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METHODOLOGICAL NOTES

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On the relation between Stokes drift and the Gerstner wave

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Contents

Abstract. **We discuss the properties of two-dimensional, nonlinear, potential, and vortex waves on the surface of an ideal liquid of infinite depth. It is shown that in the quadratic order in the amplitude, the vorticity of the Gerstner wave is equal in magnitude to and different in sign from that of the Stokes drift current in a surface layer. This allows a classic Stokes wave obtained in the framework of potential theory to be interpreted as a superposition of the Gerstner wave and Stokes drift. It is proposed that the nonlinearity coefficient in the nonlinear Schrödinger equation can be physically interpreted as the Doppler frequency shift along the vertically averaged Stokes drift current.**

Keywords: waves on water, vorticity, Stokes drift, Gerstner wave, nonlinear Schrödinger equation

1. Introduction

Gravity waves traveling on a water surface are an essential component of any course on the theory of nonlinear wave processes; the origin of this theory is closely related to the first attempts to determine the shape of stationary waves of finite amplitude propagating over the surface of a heavier fluid. Stokes and Gerstner waves are two classical examples of water waves of different physical natures. Although Stokes waves are more common in the physical literature, we begin with Gerstner waves, which represent an exact solution of the equations of vortex dynamics [1]. The profile of a stationary Gerstner wave is a trochoid; fluid particles move in this wave along circles with a radius decaying exponentially with depth (Fig. 1). For the maximum amplitude, the profile is described by a cycloid with a singularity at the crest (the cusp angle equals zero). Surprisingly, despite the nonlinearity, the dispersion relation for Gerstner waves is independent of the wave amplitude and coincides with the linear dispersion relation (we assume deep water here and hereafter). Because fluid particles move along circles, they do not drift in Gerstner waves [1–3].

In contrast, the Stokes wave is a solution of the equation of potential fluid motion [4]. Stokes waves can exist if their amplitudes (more precisely, curvatures) are less than critical. The highest Stokes wave has an angle of 120° at its crests. Nonlinearity in a Stokes wave is manifested through the occurrence of higher harmonics and a nonlinear correction to the dispersion equation. A stationary Stokes wave is unstable to smooth perturbations of its envelope, and this effect finds a firm explanation in the framework of the nonlinear Schrödinger equation (NSE), which plays a fundamental role in modern nonlinear physics [5–9]. Fluid particles in a Stokes wave also trace circles in the linear approximation, with radii exponentially decaying with depth; however, for finite amplitudes, fluid particles are displaced over a period in the direction of wave propagation (Fig. 2).

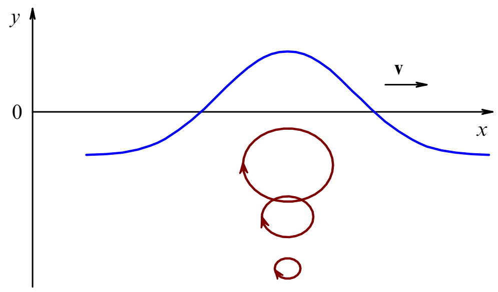

Figure 1. Profile of a Gerstner wave (upper curve) and circular trajectories of fluid particles.

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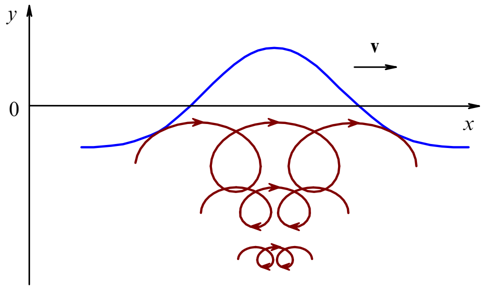

Figure 2. Profile of a Stokes wave (upper curve) and open trajectories of fluid particles.

The mean horizontal motion of fluid particles in a potential wave, known as the Stokes drift [2-4], is characterized by a vertical shear, and is therefore vortical. Thus, although the Stokes wave is potential, it generates a mean vortical shear flow. The vortical Gerstner wave, in contrast, does not generate a mean flow.

The relation among these three phenomena is the subject of this methodological note.

