D}_{i\omega\mathbf{k}} = \varepsilon_{ij}(\omega, \mathbf{k})\, E_{j\omega\mathbf{k}}, \tag{3.21}$$

where the dielectric permittivity tensor is

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \delta_{ij} + \frac{4\pi\mathrm{i}}{\omega}\sigma_{ij}(\omega, \mathbf{k}). \tag{3.22}$$

The dielectric permittivity tensor for an isotropic (non-gyrotropic) medium with the symmetry center has the form [6, § 103; 16, § 28; 17, § 2; 18, § 1]

$$\varepsilon_{ij}(\omega, \mathbf{k}) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\varepsilon^{\mathrm{tr}}(\omega, k) + \frac{k_i k_j}{k^2}\,\varepsilon^{1}(\omega, k), \tag{3.23}$$

where $\varepsilon^{\mathrm{tr}}$ and ε^{1} are functions of the absolute value of the wave vector (and frequency). The conductivity tensor is written similarly with corresponding functions $\sigma^{\mathrm{tr}}(\omega, k)$ and $\sigma^{1}(\omega, k)$. The vector $\tilde{\mathbf{D}}_{\omega\mathbf{k}}$ (3.21) with $\varepsilon_{ij}(\omega, \mathbf{k})$ from (3.23) is written in the form

$$\tilde{\mathbf{D}}_{\omega\mathbf{k}} = \frac{1}{k^2}\left[\varepsilon^{1}(\omega, k)\mathbf{k}(\mathbf{k}\mathbf{E}_{\omega\mathbf{k}}) + \varepsilon^{\mathrm{tr}}(\omega, k)\,\mathbf{k}\times(\mathbf{E}_{\omega\mathbf{k}}\times\mathbf{k})\right]. \tag{3.24}$$

The solutions of equations (3.8) with $\tilde{\mathbf{D}}_{\omega\mathbf{k}}$ from (3.24) are longitudinal and transverse waves with the corresponding dispersion equations [6, § 105, 106; 17, § 6]:

$$\varepsilon^{1}(\omega, k) = 0, \quad \mathbf{E}_{\omega\mathbf{k}} \uparrow\uparrow \mathbf{k}, \tilde{\mathbf{D}}_{\omega\mathbf{k}} = 0, \quad \mathbf{B}_{\omega\mathbf{k}} = 0, \tag{3.25}$$

$$k^2 = \frac{\omega^2}{c^2}\varepsilon^{\mathrm{tr}}(\omega, k), \quad \mathbf{E}_{\omega\mathbf{k}} \perp \mathbf{k}, \quad \mathbf{B}_{\omega\mathbf{k}} \perp \mathbf{k}, \quad \mathbf{B}_{\omega\mathbf{k}} \perp \mathbf{E}_{\omega\mathbf{k}}; \tag{3.26}$$

therefore, $\varepsilon^{1}(\omega, k)$ is called the longitudinal dielectric permittivity and $\varepsilon^{\mathrm{tr}}(\omega, k)$ is called the transverse permittivity.

Note that in the case of a gyrotropic (without the symmetry center) isotropic medium, tensor (3.23) has the additional term [6, § 104; 18, § 1]

$$\mathrm{i}\frac{c}{\omega} f(\omega, k)\, e_{ijl} k_l, \tag{3.27}$$

where f is a pseudoscalar, and $\tilde{\mathbf{D}}_{\omega\mathbf{k}}$ (3.24) has the additional term $\mathrm{i}(c/\omega) f(\omega, k)\,\mathbf{E}_{\omega k}\times\mathbf{k}$.

