

METHODOLOGICAL NOTES

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Can we refer to Hamilton's equations for an oscillator with friction?

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Contents

Abstract. A formal possibility of describing a one-dimensional dissipative problem $\ddot{x} = f(x, \dot{x})$ with completely conservative Lagrange's or Hamilton's equations is discussed. A reference case of a harmonic oscillator with linear friction is considered in detail.

Keywords: classical mechanics, Lagrange's equations, Hamilton's equations, energy integral, conservative and dissipative systems

1. Introduction

In classical physics, we are used to accounting for dissipation through explicitly given dissipative forces in generalized Euler–Lagrange's or Hamilton's equations, for example, by introducing the Rayleigh dissipative function [1–5]. The resultant equations of motion are conventionally called nonconservative to indicate that they (i) do not have the time reversal symmetry and (ii) cease to be a consequence of Hamilton's variational principle for an action functional. In this paper, we intend to remind the readers that the last assumption is not always valid, even though it is widely believed.

In 1887, Helmholtz formulated a problem known today as the inverse Lagrange problem for discrete systems. Up to the notation, Helmholtz's question was like this [6]: under which conditions is the system of n equations of the form $\ddot{x}_i - f_i(x, \dot{x}) = 0$ a system of Euler–Lagrange equations? Helmholtz succeeded in answering this question only partly: he found necessary and sufficient conditions for the functions $f_i(x, \dot{x})$ ensuring the existence of a Lagrange function, common to all equations, such that

$$\ddot{x}_i - f_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}\,.$$

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In 1957, Havas published a more complete solution to Helmholtz's problem [7]. Havas noticed that the equations of the original system can be multiplied with integrating multipliers and regrouped, which considerably broadens the class of systems that can be reduced to Lagrange's equations. In particular, it became possible to formulate necessary and sufficient conditions that should be imposed on the functions $f_i(t, x, \dot{x})$ (time dependent forces are allowed) to ensure the existence of a set of integrating multipliers $\mu_i(t, x, \dot{x})$ and a Lagrange function $L(t, x, \dot{x})$ such that

$$\mu_i(\ddot{x}_i - f_i) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}\,. \tag{1}$$

Havas found that his conditions are satisfied by a broad class of dissipative problems. Fully independently, and in a more general formulation, the "inverse problem of variational calculus" was explored by mathematicians [8].

In addition to rather general studies by Helmholtz and Havas, numerous papers have been published on various concrete approaches and techniques enabling a description of a dissipative system in the framework of a conservative formalism with the help of a Lagrange or Hamilton function. Interest in this topic is related to the following two circumstances. First, conservative equations that follow from the variational principle necessarily have properties that are nontrivial and desirable for a physical theory, such as the relation of symmetries and integrals of motion (Noether's theorem), the independence of Lagrange's equations from the choice of coordinates (covariant form), and the possibility of using the Hamilton–Jacobi method and canonical transformations in Hamilton's equations. Second, the conservative form of dissipative classical systems is a convenient departure point for developing the theory of quantum dissipative systems. In light of this last circumstance, of special importance is the case of a one-dimensional oscillator with linear friction as the simplest and simultaneously most frequently used model describing the behavior of a dissipative system in a state close to equilibrium. It is for this system that the link between the classical and quantum descriptions has been studied most thoroughly. The most complete exposition of questions just mentioned can be found in monograph [9], which considers classical and quantum systems of a general form from a common standpoint, and in reviews [10, 11] devoted almost exclu-

sively to the linear oscillator with friction. Not aiming at an exhaustive review, we formulate three main areas of this research.

The first uses the idea of an extended phase space, as can be explained with the following example. We consider a classical one-dimensional oscillator with viscous friction

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0\,. \tag{2}$$

From a physical standpoint, friction leads to irreversible energy dissipation; however, this can be formally excluded by introducing an extra dimension. For example, it suffices to consider a system of two noninteracting oscillators

$$\begin{cases} \ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0\,, \\ \ddot{y} - 2\gamma\dot{y} + \omega_0^2 y = 0\,, \end{cases} \tag{3}$$

defined by the Lagrangian [12]

$$L = \dot{x}\dot{y} + \gamma(x\dot{y} - \dot{x}y) - \omega_0^2 xy = 0\,.$$

The nonperiodic decaying and amplifying solutions $x(t)$ and $y(t)$ are intertwined in such a curious way that they preserve the integral of motion $H = \dot{x}\dot{y} + \omega_0^2 xy$. A rigorous development of the idea of extra dimensions leads to the formalism of a 'complex Hamiltonian' to describe the dynamics of classical and quantum systems with linear dissipation [13–15]. Recently, it has been shown that the hidden dimension responsible for time irreversibility can also be taken into account directly when formulating the variational principle [16]. The variational problem proposed in Ref. [16] automatically leads to Eqns (3) for the given potential and dissipative forces.

The second area is based on the possibility of accounting for time irreversibility by explicitly including it in the Lagrangian or Hamiltonian [13, 17, 18]. For example, Eqn (2) is generated by the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 - \omega_0^2 x^2)\exp(2\gamma t)\,.$$

In accounting for dissipation, we have to deal with a nonstationary 'mass'. This technically simplest approach can be elementarily generalized to multidimensional problems with linear friction. For this reason, the method, commonly referred to as the BCK Lagrangian method in honor of its discoverers Bateman, Caldirola, and Kanai, became the most widespread in the theory of quantum dissipative systems [9, 10]. In classical systems, the application of Noether's theorem allows using the hidden symmetries of the BCK Lagrangian [19].

