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Waves in inhomogeneous plasmas and liquid and gas flows. Analogies between electro- and gas-dynamic phenomena

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Contents

1. Introduction 748
2. Basic equations for equilibrium plasma and incompressible fluid 749
3. Simplest solutions for a plasma layer and their analysis 749
4. Behavior of solutions near the singular point 751
5. Plasma waves in the long-wavelength approximation 752
6. Fluid flow-induced waves in the long-wavelength approximation 753
7. Exact solution to the problem of incompressible fluid flow stability 753
8. Rayleigh theorem and some other assertions 754
9. Basic equations for nonequilibrium plasma and compressible fluid 756
10. Beam instabilities in plasma 757
11. Instabilities of inhomogeneous gas flow 759
12. Instability of plasma flows with inhomogeneous velocity 760
13. Conclusions 761
14. Appendix. Calculation of complex integrals 762

References 763

Abstract. **Two rather closely analogous classes of phenomena — plasma and hydrodynamic — are reviewed. Topics covered include: surface waves in an inhomogeneous plasma layer and in an inhomogeneous incompressible fluid flow; the instability of the tangential discontinuity of an inhomogeneous gas flow and the negative mass instability of an electron-beam-carrying plasma; radiative Cherenkov instabilities of an electron beam in a plasma and of a gas flow in a gas; the instability of plasma flows and of incompressible fluid flows with inhomogeneous velocity profiles. Special attention is given to the collisionless damping of plasma surface waves. It is shown that no such damping occurs for surface waves in gas flows.**

Keywords: plasma and hydrodynamic analogies, tangential discontinuity instability, negative mass instability, analog of collisionless plasma damping

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1. Introduction

A close analogy between electromagnetic phenomena in a plasma and mechanical phenomena in liquids and gases is sometimes clearly apparent. At first sight, this fact is not unexpected, because both the plasma and the liquids are continuous media that can be described by similar mathematical models. For example, a cold plasma is successfully described by equations of multifluid quasihydrodynamics where hydrodynamic equations (Euler's and of continuity) are written for each kind of charged particle constituting the plasma [1, 2]. As far as liquids and gases are concerned, hydrodynamic equations actually provide the sole tool for their theoretical description [3, 4].

With deeper insight, however, the similarity between plasma and hydrodynamic phenomena appears flimsy. Indeed, the main participant in the interaction between charged particles in a plasma is the self-consistent electromagnetic field created by induced plasma charges and currents. All the rest (collisions, gas-kinetic pressure) are subsidiary and do not characterize the essence of plasma as a specific constitutive medium. The main participants of liquid and gaseous media are certainly mechanical forces contributing to pressure. Moreover, it is not the kind of particle species that counts in liquids and gases, because there is no sense in talking about individual particles in these media. We mean here close physical analogies rather than an artificial or formal similarity between plasma and hydrodynamic phenomena, such as identical equations or similar wave frequency spectra. By way of example, one of the very important effects in the plasma is the stimulated Cherenkov radiation of plasma waves interpreted as a wave–particle type

of interaction [5–7]. It turns out that stimulated Cherenkov emission of acoustic waves by a gas flow is equally possible in gas dynamics, even if there are no wave–particle interactions in this medium. Of course, there are numerous plasma phenomena having no analogies whatever in gas dynamics. Suffice it to mention Landau damping directly related to Cherenkov radiation and absorption, which is a specific plasma phenomenon [1, 8].

We would like to discuss in the present article certain issues pertinent to plasma theory and gas dynamics, having both profound physical analogies and fundamental distinctions. We hope that such a discussion will be interesting to multidiscipline physicists, especially those engaged in continuous media research. An objective of this article being to focus on methodological issues, it may be of interest for those seeking a better understanding of the fundamentals of theoretical physics. It is opportune to note that some researchers thus far persevere in further attempts to develop a theory of electromagnetic fields based on the hydrodynamics of a hypothetical aether. We are not adherents of such an approach and would like to state at the very beginning that the present article is designed to consider altogether different issues.

2. Basic equations for equilibrium plasma and incompressible fluid

Potential waves in a flat layer of cold collisionless electron plasma are described by the following equation [9, 10]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\omega_{\mathrm{p}}^2(x)-\omega^2\right)\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right]-k_z^2\left(\omega_{\mathrm{p}}^2(x)-\omega^2\right)\varphi=0\,, \quad (2.1)$$

where $\omega_{\mathrm{p}}(x)\sim\sqrt{n_0(x)}$ is the electron Langmuir frequency, and $n_0(x)$ is the electron concentration distribution in a flat plasma layer infinitely extended in directions y and z. Wave $\varphi(x)\,\exp\left(-\mathrm{i}\omega t+\mathrm{i}k_z z\right)$ is assumed to propagate in the plasma layer along the z-axis, while equation (2.1) defines the complex amplitude $\varphi(x)$ of the wave scalar potential (ω is the wave frequency, and k_z is the wave number). Given that the plasma layer is placed between the grounded plates of a plane capacitor, equation (2.1) is supplemented by boundary conditions

$$\varphi(0)=\varphi(L)=0\,. \quad (2.2)$$

The solution to the boundary problem (2.1), (2.2) yields eigenfrequencies $\omega(k_z)$ and the corresponding eigenfunctions of plasma layer waves in the capacitor (planar waveguide). One more form of equation (2.1) may be needed from now on:

$$\left(\omega_{\mathrm{p}}^2(x)-\omega^2\right)\left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}-k_z^2\varphi\right)+\frac{\mathrm{d}\omega_{\mathrm{p}}^2}{\mathrm{d}x}\,\frac{\mathrm{d}\varphi}{\mathrm{d}x}=0\,. \quad (2.3)$$

Equations analogous to (2.1) are known in hydrodynamics, too. For example, the properties of plane-parallel streams of an incompressible fluid are described in the linear approximation by the Orr–Sommerfeld equation [11, 12]. In the case of a nonviscous fluid, this equation reduces to the known Rayleigh equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(U_0(x)-\frac{\omega}{k_z}\right)\frac{\mathrm{d}W}{\mathrm{d}x}-\frac{\mathrm{d}U_0}{\mathrm{d}x}\,W\right]-k_z^2\left(U_0(x)-\frac{\omega}{k_z}\right)W=0\,. \quad (2.4)$$

Equation (2.4) describes small perturbations in a fluid flow moving along the z-axis with unperturbed velocity $U_0(x)$. Perturbation is sought after in the form of a plane wave $W(x)\exp\left(-\mathrm{i}\omega t+\mathrm{i}k_z z\right)$ propagating along the flow, where $W(x)$ is the nonuniform perturbation amplitude of the velocity x-component. For a fluid flowing between absolutely hard plane-parallel walls, equation (2.4) is supplemented by the boundary conditions

$$W(0)=W(L)=0\,. \quad (2.5)$$

Yet another form of equation (2.4) may be needed:

$$\left(U_0(x)-\frac{\omega}{k_z}\right)\left(\frac{\mathrm{d}^2W}{\mathrm{d}x^2}-k_z^2W\right)-\frac{\mathrm{d}^2U_0}{\mathrm{d}x^2}\,W=0\,. \quad (2.6)$$

The boundary problem (2.4), (2.5) defines eigenfrequencies $\omega(k_z)$ of eigenwaves of a plane incompressible fluid flow. Despite the strong resemblance of boundary problems (2.1), (2.2) and (2.4), (2.5), they are essentially different.

3. Simplest solutions for a plasma layer and their analysis

It is immediately clear from equation (2.1) written as (2.3) that its general solution at $\omega_{\mathrm{p}}^2(x)=\omega_{\mathrm{p0}}^2=\mathrm{const}$ has the following form

$$\varphi(x)=A\exp\left(k_z x\right)+B\exp\left(-k_z x\right)\,, \quad (3.1)$$

where A and B are constants. Function (3.1) can, in principle, describe a certain surface wave in the plasma layer.[1] One more solution of equation (2.1) at a constant plasma frequency is given by the formulas

$$\omega^2=\omega_{\mathrm{p0}}^2\,,\qquad \varphi(x)=\varphi_0(x)\,, \quad (3.2)$$

where $\varphi_0(x)$ is the arbitrary function satisfying boundary conditions (2.2). Formulas (3.2) define a bulk Langmuir wave in a homogeneous cold electron plasma. As regards function (3.1), it can satisfy boundary conditions (2.2) only at $A=B=0$, but then $\varphi(x)\equiv 0$. Hence, an obvious result is that there are no surface waves in a plasma layer between the condenser plates, if the plasma density remains constant in the entire $(0,L)$ region.

Boundary conditions (2.2) can be satisfied if $\omega_{\mathrm{p}}^2(x)$ is a step function. Let, for instance,

$$\omega_{\mathrm{p}}^2(x)=\begin{cases}\omega_{\mathrm{p1}}^2\,, & 0<x<a\,,\\ \omega_{\mathrm{p2}}^2\,, & a<x<L\,.\end{cases} \quad (3.3)$$

Then, equation (2.3) gives two bulk Langmuir waves:

$$\begin{aligned}&\omega^2=\omega_{\mathrm{p1}}^2\,,\quad \varphi(x)=\begin{cases}\varphi_1(x)\,, & x<a\,,\\ 0, & a<x<L\,,\end{cases}\\ &\omega^2=\omega_{\mathrm{p2}}^2\,,\quad \varphi(x)=\begin{cases}0, & 0<x<a\,,\\ \varphi_2(x)\,, & a<x<L\,,\end{cases}\end{aligned} \quad (3.4)$$

co-existing in neighboring regions of the plasma layer ($\varphi_{1,2}(x)$ are the arbitrary functions satisfying the respective boundary conditions).

[1] A wave is called surface wave if its spatial structure in the direction perpendicular to propagation is given by $\exp\left(\pm k_z z\right)$ type functions with real k_z. In the case of imaginary k_z, the wave is termed a bulk wave [10, 13].

Moreover, equation (2.1) has solutions in the form of formula (3.1); they can be written, taking account of boundary conditions (2.2), as

$$\varphi(x) = \begin{cases} A \sinh(k_z x), & x < a, \\ B \sinh\big(k_z(x-L)\big), & a < x < L. \end{cases} \tag{3.5}$$

Solution (3.5) defines a surface wave. Because function $\omega_{\mathrm{p}}^2(x)$ has a discontinuity of the first kind at point $x = a$, equation (2.1) gives conditions for matching solutions (3.5) at this point:

$$\big\{\varphi(x)\big\}\big|_{x=a} = 0\,, \quad \left\{\big[\omega_{\mathrm{p}}^2(x) - \omega^2\big]\,\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\right\}_{x=a} = 0\,; \tag{3.6}$$

substituting expression (3.5) into (3.6) yields the following known expression for the frequency squared of the surface wave [10, 13]:

$$\omega^2(k_z) = \frac{\omega_{\mathrm{p1}}^2 \tanh\big(k_z(L-a)\big) + \omega_{\mathrm{p2}}^2 \tanh(k_z a)}{\tanh\big(k_z(L-a)\big) + \tanh(k_z a)}\,. \tag{3.7}$$

Now, let function $\omega_{\mathrm{p}}^2(x)$ be a piecewise constant (step) function. Namely, for $x \in G_j$, $j = 1, 2, \ldots$, we have $\omega_{\mathrm{p}}^2(x) = \omega_{\mathrm{p}j}^2 = \mathrm{const}$, where G_j denotes the continuous nonoverlapping subregions of the region $(0, L)$ partition. Evidently, the surface wave in the layer must somehow survive.[2] Its frequency is found from the boundary conditions (2.2) and the conditions for matching solutions (3.1) at the discontinuity points of function $\omega_{\mathrm{p}}^2(x)$ [see formulas (3.6)]. Bulk Langmuir waves remain as well with the frequencies and potentials given, in analogy with Eqn (3.4), by the relationships

$$\omega = \omega_{\mathrm{p}j}\,, \quad \varphi(x) = \begin{cases} \varphi_j(x), & x \in G_j, \\ 0, & x \notin G_j, \end{cases} \quad j = 1, 2, \ldots, \tag{3.8}$$

where $\varphi_j(x)$ are arbitrary functions.

Limiting transition $|G_j| \to 0$ leads from the discontinuous step function $\omega_{\mathrm{p}}^2(x)$ to a continuous dependence. That the surface wave must persist at such a transition appears from the following simple consideration. Let, for example, the plasma frequency be defined by the formula

$$\omega_{\mathrm{p}}^2(x) = \begin{cases} \omega_{\mathrm{p1}}^2, & 0 < x < a, \\ \omega_{\mathrm{p1}}^2 + (\omega_{\mathrm{p2}}^2 - \omega_{\mathrm{p1}}^2)\,\dfrac{x-a}{b-a}, & a < x < b, \\ \omega_{\mathrm{p2}}^2, & b < x < L, \end{cases} \tag{3.9}$$

instead of formula (3.3).