By using the second equation from (3.8), we can write equality (3.24) in a different form:

$$\tilde{\mathbf{D}}_{\omega\mathbf{k}} = \varepsilon^{1}(\omega, k)\,\mathbf{E}_{\omega\mathbf{k}} - \frac{\omega}{ck^2}\left[\varepsilon^{\mathrm{tr}}(\omega, k) - \varepsilon^{1}(\omega, k)\right]\mathbf{k}\times\mathbf{B}_{\omega\mathbf{k}}. \tag{3.28}$$

With the expression for $\tilde{\mathbf{D}}_{\omega\mathbf{k}}$, the fourth equation in (3.8) takes the form

$$\left[1 - \frac{\omega^2}{c^2k^2}\left(\varepsilon^{\mathrm{tr}}(\omega, k) - \varepsilon^{1}(\omega, k)\right)\right]\mathbf{B}_{\omega\mathbf{k}} = -\frac{\omega}{c}\,\varepsilon^{1}(\omega, k)\,\mathbf{E}_{\omega\mathbf{k}}. \tag{3.29}$$

Instead of two functions $\varepsilon^{1}(\omega, k)$ and $\varepsilon^{\mathrm{tr}}(\omega, k)$ characterizing the dielectric permittivity tensor of an isotropic center (3.23), we can introduce any two of their combinations. It can be seen from (3.29) that it is convenient to introduce functions $\varepsilon(\omega, k)$ and $\mu(\omega, k)$, which are called the dielectric permittivity and magnetic permeability of a medium:

$$\varepsilon(\omega, k) = \varepsilon^{1}(\omega, k), \qquad \frac{1}{\mu(\omega, k)} = 1 - \frac{\omega^2}{c^2k^2}\left(\varepsilon^{\mathrm{tr}}(\omega, k) - \varepsilon^{1}(\omega, k)\right). \tag{3.30}$$

After the introduction of $\varepsilon(\omega, k)$ and $\mu(\omega, k)$ in accordance with (3.30), we can introduce vectors

$$\mathbf{D}_{\omega\mathbf{k}} = \varepsilon(\omega, k)\,\mathbf{E}_{\omega\mathbf{k}}, \quad \mathbf{H}_{\omega\mathbf{k}} = \frac{1}{\mu(\omega, k)}\mathbf{B}_{\omega\mathbf{k}}, \tag{3.31}$$

so that [see (3.28), (3.30), and (3.31)]

$$\tilde{\mathbf{D}}_{\omega\mathbf{k}} = \mathbf{D}_{\omega\mathbf{k}} - \frac{c}{\omega}\left(\mu(\omega, k) - 1\right)\mathbf{k}\times\mathbf{H}_{\omega\mathbf{k}}. \tag{3.32}$$

[4] It is assumed that summation is performed everywhere over all doubly repeated indices $i, j, l... = x, y, z$.

(In gyrotropic media, $\mathbf{D}_{\omega\mathbf{k}}$ also contains the term $\mathrm{i}(c/\omega)f(\omega,k)\,\mathbf{E}_{\omega\mathbf{k}}\times\mathbf{k}$.)

Taking (3.32) into account, equations (3.8) are transformed as follows:

$$\mathbf{k}\mathbf{D}_{\omega\mathbf{k}}=0\,,\quad \mathbf{k}\times\mathbf{E}_{\omega\mathbf{k}}=\frac{\omega}{c}\,\mathbf{B}_{\omega\mathbf{k}}\,,$$

$$\mathbf{k}\mathbf{B}_{\omega\mathbf{k}}=0\,,\quad \mathbf{k}\times\mathbf{H}_{\omega\mathbf{k}}=-\frac{\omega}{c}\,\mathbf{D}_{\omega\mathbf{k}}\,. \tag{3.33}$$

Differential equations for the fields $\mathbf{E}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)$, $\mathbf{D}(\mathbf{r},t)$, and $\mathbf{H}(\mathbf{r},t)$ are obtained from (3.33) and have the form (1.5).

It follows from (3.11), (3.31), and (3.32) that

$$\mathbf{j}_{\omega\mathbf{k}}=-\mathrm{i}(\omega\mathbf{P}_{\omega\mathbf{k}}-c\mathbf{k}\times\mathbf{M}_{\omega\mathbf{k}})\,, \tag{3.34}$$

where the Fourier components of the polarization $\mathbf{P}_{\omega\mathbf{k}}$ and magnetization $\mathbf{M}_{\omega\mathbf{k}}$ are defined by the expressions