2. Stokes drift

We briefly recall how the Stokes drift emerges in the quadratic approximation in the wave steepness. We choose a coordinate system with the y axis directed vertically upward and the horizontal x axis aligned with the plane of unperturbed fluid motion, as shown in Fig. 2. Equations of two-dimensional fluid dynamics of an ideal incompressible fluid in the field of gravity are written as

$$\nabla \mathbf{v} = 0\,, \tag{1}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{\nabla p}{\rho} - \mathbf{g}\,, \tag{2}$$

where $\mathbf{v}(x, y, t)$ is the velocity vector, ∇ is the two-dimensional gradient operator with respect to the Cartesian coordinates x and y, t is the time, p is the pressure, and $\mathbf{g}$ is the gravity acceleration. Equation (1) is known as the continuity equation and (2) is the Euler equation [10]. Assuming that the motion is potential, we can introduce a flow potential $\varphi(x, y, t)$ such that $\mathbf{v} = \nabla\varphi$. Then Eqns (1) and (2) transform into the following equations [2, 9, 11]:

$$\Delta\varphi = 0\,, \tag{3}$$

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla\varphi)^2 + \frac{p - p_0}{\rho} + gy = 0\,. \tag{4}$$

Here, p_0 is the constant pressure on the fluid surface. Expression (4) is known as the Cauchy–Lagrange integral. The problem of computing waves on a fluid surface is thus reduced to solving Laplace equation (3) under the constraint that the pressure is constant on the free and still unknown surface $y = \eta(x, t)$, as follows from Eqn (4). The equation should be augmented with the kinematic boundary condition

$$\eta_t + \eta_x\varphi_x - \varphi_y = 0\,, \qquad y = \eta(x, t)\,, \tag{5}$$

and the condition that perturbations decay with the depth:

$$\varphi\Big|_{y=-\infty} = 0\,. \tag{6}$$

The boundary condition for pressure and the kinematic condition are nonlinear and create a major difficulty in analytic studies of water waves. Exact analytic solutions of problems (3)–(6) are still lacking. To solve this problem approximately, Stokes resorted to the method of successive approximations in the small wave steepness parameter $\varepsilon = kA$, where k is the wave number and A is the wave amplitude. Solutions derived in this framework are called 'Stokes expansions'.

Through the second order in the small wave steepness parameter, the potential of wave motion $\varphi(x, y, t)$ and the vertical free surface displacement $y = \eta(x, t)$ of stationary wave motion are written as [2, 4]

$$\varphi(x, y, t) = Ac\exp(ky)\sin\left[k(x - ct)\right]\,, \tag{7}$$

$$\eta(x, t) - \frac{1}{2}kA^2 = A\cos\left[k(x - ct)\right] + \frac{1}{2}kA^2\cos\left[2k(x - ct)\right]\,. \tag{8}$$

This solution is written in the laboratory frame of reference; the wave propagates to the right with the linear phase velocity $c = \sqrt{g/k}$. The nonlinearity leads to the generation of the second harmonic and to a shift of the mean water level above the zero level ($y = 0$). Following Ref. [2], we keep this term in the left-hand side.

System of equations (7) and (8) describes the Stokes wave in the Eulerian variables x, y. From Eqn (7), it follows that the Eulerian velocity field $\mathbf{v}(\mathbf{r}, t) = \nabla\varphi$, $\mathbf{v} = \{v_x, v_y\}$, $\mathbf{r} = \{x, y\}$ takes the form

$$\begin{aligned} v_x &= kcA\exp(ky)\cos\left[k(x - ct)\right]\,,\\ v_y &= kcA\exp(ky)\sin\left[k(x - ct)\right]\,. \end{aligned} \tag{9}$$

Field (9) is periodic in time; hence, its mean value over a period is equal to zero. However, as already found by Stokes, fluid particles do not stay at their initial positions on average, but drift in the direction of wave propagation (Stokes drift). We consider this in more detail.

We let $\mathbf{v}_{\mathrm{L}}(\mathbf{r}_0, t)$ denote the velocity of the fluid element that at the time instant $t = 0$ has the coordinate $\mathbf{r} = \mathbf{r}_0 = \{a, b\}$; the components of the vector $\mathbf{r}_0$ are called the Lagrangian coordinates and the vector $\mathbf{v}_{\mathrm{L}}$ is called the Lagrangian velocity. The position of the fluid element at subsequent time instants is described as

$$\mathbf{r} = \mathbf{r}_0 + \int_0^t \mathbf{v}_{\mathrm{L}}(\mathbf{r}_0, t')\,\mathrm{d}t'\,. \tag{10}$$

