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Operator derivation of the quasiclassical Green's function

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Contents

Abstract. The quasiclassical Green's function for the Dirac equation for an arbitrary localized potential is derived systematically using the Fock–Schwinger proper time method. The method essentially consists of exponentially parameterizing the propagator and disentangling the operator expressions. It allows calculating both the leading quasiclassical contribution and the first quasiclassical correction to the Green's function.

Keywords: proper time method, operator technique, Green's function, quasiclassical approximation

1. Introduction

Quantum electrodynamic (QED) processes in external fields can be conveniently described in terms of the Furry representation [1]. Because this representation uses Green's and wave functions in external fields to take field effects into consideration, obtaining convenient Green's function representations for wave equations in an external field becomes a major task. Simple representations significantly facilitate calculating cross sections for various processes in such fields, both analytically and numerically.

Exact Green's functions for relativistic equations in a spherically symmetric potential typically contain sums over angular momenta, which, in contrast to the nonrelativistic case, do not permit a closed-form expression. The presence of such sums greatly complicates the calculation of high-energy cross sections dominated by large (typically $l_{\rm char} \sim \varepsilon/m \gg 1$) angular momenta. The poor convergence of the angular momentum sums makes it challenging to use exact solutions. This is remarkably exemplified by the photoproduction of e^+e^- pairs in a Coulomb field, a process whose exact cross section was obtained as a sum over angular momenta [2]. Tabulating this expression is computationally challenging to the extent that only numerical results for photon energy $\omega \leqslant 12.5$ MeV have been obtained.

Fortunately, high-energy calculations can be done in the quasi-classical approximation, thus allowing external field effects to be introduced exactly. The idea is basically as follows. For higher-than-mass particle energies, the dominant contribution to the cross section comes from the small angles θ between the momenta of the initial and final particles, implying large values $l_{\rm char} \sim 1/\theta \gg 1$ ($\hbar = c = 1$) of the characteristic angular momenta that dominate the cross section. The quasiclassical Green's function includes the contribution from large ($l \gg 1$) angular momenta, but disregards the angular momenta of the order of unity, $l \sim 1$. The quasiclassical approach is remarkable in that it allows deriving not only the leading term but also the first quasiclassical correction (in θ). Developing the method of quasiclassical Dirac equation Green's functions for external atomic fields has led to a breakthrough in the theoretical description of fundamental QED processes in atomic fields. The reader is referred to Ref. [3] for a recent review of quasiclassical results.

The quasiclassical Green's function for the Coulomb field was first obtained in Refs [4, 5] (by directly summing partial contributions from large angular momenta) and for a spherically symmetric screened Coulomb potential in Refs [6, 7]. The Green's function for an arbitrary (not necessarily spherically symmetric) localized potential was obtained in [8] using a certain approximation when solving the differential equations. The first quasiclassical correction to the Coulomb potential Green's function was obtained in Ref. [9] from the exact Green's function by summing the partial contributions. Quasiclassical corrections to the Green's function for an arbitrary localized potential were obtained heuristically in Refs [8, 10].

In Ref. [11], the Green's function was evaluated in the leading quasiclassical approximation for the superposition of an atomic field and a laser field by applying the Fock–Schwinger proper time method and subsequently disentangling operators. The operator approach proved to be convenient and easy to use. In this paper, we discuss its application to evaluating the quasiclassical Dirac equation Green's function for an arbitrary atomic potential, including the first quasiclassical correction.

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2. Green's function derivation

The Dirac equation Green's function $G(\mathbf{r}, t\,|\,\mathbf{r}', t')$ in a static external potential $V(\mathbf{r})$ is the solution of the equation

$$(\hat{\mathcal{P}} - m)\, G(\mathbf{r}, t\,|\,\mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')\,, \tag{1}$$

where $\hat{\mathcal{P}} = \mathcal{P}_\mu \gamma^\mu$, γ^μ are the Dirac gamma matrices, and $\mathcal{P}_\mu = (\mathrm{i}\partial_t - V(\mathbf{r}), \mathrm{i}\boldsymbol{\nabla})$. The Green's function can be written as

$$G(\mathbf{r}, t\,|\,\mathbf{r}', t') = \left\langle \mathbf{r}, t \left| \frac{1}{\hat{\mathcal{P}} - m + \mathrm{i}0} \right| \mathbf{r}', t' \right\rangle. \tag{2}$$