$$\mathbf{P}_{\omega\mathbf{k}}=\kappa(\omega,k)\,\mathbf{E}_{\omega\mathbf{k}}\,,\quad \mathbf{M}_{\omega\mathbf{k}}=\frac{\chi(\omega,k)}{\mu(\omega,k)}\,\mathbf{B}_{\omega\mathbf{k}}\,, \tag{3.35}$$

in which

$$\kappa(\omega,k)=\frac{1}{4\pi}\bigl(\varepsilon(\omega,k)-1\bigr)\,,$$
$$\chi(\omega,k)=\frac{1}{4\pi}\bigl(\mu(\omega,k)-1\bigr) \tag{3.36}$$

are the dielectric and magnetic susceptibilities, respectively. Expression (3.34) directly gives equality (3.12), in which the polarization $\mathbf{P}(\mathbf{r},t)$ and magnetization $\mathbf{M}(\mathbf{r},t)$ are determined from (3.35).

Dispersion equation (3.26) for transverse waves in an isotropic nongyrotropic medium can be expressed in terms of $\varepsilon(\omega,k)$ and $\mu(\omega,k)$. We find from (3.30) and (3.26)

$$\mu(\omega,k)=\frac{\varepsilon^{\mathrm{tr}}(\omega,k)}{\varepsilon^{\mathrm{l}}(\omega,k)} \tag{3.37}$$

for the transverse wave, and the dispersion equation has the form

$$k^2=\frac{\omega^2}{c^2}\varepsilon(\omega,k)\,\mu(\omega,k)\,. \tag{3.38}$$

4. Dielectric permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$

Ignoring the spatial dispersion in the conductivity tensor σ_{ij} (3.20) and the dielectric permittivity tensor ε_{ij} (3.22) means that the current $\mathbf{j}(\mathbf{r},\mathbf{t})$ and induction $\tilde{\mathbf{D}}(\mathbf{r},t)$ at the point $\mathbf{r}$ are determined by the field $\mathbf{E}$ only at this point. This requires the assumption that the function $g_{ij}(t,\mathbf{r})$ in (3.18) is proportional to $\delta(\mathbf{r})$. Then, σ_{ij} and ε_{ij} are independent of $\mathbf{k}$ and are the functions σ_{ij} and ε_{ij} of only frequency ω taking the frequency dispersion into account. For isotropic media, $\varepsilon_{ij}(\omega)$ is reduced to a scalar function $\tilde{\varepsilon}(\omega)$:

$$\varepsilon_{ij}(\omega)=\tilde{\varepsilon}(\omega)\,\delta_{ij}\,. \tag{4.1}$$

Actually, the function

$$f_{ij}(\omega,\mathbf{r})=\int \mathrm{d}t\,\exp(\mathrm{i}\omega t)\,g_{ij}(t,\mathbf{r})\,, \tag{4.2}$$

standing as a factor at $\exp(-\mathrm{i}\mathbf{k}\mathbf{r})$ in (3.20), considerably decreases at some finite distance[5] $a_0(\omega)$ [6, § 103]. For $k\gg 1/a_0(\omega)$, the conductivity tensor

$$\sigma_{ij}(\omega,\mathbf{k})=\int \mathrm{d}V\exp(-\mathrm{i}\mathbf{k}\mathbf{r})f_{ij}(\omega,\mathbf{r}) \tag{4.3}$$

vanishes [see (3.22)]:

$$\sigma_{ij}(\omega,\mathbf{k})\big|_{k\to\infty}=0\,,\quad \varepsilon_{ij}(\omega,\mathbf{k})\big|_{k\to\infty}=\delta_{ij}\,. \tag{4.4}$$

For $k\ll 1/a_0(\omega)$, the exponential factor in (4.3) can be expanded in a series: $\exp(-\mathrm{i}\mathbf{k}\mathbf{r})=1-\mathrm{i}\mathbf{k}\mathbf{r}+\ldots$, and $\sigma_{ij}(\omega,\mathbf{k})$ and $\varepsilon_{ij}(\omega,\mathbf{k})$, k) can be represented in the form of a power series in a small parameter:

$$\frac{k}{k_0(\omega)}\ll 1\,,\quad k_0(\omega)=\frac{1}{a_0(\omega)}\,. \tag{4.5}$$