The Eulerian velocity $\mathbf{v}(\mathbf{r}, t)$ at the point $\mathbf{r}$ given by Eqn (10) equals the Lagrangian velocity $\mathbf{v}_{\mathrm{L}}(\mathbf{r}_0, t)$, i.e., the velocity of the particle that arrived to the point $\mathbf{r}$ at the time instant t. Then

$$\begin{aligned} \mathbf{v}_{\mathrm{L}}(\mathbf{r}_0, t) &= \mathbf{v}\left(\mathbf{r}_0 + \int_0^t \mathbf{v}_{\mathrm{L}}(\mathbf{r}_0, t')\,\mathrm{d}t', t\right)\\ &= \mathbf{v}(\mathbf{r}_0, t) + \left(\int_0^t \mathbf{v}_{\mathrm{L}}(\mathbf{r}_0, t')\,\mathrm{d}t'\right)\nabla_{\mathbf{r}_0}\mathbf{v}(\mathbf{r}_0, t) + \ldots\,, \end{aligned} \tag{11}$$

where the Taylor expansion is used and $\nabla_{\mathbf{r}_0}$ denotes the gradient operator with respect to the components of $\mathbf{r}_0$. Because only small velocities (of the order of ε) are considered, the second term in the right-hand side of Eqn (10) has the order ε^2. This implies that the Lagrange and Eulerian velocities agree in the first order,

$$\mathbf{v}_{\mathrm{L}}(\mathbf{r}_0, t) = \mathbf{v}(\mathbf{r}, t) + O(\varepsilon)\,, \tag{12}$$

just like the Eulerian and Lagrangian variables $\mathbf{r} = \mathbf{r}_0 + O(\varepsilon)$. Replacing the Eulerian coordinates with the Lagrangian ones in (9) and integrating over time, we arrive at the representation of fluid particle trajectories in the linear approximation [10]:

$$x = a - A\exp(kb)\sin\left[k(a-ct)\right],\quad y = b + A\exp(kb)\cos\left[k(a-ct)\right]. \tag{13}$$

It follows that particles in linear gravity waves describe circles around points $x_0 = a$, $y_0 = b$ with the radius decaying exponentially downward in the fluid.

What happens in the quadratic approximation? Inserting expression (12) into the integral in the right-hand side of Eqn (11), we find that up to the second-order terms (of the Stokes expansion), the Lagrangian velocity is determined by the relation (see also Ref. [12])

$$\mathbf{v}_{\mathrm{L}}(\mathbf{r}_0, t) = \mathbf{v}(\mathbf{r}_0, t) + \left(\int_0^t \mathbf{v}(\mathbf{r}_0, t')\,\mathrm{d}t'\right)\nabla_{\mathbf{r}_0}\mathbf{v}(\mathbf{r}_0, t). \tag{14}$$

To calculate it, the Eulerian coordinates must once again be replaced by the Lagrangian ones in the representation for velocity (9) and inserted into equality (14). However, we are interested not in the general expression for velocity but in its value averaged over the period. Performing the necessary calculations and averaging the expression obtained, we obtain

$$\left\langle \mathbf{v}_{\mathrm{L}}(\mathbf{r}_0, t)\right\rangle = c(kA)^2\exp(2kb)\,\mathbf{i} = U_{\mathrm{S}}(b)\,\mathbf{i}, \tag{15}$$

where the angular brackets denote averaging over the wave period, and $\mathbf{i}$ is the unit vector in the positive horizontal direction. Fluid particles, in addition to performing oscillatory motion, drift in the wave propagation direction with the velocity $U_{\mathrm{S}}(b) = c(kA)^2\exp(2kb)$. This plane-parallel flow was called the Stokes drift. In the Eulerian coordinates, with Eqn (10) taken into account, it is written as

$$U_{\mathrm{S}}(y) = c(kA)^2\exp(2ky). \tag{16}$$

Flows (15) and (16) are shear flows and hence have vorticity. How can it be explained that part of a potential Stokes wave is a vortical flow? Surprisingly, none of the authors touching on the Stokes drift has commented on this fact. Even Stokes ignored it. In our opinion, this question needs a rigorous analysis. To answer it, we turn to the other fluid wave motion featuring in the title of this note.