It is shown in [12, 13] that the amplitudes of various QED processes are conveniently calculated using the Green's function of the squared Dirac equation $D(\mathbf{r}, t\,|\,\mathbf{r}', t')$. This Green's function is related to the usual Green's function $G(\mathbf{r}, t\,|\,\mathbf{r}', t')$ by

$$\begin{aligned} G(\mathbf{r}, t\,|\,\mathbf{r}', t') &= (\hat{\mathcal{P}} + m)\, D(\mathbf{r}, t\,|\,\mathbf{r}', t')\,, \\ D(\mathbf{r}, t\,|\,\mathbf{r}', t') &= \frac{1}{\hat{\mathcal{P}}^2 - m^2 + \mathrm{i}0}\, \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')\,. \end{aligned} \tag{3}$$

To use the Fock–Schwinger proper time method, we represent the propagator in the form

$$\frac{1}{\hat{\mathcal{P}}^2 - m^2 + \mathrm{i}0} = -\mathrm{i}\int_0^\infty \mathrm{d}s\, \exp\left[\mathrm{i}s(\hat{\mathcal{P}}^2 - m^2 + \mathrm{i}0)\right]. \tag{4}$$

For convenience, we express the propagator $\exp(-\mathrm{i}s\,\hat{\mathcal{P}}^2)$ as the product of two factors,

$$\begin{aligned} &\exp(\mathrm{i}s\,\hat{\mathcal{P}}^2) = L(s)\exp(\mathrm{i}s\,\hat{\mathcal{P}}_0^2)\,, \\ &L(s) = \exp(\mathrm{i}s\,\hat{\mathcal{P}}^2)\exp(-\mathrm{i}s\,\hat{\mathcal{P}}_0^2)\,, \\ &\hat{\mathcal{P}}^2 = \big(p_0 - V(\mathbf{r})\big)^2 - \mathbf{p}^2 + \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}V(\mathbf{r}) = \hat{\mathcal{P}}_0^2 - \tilde{V}(\mathbf{r})\,, \\ &\tilde{V}(\mathbf{r}) = 2p_0 V(\mathbf{r}) - \mathrm{i}\boldsymbol{\alpha}\boldsymbol{\nabla}V(\mathbf{r}) - V^2(\mathbf{r})\,, \quad \hat{\mathcal{P}}_0^2 = p_0^2 - \mathbf{p}^2\,, \end{aligned} \tag{5}$$

where $\boldsymbol{\alpha} = \gamma_0\boldsymbol{\gamma}$. $L(s)$ is the interaction operator, which becomes unity for $V(\mathbf{r}) = 0$. We note that, unlike $\mathbf{p}$ and p_0, the operator $\boldsymbol{\alpha}\boldsymbol{\nabla}$ in Eqn (5) acts only on the potential $V(\mathbf{r})$.

The operator $L(s)$ can be found using the differential equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}L(s) &= L(s)\exp(\mathrm{i}s\,\hat{\mathcal{P}}_0^2)\big[-\mathrm{i}\tilde{V}(\mathbf{r})\big]\exp(-\mathrm{i}s\,\hat{\mathcal{P}}_0^2) \\ &= -\mathrm{i}L(s)\tilde{V}(\mathbf{r} - 2s\mathbf{p})\,. \end{aligned} \tag{6}$$

Here, the following relation has been used:

$$\exp(-\mathrm{i}\,\beta\,\mathbf{p}^2)\, g(\mathbf{r}) = g(\mathbf{r} - 2\,\beta\,\mathbf{p})\exp(-\mathrm{i}\,\beta\,\mathbf{p}^2)\,. \tag{7}$$

The solution of Eqn (6) can be represented in the form of a chronologically ordered exponential,

$$L(s) = \mathrm{P}\exp\left(-\mathrm{i}s\int_0^1 \tilde{V}_x\,\mathrm{d}x\right), \tag{8}$$

where for any operator A, $A_x = A(\mathbf{r} - 2sx\,\mathbf{p})$.