Terms not containing parameter (4.5) correspond to disregarding the spatial dispersion:

$$\varepsilon_{ij}(\omega)=\varepsilon_{ij}(\omega,0)\,. \tag{4.6}$$

For $\varepsilon^{\mathrm{l}}(\omega)\equiv\varepsilon^{\mathrm{l}}(\omega,0)$ and $\varepsilon^{\mathrm{tr}}(\omega)\equiv\varepsilon^{\mathrm{tr}}(\omega,0)$ in an isotropic medium [see (3.23)], taking (4.1) and (3.30) into account, we obtain [6, § 103; 17, § 3]

$$\varepsilon^{\mathrm{tr}}(\omega)=\varepsilon^{\mathrm{l}}(\omega)=\varepsilon(\omega)=\tilde{\varepsilon}(\omega),\ \varepsilon_{ij}(\omega)=\varepsilon(\omega)\,\delta_{ij}\,. \tag{4.7}$$

Indeed, we find from (3.23)

$$\varepsilon_{ii}(\omega,\mathbf{k})=2\,\varepsilon^{\mathrm{tr}}(\omega,k)+\varepsilon^{\mathrm{l}}(\omega,k)\,,$$
$$\frac{k_ik_j}{k^2}\,\varepsilon_{ij}(\omega,\mathbf{k})=\varepsilon^{\mathrm{l}}(\omega,k)\,, \tag{4.8}$$

and from (4.1) we obtain the relations

$$\varepsilon_{ii}(\omega)=3\tilde{\varepsilon}(\omega)\,,\quad \frac{k_ik_j}{k^2}\varepsilon_{ij}(\omega)=\tilde{\varepsilon}(\omega)\,. \tag{4.9}$$

Comparing (4.8) for $k\to 0$ with (4.9), we obtain expressions (4.7).

It follows from (3.30) that the magnetic permeability $\mu(\omega)$ in which the spatial dispersion is disregarded differs from unity only when the spatial dispersion is taken into account in $\varepsilon^{\mathrm{tr}}(\omega,k)$ and $\varepsilon^{\mathrm{l}}(\omega,k)$. By using (4.7), we find [6, § 103]

$$\frac{1}{\mu(\omega)}=1-\frac{\omega^2}{c^2}\,\frac{\partial}{\partial k^2}\bigl(\varepsilon^{\mathrm{tr}}(\omega,k)-\varepsilon^{\mathrm{l}}(\omega,k)\bigr)\big|_{k=0}\,. \tag{4.10}$$

Because $|\partial\varepsilon^{\mathrm{tr}}/\partial k^2|\sim|\partial\varepsilon^{\mathrm{l}}/\partial k^2|\sim|\varepsilon(\omega)|/k_0^2(\omega)$, we obtain

$$\left|\frac{1}{\mu(\omega)}-1\right|\sim\frac{\omega^2}{c^2k_0^2(\omega)}\,|\varepsilon(\omega)|\,. \tag{4.11}$$

Let us recall the known result by Landau and Lifshitz about the magnetic susceptibility of matter [6, § 79]: "...there is no sense using the magnetic susceptibility already beginning from the optical frequency region, and we should assume that $\mu=1$ by considering the corresponding phenomena. Taking into account a difference between $\mathbf{B}$ and $\mathbf{H}$ in this region

[5] This distance can be different for different tensor components f_{ij} [6, § 103].

would be an obvious false precision. In fact, taking into account the difference between μ and 1 is false precision for most phenomena already at frequencies much lower than optical frequencies."[6] In [6, § 103], an important remark is made concerning the relation between this result and the spatial dispersion effect. Taking into account this remark, we can assume that the restriction by optical frequencies is unnecessary: the difference between $\mu(\omega)$ and 1 can be substantial only in exclusive cases (see below). Indeed, in the general case,

$$k_0(\omega) \sim \frac{1}{a_0(\omega)} \sim \frac{\omega}{v_0}\,, \tag{4.12}$$

where v_0 is the velocity of particles (quasi-particles) making a considerable contribution to the dielectric permittivity at the given frequency ω. By substituting (4.12) into (4.11), we obtain