As $b \to a$, continuous function (3.9) turns into discontinuous function (3.3). It is clear from general considerations that spreading slightly the interface between plasmas of different densities cannot cause an appreciable change in frequency (3.7) and potential (3.5) of a real physical wave, at least till $k_z(b-a) \ll 1$. In what follows, we shall consider the limiting transition from distribution (3.9) to (3.3).

Further, in the limit $|G_j| \to 0$ for bulk Langmuir waves, we obtain the expression

$$\omega = \omega_{\mathrm{p}}(x_j)\,, \quad \varphi(x) = \psi(x_j)\,\tilde{\delta}(x - x_j)\,, \tag{3.10}$$

defining the frequencies and potential of plasma layer waves. Here, $\psi(x_j)$ is the arbitrary constant, $x_j \in (0, L)$ is the point to which the region G_j collapses as $|G_j| \to 0$, and $\tilde{\delta}(x)$ is any function other than zero only at point $x = 0$ and in its infinitesimally small vicinity (e.g., Dirac δ-function). Waves (3.10) are called local Langmuir waves of a continuous spectrum [14]. These waver will be referred to as Langmuir pseudowaves.[3] The spectrum continuity means that eigenfrequencies make up a continuous set of values $\omega_{\mathrm{p}}(x_j)$, $x_j \in (0, L)$, and $\tilde{\delta}(x - x_j)$ — the corresponding eigenfunction. Then, the eigenpseudowave has the form (taking into account time dependence)

$$\varphi_{\mathrm{S}}(t, x; x_j) = \tilde{\delta}(x - x_j)\,\exp\big(-\mathrm{i}\omega_{\mathrm{p}}(x_j)\,t\big)\,. \tag{3.11}$$

Using the definition of function $\tilde{\delta}(x)$, the arbitrary potential perturbation can be represented as an expansion into the eigenpseudowaves:

$$\varphi(t, x) = \int_0^L \psi(x_j)\,\varphi_{\mathrm{S}}(t, x; x_j)\,\mathrm{d}x_j = f(x)\exp\big(-\mathrm{i}\omega_{\mathrm{p}}(x)\,t\big)\,, \tag{3.12}$$

where $f(x)$ is the arbitrary function.

The presence of pseudowaves among normal perturbations of a plasma layer appears to be responsible for a specific mathematical peculiarity of equation (2.1) or (2.3). For $\mathrm{d}\omega_{\mathrm{p}}^2/\mathrm{d}x \neq 0$, equation (2.3) describes the surface wave interaction in the plasma layer with the Langmuir pseudowave of the continuous spectrum. As is known from the theory of coupled waves [16, 17], wave interactions are especially strong when the frequencies of waves (and wave numbers) are similar, as can be represented in our case in the form of equality $\omega(k_z) = \omega_{\mathrm{p}}(x)$ defining a certain resonance point in a plasma layer, which will be considered below.

The analogy between equations (2.1) and (2.4), as well as between (2.3) and (2.6), seems to allow the analysis of the simplest solutions for the plasma layer to be applied to a plane flow of an incompressible fluid. Indeed, if velocity $U_0(x)$ is a linear function, equation (2.6) has the solution in the form of a surface wave (3.1). In the case of a piecewise linear function $U_0(x)$, it is possible to find the solution of equation (2.6) satisfying the boundary conditions (2.5), i.e., the surface wave. It is also possible to introduce a local hydrodynamic wave (a flow pseudowave) with the continuous spectrum

$$\omega = k_z U_0(x)\,. \tag{3.13}$$

Here, the analogy ends.

The main difference between hydrodynamic equation (2.6) and its plasma counterpart (2.3) is that the second item on the left-hand side of equation (2.6) contains the second derivative $\mathrm{d}^2U_0/\mathrm{d}x^2$, whereas the second term on the left-hand side of equation (2.3) contains the first derivative $\mathrm{d}\omega_{\mathrm{p}}^2/\mathrm{d}x$. Suppose that the resonance condition for the flow surface wave and the hydrodynamic pseudowave (3.13), $\omega(k_z) = k_z U_0(x)$, is realized at a certain point $x_0 \in (0, L)$. At this point and in its vicinity, any continuous function $U_0(x)$ can be approximated by a linear dependence. But the second derivative of the linear function is zero, and the second item on the left-hand side of equation (2.6) vanishes, too. In other words, the resonant interaction between the surface wave of the flow and the hydrodynamic pseudowave (3.13) is absent.

[2] There will be as many surface waves as the discontinuity points of function $\omega_{\mathrm{p}}(x)$.

[3] The term 'pseudowave' was introduced in monograph [15] when considering resonant wave–particle interactions in a plasma.

This circumstance leads to profound consequences. It is certainly possible to go further and take into account the next terms in the expansion of function $U_0(x)$ near point x_0. The result will be discussed in Section 4. In what follows, we shall just as well consider solutions of the boundary problem (2.4), (2.5) with the piecewise linear function $U_0(x)$ [the analog of solutions (3.5), (3.7)].

4. Behavior of solutions near the singular point

For type (2.1) and (2.4) equations, the singular points are those at which the coefficient of the highest-order derivative vanishes. The singular points $x = x_0(\omega)$ are found from the equations

$$\omega_{\mathrm{p}}^2(x) - \omega^2 = 0\,, \tag{4.1}$$

$$U_0(x) - \frac{\omega}{k_z} = 0\,. \tag{4.2}$$

The singular points x_0 are also called resonance points, because relation (4.1) is the plasma resonance condition [18], while equality (4.2) is the hydrodynamic analog of the Cherenkov resonance known from electrodynamics [5, 19].

Let us consider the behavior of the solutions of equations (2.1) and (2.4) near resonance points, bearing in mind that, frequencies ω being unknown in advance, it is equally unknown whether the singular points belong to the $(0, L)$ region in which equations (2.1) and (2.4) are solved. True, this circumstance is of no significance for the formal investigation of solutions in the vicinity of the singular points.

Let us confine ourselves at the beginning to a case in which representations $\alpha(x - x_0)$, where α is a constant, hold for the left-hand sides of equations (4.1) and (4.2) near the resonance point. Due to this, equation (2.1) at $x \approx x_0$ is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x - x_0)\,\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right] - k_z^2(x - x_0)\,\varphi = 0\,. \tag{4.3}$$

The solution of equation (4.3) has the form

$$\varphi(x) = AI_0\big[k_z(x - x_0)\big] + BK_0\big[k_z(x - x_0)\big]\,, \tag{4.4}$$

where $I_0(x)$ and $K_0(x)$ are Infeld and Macdonald functions. The inequality $|k_z(x - x_0)| \ll 1$ (in the long-wavelength limit) being fulfilled, the solution (4.4) takes the form

$$\varphi(x) = \widetilde{A} + \widetilde{B}\,\ln\big[k_z(x - x_0)\big]\,, \qquad \widetilde{A}, \widetilde{B} = \mathrm{const}\,. \tag{4.5}$$

Solutions (4.4) and (4.5) are singular and have a logarithmic peculiarity. Moreover, these solutions, even at nonreal values of k_z, x, and x_0, are complex, since $\ln\big[k_z(x - x_0)\big] - \ln\big[k_z(x_0 - x)\big] = \pm\mathrm{i}\pi$.

Notice that the primary equation (2.1), as well as the simpler equation (4.3), have the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\,\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right) - \lambda q(x)\,\varphi = 0\,, \quad p(x_0) = 0\,. \tag{4.6}$$

Courses of mathematical physics and the theory of differential equations consider for equation (4.6) the boundary problem for eigenvalues λ (the Sturm–Liouville problem) [20] solved in the $x_0 < x < x_1$ (or $x_1 < x < x_0$) region under the condition of the boundedness of the solution at the singular point $x = x_0$ (the second condition is laid at point $x = x_1$). The boundedness condition means that the general solution of equation (4.6) must be written omitting its linearly independent solution, which turns into infinity at the singular point [i.e., assuming $B = 0$ in solution (4.4)]. A different situation takes place in the case of a plasma layer, namely equations (2.1) and (4.3) are solved on either side of the singular point x_0; in other words, the solution (4.4) must hold for both $x > x_0$ and $x < x_0$. In this case, the singular constituent of the solution cannot be dropped. It can be concluded that the traditional Sturm–Liouville problem for equation (4.6) and the boundary problem for a plasma layer are essentially different.

Equation (2.4) at $x \approx x_0$ transforms into

$$(x - x_0)\left(\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} - k_z^2 W\right) = 0\,. \tag{4.7}$$

The general solution of the last equation coincides with that of equation (3.1). Evidently, the solution of equation (4.7) has no singuliarities. In other words, the singular point for equation (2.4) is actually not such a point. To clarify this issue, the following expansion should be invoked near the singular point:

$$U_0(x) - \frac{\omega}{k_z} = \alpha(x - x_0) + \beta(x - x_0)^2\,. \tag{4.8}$$

Taking account of formula (4.8), equation (2.6) can be written in the form

$$(x - x_0)\left(\frac{\mathrm{d}^2 W}{\mathrm{d}x^2} - k_z^2 W\right) - 2\tilde{\beta} W = 0\,, \tag{4.9}$$

where $\tilde{\beta} = \beta/\alpha$ (the case of $\alpha = 0$ is disregarded). The general solution of equation (4.9) is expressed in terms of the hypergeometric function $\mathcal{U}(s, r, x)$ and the Laguerre function $\mathcal{L}(s, r, x)$ [21]:

$$\begin{aligned} W(x) = {} & A\exp\big[-k_z(x - x_0)\big]\,\mathcal{L}\left[-\frac{\tilde{\beta}}{k_z}, -1, 2k_z(x - x_0)\right] \\ & + B\exp\big[-k_z(x - x_0)\big]\,\mathcal{U}\left[\frac{\tilde{\beta}}{k_z}, 0, 2k_z(x - x_0)\right]. \end{aligned} \tag{4.10}$$

Solution (4.10) is regular at the singular point. Moreover, $\mathcal{U}$ is a complex-valued function. In the long-wavelength limit of $|k_z(x - x_0)| \ll 1$, solution (4.10) takes the form [cf. formula (4.5)]

$$W(x) = \widetilde{A}\,\tilde{\beta}(x - x_0) + \widetilde{B}\big\{1 + 2\tilde{\beta}(x - x_0)\,\ln\big[\tilde{\beta}(x - x_0)\big]\big\}\,. \tag{4.11}$$

Equation (2.4) has the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\,\frac{\mathrm{d}W}{\mathrm{d}x} - \frac{\mathrm{d}p(x)}{\mathrm{d}x}\,W\right) - \lambda q(x)\,W = 0\,, \quad p(x_0) = 0\,. \tag{4.12}$$

Although the last equation looks like (4.6), it possesses quite different properties. Specifically, in the statement of the boundary problem for equation (4.12), it is unessential whether or not the singular point x_0 is among inner points in the integration region. Therefore, wave properties of a plasma layer and a plane flow of an incompressible fluid are, in many respects, different (see Section 5).

5. Plasma waves in the long-wavelength approximation

In the long-wavelength approximation $k_z \to 0$, the second term on the left-hand side of equation (2.1) can be ignored. Then, a single integration of equation (2.1) yields

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = \frac{B}{\omega_{\mathrm{p}}^2(x) - \omega^2}\,. \tag{5.1}$$

From boundary conditions (2.2), it follows that $\varphi(L) - \varphi(0) = 0$. Hence, and from equation (5.1), one obtains the dispersion equation for determining frequency ω:

$$\int_0^L \frac{\mathrm{d}x}{\omega_{\mathrm{p}}^2(x) - \omega^2} = 0\,. \tag{5.2}$$

Equation (5.2) has a solution only if there is at least one singular point $x_0(\omega)$, i.e., the root of equation (4.1), in the $(0, L)$ region. But, if the domain of integration contains at least one singular point, the integral in formula (5.2) diverges. Therefore, the frequency must inevitably be complex. The sole case in which the frequency can be real is when there is coincidence of the singular point and the $\omega_{\mathrm{p}}^2(x)$ function discontinuity point (possible only in stepwise plasma filling).