Because $\tilde{V}_x$ and $\tilde{V}_y$ do not commute for $x \neq y$, transforming the chronologically ordered exponential into an ordinary exponential gives rise to an infinite series of commutators,

$$L(s) = \exp\left(-\mathrm{i}s\int_0^1 \tilde{V}_x\,\mathrm{d}x + \frac{s^2}{2}\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\,[\tilde{V}_x, \tilde{V}_y] + \ldots\right).$$

Each additional commutator is suppressed by a factor m/ε. This can be explained qualitatively as follows. Each additional commutator is proportional to Planck's constant $\hbar$, which arises from the commutation relation between the momentum and the corresponding coordinate. Because the exponential should contain a dimensionless quantity, a nondimensionalizing factor $1/(\hbar l)$ arises, where l is the dimensionless angular momentum of the electron. For ultrarelativistic particles with energy ε, the moment of momentum is typically $l_{\mathrm{char}} \sim \varepsilon/m$, giving $1/l \sim m/\varepsilon$.

To calculate the first quasiclassical correction, it suffices to consider only one commutator,

$$\begin{aligned} L(s) &= \exp\left(-\mathrm{i}s\int_0^1 \tilde{V}_x\,\mathrm{d}x + \frac{s^2}{2}\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\,[\tilde{V}_x, \tilde{V}_y]\right), \\ [\tilde{V}_x, \tilde{V}_y] &= 2\,\mathrm{i}s(x - y)\boldsymbol{\nabla}\tilde{V}_x\,\boldsymbol{\nabla}\tilde{V}_y - 2\mathrm{i}\,[\boldsymbol{\nabla}V_x \times \boldsymbol{\nabla}V_y]\,\boldsymbol{\Sigma}\,, \end{aligned} \tag{9}$$

where $\boldsymbol{\Sigma} = -\gamma^5\boldsymbol{\alpha}$, $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$.

The dominant contribution to the cross section for high energy processes comes from the following range of the arguments of the Green's function:

$$\begin{aligned} &\varepsilon_{\mathrm{char}} = (t - t')m^2 \gg m\,, \qquad z, z' \sim \frac{\varepsilon_{\mathrm{char}}}{m^2}\,, \\ &\rho, \rho' \sim \frac{1}{m}\,, \qquad t - z, t' - z' \sim \frac{1}{\varepsilon_{\mathrm{char}}}\,, \end{aligned}$$

where $\varepsilon_{\mathrm{char}}$ can be interpreted as a characteristic particle energy, the axis z is in the direction of $\mathbf{r} - \mathbf{r}'$, and $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are the coordinates of the vectors $\mathbf{r}$ and $\mathbf{r}'$ that are perpendicular to the z axis.

At this point, it is convenient to pass from the variables t and z to the light-cone variables $\phi = t - z$ and $T = (t + z)/2$. Then

$$\begin{aligned} &p^0 = \mathrm{i}\partial_t = -p_\phi - \frac{p_T}{2}\,, \qquad p^z = -\mathrm{i}\partial_z = -p_\phi + \frac{p_T}{2}\,, \\ &p_\phi = -\mathrm{i}\partial_\phi\,, \quad p_T = -\mathrm{i}\partial_T\,, \quad \mathbf{p}_\perp = -\mathrm{i}\partial_{\boldsymbol{\rho}}\,, \quad \hat{\mathcal{P}}_0^2 = 2p_\phi p_T - \mathbf{p}_\perp^2\,. \end{aligned} \tag{10}$$

Because

$$\begin{aligned} T &= z + O\left(\frac{1}{\varepsilon_{\mathrm{char}}}\right) = z\left(1 + O\left(\frac{m^2}{\varepsilon_{\mathrm{char}}^2}\right)\right), \\ T' &= z' + O\left(\frac{1}{\varepsilon_{\mathrm{char}}}\right) = z'\left(1 + O\left(\frac{m^2}{\varepsilon_{\mathrm{char}}^2}\right)\right), \end{aligned}$$

we can change the variables as $z \to T$ (in the potential V) and $\boldsymbol{\nabla} \to \boldsymbol{\nabla}_{\boldsymbol{\rho}} + \mathbf{e}_z\partial_T$. The dependence on ϕ and ϕ' is entirely contained in $\delta(\phi - \phi')$. Using the integral representation of Dirac's δ function, the Green's function $D(\phi,\, T,\, \boldsymbol{\rho}\,|\,\phi',\, T',\, \boldsymbol{\rho}')$ can be rewritten as