$$\left|\frac{1}{\mu(\omega)} - 1\right| \sim \left(\frac{v_0}{c}\right)^2, \tag{4.13}$$

if $\varepsilon(\omega) \sim 1$. The parameter $v_0/c \ll 1$ (in quantum mechanics, if v_0 is the electron velocity, then $v_0/c \sim \alpha$, where $\alpha = e^2/(\hbar c)$ is the fine structure constant) is a small parameter in the theory of matter–electromagnetic field interaction. The exclusion from result (4.13) can be related, as usual in the perturbation theory, only with the resonance situation, when the frequency ω is close enough to the frequency ω_0 of some transition between the energy levels of the medium.

The total set of quantum numbers characterizing the state of a medium contains the momentum $\mathbf{p}$ of a particle (a molecule if the medium is a gas) or the quasi-momentum of a quasi-particle (an exciton in the optical spectral region or an optical phonon in the IR region). This circumstance is manifested in the denominators of sums entering expressions for $\varepsilon_{ij}(\omega, \mathbf{k})$ (see [1; 18, § 12]). At frequency ω close to some resonance frequency ω_0 of an atom or a molecule at rest, the dominator in $\varepsilon_{ij}(\omega, \mathbf{k})$ is $\hbar(\omega - \omega_0) - [\varepsilon(\mathbf{p} + \hbar\mathbf{k}) - \varepsilon(\mathbf{p})] = \hbar(\omega - \omega_0) - \hbar\mathbf{k}\mathbf{v}(\mathbf{p})$, where $\varepsilon(\mathbf{p})$ is the kinetic energy of a particle (an atom or a molecule) and $\mathbf{v}(\mathbf{p})$ is its velocity. Now, close to the resonance, the spatial dispersion becomes substantial not for $k \geqslant \omega/v_0$ but for $k \geqslant |\omega - \omega_0|/\bar{v}$, where $\bar{v}$ is the mean velocity of the particle [6, § 103], i.e., now $k_0(\omega)$ is not determined from expression (4.12), but from the expression

$$k_0(\omega) \sim \frac{|\omega - \omega_0|}{\bar{v}}\,. \tag{4.14}$$

Now, we have from (4.11) [instead of (4.13)]

$$\left|\frac{1}{\mu(\omega)} - 1\right| \sim \left(\frac{\bar{v}}{c}\right)^2 \left(\frac{\omega}{\omega - \omega_0}\right)^2 |\varepsilon(\omega)|\,, \tag{4.15}$$

and the difference between $\mu(\omega)$ and unity can be noticeable. Of course, one should bear in mind that the detuning $|\omega - \omega_0|$ cannot be too small, because the field energy dissipation is ignored everywhere.

At frequency ω close to some resonance frequency ω_0 of a condensed medium [1; 6, § 106; 18, § 12], the dielectric permittivity tensor $\varepsilon_{ij}(\omega, \mathbf{k})$ contains a term with the dominator $\hbar(\omega - \omega_0) - \hbar^2 k^2/(2m_{\text{ex}})$, where $\hbar\omega_0$ is the energy of an immobile exciton (or an optical phonon) and m_{ex} is its effective mass. This mass can also be negative. The spatial dispersion becomes considerable for $k^2 \geqslant |m_{\text{ex}}(\omega - \omega_0)|/\hbar$, i.e., $k_0(\omega)$ is determined not from expression (4.12) but from the expression

$$k_0^2(\omega) \sim \frac{|m_{\text{ex}}(\omega - \omega_0)|}{\hbar}\,. \tag{4.16}$$

From (4.11) [instead of (4.13)], we obtain

$$\left|\frac{1}{\mu(\omega)} - 1\right| \sim \frac{\hbar\omega}{|m_{\text{ex}}|c^2} \frac{\omega}{|\omega - \omega_0|} |\varepsilon(\omega)|\,; \tag{4.17}$$

the difference between $\mu(\omega)$ and unity can also be noticeable.