3. Gerstner wave

The solution for this wave was found by the Czech scientist Franz Joseph von Gerstner in 1804. In contrast to the Stokes wave, the Gerstner wave is an exact solution of the equations of fluid dynamics (it is given below in this section). It is unique in that it is the only known exact solution of the full system of hydrodynamic equations for stationary gravity waves on deep water. This solution is less known because it is written in the Lagrangian coordinates, which are used very rarely in hydrodynamic problems because they contain a more complicated form of nonlinearity than equations in the Eulerian form. For example, the Lagrangian form of hydrodynamic equations is fully omitted in the fundamental course *Fluid Mechanics* by Landau and Lifshitz [10], and hence the Gerstner wave and Stokes drift are not mentioned there.

Equations of two-dimensional fluid dynamics have the following form in the Lagrangian variables [2, 3, 13, 14]:

$$\frac{D(X,Y)}{D(a,b)} = [X,Y] = 1, \tag{17}$$

$$X_{tt}X_a + (Y_{tt}+g)Y_a = -\frac{1}{\rho}\,p_a, \tag{18}$$

$$X_{tt}X_b + (Y_{tt}+g)Y_b = -\frac{1}{\rho}\,p_b, \tag{19}$$

where $X(a,b,t)$, $Y(a,b,t)$ are the coordinates of the trajectory of a fluid particle with the Lagrangian coordinates a and b, and the subscripts denote differentiation with respect to the appropriate variable. The b axis is directed upward and $b = 0$ corresponds to the free surface. The square brackets denote the Jacobian. In system of equations (17)–(19), the first equation is the continuity equation and the other two are momentum equations.

Using cross differentiation, we eliminate pressure from Eqns (18) and (19) [2, 14],

$$X_{ta}X_b - X_{tb}X_a + Y_{ta}Y_b - Y_{tb}Y_a = \Omega(a,b). \tag{20}$$

Equation (20) is equivalent to momentum equations (18) and (19), but it explicitly includes the vorticity of fluid particles Ω, which is a function of only Lagrangian coordinates in the case of two-dimensional flows.

Gerstner succeeded in finding an exact solution of Eqns (17) and (20). By direct substitution, it can be verified that their solution is the pair of relations

$$X = a - A\exp(kb)\sin\left[k(a-ct)\right],\quad Y = b + A\exp(kb)\cos\left[k(a-ct)\right]. \tag{21}$$

The Gerstner solution has exactly the same form as solution (13) for a linear potential wave. The dispersion equation for the Gerstner wave $\omega = \sqrt{gk}$ coincides with that for linear waves. We stress that it is independent of the wave amplitude. Fluid particles in the Gerstner wave also describe circles. However, the amplitude of the Gerstner wave is finite instead of being small. The profile of the Gerstner wave is a trochoid. For the wave amplitude $A = k^{-1}$ (a limit value), the wave has a cusp at its crest, and its shape is described by a cycloid. The angle of the cusp of the limit Gerstner wave equals zero [2, 3].

The Gerstner wave exhibits one more distinctive property: its vorticity is given by the expression

$$\Omega_{\mathrm{G}} = \frac{2k^3A^2c\exp(2kb)}{1 - k^2A^2\exp(2kb)}. \tag{22}$$

We expand vorticity (22) in a series in the small parameter kA:

$$\Omega_{\mathrm{G}} = 2k^3A^2c\exp(2kb)\left[1 + k^2A^2\exp(2kb) + \ldots\right] = \Omega_{\mathrm{G}2} + O\left((kA)^4\right). \tag{23}$$

It follows from (23) that in the linear approximation, the Gerstner wave does not carry vorticity and is therefore fully equivalent to a linear potential wave. It acquires vorticity only in the quadratic approximation. In other words, weak circular motion of fluid particles generates vorticity only in the second order in the small parameter of wave steepness.

4. Stokes wave in the Lagrangian description

We return to the question posed at the end of Section 2: How can a vortical shear flow be hidden inside a potential Stokes wave? According to the Lagrange theorem, the vorticity cannot be created in an ideal homogeneous incompressible fluid in the field of potential forces. In the case of water waves considered here, all theorem conditions are observed. Hence, in each approximation, the net flow vorticity must vanish. Eulerian representation (7), (8) of the Stokes wave in the quadratic approximation does not allow demonstrating this explicitly. But the situation changes as we pass to the Lagrangian form.

