METHODOLOGICAL NOTES

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On choosing the energy–momentum tensor in electrodynamics and on the Abraham force

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Contents

Abstract. **The highly debated problem of choosing the energy–momentum tensor in electrodynamics is examined. Electromagnetic forces in a continuum medium that follow from the Minkowski and Abraham tensors are considered. It is shown that the conservation equations for both Minkowski's momentum density and Abraham's momentum simultaneously follow from the Minkowski tensor and that they are the components of a composite electromagnetic momentum in the medium. It is shown that choosing canonical matter equations reduces the Abraham force to zero. We argue in favor of choosing the Minkowski tensor and show the incompleteness of the Abraham tensor.**

Keywords: electromagnetic force, energy–momentum tensor, Minkowski tensor, Abraham tensor, Abraham force

1. Introduction

Despite many years of research, there is still no unique understanding of the interaction of the electromagnetic field (EMF) with matter, a subject that has become highly topical owing to recent research efforts toward developing metamaterials with unique electromagnetic properties. The standard way of determining electromagnetic forces in a continuum is by expressing them as a 4-divergence of the energy–momentum tensor (EMT) [1], a quantity that plays a key role in solving the problem.

There are two parts to determining electromagnetic forces in a medium. The first is to choose the form of the EMT for the EMF interacting with matter. The second is to choose constitutive equations describing the electromagnetic properties of the medium. We discuss the first problem here. There is still no definitive answer as to which of the many EMT forms is correct; the most discussed are the Minkowski and Abraham EMTs (see, e.g., Refs [1–10]). Comparative analyses [1–3, 5, 8, 10] of the results obtained in a number of cases have favored using the Abraham tensor over the Minkowski tensor. Advantages of the Minkowski tensor and disadvantages of the Abraham tensor were highlighted in [4, 6, 7, 9]; the Abraham tensor, as argued in Refs [4, 6], is not relativistically covariant, making the Minkowski tensor preferable. It is noted in [2] that "in most situations results obtained with the Abraham tensor are completely identical to those obtained with the Minkowski tensor." The authors of Ref. [3] argue that the EMT form cannot be chosen uniquely within a purely macroscopic framework. In [11], the available experimental data are reviewed and analyzed, and Refs [3, 4, 7, 12] offer a vast bibliography on the subject.

The usual way to find the EMT is to assemble it from separate blocks. By using Maxwell's equations, the Lorentz force expression, and the Poynting theorem, we obtain what can be interpreted as the energy and momentum conservation equations, whose terms are then considered derivatives of the EMT. The building blocks of the EMT are the electromagnetic field energy and momentum densities, the energy flux density (the Poynting vector), and the three-dimensional momentum flux density tensor (or the three-dimensional stress tensor). This approach allows some freedom in choosing the EMT components, leading sometimes to 'general consideration' definitions and different interpretations of the EMT components, thus making the EMT choice a matter of debate. This is how the Minkowski, Hertz–Heaviside, Abraham, Helmholtz–Abraham, Abraham–Brillouin–Pitaevskii, and Polevoi–Rytov forms of the EMT, among others, were constructed.

A different form of electromagnetic forces corresponds to each of these EMT forms. The electromagnetic forces are obtained 'in a somewhat inconsistent way' according to Ref. [3]. However, the electromagnetic force balance equations themselves are obtained as a 4-divergence of the EMT, and the same is true for the conservation equations of energy

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quantities. This can be used to comparatively analyze tensors and obtain additional information that can allow correctly choosing the EMT form. We use this method here to provide what we consider sufficient additional arguments to finalize the EMT controversy decisively in favor of the Minkowski tensor. The medium is assumed to be homogeneous, isotropic, nonconducting, nondispersive, lossless, and at rest.

2. Energy–momentum tensors in electrodynamics

The EMT can be represented in the general canonical form

$$\mathbf{T}_{\nu\mu} = \begin{bmatrix} W & \mathrm{i}\,\dfrac{1}{c}\,\mathbf{S} \\ \mathrm{i}c\mathbf{g} & t_{ik} \end{bmatrix}, \quad \nu, \mu = 0, 1, 2, 3; \quad i, k = 1, 2, 3, \qquad (1)$$

where W is the energy density, $\mathbf{S}$ is the energy flux density (the Poynting vector), $\mathbf{g}$ is the momentum density, and t_{ik} is the momentum flux density tensor (stress tensor).