Let us transform equation (5.2) or, rather, write it down in greater detail on the assumption that frequency ω has an infinitesimally small positive imaginary part determined by the causality principle positing that any perturbations in a system must disappear as $t \to -\infty$ [22]. Because the time dependence of perturbations was taken in the form $\exp(-\mathrm{i}\omega t)$, it should be assumed that $\omega \to \omega + \mathrm{i}\delta$, $\delta \to +0$. Then, using one of the Sokhotski formulas [22, 23], dispersion equation (5.2) can be reduced to the following form:

$$\mathrm{V.p.} \int_0^L \frac{\mathrm{d}x}{\omega_{\mathrm{p}}^2(x) - \omega^2} + \mathrm{i}\pi(\operatorname{sign}\omega') \left| \frac{\mathrm{d}\omega_{\mathrm{p}}^2}{\mathrm{d}x_0} \right|^{-1} = 0\,. \tag{5.3}$$

Here, the derivative of function $\omega_{\mathrm{p}}^2(x)$ is taken at point x_0. Clearly, dispersion equation (5.3) is complex even in the case of real frequency ω (because $\delta \to 0$) by virtue of the complexity of solutions (4.4) and (4.5) found on either side of the singular point. This means that complex dispersion equation (5.1) for $x_0(\omega) \in (0, L)$ defines complex frequencies $\omega = \omega' + \mathrm{i}\omega''$.

To recall, equation (5.3) is unsuitable for finding frequencies of strongly damped waves. Indeed, this equation is actually derived from exact equation (5.2) by the limiting transition $\operatorname{Im}\omega = \omega'' = \delta \to +0$. The case of a finite and negative imaginary part of the frequency needs to be considered separately. Therefore, let us return to equation (5.2). It was mentioned in a preceding paragraph that the causality principle suggests the presence of a positive imaginary part in frequency ω that can be either infinitesimally small or arbitrarily large, which means that the integral in dispersion equation (5.2) is defined only in the upper half of the complex plane $\omega = \omega' + \mathrm{i}\omega''$. To find damping wave frequencies, the integral in equation (5.2) should also be defined in the lower half-plane $\omega'' < 0$ using the known analytic continuation procedure proposed by L D Landau for the theory of Langmuir waves in a hot plasma [10, 22, 24]. Omitting standard details of analytic continuation (passage onto complex plane x and deformation of the integration contour)[4] leads to the final form of the dispersion equation for surface wave frequencies in an inhomogeneous plasma layer:[5]

$$\int_0^L \frac{\mathrm{d}x}{\omega_{\mathrm{p}}^2(x) - \omega^2} - \mathrm{i}\pi(1 - \operatorname{sign}\omega'')\operatorname{sign} x_0'' \left(\frac{\mathrm{d}\omega_{\mathrm{p}}^2}{\mathrm{d}x_0}\right)^{-1} = 0\,. \tag{5.4}$$

Here, the integration is performed along the real axis on the complex plane $x = x' + \mathrm{i}x''$, and the derivative is taken at the complex resonance point $x_0(\omega' + \mathrm{i}\omega'') = x_0' + \mathrm{i}x_0''$. Evidently, equation (5.4) as $\omega'' \to \pm 0$ gives equation (5.3).

The frequency found from equation (5.4) has a negative imaginary part (wave damping decrement). The damping mechanism is related to resonant excitation of the pseudowave (3.11) by a surface wave in the plasma layer. Collisionless damping is well known in the plasma theory, as exemplified by Landau damping of Langmuir waves in a homogeneous hot plasma [24]. In accordance with one of the modern interpretations, it is due to resonant excitation by a Langmuir wave with frequency $\omega \approx \omega_{\mathrm{p}}$ [cf. formula (3.11)] of a pseudowave

$$\tilde{f}(t, v; v_j) = \delta(v - v_j)\exp(-\mathrm{i}k_z v_j t) \tag{5.5}$$

at Cherenkov resonance: $\omega - k_z v = 0$. Here, v is the particle velocity, and $\tilde{f}$ is the perturbation of the particle velocity distribution function.

Consider here an example. Let the Langmuir frequency of plasma electrons be given by formula (3.9) at $\omega_{\mathrm{p1}}^2 = 0$ and $\omega_{\mathrm{p2}}^2 = \omega_{\mathrm{p0}}^2$, where ω_{p0} is constant. Formula (3.9) describes a plasma with the spread-out boundary $a < x < b$. The resonance point is given by the formula

$$x_0(\omega) = \frac{\omega^2}{\omega_{\mathrm{p0}}^2}(b - a) + a\,. \tag{5.6}$$

For a real frequency $\omega < \omega_{\mathrm{p0}}$, the resonance point lies between points a and b. Substituting expression (3.9) into dispersion equation (5.4) yields, after elementary integration and taking account of formula (5.6), the following equation

$$\frac{\left[L - (b - a)\right]\omega^2 - a\omega_{\mathrm{p0}}^2}{\omega^2(\omega_{\mathrm{p0}}^2 - \omega^2)} + \frac{b - a}{\omega_{\mathrm{p0}}^2}\left(\ln\frac{\omega^2 - \omega_{\mathrm{p0}}^2}{\omega^2} + 2\mathrm{i}\pi\right) = 0\,. \tag{5.7}$$

To find the analytical solution of equation (5.7), the quantity $\mu = (b - a)/L$ is assumed to be a small parameter. In the zero-order approximation in μ, the frequency satisfies the equality $\omega^2 = \omega_{\mathrm{p0}}^2 a/L$. Then, one obtains in the first-order approximation:

$$\omega^2 = \omega_{\mathrm{p0}}^2\,\frac{a}{L} - \omega_{\mathrm{p0}}^2\,\frac{b-a}{L}\,\frac{a}{L}\left(1 - \frac{a}{L}\right)\left(\ln\frac{L - a}{a} + \mathrm{i}\pi\right). \tag{5.8}$$

Decrement of surface plasma wave damping (5.8), $\omega'' \approx -\mathrm{i}\mu\omega_{\mathrm{p0}}$, originates from resonant excitation of the local Langmuir wave in the continuous spectrum by the surface wave.

[4] The most important mathematical details of Landau's analytic continuation pertinent to the problems of interest in the present article are presented in the Appendix.

[5] See formula (A.6) in the Appendix.

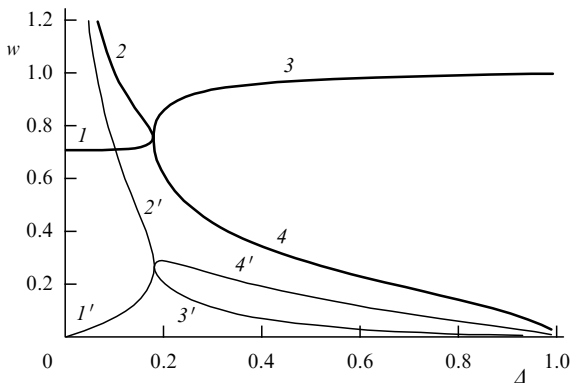

Figure 1. Complex dimensionless frequencies of surface waves in a plasma layer: solid curves (*1–4*) — real parts of the frequencies, and thin curves ($1'$–$4'$) — imaginary parts.

We have qualitatively considered in the foregoing the passage to the limit from a case with the continuous dependence of plasma density on coordinate x to a case of stepwise dependence. Such a passage to the limit is readily effected using formula (5.8). Assuming $b = a$ in formula (5.8) yields real frequency (3.7) if we put $k_z = 0$, $\omega_{\mathrm{p1}}^2 = 0$, and $\omega_{\mathrm{p2}}^2 = \omega_{\mathrm{p0}}^2$ in the latter.

Let us consider general solutions of dispersion equation (5.7). To reduce the number of free parameters, it is assumed that $a = (1 - \Delta)\, L/2$, $b = (1 + \Delta)\, L/2$, and equation (5.7) is written in the dimensionless form

$$\frac{w^2 - 1/2}{w^2(1 - w^2)} + \frac{\Delta}{1 - \Delta}\left(\ln\frac{w^2 - 1}{w^2} + 2\mathrm{i}\pi\right) = 0\,, \tag{5.9}$$

where $w = \omega/\omega_{\mathrm{p0}}$ is the dimensionless frequency. Numerical solutions of equation (5.9) are presented in Fig. 1. Clearly, there are three surface waves (curves *2*, *3*, *4*), besides the wave with frequency (5.8) (curve *1*), in different ranges of parameter values Δ; some of them are weakly damped waves.

The surface waves in the plasma layer considered above are rather well explored in a wide wavelength range [9, 25, 26]. Therefore, they are not considered in more detail in this article. Suffice it to analyze the general dispersion equation (5.2) and a concrete example (5.7).

6. Fluid flow-induced waves in the long-wavelength approximation

Let us move to considering a fluid flow, first in the long-wavelength approximation, by denoting $\omega/k_z = V$ and omitting the term proportional to k_z^2 in equation (2.4). Integration of the remaining equation by the variation of constants method gives

$$W(x) = A\big(U_0(x) - V\big) + B\big(U_0(x) - V\big)\int_0^x \frac{\mathrm{d}x'}{\big(U_0(x') - V\big)^2}\,,$$
$$0 < x < L\,. \tag{6.1}$$

Substituting the solution (6.1) into boundary conditions (2.5) yields the dispersion equation for the complex phase velocity $V = V' + \mathrm{i}V''$ of surface waves in the flow of an incompressible fluid in the long-wavelength approximation:

$$\int_0^L \frac{\mathrm{d}x}{\big(U_0(x) - V\big)^2} = 0\,. \tag{6.2}$$

Equation (6.2) is radically different from dispersion equation (5.2). Indeed, if the resonance point lies in the $(0, L)$ domain, the integral in the former is nonexistent, even in terms of a principal value (V.p.). Therefore, the roots of equation (6.2) are either complex or absent (except trivial roots $V = \pm\infty$). Moreover, if quantity $V = V' + \mathrm{i}V''$ is a root of equation (6.2), the complex conjugate quantity $V^* = V' - \mathrm{i}V''$ is a root, too. Consequently, a system for which equation (6.2) has solutions is unstable. This instability is analogous to numerous hydrodynamic instabilities in the plasma [10], such as the instability of a single-velocity cold electron beam in a cold plasma [27]. Dissipation described by the dispersion equation (5.2) is, in turn, analogous to many kinetic plasma phenomena, such as Landau damping of Langmuir waves. It should be noted that equation (6.2) excludes the analytic continuation problem. Indeed, integration of function $(x - x_0)^{-2}$ over x results in a function analytical on the entire complex plane x, barring the isolated singular point [6] $x = x_0$.

We confine ourselves to the simplest example of applying equation (6.2). Let the flow velocity be given by the formula

$$U_0(x) = U\begin{cases} 0, & 0 < x < a\,,\\ \dfrac{x - a}{b - a}\,, & a < x < b\,,\\ 1, & b < x < L\,,\end{cases} \tag{6.3}$$

where U is the constant. Substituting formula (6.3) into (6.2) yields, after simple integration, the following equation

$$\frac{a}{V^2} + \frac{L - b}{(U - V)^2} - \frac{b - a}{V(U - V)} = 0$$
$$\Rightarrow LV^2 - (a + b)\,UV + aU^2 = 0\,. \tag{6.4}$$

At $b = a$, the last equation gives

$$\omega_{1,2} = k_z U\,\frac{a}{L}\left(1 \pm \mathrm{i}\sqrt{\frac{L}{a} - 1}\right),$$
$$\text{at } L = 2a \quad \omega = k_z U\,\frac{1 \pm \mathrm{i}}{2}\,. \tag{6.5}$$

Complex frequency (6.5) describes the known instability of the tangential discontinuity in a flow of an incompressible fluid with a velocity jump [4]. At arbitrary values of a and b, the frequencies are described by the formula

$$\omega_{1,2} = k_z U\left[\frac{a + b}{2L} \pm \sqrt{\left(\frac{a + b}{2L}\right)^2 - \frac{a}{L}}\right]. \tag{6.6}$$

To recall, frequencies (6.6) turn out real if inequality $4aL < (a + b)^2$ is fulfilled. In this case, however, formula (6.6) retains meaning as well, despite the formal divergence of integral (6.2) (see Section 7).

7. Exact solution to the problem of incompressible fluid flow stability

In the case of a flow with the unperturbed velocity profile given by formula (6.3), the exact solution to the stability problem is easy to obtain [12]. Let us reproduce the essential steps for deriving the dispersion equation. For velocity profile (6.3), the solution of equation (2.6) satisfying boundary

[6] See formula (A.7) in the Appendix.

conditions (2.5) has the form

$$W(x)=\begin{cases} A\sinh(k_z x)\,, & 0<x<a\,,\\ C_1\exp(k_z x)+C_2\exp(-k_z x)\,, & a<x<b\,,\\ B\sinh\left[k_z(L-x)\right]\,, & b<x<L\,,\end{cases} \tag{7.1}$$

where $A, B, C_{1,2}$ are constants. Since function (6.3) at points $x=a$, $x=b$ undergoes discontinuities of the first derivative, equation (2.6) is invalid at these points and must be replaced by matching conditions. One of the conditions, namely continuity of the transverse velocity $W(x)$, follows from physical considerations. Another one ensues directly from equation (2.6) written in the form of equation (2.4) after integration over the infinitesimally small vicinity of the singular point. In the end, the matching conditions for the solution at discontinuity points of the derivative of function (6.3) look like

$$\left\{W(x)\right\}\big|_{x=a,b}=0\,,\quad \left[\left(U_0(x)-V\right)\frac{\mathrm{d}W}{\mathrm{d}x}-\frac{\mathrm{d}U_0}{\mathrm{d}x}\,W\right]_{x=a,b}=0\,. \tag{7.2}$$

It is easy to show that the second condition (7.2) follows as well from pressure continuity in the flow.