In the Lagrangian variables, the solution obtained by Stokes for a surface gravity wave on deep water can be expressed as [2, 15]

$$\begin{aligned} X &= a - \varepsilon k^{-1} \exp(kb) \sin\left[k(a-ct)\right] + \varepsilon^2 ct \exp(2kb)\,, \\ Y &= b + \varepsilon k^{-1} \exp(kb) \cos\left[k(a-ct)\right]. \end{aligned} \tag{24}$$

The motion of particles described by system (24) is a superposition of oscillatory and drift components. If we select a sufficiently small amplitude of Gerstner waves, the oscillatory motions of fluid particles in Eqns (21) and (24) coincide. The difference between the Stokes and Gerstner waves in the quadratic approximation is therefore due to the presence of the drift term in the potential wave (Stokes drift). The vorticity of the Gerstner wave equals Ω_{G2} in this approximation [see Eqn (23)], whereas the vorticity of the Stokes drift is

$$\Omega_S = -\frac{dU_S}{db} = -2kc\varepsilon^2 \exp(2kb) = -\Omega_{G2}\,,$$

whence the total vorticity of the Stokes wave is zero, as it should be.

This can be interpreted in the following way. The motion of fluid particles in the Stokes wave is a superposition of two flows: a rotational flow along circles (the 'Gerstner' flow) and a shear flow (the Stokes drift). Each of these flows carries vorticity, but their net vorticity is equal to zero.

Solution (24) was obtained by Stokes [4]; however, neither Stokes himself nor other researchers who reproduced his result have related the oscillatory part of this solution to the Gerstner wave. Symbolically, the Stokes result can be written as

$$\text{Stokes wave} = \text{Gerstner wave} + \text{Stokes drift.} \tag{25}$$

This result, despite its obvious character, has not been formulated in the literature. This is probably related to the fact that the solution for the potential of the Stokes wave is typically written in Eulerian variables. The representation of the Stokes solution in Lagrangian coordinates (24), in contrast, highlights our symbolic formula (25). We stress, first, its nonobvious character and, second, its nontriviality: the principle of flow superposition works in a nonlinear approximation.

Thus, there is a close connection between Stokes and Gerstner waves and the Stokes drift. However, it turns out that it is not limited to only this example. Unexpectedly for the authors, it is manifested in the modulation instability of the Gerstner wave.

5. Nonlinear Schrödinger equation for the Gerstner wave

We consider the propagation of a package of surface gravity waves in a fluid of unbounded depth. Because we are interested in the case of a Gerstner wave, we assume that the propagating wave carries a weak vorticity (of the order of the curvature squared).

We use Lagrangian variables. We introduce the complex-valued trajectory of a fluid particle $W = X + iY$ ($\overline{W} = X - iY$), where the bar denotes complex conjugation. In the new variables, Eqns (17) and (20) take the form [14, 16, 17]

$$[W, \overline{W}] = -2i\,, \tag{26}$$

$$\mathrm{Re}\,[W_t, \overline{W}] = \Omega(a,b)\,, \tag{27}$$

and system of equations (18), (19), after simple algebraic manipulations, reduces to a single equation

$$W_{tt} = -ig + i\rho^{-1}[p, W]\,. \tag{28}$$

In what follows, we use Eqns (26) and (27) to find the complex-valued coordinate of the fluid particle trajectory, and find the pressure in the fluid from Eqn (28). The boundary conditions are the impermeability condition at the bottom ($Y_t \to 0$ at $b \to -\infty$) and the constant-pressure condition on the surface (at $b = 0$).

We use the method of multiple scales. The function W is written as

$$W = a_0 + ib + w(a_l, b, t_l)\,, \quad a_l = \varepsilon^l a\,, \quad t_l = \varepsilon^l t\,, \quad l = 0, 1, 2\,, \tag{29}$$

where ε is the small parameter of wave steepness. We write the unknown functions p and w as series in this parameter,

$$w = \sum_{n=1} \varepsilon^n w_n\,, \quad p = p_0 - \rho g b + \sum_{n=1} \varepsilon^n p_n\,. \tag{30}$$

A term with hydrostatic pressure is separated in the expression for the pressure, and p_0 is the constant atmospheric pressure on the fluid surface, which can be set to zero. The vorticity Ω is assumed to be quadratic in the steepness parameter, as in the Gerstner wave:

$$\Omega = \varepsilon^2 \Omega_2(a,b)\,. \tag{31}$$

The precise form of the function Ω_2 that corresponds to the Gerstner wave is better to specify later, after obtaining the third-order evolution equation.