$$\begin{aligned} &D(\phi,\, T,\, \boldsymbol{\rho}\,|\,\phi',\, T',\, \boldsymbol{\rho}') \\ &\quad = \int\frac{\mathrm{d}\varepsilon}{2\pi}\exp\left[-\mathrm{i}\varepsilon(\phi - \phi')\right] D(T,\, \boldsymbol{\rho}\,|\,T',\, \boldsymbol{\rho}';\, \varepsilon)\,, \end{aligned} \tag{11}$$

where the function

$$\begin{aligned} &D(T,\, \boldsymbol{\rho}\,|\,T',\, \boldsymbol{\rho}';\, \varepsilon) \\ &\quad = -\frac{\mathrm{i}}{2\pi}\int_0^\infty \mathrm{d}s\, L(s)\exp\left[-\mathrm{i}s(2\varepsilon p_T + \mathbf{p}_\perp^2 + m^2)\right] \\ &\qquad \times \delta(\boldsymbol{\rho} - \boldsymbol{\rho}')\,\delta(T - T') \end{aligned} \tag{12}$$

is related to the fixed-energy Green's function by

$$D(\mathbf{r}\,|\,\mathbf{r}';\varepsilon)=\int \mathrm{d}(t-t')\exp\left[\mathrm{i}\varepsilon(t-t')\right]D(\mathbf{r},\,t\,|\,\mathbf{r}',\,t')$$
$$=\int \mathrm{d}(t-t')\exp\left[\mathrm{i}\varepsilon(t-t')\right]\int\frac{\mathrm{d}\varepsilon'}{2\pi}\exp\left[-\mathrm{i}\varepsilon'(\phi-\phi')\right]$$
$$\times D(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\,\varepsilon')=\exp\left[\mathrm{i}\varepsilon(z-z')\right]D(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\,\varepsilon)\,. \tag{13}$$

Here, we have used the fact that $D(\mathbf{r},t\,|\,\mathbf{r}',t')$ depends on t and t' only though $t-t'$, because $V(\mathbf{r})$ is independent of t. Because practical applications require knowing the Green's function $D(\mathbf{r}|\mathbf{r}';\varepsilon)$, we first calculate the function $D(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\,\varepsilon)$ and then use relation (13).

We proceed by applying the operator $\exp\left[-\mathrm{i}s(2\varepsilon p_T+\mathbf{p}_\perp^2)\right]$ in Eqn (12) to the δ function,

$$\exp\left[-\mathrm{i}s(2\varepsilon p_T+\mathbf{p}_\perp^2)\right]\delta(T-T')\,\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')$$
$$=-\frac{\mathrm{i}}{4\pi s}\,\delta(T-T'-2\varepsilon s)\exp\left[\mathrm{i}\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{4s}\right]. \tag{14}$$

Here, we have used the relations

$$\exp\left(-\mathrm{i}\tau p_T\right)f(T)=f(T-\tau)\exp\left(-\mathrm{i}\tau p_T\right), \tag{15}$$

$$\exp\left(-\mathrm{i}\beta\,\mathbf{p}_\perp^2\right)g(\boldsymbol{\rho})=\int\frac{\mathrm{d}\mathbf{q}}{\mathrm{i}\pi}\exp\left(\mathrm{i}q^2\right)g\left(\boldsymbol{\rho}+2\sqrt{\beta}\,\mathbf{q}\right), \tag{16}$$

which are valid for $\beta>0$ and for any functions $f(T)$ and $g(\boldsymbol{\rho})$, where $\mathbf{q}$ is a two-dimensional vector perpendicular to the z axis. The second, less trivial, relation can be proved using the Fourier transform of the function $g(\boldsymbol{\rho})$,