The components of EMT (1) are given by [6]

$$W = \frac{\mathbf{ED} + \mathbf{HB}}{8\pi}, \qquad \mathbf{S} = c\,\frac{\mathbf{E} \times \mathbf{H}}{4\pi},$$

$$\mathbf{g}^{\mathrm{M}} = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c}, \qquad t_{ik}^{\mathrm{M}} = \frac{E_i D_k + H_i B_k}{4\pi} - \delta_{ik}\,\frac{\mathbf{ED} + \mathbf{HB}}{8\pi}$$

for the Minkowski form and by

$$W = \frac{\mathbf{ED} + \mathbf{HB}}{8\pi}, \qquad \mathbf{S} = c\,\frac{\mathbf{E} \times \mathbf{H}}{4\pi}, \qquad \mathbf{g}^{\mathrm{A}} = \frac{\mathbf{E} \times \mathbf{H}}{4\pi c},$$

$$t_{ik}^{\mathrm{A}} = \frac{E_i D_k + E_k D_i + H_i B_k + H_k B_i}{8\pi} - \delta_{ik}\,\frac{\mathbf{ED} + \mathbf{HB}}{8\pi}$$

for the Abraham form. With Abraham's expressions for the components, the EMT becomes symmetric.

3. Electromagnetic energy and momentum conservation equations

The electromagnetic energy and momentum conservation equations are obtained by taking the 4-divergence of EMT (1). If light loses no energy, i.e., is not absorbed or scattered as it propagates through a transparent medium, then its electromagnetic radiation energy is the same before entering and after leaving the medium. The important point is that the energy exchange with the medium is reactive in nature. The divergence of the EMT should, in this case, be put to zero because there is no sink for active electromagnetic energy. Otherwise, the divergence of the electromagnetic field EMT should be put equal to that of the mechanical EMT of the medium. For simplicity, the lossless case is considered below.

In the general case, EMT (1) is nonsymmetric, and for each of its indices we can write two groups of equations (in considering the EMT in Eqn (1), we can make no distinction between covariant and contravariant indices):

$$\text{(a)}\ \partial_\nu \mathbf{T}_{\nu\mu} = 0, \quad \text{(b)}\ \partial_\mu \mathbf{T}_{\nu\mu} = 0,$$

or

$$\text{(a)}\ \frac{1}{c}\,\partial_t W + c\nabla\mathbf{g} = 0, \quad \frac{1}{c^2}\,\partial_t S_k - \partial_i t_{ik} = 0;$$

$$\text{(b)}\ \partial_t W + \nabla\mathbf{S} = 0, \quad \partial_t g_i - \partial_k t_{ik} = 0. \qquad (2)$$

Substituting the EMT components in the Minkowski form in Eqn (2) yields the following conservation equations:

- the energy density conservation equation

$$\frac{\partial_t(\mathbf{ED} + \mathbf{HB})}{8\pi c} + \frac{\nabla(\mathbf{D} \times \mathbf{B})}{4\pi} = 0, \qquad (3)$$

- the energy flux density conservation equation

$$\frac{\partial_t(\mathbf{E} \times \mathbf{H})_k}{4\pi c} - \partial_i\left(\frac{E_i D_k + H_i B_k}{4\pi} - \delta_{ik}\,\frac{\mathbf{ED} + \mathbf{HB}}{8\pi}\right) = 0; \qquad (4)$$

- the energy density conservation equation

$$\frac{\partial_t(\mathbf{ED} + \mathbf{HB})}{8\pi c} + \frac{\nabla(\mathbf{E} \times \mathbf{H})}{4\pi} = 0; \qquad (5)$$

- the momentum density conservation equation

$$\frac{\partial_t(\mathbf{D} \times \mathbf{B})_i}{4\pi c} - \partial_k\left(\frac{E_i D_k + H_i B_k}{4\pi} - \delta_{ik}\,\frac{\mathbf{ED} + \mathbf{HB}}{8\pi}\right) = 0. \qquad (6)$$

It follows from Eqns (4) and (6) that the Minkowski tensor simultaneously describes changes in the electromagnetic momentum density in the Abraham form, Eqn (4), and in the Minkowski form, Eqn (6).