Substituting solution (7.1) into matching conditions (7.2) and excluding constants $A, B, C_{1,2}$ yield, after simple calculations, the dispersion equation for determining phase velocity $V(k_z)=\omega(k_z)/k_z$ of the flow:

$$\frac{(U-V)\left\{1-\coth\left[k_z(L-b)\right]\right\}+U_0/\left[k_z(b-a)\right]}{(U-V)\left\{1+\coth\left[k_z(L-b)\right]\right\}-U_0/\left[k_z(b-a)\right]} \times\frac{V\left[1-\coth(k_z a)\right]+U_0/\left[k_z(b-a)\right]}{V\left[1+\coth(k_z a)\right]-U_0/\left[k_z(b-a)\right]} =\exp\left[2k_z(b-a)\right]. \tag{7.3}$$

At $b=a$, equation (7.3) is considerably simplified:

$$V^2\tanh\left[k_z(L-a)\right]+(U-V)^2\tanh(k_z a)=0\,. \tag{7.4}$$

In the $k_z\to 0$ limit, dispersion equation (7.3) turns into equation (6.4), while equation (7.4) in the same limit gives frequencies (6.5).

Dispersion equation (6.4) can be just as well obtained directly, without the limiting transition, in the same manner as equation (7.3), by substituting into the matching conditions (7.2) the function

$$W(x)=\begin{cases} Ax\,, & 0<x<a\,,\\ C_1x+C_2\,, & a<x<b\,,\\ B(L-x)\,, & b<x<L\,,\end{cases} \tag{7.5}$$

which is the general solution of equation (2.6) with velocity (6.3) and boundary conditions (2.5) at $k_z=0$. Because no limitations are imposed in this case, as in the derivation of the general equation (7.3), dispersion equation (6.4) and formulas for frequencies (6.6) can be regarded as valid both at complex frequencies and when frequencies ω turn out real (see the remark at the end of Section 6).

We shall not write out cumbersome solutions of quadratic equation (7.3), confining ourselves to a single typical numerical example. Let us set parameters $U=1$, $L=2$, $a=0.9$, $b=1.1$ in relative units. The dispersion curves

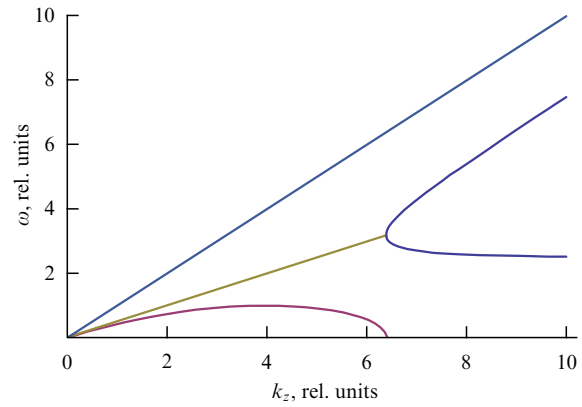

Figure 2. Dispersion curves of fluid flow waves with velocity profile (6.3).

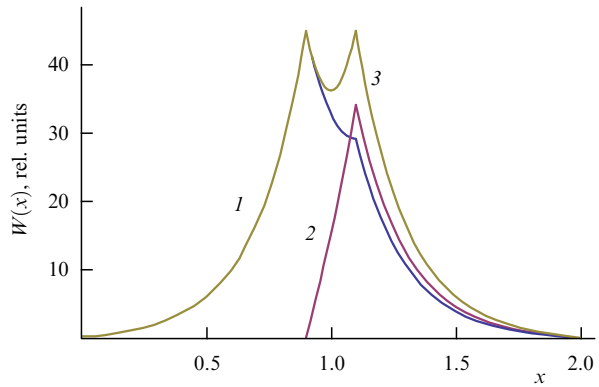

Figure 3. Velocity perturbation $W(x)$ in a fluid flow-induced wave at $k_z=5$. Re W — curve *1*, Im W — curve *2*, and $|W|$ — curve *3*.

$\omega(k_z)$ are presented in Fig. 2, where the straight (uppermost) line $\omega=k_z U$ is also drawn. The imaginary part of the frequency (lowest curve) differs from zero in the wave number range $0<k_z<k_0\approx 6.2$ (flow instability region). In a shorter wavelength region, the flow is stable and displays two surface waves with phase velocities lower than the maximum flow velocity U and higher than its minimal velocity (equaling zero in the example under consideration). The complex perturbation structure of the velocity $W(x)$ in a flow at $k_z=5$, which is typical of a surface wave, is presented in Fig. 3.

8. Rayleigh theorem and some other assertions

Let us divide both parts of equation (2.6) by $(U_0(x)-\omega/k_z)$, multiply by the complex conjugate solution $W^*(x)$, and integrate over x from 0 to L, taking account of boundary conditions (2.5). This gives

$$\int_0^L\left(\left|\frac{\mathrm{d}W}{\mathrm{d}x}\right|^2+k_z^2|W|^2\right)\mathrm{d}x =-\int_0^L\frac{\mathrm{d}^2U_0/\mathrm{d}x^2}{\left[U_0(x)-\omega/k_z\right]}\,|W|^2\,\mathrm{d}x\,. \tag{8.1}$$

Notice that the formula for integration by parts was applied to functions $W^*(x)$ and $\mathrm{d}W/\mathrm{d}x$ to obtain relationship (8.1), which at least suggests the absence of discontinuities of the

second kind in these functions in the $(0, L)$ domain. This situation was confirmed in Section 4 [formulas (4.10) and (4.11)]. Relation (8.1) is used to prove the known Rayleigh inflection-point theorem [11, 12], according to which the frequency $\omega(k_z)$ defined by equation (8.1) can have a nonzero imaginary part only under the condition that the second derivative $\mathrm{d}^2 U_0/\mathrm{d}x^2$ in the $(0, L)$ domain change sign, i.e., function $U_0(x)$ has an inflection point. The proof is based on the reality of the left-hand side of relationship (8.1).

The Rayleigh theorem can be slightly complemented. Let x_s be the inflection point of function $U_0(x)$. Suppose that the real singular point x_0 is the inner point of the $(0, L)$ domain. Then, the integral on the right-hand side of relationship (8.1) can be convergent only if the singular point and the point of inflection coincide. In other words, the presence of the inflection point makes possible the following real spectrum (neutral mode):

$$\omega = k_z U_0(x_\mathrm{s})\,. \tag{8.2}$$

In the absence of an inflection point, there can be no singular point in the $(0, L)$ domain and frequency ω can only be real in accordance with Rayleigh's theorem. Then, equation (6.2) has no solution. Consequently, the normal modes (eigenwaves) are absent in a fluid flow if the flow velocity profile $U_0(x)$ of the incompressible fluid has no point of inflection.

Let us consider a fluid flow with velocity profile (6.3) in the context of Rayleigh's theorem. Function (6.3) has the discontinuous first derivative, while its second derivative is naturally defined as

$$\frac{\mathrm{d}^2 U_0(x)}{\mathrm{d}x^2} = \frac{U}{b-a}\left(\delta(x-a) - \delta(x-b)\right). \tag{8.3}$$

Function (8.3) changes sign in the $(0, L)$ domain, but there is a whole domain of inflection instead of an inflection point, i.e., interval (a, b) in which the second derivative of velocity is identical to zero. Therefore, complex frequencies proved possible and were found as described in Sections 6 and 7 [see formulas (6.5), (6.6) and Fig. 2]. Next, it turns out that in the case of distribution (6.3), frequency (8.2) can be replaced by a whole 'cone' of frequencies:

$$0 = k_z U_0(a) < \omega < k_z U_0(b) = U\,. \tag{8.4}$$

Both real frequencies shown in Fig. 2 fall within this cone, in excellent agreement with formula (8.2).

To obtain a (8.1) type relation for a plasma layer, equation (2.2) should be used. Multiplying this equation by the complex conjugate solution $\varphi^*(x)$ and integrating it over x from 0 to L with due regard for boundary conditions (2.2) lead to

$$\int_0^L \left(\omega_\mathrm{p}^2(x) - \omega^2\right) \left|\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right|^2 \mathrm{d}x + k_z^2 \int_0^L \left(\omega_\mathrm{p}^2(x) - \omega^2\right) |\varphi|^2\, \mathrm{d}x = 0\,. \tag{8.5}$$

Because the plasma resonance point $x_0(\omega)$ given by equation (4.1) is inside the plasma layer, the first integral in equation (8.5) is logarithmically divergent [as follows from solutions (4.4) and (4.5)] and the second one is convergent; thus, expression (8.5) makes no sense whatever. The fact is that application of the formula for integration by parts to functions $\varphi^*(x)$ and $(\omega_\mathrm{p}^2 - \omega^2)\,\mathrm{d}\varphi/\mathrm{d}x$ (the former having a discontinuity of the second kind) is incorrect. Thus, there is no analog of the Rayleigh theorem for a plasma layer. Also, substituting expression (5.1) into (8.5) at $k_z = 0$ leads to dispersion equation (5.2), which makes sense (see above) only in the presence of an imaginary part in the frequency. This is fully applied to general relation (8.5). In any case, dispersion equation (5.2) and relation (8.5) taken together allow formulating the following alternative: either not bypassing the pole and understanding the divergent integrals in terms of a principal value or using the analytic continuation procedure for calculating the divergent integrals. The latter approach is currently universally accepted.

As was mentioned above, the differential equation of a plasma layer is a (4.6) type equation. Let $\varphi_{1,2}(x)$ be linearly independent solutions of this equation, with function $\varphi_1(x)$ being regular at the singular point x_0. It follows from equation (4.6) for the Wronskian of $\varphi_{1,2}(x)$ functions that

$$\varphi_1\varphi_2' - \varphi_2\varphi_1' = \varphi_1^2\left(\frac{\varphi_2}{\varphi_1}\right)' = \frac{C}{p(x)}\,, \tag{8.6}$$

where C is the constant, and the prime denotes differentiation with respect to x. Using relationship (8.6), it is possible to express the solution $\varphi_2(x)$ via $\varphi_1(x)$ and in the end write down the following general solution to equation (4.6) [20]:

$$\varphi(x) = A\varphi_1(x) + B\varphi_1(x)\int_0^x \frac{\mathrm{d}x'}{p(x')\,\varphi_1^2(x')}\,. \tag{8.7}$$

Substituting solution (8.7) into boundary conditions (2.2) yields the relationship

$$\int_0^L \frac{\mathrm{d}x}{p(x)\,\varphi_1^2(x)} = 0\,, \tag{8.8}$$

which should be regarded, taking account of the formula $p(x) = \omega_\mathrm{p}^2(x) - \omega^2$, as a generalization for the case of $k_z \neq 0$ of dispersion equation (5.2) for a plasma layer. The regular solution $\varphi_1(x)$ of equation (4.6) at $k_z = 0$ for the plasma layer is a constant, and equation (8.8) turns into equation (5.2). Because the function $\varphi_1(x)$ remains regular at any k_z, the imaginary part of dispersion equation (8.8) is given by residue of the integrand at the singular point x_0, where $p(x) = 0$. Therefore, surface waves in the layer of a smoothly inhomogeneous plasma inevitably damp down in any wavelength range. The imaginary part of dispersion equation (8.8) is determined by the analytic continuation method, just like the imaginary part in equation (5.4).