We insert representations (29)–(31) into Eqns (26)–(28). In the first approximation, the solution becomes

$$\begin{aligned} w_1 &= A(a_1, a_2, t_1, t_2) \exp\left[i(ka_0 - \omega t_0) + kb\right] \\ &\quad + \psi_1(a_1, a_2, b, t_1, t_2)\,. \end{aligned} \tag{32}$$

Here and hereafter, A is the complex-valued wave amplitude of a wave propagating to the right. The function ψ_1 is real valued, and its form is determined by considering the next approximation. Expression (32) describes the wave motion in the laboratory frame. Just as for stationary potential wave (24), it consists of oscillatory motion of fluid particles along circles and the mean flow.

We skip the details of computations, giving only the main results in higher-order approximations. There are two equations in the second-order approximation:

$$A_{t_1} + c_g A_{a_1} = 0\,, \qquad (33)$$

$$\psi_{1t_1} = k\omega |A|^2 \exp(2kb) - \int_{-\infty}^{b} \Omega_2(a_1, b')\,\mathrm{d}b'\,, \qquad (34)$$

where $c_g = g/(2\omega)$ is the group velocity of linear gravity waves. We note that the first term in Eqn (34) coincides with the expression for the Stokes drift. Using Eqns (33) and (34), in the third order, we arrive at the evolution equation

$$\mathrm{i}\,\frac{\partial A}{\partial t_2} - \frac{\omega}{8k^2}\frac{\partial^2 A}{\partial a_1^2} - 2k^2 A \int_{-\infty}^{0} \psi_{1t_1} \exp(2kb)\,\mathrm{d}b = 0\,, \qquad (35)$$

written in the reference frame moving with the group velocity c_g.

System (34), (35) is the modified nonlinear Schrödinger equation (NSE) for weakly vortical waves, which requires a special investigation in the general case. We limit ourselves to its analysis as applied to the Gerstner wave.

In representation (29), (30), the function w_1 for the Gerstner wave is described by expressions that are analogous to relations (21) but for the amplitude A times the parameter ε. Using Eqn (21) for the vorticity, we find that for the Gerstner wave,

$$\Omega_2 = 2k^2\omega |A|^2 \exp(2kb) = \Omega_{G2}\,.$$

Inserting this into Eqn (34), we find that $\psi_{t_1} = 0$. The effective shear flow related to the vorticity of the Gerstner wave [the second term in the right-hand side of Eqn (34)] exactly compensates the Stokes drift. Thus, a package of weakly nonlinear Gerstner waves is not affected by the nonlinearity in this approximation, and the effect of modulation instability is absent for it.

This is the second aspect of the connection of the Gerstner wave and Stokes drift, this time related to the NSE.

6. Physical meaning of the nonlinearity coefficient in the Schrödinger equation

If $\Omega_2 = 0$, Eqn (35) transforms into the classic NSE for potential waves:

$$\mathrm{i}\,\frac{\partial A}{\partial t_2} - \frac{\omega}{8k^2}\frac{\partial^2 A}{\partial a_1^2} - \frac{1}{2}\,\omega k^2 |A|^2 A = 0\,. \qquad (36)$$

This equation was first derived for potential waves on deep water by Zakharov, who used the Hamiltonian formalism [5] (see also Ref. [16]). This same result was obtained independently by Hasimoto and Ono [15] and Davey [18] with the help of the multiple expansion method, and by Yuen and Lake based on the method of the averaged Lagrangian [19]. Here, we describe how the NSE can be derived in Lagrangian coordinates.

To write Eqn (36) in Eulerian coordinates, we need to express the horizontal Lagrangian coordinate a through the horizontal Eulerian coordinate X. It follows from relations (29) and (30) that

$$X = a + \varepsilon \operatorname{Re}\left(w_1 + \sum_{n=2} \varepsilon^{n-1} w_n\right) = a + O(\varepsilon)\,,$$

and hence, in order to pass to the Eulerian form, we should simply replace the Lagrangian coordinate by the respective Eulerian variable ($a_l \to X_l$). It is apparent that the inverse coordinate transformation is also possible; hence, all known solutions of the NSE in the Eulerian variables can be written in the Lagrangian coordinates.