From Eqns (3) and (5),

$$\frac{\nabla(\mathbf{D} \times \mathbf{B})}{4\pi} = \frac{\nabla(\mathbf{E} \times \mathbf{H})}{4\pi},$$

or

$$\frac{\nabla(\mathbf{D} \times \mathbf{B})}{4\pi c} = \frac{\nabla(\mathbf{E} \times \mathbf{H})}{4\pi c},$$

or

$$\nabla\mathbf{g}^{\mathrm{M}} = \nabla\mathbf{g}^{\mathrm{A}}.$$

This implies the same divergences of the momentum density in the Minkowski and Abraham forms. Taking the time derivative of both sides of the last equation yields

$$\nabla\,\partial_t\mathbf{g}^{\mathrm{M}} = \nabla\,\partial_t\mathbf{g}^{\mathrm{A}},$$

or

$$\nabla(\partial_t\mathbf{g}^{\mathrm{M}} - \partial_t\mathbf{g}^{\mathrm{A}}) = 0.$$

The bracketed expression represents the Abraham force. Thus, the Minkowski EMT expression implies that the divergence of the Abraham force is zero. implying further that the Abraham force is vortical in nature.

Substituting components of the Abraham EMT in Eqn (2) and considering the symmetry of the tensor, we obtain two conservation equations for energy and momentum:

- the energy density conservation equation

$$\frac{\partial_t(\mathbf{ED} + \mathbf{HB})}{8\pi c} + \frac{\nabla(\mathbf{E} \times \mathbf{H})}{4\pi} = 0,$$

identical to Eqn (5), which follows from the Minkowski tensor; and

- a conservation equation for momentum density in the Abraham form,

$$\frac{\partial_t(\mathbf{E}\times\mathbf{H})_i}{4\pi c} - \partial_k\left(\frac{E_i D_k + E_k D_i + H_i B_k + H_k B_i}{8\pi} - \delta_{ik}\,\frac{\mathbf{ED}+\mathbf{HB}}{8\pi}\right) = 0,$$

or

$$\partial_t g_i^{\mathrm{A}} = \partial_k \left(\frac{E_i D_k + E_k D_i + H_i B_k + H_k B_i}{8\pi} - \delta_{ik} \frac{\mathbf{ED} + \mathbf{HB}}{8\pi} \right);$$

• the energy density flux conservation equation for the Abraham tensor

$$\frac{1}{c^2} \partial_t S_k - \partial_i \left(\frac{E_i D_k + E_k D_i + H_i B_k + H_k B_i}{8\pi} - \delta_{ik} \frac{\mathbf{ED} + \mathbf{HB}}{8\pi} \right) = 0,$$

or

$$\partial_t g_k^{\mathrm{A}} = \partial_i \left(\frac{E_i D_k + E_k D_i + H_i B_k + H_k B_i}{8\pi} - \delta_{ik} \frac{\mathbf{ED} + \mathbf{HB}}{8\pi} \right).$$

This equation is also the conservation equation for the momentum density in the Abraham form.

Thus, the Minkowski tensor leads to conservation equations for the momentum density in the Minkowski and Abraham forms, whereas the Abraham tensor leads to conservation equations for the momentum density in the Abraham form alone. This means that unlike the Minkowski tensor, the Abraham tensor does not provide a comprehensive description of the processes of EMF propagation.

For some authors, the Minkowski EMT is disadvantageous due to its lack of symmetry, which, as they argue, results in the nonconservation of the angular momentum in the medium. But this argument is disproved in [13]. All divergence-form conservation equations are derived only from the symmetric part of a nonsymmetric EMT, because the total divergence of its antisymmetric part is zero.

