

METHODOLOGICAL NOTES

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Electromagnetic analogies in electro- and magnetostatics problems

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Contents

Abstract. **A relation underlying the electromagnetic analogy method is derived from the Biot–Savart and Coulomb laws: the magnetic field B generated by a current I in a conducting loop L and the electric field E of a thin double charged layer of area S bounded by the same loop L are linked by the formula $\mathbf{B} = (\varepsilon_0 \mu_0 I / p_{1e})\,\mathbf{E}$, where p_{1e} is the electric moment per unit area S, and ε_0 and μ_0 are the permittivity and permeability of free space. Examples are given where this electromagnetic analogy can be used for solving electro- and magnetostatics problems.**

Keywords: electrostatics, magnetostatics, electric dipole, magnetic dipole, capacitance, inductance

1. Introduction

The analogy between magnetic and electric fields has been noted in a number of textbooks [1–5]. For example, it is mentioned in Stratton's textbook [2] that every magnetostatic field can be replaced with an electrostatic field of the identical structure created by the corresponding distribution of dipoles and double layers of charge. In his textbook [4], Smythe notes an experimental fact that two small current loops interact with the same forces and torques as electric dipoles situated in place of the loops and oriented normal to their planes. In Purcell's book [5], the expression for the **B** field of a magnetic dipole is compared with the expression for the **E** field of the electric dipole: these expressions are the same up to a constant factor. When comparing the **B** field of a homogeneously magnetized cylinder and the **E** field of a uniformly polarized cylinder, Purcell notes that these fields coincide outside the cylinder, and the **B** field can be calculated in the same way as in electrostatics, if one introduces magnetic charges or the scalar magnetic potential related to currents.

In this article, we derive a relation between the magnetic field **B** created by a current I in a conducting loop L and the electric field **E** of a thin double layer of charge with an area S bounded by the same loop L. This relation is then used to derive the expression for the **B** field of a point magnetic dipole to prove the theorem about the magnetic curl and flux in magnetostatics and to derive an expression for the solenoid inductance with edge effects taken into account. Each of the problems mentioned can be solved based on the laws of magnetostatics, but the electromagnetic analogy method suggested in this article allows an easier solution based on the corresponding problems in electrostatics.

The electromagnetic analogy "works both ways," and some problems of electrostatics can be solved via simpler problems of magnetostatics. As an example, we consider the problem of calculating the capacitance of a planar capacitor and the electric field outside it.

2. Electromagnetic analogy

We consider a double layer of charge with one surface charged uniformly with a surface density $+\sigma$ and the other one with a surface density $-\sigma$. The layer thickness d is assumed to be small compared with its lateral dimensions. We can then consider this system as a surface S with a uniformly distributed dipole moment with the surface density $p_{1e} = |\mathbf{p}_{1e}| = \sigma d$. The dipole moment of an elementary area d**S** is directed along its normal and is equal to $\mathrm{d}\mathbf{p}_e = p_{1e}\,\mathrm{d}\mathbf{S}$. According to Coulomb's law, the electrostatic field potential created by an elementary 'dipole' area dS can

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be expressed at a point A as [1–3]

$$d\varphi = \frac{\mathbf{r}\, d\mathbf{p}_e}{4\pi\varepsilon_0 r^3}\,,$$

where $\mathbf{r}$ is the vector directed from the elementary area towards the point A. Then the field potential at A created by the whole double layer is

$$\varphi = \frac{p_{1e}}{4\pi\varepsilon_0}\int_S \frac{\mathbf{r}\, d\mathbf{S}}{r^3} = \frac{p_{1e}}{4\pi\varepsilon_0}\,\Omega\,,$$

where $\Omega = \int_S \mathbf{r}\, d\mathbf{S}/r^3$ is the solid angle of the surface S with respect to the point A and the sign of Ω is chosen so as to coincide with the sign of the charge of the surface side closest to A. The electric field vector of a double layer can be calculated using the expression

$$\mathbf{E} = -\mathrm{grad}\,\varphi = -\frac{p_{1e}}{4\pi\varepsilon_0}\,\mathrm{grad}\,\Omega\,.$$

We find the variation of the solid angle $d\Omega$ caused by the translation of the point A by $d\boldsymbol{\rho}$ [1]. We assume that the point A does not change its position, while every point of the surface S is translated by $-d\boldsymbol{\rho}$, as shown in Fig. 1. Then $d\Omega = \int_{\Delta S} \mathbf{r}\, d\mathbf{S}/r^3$, where ΔS is the side face of the skew cylinder defined by $d\rho$ (see Fig. 1). The elementary area of this surface can be expressed as $d\mathbf{S} = [-d\boldsymbol{\rho}\, d\mathbf{l}]$. Using a cyclic rearrangement in the mixed product, we obtain $\mathbf{r}\, d\mathbf{S} = -d\boldsymbol{\rho}\,[d\mathbf{l}\mathbf{r}]$ and $d\Omega = -d\boldsymbol{\rho}\oint_L [d\mathbf{l}\mathbf{r}]/r^3$. Because $d\Omega = d\boldsymbol{\rho}\,\mathrm{grad}\,\Omega$, we have $\mathrm{grad}\,\Omega = -\oint_L [d\mathbf{l}\mathbf{r}]/r^3$ and

$$\mathbf{E} = \frac{p_{1e}}{4\pi\varepsilon_0}\oint_L \frac{[d\mathbf{l}\mathbf{r}]}{r^3}\,. \tag{1}$$