The low-frequency magnetic susceptibility $\chi_0 \equiv \chi(\omega)|_{\omega=0}$ is not necessarily equal to the static magnetic susceptibility $\chi_{\text{st}} \equiv \chi(k)|_{k=0}$, where $\chi(k) = \chi(\omega, k)|_{\omega=0}$, because functions $\chi(k)$ and $\chi(\omega) = \chi(\omega, k)|_{k=0}$ are obtained from $\chi(\omega, k)$ by expanding in powers of different small parameters. For example, $\chi(\omega)$ for monatomic gas [20] is obtained by disregarding a small parameter $ka_{\text{B}}/(\omega/\omega_{\text{R}}) \ll 1$, where $a_{\text{B}} = \hbar^2/(me^2)$ is a quantity of the order of the atomic 'radius' and $\hbar\omega_{\text{R}} = \hbar^2/(ma_{\text{B}}^2)$ is the characteristic electron energy. The limiting value of the susceptibility χ_0 is then obtained by disregarding the small parameter $\omega/\omega_{\text{R}} \ll 1$ and consists of two temperature-independent parts corresponding to the Langevin diamagnetism and Van Vleck paramagnetism [20, 21]. The susceptibility $\chi(k)$ is obtained by ignoring the small parameter

$$\frac{\hbar\omega}{(\hbar k)^2/M} \ll 1\,,$$

where M is the molecule mass. The static susceptibility χ_{st} is then obtained by disregarding the small parameter $\hbar k/(M\bar{v}) \ll 1$, where $\bar{v}$ is the mean velocity of molecules. It differs from χ_0 by the term $\sim 1/T$, corresponding to the Langevin paramagnetism [20, 21].

In the electron collisionless plasma [17, § 26], $\chi(\omega)$ is obtained from $\chi(\omega, k)$ by expanding in powers of the small parameter $k\bar{v}/\omega \ll 1$, where $\bar{v}$ — mean velocity of electron [see (4.5) and (4.12)], and static susceptibility $\chi(k)$ is calculated from $\chi(\omega, k)$ by small parameter $\omega/(k\bar{v}) \ll 1$ expansion. In a homogeneous field, $\chi_{\text{st}} = \chi(k)|_{k=0}$ is obtained from $\chi(k)$ by expanding in powers of another small parameter $\hbar k/(m\bar{v}) \ll 1$, where m is the electron mass, and corresponds to the Pauli paramagnetism and Landau diamagnetism [17, § 26].

Consider dispersion equation (3.38) for transverse waves. It follows from (3.37) and (4.7) that, by ignoring the spatial dispersion, we should assume that $\mu(\omega) = 1$ in this equation and write it in the form

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\,. \tag{4.18}$$

Correspondingly, the phase velocity of the wave is $v_{\text{ph}} = c/(\varepsilon(\omega))^{1/2}$ (and, therefore, the refractive index is $n = (\varepsilon(\omega))^{1/2}$). Equation (4.18) is also obtained from (3.26) taking into account (4.7). In this case, of course, condition (4.5) should be fulfilled, which is used to obtain (4.18):

$$\left(\frac{k}{k_0(\omega)}\right)^2 = \varepsilon(\omega)\,\frac{\omega^2}{c^2 k_0^2(\omega)} \ll 1\,. \tag{4.19}$$

[6] This result is presented already in the first (1957) issue of *Electrodynamics of Continuous Media* [6, § 60]).

At frequencies ω far from the resonance frequencies ω_0 of the medium, $k_0(\omega)$ is determined from (4,12), $\varepsilon(\omega) \sim 1$ and condition (4.19) is reduced to the inequality $(v_0/c)^2 \ll 1$, which is always fulfilled. At frequency ω close to some transition frequency ω_0 in a gas medium, $k_0(\omega)$ is determined from (4.14), and condition (4.19) restricts the minimal value of $|\omega - \omega_0|$:

$$\frac{|\omega - \omega_0|}{\omega} \gg \sqrt{\varepsilon(\omega)}\,\frac{\bar{v}}{c}\,. \tag{4.20}$$

A comparison with (4.15) shows that, in this case, as expected, $|\mu(\omega) - 1| \ll 1$. At frequency ω close to some transition frequency ω_0 in a condensed medium, $k_0(\omega)$ is determined from (4.16), and condition (4.19) also restricts the minimal value of $|\omega - \omega_0|$:

$$\frac{|\omega - \omega_0|}{\omega} \gg \varepsilon(\omega)\frac{\hbar\omega}{|m_{\rm ex}|c^2}\,. \tag{4.21}$$

A comparison with (4.17) shows that, in this case, as should be, $|\mu(\omega) - 1| \ll 1$. If condition (4.20) [or (4.21)] is not fulfilled, dispersion equation (4.18) is not valid, and exact dispersion equation (3.26) or (3.38) should be solved.