4. Electromagnetic forces in a nonconducting continuum

Electromagnetic forces (or, more precisely, their density) in a continuous nonconducting medium are defined as time derivatives of the electromagnetic momentum density $\partial_t \mathbf{g}$. If no external forces, charges, or currents are present, Eqns (4) and (6), which follow from the Minkowski tensor, can be regarded as electromagnetic force balance equations for the medium. Equation (4) can be written in the form

$$\partial_t g_k^{\mathrm{A}} = \frac{\partial_t (\mathbf{E} \times \mathbf{H})_k}{4\pi c} = \partial_i \left(\frac{E_i D_k + H_i B_k}{4\pi} - \delta_{ik} \frac{\mathbf{ED} + \mathbf{HB}}{8\pi} \right). \quad (7)$$

Equation (6) can be written as

$$\partial_t g_i^{\mathrm{M}} = \frac{\partial_t (\mathbf{D} \times \mathbf{B})_i}{4\pi c} = \partial_k \left(\frac{E_i D_k + H_i B_k}{4\pi} - \delta_{ik} \frac{\mathbf{ED} + \mathbf{HB}}{8\pi} \right). \quad (8)$$

Electromagnetic forces in a medium are determined by two quantities, the electric field displacement $\mathbf{D}$ and the magnetic field strength $\mathbf{H}$, which respectively depend on the electric and magnetic characteristics of the medium. Then Eqn. (7), with the momentum density in the Abraham form (which involves the magnetic field strength $\mathbf{H}$), determines electromagnetic forces related to the magnetic characteristics of the medium, whereas Eqn (8), with the momentum density in the Minkowski form (which involves the electric field displacement $\mathbf{D}$) describes electromagnetic forces related to the electric characteristics of the medium. For brevity, these electromagnetic force densities can be loosely termed magnetic and electric forces. Hence, we can conclude that the EMT in the Minkowski form provides a description of both electric and magnetic forces in the medium, i.e., electromagnetic forces are described comprehensively, whereas the EMT in the Abraham form provides a description of magnetic forces only. This is another indication of the incomplete nature of the Abraham tensor. The electric and magnetic forces are generally different in magnitude, and the difference in the electromagnetic force is the Abraham force. Because the Abraham tensor does not contain this force, it follows that in order to obtain correct results, it should be supplemented by the Abraham force [2, p. 315]. The most general form of the Abraham force can be written as the difference between the expressions for the change in momentum in the Minkowski form and in the Abraham form [3]:

$$\mathbf{F}^{\mathrm{A}} = \partial_t \mathbf{g}^{\mathrm{M}} - \partial_t \mathbf{g}^{\mathrm{A}} = \frac{\partial_t (\mathbf{D} \times \mathbf{B})}{4\pi c} - \frac{\partial_t (\mathbf{E} \times \mathbf{H})}{4\pi c}$$
$$= \frac{1}{4\pi c} \partial_t (\mathbf{D} \times \mathbf{B} - \mathbf{E} \times \mathbf{H}). \quad (9)$$

Equations (7) and (8) that follow from the Minkowski tensor allow writing the Abraham force as the difference between the divergences of its stress tensor t_{ik}:

$$F_i^{\mathrm{A}} = \partial_t g_i^{\mathrm{M}} - \partial_t g_i^{\mathrm{A}} = \partial_k t_{ik} - \partial_k t_{ki},$$

or

$$F_i^{\mathrm{A}} = \partial_k \left(\frac{E_i D_k + H_i B_k}{4\pi} - \delta_{ik} \frac{\mathbf{ED} + \mathbf{HB}}{8\pi} \right)$$
$$- \partial_k \left(\frac{E_k D_i + H_k B_i}{4\pi} - \delta_{ik} \frac{\mathbf{ED} + \mathbf{HB}}{8\pi} \right) = \frac{1}{4\pi} [\nabla \times (\mathbf{D} \times \mathbf{E} + \mathbf{B} \times \mathbf{H})]_i.$$

Finally, the equation for the Abraham force takes the form

$$\mathbf{F}^{\mathrm{A}} = \frac{1}{4\pi c} \partial_t (\mathbf{D} \times \mathbf{B} - \mathbf{E} \times \mathbf{H}) = \frac{1}{4\pi} \nabla \times (\mathbf{D} \times \mathbf{E} + \mathbf{B} \times \mathbf{H}). \quad (10)$$

Equation (10) supports the conclusion made in Section 3 that the Abraham force has a vortical nature and that its divergence is zero. Equation (10) that follows from the Minkowski tensor imposes no restrictions on the constitutive equations and is universal for any nonconducting continuous medium. Because Eqn (10) follows from the momentum density conservation equations (4) and (6), it can also be viewed as a momentum density conservation equation.