We note that expression (1) is equivalent up to a constant factor to the expression

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\oint_L \frac{[d\mathbf{l}\mathbf{r}]}{r^3}\,, \tag{2}$$

which, according to the Biot–Savart law, gives the $\mathbf{B}$ field created by the current I in a thin wire placed along the loop L at the point A. This means that the magnetic field $\mathbf{B}$ of a current I running through a thin wire along the loop L and the electric field $\mathbf{E}$ of a thin double layer of charge S bounded by L

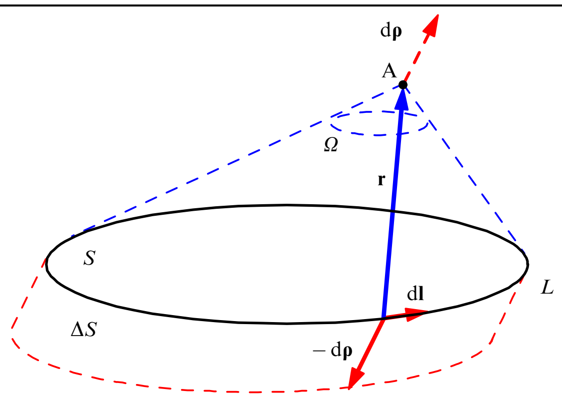

Figure 1. Illustration of the derivation of Eqn (1). L — closed loop around the surface S, $d\mathbf{l}$ — elementary segment of this loop, $\mathbf{r}$ — vector directed from $d\mathbf{l}$ to point A. Direction of circulation around the loop is related to the direction $\mathbf{p}_{1e}$ by the right-hand screw rule.

are related as

$$\frac{\mathbf{B}}{\mu_0 I} = \frac{\varepsilon_0}{p_{1e}}\,\mathbf{E}\,. \tag{3}$$

Equation (3) follows form the Biot–Savart and Coulomb laws, and it is valid for an infinitely thin wire with a current and an infinitely thin double layer of charge with a dipole moment p_{1e} uniformly distributed over its surface. If the closed wire has a finite cross section with the characteristic dimension Δ, then it can be split into infinitely thin closed conductors with corresponding infinitely thin double layers of charge. The superposition of these layers leads to the formation of a double electric layer with the thickness Δ. Due to the superposition principle, Eqn (3) remains valid for all points located outside the double layer with thickness Δ at a distance $r \gg \Delta$ from the wire. Equation (3) can be used as the basis for the electromagnetic analogy method, which allows replacing magnetostatic problems with electrostatic ones and vice versa. We discuss examples where this method is used in what follows.

3. Field of a point magnetic dipole

As the first example for the application of the electromagnetic analogy method, we use Eqn (3) to derive the expression for the B field of a point magnetic dipole. We assume that the expression

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left(3(\mathbf{p}_e\mathbf{r})\,\frac{\mathbf{r}}{r^2} - \mathbf{p}_e\right) \tag{4}$$

for the E field of the electrostatic field created by a point electric dipole is already known (see, e.g., [3]). We consider the point electric dipole as an elementary dipole area ΔS with the dipole moment $\mathbf{p}_e = p_{1e}\,\Delta S\mathbf{n}$, and the point magnetic dipole as a current loop I along the boundary of the area ΔS. The magnetic moment of such a dipole is $\mathbf{p}_m = I\Delta S\mathbf{n}$, where $\mathbf{n}$ is the normal to ΔS. Equations (3) and (4) imply that

$$\mathbf{B} = \frac{\mu_0}{4\pi r^3}\left(3(\mathbf{p}_m\mathbf{r})\,\frac{\mathbf{r}}{r^2} - \mathbf{p}_m\right). \tag{5}$$

The derivation of expression (5) directly from the Biot–Savart law is much more complicated [5].

4. Proof of theorems on the magnetic curl and flux

The proof of the magnetic curl theorem

$$\oint_C \mathbf{B}\, d\mathbf{l} = \mu_0 I \tag{6}$$

in magnetostatics is much more complicated than the proof of the corresponding theorem

$$\oint_C \mathbf{E}\, d\mathbf{l} = 0 \tag{7}$$

in electrostatics (here, C is an arbitrary geometric contour). This is the reason why in general physics textbooks the proof of theorem (6) is often either omitted [6, 7] or considered in the specific case where the currents I run in infinitely long straight wires [5, 8]. The electromagnetic analogy method allows proving theorem (6) based on electrostatics theorem (7). As