For the Wronskian of differential equation (4.12), one finds

$$W_1 W_2' - W_2 W_1' = W_1^2\left(\frac{W_2}{W_1}\right)' = C\,, \tag{8.9}$$

with both functions, $W_{1,2}(x)$, being regular in the $(0, L)$ domain (constant α in expansion (4.8) is nonzero). Using relationship (8.9), the general solution of equation (4.12) can be written down as

$$W(x) = AW_1(x) + BW_1(x)\int_0^x \frac{\mathrm{d}x'}{W_1^2(x')}\,. \tag{8.10}$$

Substituting solution (8.10) into the boundary conditions (2.5) yields

$$\int_0^L \frac{\mathrm{d}x}{W_\mu^2(x)} = 0\,. \tag{8.11}$$

Here, $W_\mu(x)$ is any of the linearly independent solutions of equation (4.12), i.e., $\mu = 1$ or $\mu = 2$. Moving from the general equation (4.12) to the concrete equation (2.4) gives evidence that one of the linearly independent solutions in the long-wavelength limit is $W_\mu(x) = U_0(x) - V$, with relationship (8.11) turning into (6.2). If frequency ω is real and the $(0, L)$ domain has no singular point, functions $W_\mu(x)$ are real. Then, equality (8.11) is impossible. We again return to the assertion that the boundary problem (2.4), (2.5) has no solutions in the absence of an inflection point in velocity $U_0(x)$.

9. Basic equations for nonequilibrium plasma and compressible fluid

The main feature of a nonequilibrium plasma is known to be its instability to certain minor perturbations. An important and interesting class of instabilities developed in the nonequilibrium plasma are those associated with excitation (emission) of electromagnetic eigenwaves. These are the so-called radiative instabilities [5, 28] considered below. A source of the plasma nonequilibrium state can be the directed motion of charged particles of one kind relative to particles of other kinds, i.e., beams of charged particles. As regards the presence of proper waves in a plasma, it has a variety of electromagnetic eigenwaves, including Langmuir waves, cyclotron waves, ion acoustic waves and so forth.

In the potential approximation to which we chiefly confine ourselves, the most general differential equation of the plasma layer is the following [9, 10]:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\varepsilon_\perp(\omega, x)\,\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right) - k_z^2\varepsilon_{||}(\omega, x)\,\varphi = 0\,. \tag{9.1}$$

Here, $\varepsilon_\perp(\omega, x)$ and $\varepsilon_{||}(\omega, x)$ are the transverse and longitudinal permittivities, respectively. Equation (2.1) is a special case of equation (9.1).

Let a plasma consist of two sorts of particles: cold immobile electrons with nonuniform density, and beam electrons with nonuniform density and velocity directed along the layer (movements of heavy ions are disregarded). Also let an external uniform magnetic field be applied along the layer. The beam is assumed to be nonrelativistic. Dielectric constants of such a layer are defined by the known formulas [10, 29]

$$\begin{aligned}\varepsilon_\perp(\omega, x) &= 1 - \frac{\omega_\mathrm{p}^2(x)}{\omega^2 - \Omega_\mathrm{e}^2} - \frac{\omega_\mathrm{b}^2(x)}{\left(\omega - k_z U_0(x)\right)^2 - \Omega_\mathrm{e}^2}\,,\\ \varepsilon_{||}(\omega, x) &= 1 - \frac{\omega_\mathrm{p}^2(x)}{\omega^2} - \frac{\omega_\mathrm{b}^2(x)}{\left(\omega - k_z U_0(x)\right)^2}\,,\end{aligned} \tag{9.2}$$

where $\omega_\mathrm{b}(x)$ is the Langmuir frequency of beam electrons, and Ω_e is the electron cyclotron frequency. Suppose that the plasma layer is bounded by grounded conducting planes with coordinates given by the equation $G_0(x) = 0$, while planes $G(x) = 0$ specify the coordinates of dielectric constant discontinuity points (9.2). The following boundary conditions take place on these planes:

$$\begin{aligned}&\varphi(x)|_{G_0(x)=0} = 0\,, \qquad \left\{\varphi(x)\right\}|_{G(x)=0} = 0\,,\\ &\left\{\varepsilon_\perp(\omega, x)\,\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\right\}_{G(x)=0} = 0\,.\end{aligned} \tag{9.3}$$

Let us consider the simplest plasma waves defined by the boundary problem (9.1), (9.3). Let $\omega_\mathrm{b}^2(x) \equiv 0$, $\omega_\mathrm{p}^2 = \omega_{\mathrm{p}0}^2 = \mathrm{const}$, and $G_0(x) = x \pm L$. In other words, the case in hand is a layer of a homogeneous magnetically active plasma as thick as $2L$ (or a planar plasma waveguide). The general solution of equation (9.1) has the form $A\sin(\chi x) + B\cos(\chi x)$, where $\chi^2 = -k_z^2\varepsilon_{||}/\varepsilon_\perp$. Substituting the general solution into first boundary condition (9.3) yields the dispersion equation for plasma layer frequencies:

$$\left(\frac{\pi n}{2Lk_z}\right)^2\left(1 - \frac{\omega_{\mathrm{p}0}^2}{\omega^2 - \Omega_\mathrm{e}^2}\right) + 1 - \frac{\omega_{\mathrm{p}0}^2}{\omega^2} = 0\,, \quad n = 1, 2, \ldots, \tag{9.4}$$

and the respective eigenfunctions

$$\varphi(x) = \begin{cases}\cos\left(x\,\dfrac{\pi n}{2L}\right), & n = 1, 3, 5, \ldots\,,\\ \sin\left(x\,\dfrac{\pi n}{2L}\right), & n = 2, 4, 6, \ldots\,.\end{cases} \tag{9.5}$$

The waves under consideration are divided in terms of the eigenfunction structure into odd and even in coordinate x $(n = 2, 4, \ldots)$ and $(n = 1, 3, \ldots)$, respectively. Their frequencies in the short-wavelength region $(k_z L \gg 1)$ are defined by the formulas

$$\omega = \sqrt{\omega_{\mathrm{p}0}^2 + \left(\frac{\pi n}{2Lk_z}\right)^2 \frac{\omega_{\mathrm{p}0}^2\Omega_\mathrm{e}^2}{\omega_{\mathrm{p}0}^2 - \Omega_\mathrm{e}^2}} \equiv \omega_1(k_z)\,, \tag{9.6}$$

$$\omega = \sqrt{\Omega_\mathrm{e}^2 + \left(\frac{\pi n}{2Lk_z}\right)^2 \frac{\omega_{\mathrm{p}0}^2\Omega_\mathrm{e}^2}{\Omega_\mathrm{e}^2 - \omega_{\mathrm{p}0}^2}} \equiv \omega_2(k_z)\,. \tag{9.7}$$

Formulas (9.6) and (9.7) are valid as long as the second items under the root sign are small compared with the first ones. In the opposite long-wavelength limit $k_z L \ll 1$, one finds

$$\omega = \frac{k_z 2L}{\pi n}\sqrt{\frac{\omega_{\mathrm{p}0}^2\Omega_\mathrm{e}^2}{\omega_{\mathrm{p}0}^2 + \Omega_\mathrm{e}^2}} \equiv \omega_1(k_z)\,, \tag{9.8}$$

$$\omega = \sqrt{\omega_{\mathrm{p}0}^2 + \Omega_\mathrm{e}^2 - \left(\frac{k_z 2L}{\pi n}\right)^2 \frac{\omega_{\mathrm{p}0}^2\Omega_\mathrm{e}^2}{\omega_{\mathrm{p}0}^2 + \Omega_\mathrm{e}^2}} \equiv \omega_2(k_z)\,. \tag{9.9}$$

In what follows, these waves are referred to as low-frequency [spectrum (9.8)] and high-frequency [spectrum (9.9)] plasma waves. Which of the frequencies, either (9.6) or (9.7), is higher depends on the relationship between frequencies $\omega_{\mathrm{p}0}$ and Ω_e, which does not matter in our case. It is shown below that waves in a plasma layer can be excited (emitted) by an electron beam passing through the layer.

To recall, the primary equation (9.1) is written down in the potential approximation; therefore, the expressions (9.6)–(9.9) for frequencies are valid only if inequalities $\omega_{1,2}(k_z)/k_z \ll c$ are satisfied, where c is the speed of light in a vacuum. The nonpotential theory of plasma waves (9.6)–(9.9), usually called oblique Langmuir waves or Trivelpiece–Gould modes, is fairly well elaborated [13, 16, 27] and not discussed here, because nonpotential effects are immaterial with respect to excitation of these waves by an electron beam. However, one more example from electrodynamics will be considered below, in which nonpotential effects are crucial.

Let us turn now to the formulation of the respective hydrodynamic equations. The equations for a fluid flow with the transversely nonuniform longitudinal velocity $U_0(x)$ are suitable for the purpose. It is the nonuniformity of flow velocity that serves as the source of the system's nonequilibrium state.[7] The fluid must necessarily be compressible, because an incompressible fluid has no eigenwaves excitable (emittable) under instability conditions. Therefore, as an analogue of the above electrodynamic equations of nonequilibrium plasma layer we take the following equation of a compressible fluid (or gas) plane flow [4, 30]:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(k_z^2-\frac{\varDelta^2(x)}{c_\mathrm{s}^2}\right)^{-1}\left(\varDelta(x)\,\frac{\mathrm{d}W}{\mathrm{d}x}+k_z\,\frac{\mathrm{d}U_0}{\mathrm{d}x}\,W\right)\right] - \varDelta(x)\,W=0\,, \tag{9.10}$$

where $\varDelta(x)=\omega-k_zU_0(x)$, and c_s is the speed of sound. In the case of an incompressible fluid, a passage to the limit takes place: $c_\mathrm{s}\to\infty$, and equation (9.10) turns into (2.4). It is convenient to write down equation (9.10) with respect to pressure perturbation $p(x)$ in the flow:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\varDelta(x)}\,\frac{\mathrm{d}p}{\mathrm{d}x}\right)-\frac{1}{\varDelta(x)}\left(k_z^2-\frac{\varDelta^2(x)}{c_\mathrm{s}^2}\right)p + k_z\,\frac{1}{\varDelta^2(x)}\,\frac{\mathrm{d}U_0}{\mathrm{d}x}\,\frac{\mathrm{d}p}{\mathrm{d}x}=0\,, \tag{9.11}$$

with pressure perturbation being linked with the transverse velocity by the formula

$$W=-\mathrm{i}\,\frac{1}{\varDelta(x)}\,\frac{\mathrm{d}p}{\mathrm{d}x}\,. \tag{9.12}$$

Equations (9.10) and (9.11) were obtained on the assumption that the speed of sound is independent of coordinate x.

Let us suppose that a gas flow is restricted in the transverse direction by absolutely hard walls with coordinates given by equations $G_0(x)=0$. The flow transverse velocity vanishes at these walls, which leads, taking account of formula (9.12), to the following boundary condition:

$$\left.\frac{\mathrm{d}p}{\mathrm{d}x}\right|_{G_0(x)=0}=0\,. \tag{9.13}$$

Let also planes $G(x)=0$ provide coordinates of $U_0(x)$-function discontinuity points (a flow with a piecewise inhomogeneous velocity profile). At these points, pressure perturbations and the transverse component of the flow velocity $W(x)$ must be continuous, namely

$$p(x)\big|_{G(x)=0}=0\,,\quad \left(\frac{1}{\varDelta(x)}\,\frac{\mathrm{d}p}{\mathrm{d}x}\right)_{G(x)=0}=0\,. \tag{9.14}$$

By way of example, let us find the solution of the boundary problem (9.11), (9.13), and (9.14) for a homogeneous gas layer with zero directed velocity $U_0(x)=0$ and $G_0(x)=x\pm L$ (a flat acoustic waveguide). The general solution of equation (9.11) can be written down in the form $A\sin(\chi x)+B\cos(\chi x)$, where $\chi^2=\omega^2/c_\mathrm{s}^2-k_z^2$. Substituting this solution into boundary conditions (9.13) yields the sound wave spectra of the acoustic waveguide:

$$\omega=c_\mathrm{s}\sqrt{k_z^2+\left(\frac{\pi n}{2L}\right)^2}\equiv\omega_\mathrm{s}(k_z)\,,\quad n=1,2,\ldots\,, \tag{9.15}$$

and the respective eigenfunctions

$$p(x)=\begin{cases}\sin\left(x\,\dfrac{\pi n}{2L}\right), & n=1,3,5,\ldots\\[2mm] \cos\left(x\,\dfrac{\pi n}{2L}\right), & n=2,4,6,\ldots\end{cases} \tag{9.16}$$

Sound waves (9.15) are the sole waves that can be emitted in the acoustic waveguide.