An important conclusion from Eqn (10) is that if the constitutive equations of the medium have the canonical forms $\mathbf{D} = \varepsilon\varepsilon_0 \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/(\mu\mu_0)$ with ε and μ constants or scalar functions, then the vectors $\mathbf{D}$ and $\mathbf{E}$, $\mathbf{H}$ and $\mathbf{B}$ are collinear and the right-hand side of Eqn (10), containing the vector products of these vectors, is zero. The Abraham force is then zero, and the EMT (1) is symmetric. This suggests equating the Abraham force

$$\mathbf{F}_{\mathrm{A}} = \frac{\varepsilon\mu - 1}{4\pi c} \partial_t (\mathbf{E} \times \mathbf{H}),$$

which is often used in this case, to zero. Then the off-diagonal components of the stress tensor t_{ik} also vanish, and the electromagnetic forces acting on the medium are determined only by its diagonal terms. The equation for the electromagnetic forces can then be written in the form

$$\mathbf{F} = \partial_t \mathbf{g}^{\mathrm{M}} = \partial_t \mathbf{g}^{\mathrm{A}} = \frac{\partial_t(\varepsilon\varepsilon_0 \mathbf{E}\times\mathbf{B})}{4\pi c} = \frac{\partial_t\big((\mathbf{E}\times\mathbf{B})/\mu\mu_0\big)}{4\pi c}$$

$$= -\frac{1}{8\pi\mu_0}\,\nabla\left(\frac{\varepsilon\mathbf{E}^2}{c^2} + \frac{\mathbf{B}^2}{\mu}\right).$$

This last equation for electromagnetic forces also follows from the Abraham EMT. While the above results are obtained for media at rest, the relativistic covariance of the Minkowski EMT has the consequence that the equations that follow from it also hold for a uniformly moving medium, if well-known transformation formulas are used.

5. Conclusion

Thus, the Minkowski EMT leads to conservation equations for the momentum density in both Minkowski form (6) and Abraham form (4) and also produces Eqn (10) for the Abraham force, which shows that this force is vortical and has zero divergence. Hence, discussing which form of the momentum density is 'correct' is addressing a problem that does not exist: both forms are correct and equal in value, because both follow from the Minkowski tensor (not from the Abraham tensor). Figuratively speaking, both forms of the momentum density are two sides of the same coin and complement each other. Because the Abraham tensor, unlike the Minkowski tensor, is symmetric, it cannot directly lead to an expression for the Abraham force in terms of the stress tensor (which is possible in the case of the Minkowski tensor). Therefore, the Abraham tensor is augmented by the Abraham force obtained in a different way, after which it becomes identical to the Minkowski tensor [2, p. 317]. Only then can it be used and the correct result can be obtained.

In this connection, it was noted in [3, p. 185] that the Abraham force should be found "from experimental data or by going beyond the macroscopic field determining equations themselves that yield only the conservation law or its immediate consequences." This statement refers to the Abraham tensor but not to the Minkowski tensor containing all the necessary information. Thus, as far as the Abraham tensor is concerned, the Abraham force is an extraneous construct needed to ensure the correctness of the Abraham tensor but whose physical nature is unclear and can only be determined by going beyond the macroscopic field equations. As far as the Abraham tensor is concerned, the Abraham force is, in effect, its component part: its physical essence and description are defined precisely and follow immediately from the Minkowski tensor given by Eqn (10), and there is no need to go beyond the EMT.

It follows from the foregoing that preference should definitely be given to the Minkowski tensor, which provides a comprehensive description of energy processes and electromagnetic forces in a medium, including the description of the Abraham force. For material media with no Abraham force, i.e., with the vectors **D** and **E**, **H** and **B** collinear, the Abraham tensor can also be used.

Because the choice of the EMT form is not yet a settled issue, some researchers use the Abraham tensor in their treatment of the electrodynamics of continuum media, which leads to potentially incorrect results and conclusions. We hope that this paper will be useful in correctly choosing the EMT form and using the right approach for further research.

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