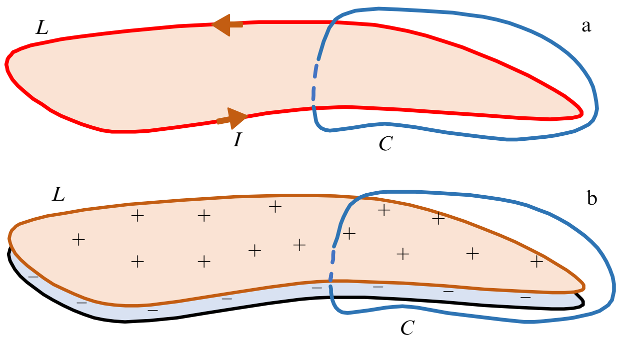

Figure 2. Illustration of the proof of the theorem about the B field curl. (a) L — closed conducting loop with the current I, C — an arbitrary geometric contour. (b) Double electric layer bounded by the loop L and the geometric contour C.

the first step, the arbitrary closed wire with a current I is replaced with a double electric layer having the surface density $\pm\sigma$ and the thickness d, which we further send to zero such that the surface density of the electric moment $p_{1e} = (\sigma d)_{d\to 0}$ remains finite. Let a geometric contour C cross the current loop and hence the double electric layer (Fig. 2). Integral (7) can be expressed as the sum

$$\int_{ext} \mathbf{E}_{ext}\, d\mathbf{l} + \int_{int} \mathbf{E}_{int}\, d\mathbf{l} = 0\,.$$

The first integral is taken along the part of the contour C that is located outside the double electric layer, and the field $\mathbf{E}_{ext}$ is the 'dipole field' calculated as the vector sum of single dipole fields (4). The second integral is taken along the part of the contour located inside the electric layer:

$$\int_{int} \mathbf{E}_{int}\, d\mathbf{l} = -\frac{\sigma d}{\varepsilon_0}\bigg|_{d\to 0} = -\frac{p_{1e}}{\varepsilon_0}\,.$$

Hence,

$$\int_{ext} \mathbf{E}_{ext}\, d\mathbf{l} = \frac{p_{1e}}{\varepsilon_0}\,.$$

When calculating the $\mathbf{B}$ field curl for the corresponding current loop and the same geometric contour C (see Fig. 2), we take into account that relation (3) holds at every point outside the double electric layer. Therefore,

$$\int_{ext} \mathbf{B}\, d\mathbf{l} = \varepsilon_0\mu_0 \frac{I}{p_{1e}} \int_{ext} \mathbf{E}_{ext}\, d\mathbf{l} = \mu_0 I\,.$$

According to the Biot–Savart law, the magnetic field is finite in regions without currents and, because $d \to 0$, the integral $\int_{int} \mathbf{B}\, d\mathbf{l} = 0$. This means that $\oint_C \mathbf{B}\, d\mathbf{l} = \int_{ext} \mathbf{B}\, d\mathbf{l} = \mu_0 I$. If the geometric contour C does not encompass the current loop L, then the curl of $\mathbf{B}$ is obviously zero.

We now prove the theorem on the $\mathbf{B}$ field flux. As the magnetic field source, we also consider a stationary current I in a thin closed wire placed along the contour L. We consider an arbitrary simply connected closed surface S_0 that does not intercept the contour L. We can choose the surface of the double layer S bounded by L such that all of it is located outside a closed surface S_0. The electric field flux through the closed surface S_0 is zero according to the Gauss theorem:

$$\oint_{S_0} \mathbf{E}\, d\mathbf{S} = 0\,.$$

Because the $\mathbf{B}$ field of the current and the $\mathbf{E}$ field of the dipole surface S are related by Eqn (3), it follows that

$$\oint_{S_0} \mathbf{B}\, d\mathbf{S} = 0\,. \tag{8}$$

To consider the case of a multiply connected surface S_0, we need an additional analysis, which we omit here. Using the superposition principle, the proofs of theorems (6) and (8) can be generalized to an arbitrary system of closed stationary currents.

5. When do magnetic field lines form closed curves?

It is known [1, 9, 10] that even in simple current configurations, the magnetic field lines do not necessarily form closed loops. We can pose the question: in which cases do magnetic field lines form closed curves? The electromagnetic analogy method provides a broad variety of current configurations in which magnetic field lines are closed curves: such is any system of closed currents located in one plane.

This statement can be derived directly from magnetostatics, but switching from currents to corresponding double charged layers makes the proof easy and clear. Indeed, it follows from the symmetry of the electrostatics problem that if a magnetic field line originates from a point on a flat double electric layer, then it is mirror symmetric with respect to the layer plane and must therefore return to the initial point from the other side of the layer. According to (3), the magnetic line of the contour with the same configuration must follow the same pattern. We note that the condition mentioned above is sufficient, but not necessary for the magnetic field lines to be closed.

6. Magnetic field of a solenoid

If a solenoid has a round cross section, then it is not difficult to use the Biot–Savart law to calculate the magnetic field at points on the solenoid axis [5]. For other points, the integration becomes complicated, and even greater problems arise if the solenoid cross section is not round.

If we assume that the solenoid is infinitely long, the theorem on the magnetic curl can be used to calculate the field at any point located inside the solenoid: $B_0 = \mu_0 n I$, where n is the number of coils per unit length. A derivation of this expression is demonstrated in almost all textbooks under the assumption that the magnetic field outside the solenoid is zero. However, it is not easy to rigorously prove this fact in the framework of a general physics course [11].