10. Beam instabilities in plasma

In the simplest case of a homogeneous plasma and a beam ($\omega_\mathrm{p}^2(x)=\omega_\mathrm{p0}^2$, $\omega_\mathrm{b}^2(x)=\omega_\mathrm{b0}^2$, $U_0(x)=U$), the dispersion equation for eigenfrequencies of the plasma–beam layer derived in the same way as equation (9.4) has the form

$$\left(\frac{\pi n}{2Lk_z}\right)^2\left(1-\frac{\omega_\mathrm{p0}^2}{\omega^2-\varOmega_\mathrm{e}^2}-\frac{\omega_\mathrm{b0}^2}{(\omega-k_zU)^2-\varOmega_\mathrm{e}^2}\right) + 1-\frac{\omega_\mathrm{p0}^2}{\omega^2}-\frac{\omega_\mathrm{b0}^2}{(\omega-k_zU)^2}=0\,. \tag{10.1}$$

Let us consider, assuming inequalities

$$\omega_\mathrm{b0}^2\ll\omega_\mathrm{p}^2(x),\ \varOmega_\mathrm{e}^2 \tag{10.2}$$

to be fulfilled, only such solutions of equation (10.1) that satisfy the condition

$$|\omega-k_zU|^2\ll\varOmega_\mathrm{e}^2\,. \tag{10.3}$$

Then, equation (10.1) can be represented as

$$D_0(\omega,k_z)=Q(\omega,k_z)\,\frac{\omega_\mathrm{b0}^2}{(\omega-k_zU)^2}\,, \tag{10.4}$$

where $D_0(\omega,k_z)$ is the dispersion function whose zeros determine eigenfrequencies of the plasma layer without a beam, and $Q(\omega,k_z)$ is some function not disappearing together with D_0. In our case, $D_0(\omega,k_z)$ coincides with the left-hand side of equation (9.4), and $Q=1$. Equation (10.4) also takes place in the case of a system with more complicated geometry [27, 31], for example, if a beam propagates in a channel with thickness $a\ll L$, then $Q\sim a\varphi^2(x_0)/L$, where $\varphi(x)$ is the eigenfunction (9.5), and x_0 is the beam average coordinate.

If k_zU does not coincide with any of the eigenfrequencies $\omega_{1,2}(k_z)$ of the plasma layer, solutions of equation (10.4), taking account of inequalities (10.2), have the form

$$\omega=k_zU\pm\omega_\mathrm{b0}\sqrt{\frac{Q(k_zU,k_z)}{D_0(k_zU,k_z)}}\,. \tag{10.5}$$

For a negative radicand, frequencies (10.4) are complex, which suggests instability of the plasma–beam system. This instability, unrelated to plasma wave emission, is called negative mass instability or beam instability in a medium

[7] To recall, the nonuniformity of electron longitudinal velocity, i.e., the presence of a beam, is the main source of the nonequilibrium state in a plasma.

with negative dielectric constant [27, 32]. The latter definition arises from the fact that the dispersion function D_0 in a boundless system ($k_z L \gg 1$) reduces to plasma longitudinal dielectric constant negative at $\omega \approx k_z U < \omega_{p0}$. For this special case, formula (10.5) takes the form

$$\omega = k_z U \pm \mathrm{i}\sqrt{\frac{\omega_{b0}^2}{\omega_{p0}^2}}\left(1 - \frac{U^2}{c_p^2}\right)^{-1/2} k_z U\,, \tag{10.6}$$

where $c_p = \omega_{p0}/k_z$ is the phase velocity of a Langmuir wave in a boundless plasma. Instability takes place only for $U < c_p$. It will be shown below that such instability is possible in a gas flow, too.

In the presence of Cherenkov resonance $\omega_{1,2}(k_z) = k_z U$ between the beam and any plasma wave, the solution of the dispersion equation (10.4) should be sought in the form

$$\omega = \omega_{1,2}(k_z) + \delta\omega = k_z U + \delta\omega\,. \tag{10.7}$$

Then, it is easy to obtain from equation (10.4) the following expression for the complex increment:

$$\delta\omega = \frac{-1 \pm \mathrm{i}\sqrt{3}}{2}\left[\omega_{b0}^2 Q_{1,2}\left(\frac{\partial D_0}{\partial \omega_{1,2}}\right)^{-1}\right]^{1/3}, \tag{10.8}$$

where function Q and the derivative of the dispersion function D_0 are taken at $\omega = \omega_1$ or $\omega = \omega_2$. Instability with increment (10.8) is known in the literature as the single-particle stimulated Cherenkov effect [28].

Substituting $k_z = \omega/U$ into dispersion equation (9.4) yields the equation for determining Cherenkov resonance frequency. There are two resonance frequencies. The higher one always exists; it is the frequency of resonance between a beam and a high-frequency wave (9.9). The other resonance frequency exists only if inequality

$$\sqrt{\frac{\omega_{p0}^2 \Omega_e^2}{\omega_{p0}^2 + \Omega_e^2}} > \frac{\pi n}{2L}\,U\,, \tag{10.9}$$

which can be written down as $U < c_p$, is satisfied. Here, $c_p = \omega_1(k_z)/k_z$ is the phase velocity of the low-frequency plasma wave (9.8). If $U > c_p$, the low-density beam in a layer of the magnetoactive plasma is resistant to excitation of a low-frequency plasma wave, as dictated by the characteristic dispersion law for this wave, i.e., the law of acoustic type dispersion.

Let us consider an example illustrating another law of dispersion of a wave emitted by an electron beam. Let there be a dielectric layer, e.g., a planar waveguide with dielectric filling instead of a plasma layer. Let us further suppose that the dielectric is isotropic and its permittivity ε_0 is frequency-independent. Because waves in such a dielectric are of a purely electromagnetic nature, the potential approximation is inapplicable. Suppose also that a beam propagates in a narrow channel $-a < x < a$ in the dielectric. Finally, let us assume that inequality (10.3) is satisfied. In the framework of these assumptions, the boundary problem for determining eigenfrequencies of the dielectric waveguide with the beam is formulated as [27]

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\varepsilon_\perp(x)}{\chi^2(\omega, x)}\,\frac{\mathrm{d}E_z}{\mathrm{d}x}\right) - \varepsilon_{||}(\omega, x)\,E_z = 0\,, \tag{10.10}$$

$$E_z(-L) = E_z(L) = 0\,.$$

Here, E_z is a component of the electric field strength, $\chi^2(\omega, x) = k_z^2 - \varepsilon_\perp(x)\,\omega^2/c^2$, $\varepsilon_\perp = \varepsilon_0$ in the dielectric volume and $\varepsilon_\perp = 1$ in the beam channel [for simplicity, the beam is assumed to be magnetized, i.e., $\Omega_e \gg \omega_{b0}$ in formula (9.2)], $\varepsilon_{||}(\omega, x) = 1 - \omega_{b0}^2/(\omega - k_z U)^2 \equiv \varepsilon_b$ in the beam channel, and $\varepsilon_{||}(\omega, x) = \varepsilon_0$ in the dielectric volume, while c is the speed of light).

A dispersion equation is derived from relationship (10.10) in the standard way. Omitting the simple derivation procedure, we arrive at

$$\frac{\varepsilon_0}{q_0}\cot\left[q_0(L - a)\right] - \frac{\sqrt{\varepsilon_b}}{q}\,\mathrm{Hth}\,(q\sqrt{\varepsilon_b}a) = 0\,, \tag{10.11}$$

where $q = \sqrt{k_z^2 - \omega^2/c^2}$, $q_0 = \sqrt{\omega^2/c_0^2 - k_z^2}$, $c_0 = c/\sqrt{\varepsilon_0}$, and $\mathrm{Hth}(x) = \coth(x)$ in the case of an odd wave, and $\mathrm{Hth}(x) = \mathrm{th}\,(x)$ for an even wave. Suppose that inequalities $a \ll L$ and $k_z a \ll 1$ indicating that the beam channel is narrow are fulfilled. It then follows from equation (10.11) that the beam does not interact with the odd wave in the zero-order approximation in a/L, because its field E_z in the beam channel is close to zero.

For the even wave, equation (10.11) is reduced to

$$\left[q_0 L \tan\,(q_0 L)\right]^{-1} = -\frac{a}{L}\,\frac{\omega_b^2}{\varepsilon_0(\omega - k_z U)^2}\,, \tag{10.12}$$

coinciding with formula (10.4) at $D_0(\omega, k_z) = \left[q_0 L \tan\,(q_0 L)\right]^{-1}$, $Q = -a/(L\varepsilon_0)$. The following equation for the frequency follows from (10.5):

$$\omega = k_z U \pm \mathrm{i}\sqrt{\frac{a}{L\varepsilon_0}} \times \omega_{b0}\sqrt{k_z L\sqrt{\frac{U^2}{c_0^2} - 1}\,\tan\left[k_z L\sqrt{\frac{U^2}{c_0^2} - 1}\right]}\,. \tag{10.13}$$

For $k_z L \ll 1$, formula (10.13) is simplified to

$$\omega = k_z U \pm \mathrm{i}\sqrt{\frac{a}{L\varepsilon_0}}\,\omega_{b0} k_z L\left(\frac{U^2}{c_0^2} - 1\right)^{1/2}. \tag{10.14}$$

The imaginary parts of frequencies (10.13) and (10.14) determine increments of beam nonresonant instability in the dielectric waveguide with respect to even perturbations (negative-mass type instability). Instability with increments (10.13) and (10.14) takes place only for $U > c_0$. For $U < c_0$, frequencies (10.13) are real and, together with frequencies of conventional electromagnetic waves slightly disturbed by the beam, form the total spectrum of the dielectric waveguide with the beam.

Formula (10.13) is invalid at $k_z L\sqrt{U^2/c_0^2 - 1} = \pi n/(2L)$, $n = 1, 3, \ldots$, which takes place at the Cherenkov resonance of a beam with an even electromagnetic wave associated with resonant emissive instability. The general expression for the increment of this instability is presented in formula (10.8). Hence, and from equation (10.12), we obtain for the increment the following expression

$$\delta\omega = \frac{-1 + \mathrm{i}\sqrt{3}}{2}\left[\frac{a}{L}\,\frac{\omega_{b0}^2}{\varepsilon_0\omega_0^2}\left(\frac{\pi n c_0}{2L\omega_0}\right)^2\right]^{1/3}\omega_0\,. \tag{10.15}$$

The resonance frequency is found as

$$\omega_0 = \frac{\pi n}{2L}\,c_0\left(1 - \frac{c_0^2}{U^2}\right)^{-1/2}. \tag{10.16}$$

Frequency (10.16) exists only for $U > c_0$, which is the beam instability condition in the dielectric waveguide (see above). If $U < c_0$, the beam in the waveguide with dielectric is stable by virtue of the characteristic wave dispersion law, i.e., the optical type dispersion law.

It stands to reason that we refer to the dielectric waveguide to avoid consideration of a purely plasma system. This will enable us to compare gas-dynamic phenomena with electromagnetic processes, which are more general than plasma ones only (see Section 11).

11. Instabilities of inhomogeneous gas flow

Let us consider a quiescent gas enclosed between hard surfaces with a homogeneous gas flow in its center, i.e., when the directed velocity distribution in a gas channel takes the form

$$U_0(x) = \begin{cases} 0, & x \in (-L, -a)\,, \\ U = \text{const}\,, & x \in [-a, a]\,. \\ 0\,, & x \in (a, L)\,. \end{cases} \tag{11.1}$$

It follows from the symmetry of distribution (11.1) with respect to plane $x = 0$ that gas perturbations fall into even (pressure — even function of x) and odd (pressure — odd function of x). Therefore, the solution of equation (9.11) in different regions of the gas channel cross section for $x > 0$ should be sought in the form

$$p(x) = \begin{cases} A \begin{cases} \sinh(q_1 x) \text{ — odd wave}, & 0 < x < a\,, \\ \cosh(q_1 x) \text{ — even wave}\,, & 0 < x < a\,, \end{cases} \\ B\cosh\left[q_2(L - x)\right], & a < x < L\,, \end{cases} \tag{11.2}$$

where $q_1^2 = k_z^2 - (\omega - k_z U)^2/c_s^2$, $q_2^2 = \omega^2/c_s^2 - k_z^2$. Solution (11.2) satisfies the last boundary condition (9.13). Substituting distributions (11.2) into the first two conditions (9.14) and excluding constants A and B yield the following dispersion equation [30]:

$$\frac{1}{\omega^2}\, q_2 \tan\left[q_2(L - a)\right] - \frac{1}{(\omega - k_z U)^2}\, q_1\, \mathrm{Hth}\,(q_1 a) = 0\,. \tag{11.3}$$

Function $\mathrm{Hth}(x)$ is the same as in equation (10.11).