$$\exp(-\mathrm{i}\beta\,\mathbf{p}_\perp^2)\,g(\boldsymbol{\rho})=\exp\left(-\mathrm{i}\beta\,\mathbf{p}_\perp^2\right)\int\mathrm{d}\mathbf{q}\,\exp\left(\mathrm{i}\mathbf{q}\boldsymbol{\rho}\right)\tilde{g}(\mathbf{q})$$
$$=\int\mathrm{d}\mathbf{q}\exp\left(-\mathrm{i}\beta q^2+\mathrm{i}\mathbf{q}\boldsymbol{\rho}\right)\tilde{g}(\mathbf{q})$$
$$=\int\frac{\mathrm{d}\boldsymbol{\varrho}}{(2\pi)^2}\,g(\boldsymbol{\varrho})\int\mathrm{d}\mathbf{q}\,\exp\left[-\mathrm{i}\beta q^2+\mathrm{i}\mathbf{q}(\boldsymbol{\rho}-\boldsymbol{\varrho})\right]$$
$$=-\frac{\mathrm{i}}{\beta}\int\frac{\mathrm{d}\boldsymbol{\varrho}}{4\pi}\,g(\boldsymbol{\varrho})\exp\left[\frac{\mathrm{i}(\boldsymbol{\varrho}-\boldsymbol{\rho})^2}{4\beta}\right]. \tag{17}$$

Passing from the variable $\boldsymbol{\varrho}$ to $\mathbf{q}=(\boldsymbol{\varrho}-\boldsymbol{\rho})/(2\sqrt{\beta}\,)$, we obtain identity (16).

For compactness in what follows, instead of $L(s)$, we consider the operator

$$L_0(s)=\exp\left(-2\mathrm{i}s\varepsilon\int_0^1 V_x\,\mathrm{d}x\right), \tag{18}$$

which corresponds to the leading quasiclassical approximation for the Green's function of the Klein–Fock–Gordon equation. All of the algebra involved is also valid for $L(s)$, but leads to more cumbersome intermediate results.

Substituting $L_0(s)$ as given in Eqn (18) and Eqn (14) in Eqn (12), we obtain the Green's function for the Klein–Fock–Gordon equation

$$D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)$$
$$=-\int_0^\infty\frac{\mathrm{d}s}{4\pi s}\,\delta(T-T'-2\varepsilon s)\exp\left[\mathrm{i}\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{4s}-\mathrm{i}sm^2\right]$$
$$\times\exp\left[-2\mathrm{i}s\varepsilon\int_0^1 V\big(\boldsymbol{\rho}-x(\boldsymbol{\rho}-\boldsymbol{\rho}')-2sx\mathbf{p}_\perp,\,T-x(T-T')\big)\,\mathrm{d}x\right]. \tag{19}$$

Next, we use Eqn (7) to obtain

$$\int_0^1 V\Big(\boldsymbol{\rho}-x(\boldsymbol{\rho}-\boldsymbol{\rho}')-2sx\mathbf{p}_\perp,\,T-x(T-T')\Big)\,\mathrm{d}x$$
$$=\int_0^1\exp\left(-\frac{\mathrm{i}sx\mathbf{p}_\perp^2}{1-x}\right)V\Big(\boldsymbol{\rho}-x(\boldsymbol{\rho}-\boldsymbol{\rho}'),\,T-x(T-T')\Big)$$
$$\times\exp\left(\frac{\mathrm{i}sx\mathbf{p}_\perp^2}{1-x}\right)\mathrm{d}x\,. \tag{20}$$

The arguments of the Green's function $D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)$ for an ultrarelativistic energy ε typically have the values $|T|,|T'|\sim(\varepsilon/m^2)\gg|\rho|,|\rho'|\sim 1/m$. Because the operator $\mathbf{p}_\perp$ in Eqn (20) acts only on the atomic potential, it is only near the origin that the operator $\exp\left[\pm\mathrm{i}sx\mathbf{p}_\perp^2/(1-x)\right]$ contributes significantly. If T and T' have the same sign, then the particle does not fly near the center of the potential and the operators $\exp\left[\pm\mathrm{i}sx\mathbf{p}_\perp^2/(1-x)\right]$ can be replaced by unity, which corresponds to the eikonal approximation. For $T>0$ and $T'<0$, the above arguments allow replacing x by $x_0=T/(T-T')$ in the operator $\exp\left[\pm\mathrm{i}sx\mathbf{p}_\perp^2/(1-x)\right]$. Then, $D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)$ becomes