If the spatial dispersion is small, it can be taken into account in (3.26) or (3.38) by the perturbation theory, thereby refining dispersion equation (4.18). Not increasing the order of algebraic equation (3.26) determining k^2, we substitute into it the expression for the transverse dielectric permittivity,

$$\varepsilon^{\rm tr}(\omega, k) = \varepsilon(\omega) + \left.\frac{\partial\varepsilon^{\rm tr}(\omega, k)}{\partial k^2}\right|_{k=0} k^2\,. \tag{4.22}$$

As a result, we obtain instead of (4.18) a more accurate equation,

$$k^2 = \frac{\omega^2}{c^2}\,\varepsilon(\omega)\,\tilde{\mu}(\omega)\,, \tag{4.23}$$

where the function

$$\tilde{\mu}(\omega) = 1 + \frac{\omega^2}{c^2}\left.\frac{\partial\varepsilon^{\rm tr}(\omega, k)}{\partial k^2}\right|_{k=0}, \tag{4.24}$$

which is close to unity, does not coincide with the magnetic permeability determined from the expression [see (4.10) for $|\mu(\omega) - 1| \ll 1$]

$$\mu(\omega) = 1 + \frac{\omega^2}{c^2}\,\frac{\partial}{\partial k^2}\left(\varepsilon^{\rm tr}(\omega, k) - \varepsilon^{\rm l}(\omega, k)\right)\Big|_{k=0}. \tag{4.25}$$

Thus, we should assume that $\mu(\omega) = 1$ in the generally accepted dispersion equation for transverse waves (1.1) and in the expression for the phase velocity (1.2), and accounting for the difference between $\mu(\omega)$ and unity is 'the obvious false precision' [6, § 79]. However, the authors of review [1] obtain the usual equation (1.1) as the dispersion equation for transverse waves. How do they do it? For the convenience of readers, we will attach to formulas from [1] the numbers of the same formulas from the present paper. For example, (19)–(4.10) denotes formula (19) from [1] and the same formula (4.10) from our paper. The authors of [1, p. 1056] write: "It is easy to see that, if we set

$$\varepsilon^{\rm tr}(\omega, k) = \varepsilon(\omega) + \frac{c^2k^2}{\omega^2}\left(1 - \frac{1}{\mu(\omega)}\right), \qquad (20)-(4.26)$$

in the dispersion equation (16)–(3.26) for transverse polaritons, then (16)–(3.26) becomes *identical* to equation (3)–(1.1)." This is really easily seen: by substituting (4.26) into (3.26), we obtain (1.1). But is it possible to 'set' (20)–(4.26)? Taking into account (18)–(4.7), we can rewrite equality (20)–(4.26) used in [1] in the form

$$1 - \frac{1}{\mu(\omega)} = \frac{\omega^2}{c^2k^2}\left(\varepsilon^{\rm tr}(\omega, k) - \varepsilon(\omega)\right) = \frac{\omega^2}{c^2}\left.\frac{\partial\varepsilon^{\rm tr}(\omega, k)}{\partial k^2}\right|_{k=0}, \tag{4.27}$$

which shows that this equality does not contradict the definition of $\mu(\omega)$ given by expression (19)–(4.10) only when

$$\left|\left.\frac{\partial\varepsilon^{\rm tr}(\omega, k)}{\partial k^2}\right|_{k=0}\right| \gg \left|\left.\frac{\partial\varepsilon^{\rm l}(\omega, k)}{\partial k^2}\right|_{k=0}\right|. \tag{4.28}$$

Note that only when this inequality is valid does function $\tilde{\mu}(\omega)$ (4.24) not differ from the magnetic permeability $\mu(\omega)$ (4.25). However, there are no grounds for fulfilling inequality (4.28). In fact, the authors of [1] introduce by their expression (20)–(4.26) a new function, denoting it as the magnetic permeability. By comparing (4.27) and (4.24), we see that their function coincides with our function $\tilde{\mu}(\omega)$ (4.24). However, the authors themselves [1, p. 1065] write then that "equation (20)–(4.26) specifies already some effective permeability." Therefore, we believe that in fact our result concerning a transverse wave in an isotropic medium in the case of a small spatial dispersion does not contradict the corresponding result in [1].