Prior to analyzing equation (11.3) for our purposes, we present its solution for the case of an incompressible fluid ($c_s \to \infty$) in the form

$$\omega = \frac{k_z U}{1 \pm \mathrm{i}\sqrt{\mathrm{Hth}\,(k_z a)\coth\left[k_z(L - a)\right]}}\,. \tag{11.4}$$

The imaginary part of frequency (11.4) defines the increment of development of tangential discontinuity instability. Specifically, in the $k_z a \gg 1$, $k_z(L - a) \gg 1$ limits, Eqn (11.4) for complex frequency gives the known formula (6.5). Solutions in the form (11.4) to dispersion equation (11.3) exist at any relationship between the flow velocity U and the speed of sound c_s. In what follows, we shall consider other solutions describing instability differing from that of tangential discontinuity of the flow velocity.

Let us turn now to the analysis of equation (11.3) proper, assuming that the gas flow thickness is small in comparison with the entire channel size, i.e., $a \ll L$, and that the following inequalities are fulfilled:

$$k_z a \ll 1\,, \quad \frac{a|\omega|}{c_s} \ll 1\,. \tag{11.5}$$

Then, dispersion equation (11.3) is simplified. For an odd wave, one finds

$$\left(q_2 L \tan(q_2 L)\right)^{-1} = \frac{a}{L}\,\frac{(\omega - k_z U)^2}{\omega^2}\,, \tag{11.6}$$

and for an even wave, the dispersion relation reads as

$$q_2 L \tan(q_2 L) = \frac{a}{L}\, q_1^2 L^2\, \frac{\omega^2}{(\omega - k_z U)^2}\,. \tag{11.7}$$

At $a = 0$, when the gas in the entire channel is quiescent, dispersion equations (11.6) and (11.7) are reduced to $\cos(q_2 L) = 0$ and $\sin(q_2 L) = 0$, which entails frequency spectra (9.15) of odd and even sound waves propagating along the planar acoustic waveguide.

For $a \neq 0$, equations (11.6) and (11.7) have complex roots, besides real solutions. Equation (11.6) acquires a complex frequency associated with the instability of tangential discontinuity in the acoustic waveguide with a gas flow relative to odd perturbations:

$$\omega = \pm\mathrm{i}\sqrt{\frac{a}{L}}\, k_z U \sqrt{k_z L \tanh(k_z L)}\,. \tag{11.8}$$

Of course, this formula enters general expression (11.4). Dispersion equation (11.6) does not describe other instabilities. This result follows from the fact that pressure perturbation in the odd wave in plane $x = 0$, i.e., immediately in the gas flow, vanishes. We have already encountered a similar situation when considering the odd waves in the dielectric waveguide with an electron beam.

Let us turn now to the analysis of the dispersion equation for even waves (11.7). It is immediately clear that equations (11.7) and (10.12) are alike. The difference between their left-hand sides is attributable to different eigenfunctions of dielectric and acoustic waveguides, which in turn is due to the difference between boundary conditions (10.10) and (9.13). The difference between the right-hand sides arises from different mechanisms underlying the interaction of the electron beam and the gas flow with the waveguide: in one case, these are electromagnetic field forces, whereas in the other mechanical pressure forces. For $a/L \ll 1$, one of the solutions to equation (11.7) has the form

$$\omega = k_z U \times \left(1 \pm \mathrm{i}\sqrt{\frac{a}{L}}\sqrt{\frac{k_z L}{(1 - U^2/c_s^2)^{1/2}\,\tanh\left[k_z L(1 - U^2/c_s^2)^{1/2}\right]}}\right). \tag{11.9}$$

For $k_z L \ll 1$, expression (11.9) is simplified:

$$\omega = k_z U \pm \mathrm{i}\sqrt{\frac{a}{L}}\left(1 - \frac{U^2}{c_s^2}\right)^{-1/2} k_z U\,. \tag{11.10}$$

Evidently, for $U < c_s$, one of the frequencies (11.10) has a positive imaginary part, suggesting the development of tangential discontinuity instability with respect to even

perturbations in the acoustic waveguide with a gas flow. Comparing formulas (11.10) and (10.6) gives evidence of a strong resemblance between instability of tangential discontinuity in a gas and instability of the electron beam in a plasma with negative dielectric constant.

Moreover, it is beyond argument that formulas (11.9) and (11.10), on the one hand, and formula (10.13) and (10.14), on the other hand, have much in common. The main difference lies in the fact that instability with increment (11.10) takes place only for $U < c_s$, whereas the dielectric waveguide with the beam is on the whole stable, when the analogous inequality $U < c_0$ is fulfilled. To recall, dispersions of electromagnetic waves in a dielectric waveguide and sound waves in an acoustic waveguide relate to the same optical type.

However, frequencies (11.9) in certain ranges of wave numbers k_z happen to be complex, even if the inequality $U > c_s$ is satisfied. In other words, there is one more instability of tangential discontinuity that can be regarded as analogous to instability with increment (10.13). Moreover, at $k_z L\sqrt{U^2/c_s^2 - 1} = \pi n/(2L)$, $n = 2, 4, \ldots$, formula (11.9) becomes inapplicable. It happens when there is Cherenkov resonance of the gas flow with an even sound wave of the acoustic waveguide, when the resonant emissive instability develops in the system. Substituting

$$\omega = \omega_s(k_z) + \delta\omega = k_z U + \delta\omega\,, \tag{11.11}$$

where $\omega_s(k_z)$ is given by formula (9.15) at even n, into dispersion equation (11.7) yields for increment $\delta\omega$:

$$\delta\omega = \frac{-1 + \mathrm{i}\sqrt{3}}{2}\left(\frac{a}{L}\,\frac{k_z^2 c_s^2}{\omega_0^2}\right)^{1/3}\omega_0\,. \tag{11.12}$$

The resonance frequency is defined by the formula

$$\omega_0 = \frac{\pi n}{2L}\,c_s\left(1 - \frac{c_s^2}{U^2}\right)^{-1/2} \equiv \omega_{0n}\,. \tag{11.13}$$

Instability with increment (11.12) develops at each even in x transverse mode of the nonequilibrium acoustic waveguide. This instability is due to Cherenkov emission of sound waves in a gas by the supersonic gas flow. It can be shown that the increment defined by formula (11.12) is the maximally possible one for $a \ll L$ [27, 32]. It follows from relationship (11.11) that instability with increment (11.12) develops when the resonance condition $\omega_s(k_z) = k_z U$ (a known condition of Cherenkov radiation) is fulfilled, and ω_{0n} is the frequency of emitted waves. Formulas (10.15), (10.16) and (11.12), (11.13) are more than outwardly similar: there is a deep physical analogy between them.

Cherenkov radiation in a medium is possible only if the condition

$$V_{\max} > U > V_{\min} \tag{11.14}$$

is fulfilled, where U is the radiator speed, and $V_{\max}$ and $V_{\min}$ are the maximum and minimum phase velocities of waves in the medium. In a plasma waveguide, $V_{\min} = 0$ and $V_{\max} = \infty$ at high-frequency radiation (9.9) or is calculated from formula (9.8) in the case of low-frequency radiation. In a dielectric waveguide, $V_{\max} = \infty$ and $V_{\min} = c_0$. In an acoustic waveguide, $V_{\max} = \infty$ and $V_{\min} = c_s$. There is a deep analogy between emission of electromagnetic waves by a beam of charged particles in electrodynamics and emission of acoustic waves by a gas flow in gas dynamics (see above). However, this is true only for the stimulated Cherenkov radiation effect. The spontaneous Cherenkov radiation effect is known in electrodynamics [33]. It is actually the effect that was discovered and explained in Ref. [19] and consists in creating a field by a moving charge that can detach itself from the charge in the medium. There is no spontaneous emission of a sound in gas dynamics analogous to spontaneous charge emission in electrodynamics. The supersonic motion of a body traveling in a gas is accompanied by emission of the gas flow created by the body. Emission of the flow is actually the stimulated emission.

12. Instability of plasma flows with inhomogeneous velocity

Let us consider one more instability of nonequilibrium plasma, having an analogy in incompressible fluid hydrodynamics. What is meant is instability of a plasma flow with inhomogeneous velocity (slipping instability) [29, 34]. This instability is due to slipping electron layers relative to one another. It is assumed that an external magnetic field is absent. Therefore, equation (9.1) can be applied, in which [29]

$$\varepsilon_\perp(\omega, x) = \varepsilon_{||}(\omega, x) = \varepsilon(\omega, x) = 1 - \frac{\omega_b^2(x)}{(\omega - k_z U_0(x))^2}\,. \tag{12.1}$$

In the long-wavelength approximation, the second term on the left-hand side of equation (9.1) can be disregarded, which leads to the following dispersion equation

$$\int_0^L \frac{\mathrm{d}x}{\varepsilon(\omega, x)} = 0 \Rightarrow \int_0^L \frac{\left[V - U_0(x)\right]^2}{\left[V - U_0(x)\right]^2 - \omega_b^2(x)/k_z^2}\,\mathrm{d}x = 0\,, \tag{12.2}$$

where $V = \omega/k_z$. Notice that the second term on the left-hand side of Eqn (9.1) is ignored on the assumption of fulfilment of inequality

$$k_z L \ll 1\,, \tag{12.3}$$

without imposing any substantial limitations on the permissible values of k_z and ω in dispersion equation (12.2). It refers to the full extent to dispersion equations (5.2) and (6.2). In the quasistatic limit

$$\lim_{k_z \to 0} V(k_z) = \text{const}\,, \tag{12.4}$$

to which we confine ourselves, dispersion equation (12.2) reduces to the form

$$\int_0^L \left[V - U_0(x)\right]^2 \omega_b^{-2}(x)\,\mathrm{d}x = 0\,. \tag{12.5}$$

Solutions of equation (12.5) can be only complex and, more than that, only complex conjugate. In other words, dispersion equation (12.5) describes definitively unstable systems.

Let electron density be constant and the velocity profile be given by formula (6.3). Substituting formula (6.3) into (12.5) yields the following dispersion equation:

$$LV^2 + (a + b - 2L)\,UV - \frac{1}{3}\,(a + 2b - 3L)\,U^2 = 0\,. \tag{12.6}$$

This equation does not coincide with equation (6.4). Therefore, an incompressible fluid flow and an electron flow, even in the quasistatic limit, have different sets of normal perturbations. In particular, an electron flow is always unstable, unlike a flow of an incompressible fluid, which is stable at certain relationships between a, b, and L. In a special case of $a = b = L/2$, equations (12.4) and (6.4) coincide and their solutions are expressed as given by formula (6.5). In another special case, $a = 0$, $b = L$ (electron flow with a linear velocity profile), the solution of equation (12.4) is described by the formula

$$V = \frac{\omega}{k_z} = \frac{1}{2}\, U\left(1 \pm \mathrm{i}\frac{1}{\sqrt{3}}\right), \tag{12.7}$$

and the incompressible fluid flow is stable at $a = 0$, $b = L$.

For a piecewise constant permittivity of the electron flow in a waveguide, it is easy to obtain the dispersion equation, even at an arbitrary value of parameter $k_z L$. Let, for example, the permittivity of a flow at $x \in (0, a)$ be ε_1, and at $x \in (a, L)$ be ε_2. Then, the dispersion equation takes the form

$$\varepsilon_1 \tanh\left[k_z(L-a)\right] + \varepsilon_2 \tanh\left(k_z a\right) = 0\,,$$
$$\varepsilon_{1,2} = 1 - \frac{\omega_{\mathrm{b}1,2}^2}{\left(\omega - k_z U_{1,2}\right)^2}\,. \tag{12.8}$$

It seems that the analog of equation (12.8) in the hydrodynamics of an incompressible fluid should be dispersion equation (7.4). However, these equations appear to have nothing in common, because there is no analog of displacement current in hydrodynamics. In the quasistatic long-wavelength limit, when $|\omega - k_z U_{1,2}| \ll \omega_{\mathrm{b}1,2}$ and the polarization current is much higher than the displacement current, equation (12.8) takes the form

$$\omega_{\mathrm{b}1}^2 (U_2 - V)^2 \tanh\left[k_z(L-a)\right] + \omega_{\mathrm{b}2}^2 (U_1 - V)^2 \tanh(k_z a) = 0\,. \tag{12.9}$$

At $\omega_{\mathrm{b}1} = \omega_{\mathrm{b}2}$, $U_1 = 0$, and $U_2 = U$, equations (7.4) and (12.9) have an identical structure, even if they do not coincide completely. Equation (12.9), like (7.4), has only complex roots, i.e., the electron flow with a velocity jump is definitively unstable. The similarity of dispersion equations for a fluid flow and electron flow with a velocity jump has physical causes. Both tangential discontinuity instability and slipping instability are due to transverse displacement of the flow under the action of pressure forces ($\sim \mathrm{d}W/\mathrm{d}x$) or electrical forces ($\sim \mathrm{d}\varphi/\mathrm{d}x$). The energetic source of these instabilities is the energy of relative movements of flow layers. Therefore, an electron flow with the inhomogeneous velocity profile in the quasistatic long-wavelength limit is analogous to an inhomogeneous incompressible fluid flow. However, this inference is not universal, because much depends on the velocity profile.