The electromagnetic analogy allows replacing the problem of the solenoid magnetic field with an electrostatic problem. As always, we ignore the discreteness of the coils and consider a solenoid to be a straight cylinder with the current flowing around it. We assume that the cylinder axis is parallel to the x axis and the transverse cross section has an area S with an arbitrary shape. Stationary current is flowing along the side face of the cylinder perpendicular to the x axis with a constant linear density $I_S = IN/l = In$, where N is the

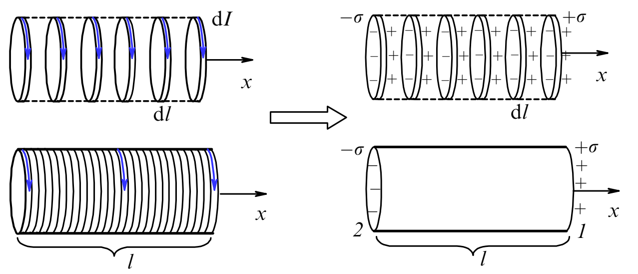

Figure 3. Illustration of the calculation of the magnetic field outside a solenoid.

number of coils, l the length of the solenoid, and $n = N/l$. We conceptually split the solenoid into narrow layers perpendicular to the x axis, each with a thickness dl. Each layer carries the current $\mathrm{d}I = In\,\mathrm{d}l$, running along its perimeter, and can be replaced by a double charge layer with a surface charge density $\pm\sigma$ and the electric dipole moment density $\mathrm{d}p_{1\mathrm{e}} = \sigma\,\mathrm{d}l$ (Fig. 3). The array of such layers creates the same electric field as two bases of a cylinder uniformly charged with densities $\pm\sigma$. According to (3), the magnetic field of the solenoid at the points located outside the cylinder can be calculated as

$$\mathbf{B} = \beta(\mathbf{E}_1 + \mathbf{E}_2)\,, \tag{9}$$

where $\beta = \varepsilon_0\mu_0 nI/\sigma$ and $\mathbf{E}_1 + \mathbf{E}_2$ in the electric field at the same points created by two bases of the cylinder with the respective surface charge density $+\sigma$ and $-\sigma$. For points located far from the cylinder bases (at a distance much longer than the characteristic dimension of the cylinder cross section), the charges on the cylinder bases can be regarded as point charges, and

$$\mathbf{E} = \frac{\sigma S}{4\pi\varepsilon_0}\left(\frac{\mathbf{r}_1}{r_1^3} - \frac{\mathbf{r}_2}{r_2^3}\right), \quad \mathbf{B} = B_0\frac{S}{4\pi}\left(\frac{\mathbf{r}_1}{r_1^3} - \frac{\mathbf{r}_2}{r_2^3}\right), \tag{10}$$

where the vector $\mathbf{r}_1$ is directed towards the 'observation point' from the north pole of the solenoid, and $\mathbf{r}_2$ from the south pole. We note that expression (10) is valid for a cross section of any shape.

If the solenoid is long, the magnetic field $\mathbf{B}$ near its edge in the outer space coincides up to a constant factor with the electric field $\mathbf{E} \approx \mathbf{E}_1$ of a uniformly charged surface aligned with the solenoid base, because the field of the second charged base can be disregarded. In particular, this implies that 1) the normal component of the magnetic field near the edge of a long solenoid is uniform in its plane; 2) magnetic field lines outside a long solenoid are symmetric with respect to the plane of the solenoid edge; 3) magnetic field lines originating from the boundary points on the solenoid edge are straight and perpendicular to its axis. The magnetic field lines $\mathbf{B} = \beta\mathbf{E}_1$ of a semi-infinite solenoid are shown in Fig. 4. The field $\mathbf{E}_1$ of a uniform charged disk was calculated using numerical integration, and the method for the calculation of the $\mathbf{B}$ field inside the solenoid is discussed in what follows.

To calculate the magnetic field inside a solenoid with the shape of a straight cylinder (not necessarily with a round cross section), we conceptually make a narrow slit in it with a thickness $\varDelta \ll \sqrt{S}$, perpendicular to the cylinder axis (Fig. 5). The magnetic field inside the slit is then almost the same as inside the uncut solenoid (except for a narrow region with a

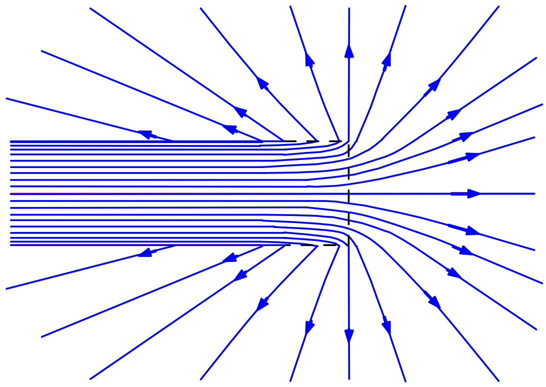

Figure 4. B field lines near the edge of a semi-infinite solenoid.