In deriving the NSE in Section 5, we mentioned that the first term in Eqn (34) coincides with the expression for the Stokes drift, but with the amplitude A in it being a variable quantity. In the case of zero vorticity of waves ($\Omega_2 = 0$), that term alone determines the form of the function ψ_{1t_1} and hence the nonlinearity coefficient in the NSE [see Eqn (35)]. The derivation of the NSE is often carried out formally, and we therefore believe that it is important to relate one of its terms to the Stokes drift. To our knowledge, this point has not been mentioned in the literature dealing with waves on water.

Very popular among physicists is a heuristic derivation of the NSE based on the nonlinear dispersion equation for the Stokes wave. We recall it. The nonlinear dispersion relation for the Stokes wave is obtained in the third order of the Stokes expansion and takes the form [2, 4, 9]

$$\omega = \sqrt{gk}\left(1 + \frac{1}{2}\,k^2 A^2\right). \qquad (37)$$

For a narrow wave package, we expand expression (37) in the vicinity of some constant k_0, keeping terms of the second order in the wave number and nonlinearity. Then the perturbations of the wavenumber k' and frequency ω' satisfy the equation

$$\omega' - \frac{\omega_0}{2k_0}\,k' + \frac{\omega_0}{8k_0^2}\,k'^2 - \frac{1}{2}\,\omega_0 k_0^2 A^2 = 0\,, \quad \omega_0 = \sqrt{gk}\,. \qquad (38)$$

Considering the frequency and wavenumber in Eqn (38) to be operators in accordance with

$$-\mathrm{i}\omega' \to \frac{\partial}{\partial t}\,, \qquad \mathrm{i}k' \to \frac{\partial}{\partial x}\,,$$

we arrive at the nonlinear Schrödinger equation [20–23]:

$$\mathrm{i}\left(\frac{\partial A}{\partial t} + c_g\,\frac{\partial A}{\partial X}\right) - \frac{\omega}{8k^2}\frac{\partial^2 A}{\partial X^2} - \frac{1}{2}\,\omega k^2 |A|^2 A = 0\,.$$

This equation is analogous to Eqn (36) if it is written in the reference frame moving to the right with the group velocity, and the transformation

$$A \to \frac{A}{\varepsilon}\,, \quad t_2 \to \varepsilon^2 t\,, \quad a_1 \to \varepsilon X$$

is done. This heuristic derivation explicitly indicates that the nonlinearity coefficient in the NSE coincides with the nonlinear correction to the dispersion relation for linear waves. This derivation is well known, and is reproduced here for completeness.

We write dispersion relation (37) in a somewhat different form,

$$\omega - ku = \omega_0\,, \qquad (39)$$

where the quantity $u = \omega_0 k A^2/2$ is introduced that corresponds to a nonlinear correction to the phase velocity. From Eqn (39), it follows that it can be interpreted as a surface flow ensuring the Doppler frequency shift. This velocity satisfies the relations

$$u = k\int_{-\infty}^{0} U_S(y)\,\mathrm{d}y = k\int_{-\infty}^{0} U_S(b)\,\mathrm{d}b\,,$$

and has the meaning of bulk horizontal transport through a cross section of unit length (times k). Taking into account that the drift velocity magnitude decays exponentially with depth, being substantial only in the near-surface layer with a thickness of the order of the wavelength, we can conjecture that it is approximately equal to the Stokes drift velocity averaged along the vertical.

We recall in this respect that the drift motion of fluid particles is absent in the Gerstner wave ($u = 0$) and that its dispersion relation coincides with that for linear potential waves. Thus, following the heuristic approach of deriving the NSE based solely on the dispersion equation, we could argue from the very beginning that it would contain no nonlinearity. However, the reason would remain obscure. Furthermore, as it seems to us, it would hardly be possible to elucidate it without resorting to the NSE for weakly vortical waves.

7. Conclusions

This note is devoted to the analysis of classic examples of waves on deep water. We draw attention to the fact that in the Lagrangian description, the solution for the Stokes wave in the quadratic approximation can be viewed as a superposition of two flows: the vortical Gerstner wave and the shear flow of the Stokes drift. The vorticity of these flows is the same in absolute value, but differs in sign. It follows that this same feature explains the absence of a nonlinear term in the nonlinear Schrödinger equation for the Gerstner wave. The proof is based on the derivation of the NSE for weakly vortical waves in Lagrangian variables. It is shown that the nonlinearity coefficient in the NSE for potential waves coincides in magnitude with the vertically averaged Stokes drift.

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