$$D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)=-\int_0^\infty\frac{\mathrm{d}s}{4\pi s}\,\delta(T-T'-2\varepsilon s)$$
$$\times\exp\left[\mathrm{i}\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{4s}-\mathrm{i}sm^2\right]\exp\left(-\mathrm{i}s\frac{x_0}{1-x_0}\,\mathbf{p}_\perp^2\right)$$
$$\times\exp\left[-2\mathrm{i}s\varepsilon\int_0^1 V\Big(\boldsymbol{\rho}-x(\boldsymbol{\rho}-\boldsymbol{\rho}'),\,T-x(T-T')\Big)\,\mathrm{d}x\right]. \tag{21}$$

Using relation (16), we obtain the following expressions for $D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)$:

$$D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)=\mathrm{i}\int_0^\infty\frac{\mathrm{d}s}{4\pi^2 s}\,\delta(T-T'-2\varepsilon s)$$
$$\times\exp\left[\mathrm{i}\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{4s}-\mathrm{i}sm^2\right]\int\mathrm{d}\mathbf{q}\,\exp\left(\mathrm{i}q^2-2\mathrm{i}s\varepsilon\int_0^1 V_x\,\mathrm{d}x\right),$$
$$V_x=V\Big(\boldsymbol{\rho}-x(\boldsymbol{\rho}-\boldsymbol{\rho}')+2\sqrt{x_0(1-x_0)s}\,\mathbf{q}\,,\,T-x(T-T')\Big). \tag{22}$$

We note that momentum operators are already absent from this expression. Integrating over s, we obtain the leading quasiclassical approximation for the Klein–Fock–Gordon equation Green's function $D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\,\varepsilon)$,

$$D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\,\varepsilon)=\frac{\mathrm{i}\theta(s_0)}{4\pi^2|T-T'|}\exp\left[\mathrm{i}\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')^2}{4s_0}-\mathrm{i}m^2 s_0\right]$$
$$\times\int\mathrm{d}\mathbf{q}\exp\left(\mathrm{i}q^2-2\mathrm{i}s_0\varepsilon\int_0^1 V_x\,\mathrm{d}x\right), \tag{23}$$

where $s_0=(T-T')/(2\varepsilon)$.

It is thus seen that the term proportional to $\mathbf{q}$ in the atomic potential arises because the operators $\boldsymbol{\rho}$ and $\mathbf{p}_\perp$ do not commute. Therefore, integration over the two-dimensional vector $\mathbf{q}$ can be interpreted as taking the quantum fluctuations near the rectilinear quasiclassical trajectory of an unltrarelativistic electron into account. We recall that the term proportional to $\mathbf{q}$ in the atomic potential should be omitted for $x_0<0$ and $x_0>1$; in this case, the quasiclassical Green's function changes to an eikonal one.

Similar arguments can be used to find the Green's function for the squared Dirac equation with the first

quasiclassical correction, giving

$$
\begin{aligned}
&D(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon) = d_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)\\
&\quad + \boldsymbol{\alpha}\,\mathbf{d}_1(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon) + \boldsymbol{\Sigma}\,\mathbf{d}_2(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)\,,\\
&d_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon) = D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)\\
&\quad \times\left[1 + \mathrm{i}4\varepsilon^2 s^3\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\,(x-y)\boldsymbol{\nabla}_\perp V_x\,\boldsymbol{\nabla}_\perp V_y\right],\\
&\mathbf{d}_1(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon) = -\frac{\mathrm{i}}{2\varepsilon}\boldsymbol{\nabla} d_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)\\
&\quad -\frac{s}{2\varepsilon}D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)\int_0^1 \mathrm{d}x\,\boldsymbol{\nabla}V_x^2\,, \qquad (24)\\
&\mathbf{d}_2(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon) = -\mathrm{i}s^2 D_0(T,\,\boldsymbol{\rho}\,|\,T',\,\boldsymbol{\rho}';\varepsilon)\\
&\quad \times\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\,[\boldsymbol{\nabla}V_x\times\boldsymbol{\nabla}V_y]\,,\\
&\boldsymbol{\nabla} = \boldsymbol{\nabla}_{\boldsymbol{\rho}} + \boldsymbol{\nabla}_{\boldsymbol{\rho}'} + \mathbf{e}_z(\partial_T + \partial_{T'})\,,\\
&V_x = V\big(\boldsymbol{\rho} - x(\boldsymbol{\rho}-\boldsymbol{\rho}') + 2\sqrt{x_0(1-x_0)s}\,\mathbf{q}\,,\;\; T - x(T-T')\big)\,,
\end{aligned}
$$