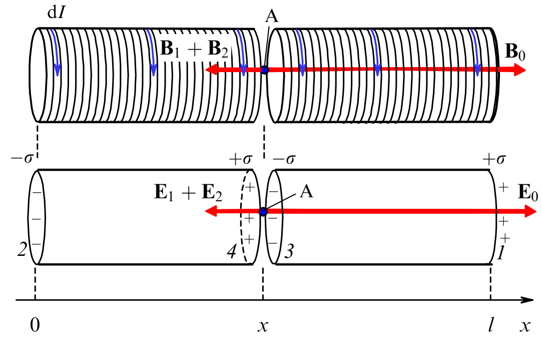

Figure 5. Illustration of the calculation of the magnetic field inside a solenoid.

thickness $\sim \varDelta \to 0$ near the cylinder surface). According to the electromagnetic analogy, the problem corresponds to the calculation of the electric field induced at a point A by four uniformly charged flat edges of two solenoids (denoted as 1–4 in Fig. 5), with the subsequent recalculation using the expression

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4 = \beta(\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4)\,.$$

The electric field $E_0 = |\mathbf{E}_3 + \mathbf{E}_4| = \sigma/\varepsilon_0$ is induced by two uniformly charged surfaces with charge densities $+\sigma$ and $-\sigma$ located infinitely close to the point A, while the fields $\mathbf{E}_1$ and $\mathbf{E}_2$ are induced by the charged edges of the initial solenoid. The magnetic field $|\mathbf{B}_0| = \beta E_0 = \mu_0 In$ is the same in all cross sections of the solenoid, while the calculation of the field $\mathbf{B}_1 + \mathbf{B}_2 = \beta(\mathbf{E}_1 + \mathbf{E}_2)$ in general requires the use of numerical methods. In the specific case of a round cylinder, the field $\mathbf{E}_1 + \mathbf{E}_2$ on its symmetry axis can be expressed analytically. The electric fields of a negatively charged disk with radius R located at $x = 0$ and of a positively charged disk with the same radius located at $x = l$ are given by the respective expressions

$$E_{2x} = -\frac{\sigma}{2\varepsilon_0}\left(\frac{|x|}{x} - \frac{x}{\sqrt{x^2 + R^2}}\right),$$

$$E_{1x} = \frac{\sigma}{2\varepsilon_0}\left(\frac{|x-l|}{x-l} - \frac{x-l}{\sqrt{(x-l)^2 + R^2}}\right).$$

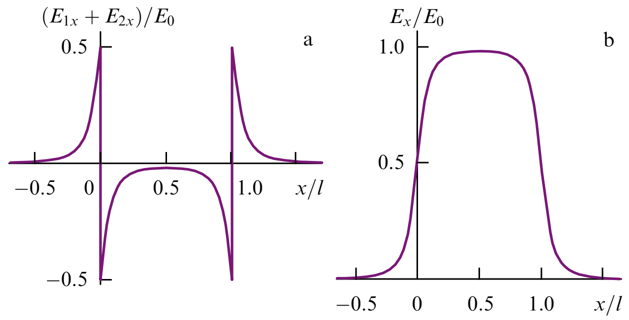

Figure 6. (a) Electric field on the axis of two oppositely charged parallel disks. (b) Field of the same disks with an added uniform field E_0 ($l/R = 5$) in the region between the disks.

Figure 6a shows the dependence of the field $E_{1x} + E_{2x}$ on the position x. Adding a uniform field E_0 to $E_{1x} + E_{2x}$ in the region $0 < x < l$, we obtain an expression valid for any x:

$$E_x = \frac{\sigma}{2\varepsilon_0}\left(\frac{x}{\sqrt{x^2+R^2}} - \frac{x-l}{\sqrt{(x-l)^2+R^2}}\right),$$

which does not have discontinuity points (Fig. 6b). According to the magnetoelectric analogy, the magnetic field on the axis of a round solenoid can be expressed as

$$B_x = \beta E_x = \frac{\mu_0 In}{2}\left(\frac{x}{\sqrt{x^2+R^2}} - \frac{x-l}{\sqrt{(x-l)^2+R^2}}\right).$$

Of course, this expression can also be obtained directly from the Bio–Savart law [5].

We note that the field $\mathbf{E}_1 + \mathbf{E}_2$ is an electrostatic one, while the 'slightly' corrected field $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \gamma\mathbf{E}_0$ (where $\gamma = 1$ inside and $\gamma = 0$ outside the cylinder) is not. The latter field is nonpotential and divergence-free: its tangential component is discontinuous on the cylinder surface, while the normal component changes continuously across the charged surfaces at $x = 0$ and $x = l$.

7. Inductance of a solenoid

We have shown that the magnetic field inside a solenoid with the shape of a straight cylinder can be expressed as $\mathbf{B}_{\text{int}} = \beta(\mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2)$, where $|\mathbf{E}_0| = \sigma/\varepsilon_0$ and $\mathbf{E}_1 + \mathbf{E}_2$ is the electric field induced by the cylinder bases uniformly charged with surface densities $+\sigma$ and $-\sigma$. We now calculate the inductance L of this solenoid with a length l, a cross section area S, and the winding density $n = N/l$. The magnetic flux through the solenoid coils is $\Phi = \int_0^l \Phi_1(x)\,n\,\mathrm{d}x$, where the magnetic flux through the cross section at the coordinate x (see Fig. 5) is

$$\Phi_1(x) = \beta E_0 S + \beta\int_S (\mathbf{E}_1 + \mathbf{E}_2)\,\mathrm{d}\mathbf{S}.$$