Here is one more characteristic example. Let the flow velocity profile have the form

$$U_0(x) = U\left(\frac{x}{L}\right)^2, \quad -L < x < L\,. \tag{12.10}$$

In hydrodynamics, a flow of a fluid with such a velocity profile is called the Poiseuille flow. It follows from Rayleigh's theorem that such a flow is stable, as is easily seen after substituting velocity profile (12.10) into (6.2): the resulting dispersion equation has no solutions whatever. That the statement of stability of the Poiseuille flow of a nonviscous fluid is in conflict with experimental findings was recognized only after the work of Werner Heisenberg, who considered the problem taking into consideration viscosity [11, 12]. In the case of an electron flow, substituting formula (12.10) into equation (12.5) yields

$$V^2 - \frac{2}{3}\,V + \frac{1}{5} = 0 \rightarrow V_{1,2} = \frac{1}{3}\left(1 \pm \mathrm{i}\,\frac{2}{\sqrt{5}}\right), \tag{12.11}$$

which suggests instability of the electron flow with the 'Poiseuille' velocity profile. This means that the analogy between the hydrodynamics of an incompressible fluid flow and the electrodynamics of an inhomogeneous plasma flow holds only up to a certain limit: some physical processes in plasma and fluid (gas) flows are very similar and are described by similar equations, but the concrete solutions may be essentially different.

13. Conclusions

To sum up, the results of the above comparative analysis of plasma and gas-dynamic phenomena allow the following conclusions:

(1) The principal properties of a quasineutral plasma are due to the self-consistent electromagnetic field that evolves from quasineutrality breaking. In the simplest potential case, the strength of the self-consistent electric field is proportional to the integral of the density perturbation of charges induced in a plasma (in accordance with the Poisson equation). In other words, the forces acting in the plasma are determined by integrals of density perturbations of the particles composing the plasma. Pressure forces determined by derivatives of density perturbations prevail in neutral gases and fluids. Despite the different physical natures of forces in gases and plasmas, they are similarly determined by linear operators on density perturbations. This accounts for numerous analogies between plasma and gas-dynamic phenomena. Certainly, the application of integral and differential operators to the same function in a multidimensional case may lead to quite different results which, in turn, accounts for sometimes different plasma and gas-dynamic phenomena.

(2) In both a transversely inhomogeneous layer of an equilibrium electron plasma and a transversely inhomogeneous flow of an incompressible neutral gas, surface waves similar in terms of dispersion law and transverse structure of perturbations (of the field and velocity) may occur. Equally possible are local bulk waves with a continuous spectrum (pseudowaves). The resonant interaction between surface and local bulk waves in the plasma leads to collisionless damping of the former, whereas there is no analogous damping of surface waves in a gas flow. Such a global result follows from the distinction (insignificant at first sight) between differential equations describing the plasma layer and gas flow, i.e., equations (2.1) and (2.4).

(3) One of the most important properties of a plasma is the existence of electromagnetic waves in it with phase velocities lower than the speed of light in a vacuum. The same property is inherent in many other constitutive media (or electrodynamic systems). An electron moving in a medium with a velocity higher than the minimal phase velocity of the wave emits such a wave. The mechanism of this emission, known as Vavilov–Cherenkov radiation, consists in medium polarization by an electron that thereby creates an electro-

magnetic field falling behind (becoming detached from) the moving electron. It gives rise to a free electron-unrelated field, i.e., radiation. Emission of an individual electron is spontaneous. In the presence of an electron flow in a medium, stimulated Cherenkov radiation develops, regarded in the plasma theory as one of the forms of beam instability. An analog of slow electromagnetic waves in plasma and gases and compressible fluids are acoustic waves. A gas flow traversing a gas with a velocity higher than the minimal phase velocity of an acoustic wave causes sound emission, analogous to stimulated Cherenkov radiation. Such emission can be interpreted as some hydrodynamic instability. In this case, the resemblance between plasma electrodynamics (and even linear electrodynamics of constitutive media with dispersion as a whole) and gas dynamics is so strong that the dispersion equations for increments of instabilities and increments themselves almost coincide. However, there is no spontaneous sound emission by a moving body, resembling spontaneous electron emission in gas dynamics: the body, unlike an electron, does not create a field. In the case of supersonic movement of a body in a gas, the gas flow created by a moving body emits. This is stimulated emission.

(4) One of the best known hydrodynamic instabilities is tangential discontinuity instability in a gas or fluid flow with a velocity jump. Under certain conditions, instability takes place even in the absence of a jump (discontinuity) in the flow velocity. Analogous instabilities are known in transversely inhomogeneous plasma flows (beams of charged particles). Unlike fluid and gas flows, plasma flows are always unstable, irrespective of the degree of velocity inhomogeneity. The especially striking difference between inhomogeneous plasma and gas flows emerges from a comparison of the main dispersion equation of tangential discontinuity instability (6.2) and the dispersion equation of slipping instability (12.5) in the long-wavelength approximation. One more plasma analog of hydrodynamic tangential discontinuity instabilities is nonresonant nonemissive beam instabilities of negative mass (instabilities in a medium with negative dielectric permittivity).

We tried to show that electrodynamic and hydrodynamic phenomena have radically different physical natures, despite the similarity in their mathematical descriptions. Soon after the appearance of the Maxwell theory, followed by the electrodynamics of continuous media, researchers sought to reduce electrodynamic phenomena to gas-hydrodynamic ones. The cause was the presence of the linear spectrum of electromagnetic waves in dispersion-free media, similar to the spectrum of acoustic waves with the phase velocity of light (on the order of 10^8 m s^{-1}) much higher than the speed of sound in gases and liquids ($10^2 - 10^4$ m s^{-1}). True, the exists of sound in media only if their compressibility is taken into account. Scientists then invented an incomprehensible aether, imparting to it supernatural properties that varied as experimental electrodynamics developed. The most important of them was the wave spectrum identified with sound, because the speed of light in the aether is the speed of sound conditioned by aether compressibility. However, no single real body with a finite mass proved capable of developing a speed higher than the speed of sound in the aether (or the speed of light in a vacuum); even the elastic wave energy transfer rate in gases and fluids must always be lower than the speed of light in a vacuum.

The first surprise came from experiments by Irving Langmuir [35–39] in 1925 that demonstrated the emergence of high-frequency oscillations in a gas-discharge plasma with a long-wavelength limit in frequency independent of electron density and charge. No such oscillations were observed in neutral particle gases and fluids. Everything suggested collective particle–particle interactions in a plasma governed by long-range forces, the remote action of which was apparent at distances much greater than the mean interparticle distance (in the case of a gas, the notion of particle being inapplicable to fluids).

In 1929, I Langmuir and Lewi Tonks [40–42] proposed a simple mechanical theory of plasma in which interactions between particles are determined by the electromagnetic fields they generate. In other words, these authors constructed a simple plasma theory with self-consistent field interaction totally neglecting pairwise collisions lying at the bottom of Ludwig Boltzmann's kinetic theory of gases [43].

Langmuir and Tonks explained low-frequency oscillations in a nonisothermal plasma with hot electrons; in such a plasma, pressure is determined by the electron component (electron temperature), while inertia by the ion component (ion mass). Such oscillations also have a limiting frequency, but in the short-wavelength limit, and are just as well determined by charged particle density and electron temperature (and pressure). Unlike high-frequency oscillations, named Langmuir oscillations after their original discoverer, low-frequency waves are called ion-acoustic waves.

In these Conclusions, we intentionally departed from the subject matter of the article in order to emphasize that various 'similar' phenomena in plasma and gases have basically different physical natures, despite the aforementioned analogies and mathematical likeness of their description.

14. Appendix. Calculation of complex integrals

Integrals in the form

$$J_n(w) = \int_a^b \frac{f(z)}{(z-w)^n}\,\mathrm{d}z\,, \tag{A.1}$$

where w is a complex variable, $n = 1, 2$, and integration is over a segment of the real axis $a < z < b$, frequently occur in the theory of plasma and hydrodynamics. The limits of integration can also be infinite: $a = -\infty$ and $b = +\infty$. In the kinetic theory of plasma, integrals (A.1) emerge in the calculation of dielectric permittivities, with w being proportional to the frequency, z is the velocity, and $f(z)$ is the distribution function. In the theory of inhomogeneous plasma, integrals (A.1) appear in the integration of field equations [see formula (5.2)], with z being the spatial coordinate, w proportional to the frequency acquired, and $f(z)$ determined by geometry. In the hydrodynamics of inhomogeneous flows, the (A.1) type integral appears in equation (6.2). Let us assume function $f(z)$ to be analytical over the entire complex plane $z = z' + \mathrm{i}z''$, barring sections $z' < a$ and $z' > b$ along the real axis $z'' = 0$. It is possible to set $f(z) \equiv 0$ in sections and calculate integral (A.1) within the infinite limits.

Let integral $J_n(w)$ be an analytical function in the upper half-plane $w'' > 0$ of complex variable $w = w' + \mathrm{i}w''$. Let us consider the problem of analytical continuation of function $J_n(w)$ into the lower half-plane $w'' < 0$. This problem arises from the nonexistence of integral $J_n(w)$ for real values of $w \in [a, b]$. We can only talk about the limit

$$\lim_{\substack{w''\to\pm 0\\ w'\in[a,b]}} J_n(w' + \mathrm{i}w'')\,. \tag{A.2}$$

Let us begin with the case of $n = 1$. According to the Sokhotski formulas, one has

$$J_1(w' + \mathrm{i}0) - J_1(w' - \mathrm{i}0) = 2\pi\mathrm{i} f(w') . \tag{A.3}$$

This means that function $J_1(w)$ is not single-valued along the segment $a < w' < b$ of the real axis. The natural way to analytically continue the function of complex variable $F(w)$, analytical in region W_1, into the W_2 region is transposition of point w from region W_1 to W_2 bypassing all peculiarities of function $F(w)$. The problem of analytic continuation just arises in the absence of such a route. Let us turn to our case.

For $w'' > 0$, the singular point $z = z_0 = w$ of the subintegral function in (A.1) is located in the upper half-plane of the complex plane z. As the imaginary part of w changes from positive values to negative, the singular point z_0 goes to the lower half-plane of the complex plane z (Fig. 4). Three routes of movement are conceivable for point z_0, which are indicated by arrows in Fig. 4. Two of them go left of point $z = a$ and right of point $z = b$, while the third crosses a segment $a < z < b$ of the real axis, i.e., the integration contour in (A.1). However, none of the three routes is appropriate; the former two cross sections on the complex plane z, determined by function $f(z)$, and the last one crosses the region of integral (A.1) divergence. Consequently, there is no natural method for conducting analytic continuation.

One way out is to deform the integration contour in (A.1) so that it always passes beneath the singular point $z_0 = w$, as proposed by Landau, i.e., to replace contour $a0b$ by contour $acdb$ for $w'' < 0$ (see Fig. 4). This Landau rule is substantiated by the fact that a change in w'' from positive to negative values displaces the singular point z_0 from the upper half-plane of the complex plane z to the lower half-plane, engaging and deforming the integration contour. Because the integral along contour $acdb$ is equal to that along contour $a0b$ plus the residue of the integrand at the singular point multiplied by $2\pi\mathrm{i}$, we get the following main formula

$$J_1(w) = \left(\int_a^b \mathrm{d}z \, \frac{f(z)}{z - w} \right) + \begin{cases} 0 , & w'' > 0 , \\ 2\pi\mathrm{i} f(w) , & w'' < 0 . \end{cases} \tag{A.4}$$

This line of reasoning remains in force at $n = 2$. The final result can be presented, omitting details, as

$$J_2(w) = \left(\int_a^b \mathrm{d}z \, \frac{f(z)}{(z - w)^2} \right) + \begin{cases} 0 , & w'' > 0 , \\ 2\pi\mathrm{i} f'(w) , & w'' < 0 , \end{cases} \tag{A.5}$$

where $f'(w)$ is the derivative of function $f(z)$. Formulas (A.4) and (A.5) are special cases of the more general formula

$$J_n(w) = \left(\int_a^b \mathrm{d}z \, \frac{f(z)}{(z - w)^n} \right) + \begin{cases} 0 , & w'' > 0 , \\ \dfrac{2\pi\mathrm{i}}{(n-1)!} \, f^{(n-1)}(w) , & w'' < 0 . \end{cases} \tag{A.6}$$

It is easily seen that dispersion equation (5.4) follows from (5.2) using formula (A.4), while equation (6.2) ensues from formula (A.5) (at $f(z) \equiv 1$).

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