5. Conclusions

The main results of our paper are as follows:

1. It is shown how within the framework of linear material equations for a homogeneous isotropic medium we can rigorously move from electrodynamic Maxwell–Lorentz equations for fields **E** and **B** (3.1) to Maxwell's equations for four fields, **E**, **B**, **D**, and **H** (1.5). This possibility is substantially related to the fact that the number of independent scalar functions characterizing the electrodynamic properties of such a medium ($\varepsilon^{\rm l}(\omega, k)$, $\varepsilon^{\rm tr}(\omega, k)$ or $\varepsilon(\omega, k)$, $\mu(\omega, k)$) is equal to the number of new vectors (**P**, **M** or **D**, **H**) according to (3.12) and (3.23). In this case, first the dielectric permittivity $\varepsilon(\omega, k)$ and magnetic permeability $\mu(\omega, k)$ are determined taking into account the frequency and spatial dispersion, and only then are the Fourier components of fields $\mathbf{D}_{\omega\mathbf{k}}$ and $\mathbf{H}_{\omega\mathbf{k}}$ determined by expressions (3.31) (and, therefore, the fields $\mathbf{D}(\mathbf{r}, t)$ and $\mathbf{H}(\mathbf{r}, t)$ themselves).

2. A similar problem for anisotropic media is much more complicated. It is necessary to know the principle of dividing one vector (the current density in the medium) into two vectors (called the polarization and magnetization) [see expression (3.4)]. Or, which, in fact, is the same thing, how to introduce unambiguously, instead of the conductivity tensor $\sigma_{ij}(\omega, \mathbf{k})$ (3.20), two tensors, one determining the electric induction $D_{i\omega\mathbf{k}} = \varepsilon_{ij}(\omega, \mathbf{k})\,E_{j\omega\mathbf{k}}$ and the other determining the magnetic field strength $B_{i\omega\mathbf{k}} = \mu_{ij}(\omega, \mathbf{k})\,H_{j\omega\mathbf{k}}$. We do not know the solution to this problem and because of this considered only isotropic media.

3. The known result of Landau and Lifshitz that the magnetic permeability is $\mu(\omega) \approx 1$, related, as pointed out in

[6, §79], to a small spatial dispersion, is valid, generally speaking, not only at optical frequencies but also for all frequencies. Exclusions can be at resonance frequencies close to transition frequencies between the corresponding states of the medium.

4. Transverse waves with a negative group velocity do not exist within the framework of a theory dealing with the dielectric permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$ (i.e., by ignoring the spatial dispersion in them). This conclusion is in no way connected to the Rautian postulate [7] about the absence of optical frequencies at which the dielectric permittivity $\varepsilon(\omega)$ and magnetic permeability $\mu(\omega)$ take negative values. The conclusion directly follows from the correct dispersion equation for transverse waves, which is described by expression (4.18) for all isotropic nongyrotropic media (including so-called metamaterials) by ignoring the spatial dispersion. This equation can be obtained from known equation (1.1) assuming in it that $\mu(\omega) = 1$. The consideration of a weak spatial dispersion leads to dispersion equation (4.23) containing some function $\tilde{\mu}(\omega)$ (4.24) different from the magnetic permeability $\mu(\omega)$ (4.25). Therefore, the propagation of electromagnetic waves is a process for which accounting for the difference between $\mu(\omega)$ and unity is a false precision [6, §79]. Waves with the negative group velocity can exist only at frequencies for which the spatial dispersion becomes considerable [1, 8, 20]. In this case, the dispersion equation is written in form (3.26) or (3.38).

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