Introducing the notation

$$F_{1x} = -\sigma\int_S \mathbf{E}_1\,\mathrm{d}\mathbf{S}, \quad F_{2x} = \sigma\int_S \mathbf{E}_2\,\mathrm{d}\mathbf{S},$$

we can write

$$\Phi = \beta E_0 Snl + \frac{\beta}{\sigma}\int_0^l (F_{2x} - F_{1x})\,\mathrm{d}x. \tag{11}$$

The quantity $F_{2x}(x)$ has a simple physical meaning: it is equal to the projection of the force with which the negatively charged cylinder base acts on the positively charged one if the cylinder length is x. This means that the integral $-\int_0^l F_{2x}(x)\,\mathrm{d}x$ is equal to the work of the external forces that is needed to slowly separate two oppositely charged plates (cylinder bases) initially separated by a distance l. This work equals the change in the electric energy of the system. The initial energy of the system with two uniformly charged plates carrying surface densities $\pm\sigma$ pressed against each other is zero. Thus, we can write the energy W_C of two plates separated by a distance l as

$$W_C = -\int_0^l F_{2x}(x)\,\mathrm{d}x.$$

In the same manner, we obtain

$$W_C = -\int_l^0 F_{1x}(x)\,\mathrm{d}x = \int_0^l F_{1x}(x)\,\mathrm{d}x.$$

Substituting the calculated integrals in (11), we obtain the magnetic flux

$$\Phi = \beta E_0 Snl - 2\,\frac{\beta n}{\sigma}\,W_C.$$

We now substitute the expressions for E_0 and β, and after simple transformations obtain the inductance $L = \Phi/I$ of the solenoid

$$L = L_0\left(1 - \frac{W_C}{q^2/2C_0}\right), \tag{12}$$

where $L_0 = \mu_0 N^2 S/l$ and $C_0 = \varepsilon_0 S/l$. The meaning of this notation is clear: L_0 is the inductance of a solenoid in the limit case where $l \gg \sqrt{S}$ and the edge effects for the $\mathbf{B}$ field are negligible, and C_0 is the capacitance of the capacitor formed by the plates coinciding with the solenoid edges when $l \ll \sqrt{S}$, and with edge effects disregarded. Equation (12) can also be rewritten in a symmetric way:

$$\frac{W_L}{(L_0 I^2/2)} + \frac{W_C}{(q^2/2C_0)} = 1, \tag{13}$$

where $W_L = \Phi I/2$ is the energy of the solenoid magnetic field for the current I and W_C is the electrostatic field energy of two plates coinciding with the solenoid edges with uniformly distributed charges q and $-q$. Expressions (12) and (13) are valid for a broad range of parameters. The only assumptions are that the solenoid has the shape of a straight cylinder and its winding is thin and dense. The cross section of the cylinder can have an arbitrary shape and the ratio between the cylinder length l and its cross section diameter can also be arbitrary.

We use Eqn (12) to calculate the inductance of a long solenoid with edge effects taken into account. We assume that the solenoid has a round cross section with radius R and length $l \gg R$. In this case, the energy W_C can be represented as $W_C = 2W_1 + W_{12}$, where $W_1 = 2\sigma^2 R^3/3\varepsilon_0$ is the energy of a single uniformly charged disk, and the energy of interaction between two oppositely charged distant disks is

$$W_{12} \approx -\frac{q^2}{4\pi\varepsilon_0 l} = -\frac{\sigma^2\pi^2 R^4}{4\pi\varepsilon_0 l}.$$

These relations can be used to obtain W_C, which we then substitute in (12):

$$L \approx L_0\left(1 - \frac{8}{3\pi}\frac{R}{l} + \frac{1}{2}\frac{R^2}{l^2}\right). \quad (14)$$

Expression (14) gives the inductivity of a long solenoid with edge effects taken into account and coincides in the first- and second-order approximations with the expression obtained by directly solving a much more complicated magnetostatics problem [12].

Using (13), we can calculate the inductance of a short solenoid ($R \gg l$):

$$L = \mu_0 N^2 R\left(\ln\frac{8R}{l} - \frac{1}{2}\right). \quad (15)$$

However, in this case, the calculation of W_C involves elliptic integrals and the derivation is not so simple anymore (see the Appendix).

8. Electric field and capacitance of a capacitor

We consider a flat capacitor with the plate area S and separation l. At a remote point ($r \gg \sqrt{S}$), the electric field of the capacitor coincides with the field of a point dipole with the dipole moment $p_e = \varepsilon_0 SU$, where U is the voltage on the capacitor. The electrostatic field near a charged capacitor was studied in [13]. When the distance between the plates is much less than all other characteristic dimensions ($l \ll r, \sqrt{S}$), the boundary value problem can be solved and expressions for the electric field potential and strength can be obtained for points near the capacitors with round and rectangular plates.