where V_y is obtained from V_x by replacing $x \to y$, and the function $D_0(T,\,\boldsymbol{\rho}|\,T',\,\boldsymbol{\rho}';\,\varepsilon)$ is defined in Eqn (23). We note that, unlike Eqn (9), the first quasiclassical correction to d_0 does not contain longitudinal gradients, because the term $\partial_T V_x \partial_T V_y$ cancels the term V_x^2 in $\tilde{V}_x$ after integration by parts [see Eqn (5)].

For practical calculations, it is convenient to change to space–time coordinates, leading to the following form of the Green's function for the squared Dirac equation:

$$
\begin{aligned}
&d_0(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) = \frac{\mathrm{i}\exp(\mathrm{i}\kappa r)}{4\pi^2 r}\\
&\times\int \mathrm{d}\mathbf{Q}\left\{1 + \frac{\mathrm{i}r^3}{2\kappa}\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\,(x-y)\boldsymbol{\nabla}_\perp V(\mathbf{R}_x)\boldsymbol{\nabla}_\perp V(\mathbf{R}_y)\right\}\mathcal{T},\\
&\mathbf{d}_1(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) = -\frac{\mathrm{i}}{2\varepsilon}\,\boldsymbol{\nabla} d_0(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) - \frac{\mathrm{i}\exp(\mathrm{i}\kappa r)}{16\pi^2\varepsilon^2}\\
&\quad\times\int \mathrm{d}\mathbf{Q}\int_0^1 \mathrm{d}x\,\boldsymbol{\nabla}V^2(\mathbf{R}_x)\,\mathcal{T}\,, \qquad (25)\\
&\mathbf{d}_2(\mathbf{r}_2,\mathbf{r}_1|\varepsilon) = -\frac{r\exp(\mathrm{i}\kappa r)}{16\pi^2\varepsilon^2}\\
&\quad\times\int \mathrm{d}\mathbf{Q}\int_0^1 \mathrm{d}x\int_0^x \mathrm{d}y\,\big[\boldsymbol{\nabla}V(\mathbf{R}_x)\times\boldsymbol{\nabla}V(\mathbf{R}_y)\big]\,\mathcal{T}\,,\\
&\mathcal{T} = \exp\left[\mathrm{i}Q^2 - \mathrm{i}r\int_0^1 \mathrm{d}x\,V(\mathbf{R}_x)\right],\quad \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1\,,\\
&\boldsymbol{\nabla} = \frac{\partial}{\partial\mathbf{r}_1} + \frac{\partial}{\partial\mathbf{r}_2}\,,\quad \mathbf{R}_x = \mathbf{r}_1 + x\mathbf{r} + \mathbf{Q}\sqrt{\frac{2r_1 r_2}{\kappa r}},\quad \kappa = \sqrt{\varepsilon^2 - m^2}\,.
\end{aligned}
$$

This result is the same as the one obtained in Ref. [10]. The accuracy of the quasiclassical Green's function (25) is as follows: the functions $d_0(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)$ and $d_1(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)$ are found including the first correction in m/ε and $\rho_{1,2}/r_{1,2}$, and the function $d_2(\mathbf{r}_2,\mathbf{r}_1|\varepsilon)$ is obtained in the first-order approximation, because this term is suppressed in itself. This accuracy is sufficient for evaluating the cross section of various QED processes with the first correction in m/ε_i and θ_i included, where ε_i is the particle energy and θ_i are the angles between the momenta of the initial and final particles.

3. Conclusion

The Fock–Schwinger proper time method has been used here to derive the quasiclassical Green's function for the Dirac equation in an arbitrary atomic potential with the inclusion of the first quasiclassical correction. Rather simple and user-friendly, the method also allows obtaining the high-energy Green's function for other external field configurations.

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