The electromagnetic analogy method allows obtaining the main results in [13] much easier. A planar capacitor in the approximation $l \ll r, \sqrt{S}$ can be regarded as a double electric layer with the area S and dipole moment density $p_{1e} = ql/S = \varepsilon_0 U$, bounded by a contour L. It follows from (3) that the electric field outside the capacitor is

$$\mathbf{E} = \left(\frac{U}{\mu_0 I}\right)\mathbf{B}, \quad (16)$$

where $\mathbf{B}$ is the magnetic field of the current I running along the edges of the plates. Expression (16) implies that the $\mathbf{E}$ field lines of a planar capacitor with an arbitrary shape at a distance r much longer than l coincide with the $\mathbf{B}$ field lines of the current I that flows along the edges of the plates. Notably, close to the edges of a planar capacitor ($\sqrt{S} \gg r \gg l$), the lines of the electric field $\mathbf{E}$ are circles. A similar approach was used previously in [14], where it was shown that if constant potentials are set on a plane inside and outside some contour, then in the half-space bounded by this plane, the conventional Dirichlet problem can be reduced to the Biot–Savart problem.

We find the electric field on the symmetry axis of a planar capacitor with round plates. First, using expression (2), we calculate the magnetic field on the axis of a circular current loop

$$B = \frac{\mu_0 I}{2}\frac{R^2}{(z^2+R^2)^{3/2}},$$

and then, using relation (16), we can find the electric field on the capacitor axis

$$E = \frac{UR^2}{2(z^2+R^2)^{3/2}},$$

where z is the distance from the center of the plates to the observation point, measured along the symmetry axis, and R the plate radius. This expression coincides with the one obtained in [13]. If the capacitor plates are square, then, using the same method, we obtain the expression

$$E = \frac{Ua^2}{2\pi(z^2+a^2/4)\sqrt{z^2+a^2/2}}$$

for the electric field outside the capacitor on the z axis that passes through the centers of the plates (a is the square side).

We now estimate the correction to the planar capacitor capacitance due to edge effects. We first consider two uniformly charged plates, one of which has charge q and the other $-q$. The plates are located, as in a planar capacitor, parallel to each other at a distance l. The electric energy of such a system can be represented using (12) as

$$W_C = \frac{q^2}{2C_0}\left(1 - \frac{L}{L_0}\right). \quad (17)$$

If we now 'free' the charges by making the plates conductive, the charges redistribute such that each plate becomes equipotential. After this process, the electric energy of the system decreases and reaches the value $q^2/2C$, where C is the capacitance of the capacitor. Knowing that $q^2/2C < W_C$ and using expression (17), we can derive the inequality

$$C > \frac{C_0}{1-(L/L_0)}, \quad (18)$$

which can be represented in a symmetric way:

$$\frac{C_0}{C} + \frac{L}{L_0} < 1. \quad (19)$$

Inequalities (18) and (19), as well as expressions (12) and (13), are valid for any shape of the plates and any distance between them.

In the special case where the capacitor plates are round and the distance between them is much shorter than their radius R, it follows from (15) and (18) that

$$C > C_0\left[1 + \frac{l}{\pi R}\left(\ln\frac{8R}{l} - 0.5\right)\right].$$

The capacitance correction caused by edge effects can now be estimated as

$$\delta C = \frac{C - C_0}{C_0} > \frac{l}{\pi R}\left(\ln\frac{8R}{l} - 0.5\right).$$

This result is in agreement with the well-known Kirchhoff formula [15]

$$\delta C = \frac{l}{\pi R}\left(\ln\frac{16\pi R}{l} - 1\right) = \frac{l}{\pi R}\left(\ln\frac{8R}{l} + 0.84\right).$$

9. Conclusions

The electromagnetic analogy manifested by relation (3) can be compared to Ampère's theorem [1–4], which states the equivalence of the magnetic fields created by a magnetic sheet and a constant electric current flowing along the edge of this sheet. Conventionally, the magnetic sheet is a surface two sides of which are uniformly covered with respective 'north' and 'south' magnetic charges of the same surface density. Hypothetical magnetic charges are in many ways equivalent to electric ones and, in particular, satisfy 'Coulomb's magnetic law' [1–4], and hence it is not a surprise that there is an electrostatic analogue of Ampère's theorem.

The electromagnetic analogy method does not require complicated mathematics, and it can be justified without rigorous derivation of expression (1) by just noting the remarkable fact that the **E** and **B** fields of point electric and magnetic dipoles are similar, and their expressions differ by only a constant factor. This method can then be used to prove the theorem on the magnetic curl in magnetostatics at a level accessible to first year students.

Applying the electromagnetic analogy method to the investigation of a solenoid magnetic field revealed a number of interesting features. The magnetic field **B** outside a straight solenoid is similar to the electric field **E** of two oppositely charged plates located at the edges of the solenoid, while the magnetic field **B** inside the solenoid is similar to the electric field of these plates with an additional uniform field. These properties persist for any length of the solenoid and any shape of its cross section.

To summarize, the electromagnetic analogy method was used to obtain a formula that relates the magnetic energy of a solenoid and the electric energy of two oppositely charged plates located at the edges of the solenoid. This formula was used to find an expression for the inductance of a long solenoid with edge effects taken into account. The expression obtained coincides with the known formula in the first- and second-order approximations. The electromagnetic analogy method was also used to derive expressions for the electrostatic field outside a planar capacitor, which had previously been derived in a much more complicated way. An inequality was derived for the capacitance of a capacitor and the inductance of a coil with the same geometry, and it is valid for any shape of the plates and any distance between them. This inequality was used to estimate the correction to the capacitance of a planar capacitor due to edge effects.

Most probably, the electromagnetic analogy method can also be efficiently applied to other problems of electro- and magnetostatics.

10. Appendix. Inductance of a short solenoid

We calculate the energy W_C by considering the process of capacitor charging as the transfer of positive charge by infinitely small portions $\mathrm{d}q = 2\pi r\sigma\,\mathrm{d}r$ from the negative plate to the positive one. At each step, the transferred charge is assumed to be distributed with a constant density σ over a ring with radius r and thickness $\mathrm{d}r$. The energy increase is $\mathrm{d}W_C = (\varphi_A - \varphi_B)\,\mathrm{d}q$, where φ_A and φ_B are potentials at the edges of positively and negatively charged disks, each with the radius r, and $\varphi_B = -\varphi_A$. Due to the superposition principle, $\varphi_A = \varphi_1 - \varphi_2$, where $\varphi_1 = \sigma r/\pi\varepsilon_0$ is the potential at point *1* on the edge of a single uniformly charged disk with the radius r, and φ_2 is the potential of the field created by the same disk at point *2* located at a distance l from the disk plane and at the distance r from the disk axis (Fig. 7a).

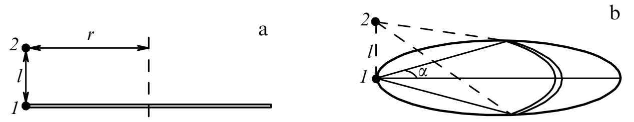

Figure 7. Derivation of expression (15).

By splitting the disk into annular zones centered at point *1* (Fig. 7b) and using the superposition principle, we obtain the expression for the potential at point *2*:

$$\varphi_2 = \varphi_1 \int_0^{\pi/2} \frac{\alpha \sin\alpha \cos\alpha}{\sqrt{\delta^2 + \cos^2\alpha}}\,\mathrm{d}\alpha\,,$$

where $\delta = l/2r$. After integrating by parts, this expression takes the form

$$\varphi_2 = \varphi_1\left(\mathrm{E}(r)\sqrt{1+\delta^2} - \frac{\pi\delta}{2}\right),$$

$$\mathrm{E}(r) = \int_0^{\pi/2}\sqrt{1 - \frac{\sin^2\alpha}{1+\delta^2}}\,\mathrm{d}\alpha\,.$$

The energy of two oppositely charged plates is given by the integral

$$W_C = 4\pi\sigma\int_0^R (\varphi_1 - \varphi_2)\,r\,\mathrm{d}r = W_0\left(1 + \frac{8R}{3\pi l} - \frac{8}{\pi l R^2}\int_0^R \mathrm{E}(r)\,r^2\sqrt{1+\delta^2}\,\mathrm{d}r\right),$$

where $W_0 = q^2/(2C_0)$ and $q = \sigma\pi R^2$. Changing the integration order in the right-hand side, we obtain

$$\frac{W_C}{W_0} = 1 + \frac{8R}{3\pi l} - \frac{l^2}{3\pi R^2\delta_0^3}\int_0^{\pi/2}\left[(\cos^2\alpha + \delta_0^2)^{3/2} - \delta_0^3\right]\frac{\mathrm{d}\alpha}{\cos^2\alpha}\,,$$

where $\delta_0 = l/(2R)$. The integration result is valid for any value of l/R, and it can be represented as

$$\frac{W_C}{W_0} = 1 + \frac{4}{3\pi\delta_0} - \frac{4}{3\pi}\left(\kappa\mathrm{D} + \frac{1}{\kappa}\,\mathrm{E}\right), \tag{20}$$

where D and E are the elliptic integrals

$$\mathrm{D} = \int_0^{\pi/2}\frac{\sin^2\alpha\,\mathrm{d}\alpha}{\sqrt{1 - k^2\sin^2\alpha}}\,, \quad \mathrm{E} = \int_0^{\pi/2}\sqrt{1 - k^2\sin^2\alpha}\,\mathrm{d}\alpha\,,$$

with $k = 1/(1+\delta_0^2)$ and $\kappa = \sqrt{1-k^2}$. We are interested in the case where $\delta_0 \ll 1$ and the elliptic integrals can be represented as [16]

$$\mathrm{E} \approx 1 + \frac{1}{2}\left(\lambda - \frac{1}{2}\right)\kappa^2\,, \quad \mathrm{D} \approx \lambda - 1 + \frac{3}{4}\left(\lambda - \frac{4}{3}\right)\kappa^2\,,$$

where $\lambda = \ln(4/\kappa)$ and expression (20), after some transformations, implies

$$\frac{W_C}{W_0} = 1 - \frac{l}{\pi R}\left(\ln\frac{8R}{l} - \frac{1}{2}\right).$$

Using (12), we can now easily obtain expression (15) for the inductance of a short solenoid. This formula coincides with the corresponding expression obtained by directly solving the magnetostatics problem [12].

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