The third area is based on an attempt to describe irreversible dissipation from first principles as a 'thermodynamic' limit of interaction with a conservative system of very many dimensions [20–26] (see also [10, § 5]). As a rule, such an approach leads to a nonlocal dissipation operator, with the result being weakly sensitive to the particulars of the selected micromodel. For example, for the Lagrangian describing the interaction of an individual harmonic oscillator with an ensemble of harmonic oscillators,

$$L = \frac{1}{2}(\dot{x}^2 - \omega_0^2 x^2) + \sum_{j=1}^{N}\left(\frac{1}{2}\,\dot{y}_j^2 - \frac{1}{2}\,\omega_j^2 y_j^2 + \kappa_j xy_j\right),$$

under the condition that at $t = 0$ all the oscillators y_i are at rest with zero displacement, the equation for x takes the form

$$\ddot{x} + \omega_0^2 x + \int_0^t K(t - t')\,x(t')\,\mathrm{d}t' = 0\,,$$

where

$$K(t - t') = \sum_{j=1}^{N}\frac{\kappa_j^2}{\omega_j}\sin\left[\omega_j(t - t')\right].$$

The standard viscous friction corresponds to the limit $K = 2\gamma\delta'(t - t')$, which is realized for $N \gg 1$ and a special choice $\kappa_j \propto \omega_j$. An example of the recent application of this approach for quantum systems can be found in Ref. [27]. From the mathematical standpoint, this approach resembles the method of fractional derivatives, also proposed for the description of nonlocal dissipation in the framework of the conservative formalism [28, 29].

In this study, we consider in detail one more example of the conservative description of the one-dimensional motion of a classical particle with linear friction, which is not well known, is simple, and is rather insightful for physicists in our opinion.

2. Conservative description

An impetus to write this paper was exercise no. 12.95 in the well-known book of exercises on theoretical mechanics by Pyatnitsky et al. [30], which proposes to solve the one-dimensional problem of Helmholtz and Havas. More precisely, it is proposed to prove that *any one-dimensional equation of the form* $\ddot{x} = f(t, x, \dot{x})$ *can be reduced to the Lagrangian form by multiplying its right-hand and left-hand sides by some function* $\mu(t, x, \dot{x})$. Here, what is meant is that the new equation must be fully determined by the Lagrangian $L(t, x, \dot{x})$, i.e., relation (1) must hold even if dissipative forces are present in the original equation.

A formal solution of this problem takes just several lines. We assume that there is a Lagrangian such that the original equation is equivalent to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{x}}\right) \equiv \frac{\partial^2 L}{\partial t\,\partial\dot{x}} + \dot{x}\,\frac{\partial^2 L}{\partial x\,\partial\dot{x}} + \ddot{x}\,\frac{\partial^2 L}{\partial\dot{x}^2} = \frac{\partial L}{\partial x}\,.$$

We differentiate this equation with respect to velocity keeping the coordinate fixed:

$$\frac{\partial^3 L}{\partial t\,\partial\dot{x}^2} + \dot{x}\,\frac{\partial^3 L}{\partial x\,\partial\dot{x}^2} + \frac{\partial}{\partial\dot{x}}\left(\ddot{x}\,\frac{\partial^2 L}{\partial\dot{x}^2}\right) = 0\,.$$

Hence, it follows that Lagrange's equations are equivalent to the original equation $\ddot{x} = f$ if and only if the quantity

$$\mu(t, x, \dot{x}) \equiv \frac{\partial^2 L}{\partial\dot{x}^2}$$

satisfies the equation

$$\frac{\partial}{\partial t}\,\mu + \frac{\partial}{\partial x}(\dot{x}\mu) + \frac{\partial}{\partial\dot{x}}(f\mu) = 0\,. \tag{4}$$

The integrating multiplier $\mu(t, x, \dot{x})$ can be interpreted as an 'effective mass' such that multiplying the original equation with it yields the equation of motion under the action of the

'Lagrangian' force μf. The effective mass obeys the continuity equation in the two-dimensional space $(x, \dot{x})$, i.e., it is carried along the streamlines $\mathbf{j} = (\dot{x}, f)$, which are uniquely determined by the original equation of motion. From this condition, the Lagrangian can be recovered for any f.

In particular cases, relation (4) allows finding the simplest classes of f that admit the Lagrangian description. The substitution $\mu = 1$ gives rise to the obvious family of potential forces $f = f_0(t, x)$. The substitution $\mu = \mu(t)$ leads to the problem with linear friction, $f = f_0 - (\dot{\mu}/\mu)\dot{x}$. Linear friction with a constant coefficient gives rise to the BCK Lagrangian. The substitution $\mu = \mu(x)$ gives the known class of problems with friction quadratic in velocities, $f = f_0 - (\mu'/2\mu)\dot{x}^2$. Such friction with a coefficient that is constant in time leads to an autonomous Lagrange function, preserving the energy integral, and the respective force is therefore frequently referred to as pseudofriction.

A stricter statement is valid: *if the right-hand side of a one-dimensional equation of motion does not depend explicitly on time, the problem can always be reduced to autonomous Lagrange's and Hamilton's equations*.[1]

We can arrive at this conclusion by introducing a time-independent effective mass in the manipulations above from the very beginning. More revealing would be to use the general property of conservative systems to conserve the integral of generalized energy. We suppose that the equation of motion is generated by a Lagrangian $L(x, \dot{x})$, which does not depend on time explicitly. Then the generalized energy [1–5]

$$H(x, \dot{x}) = \dot{x}\,\frac{\partial L}{\partial \dot{x}} - L \tag{5}$$

is conserved on solutions of the original equation,

$$\dot{x}\,\frac{\partial H}{\partial x} + f(x, \dot{x})\,\frac{\partial H}{\partial \dot{x}} = 0\,. \tag{6}$$

A solution of (6) is any time-independent integral of motion $H(x, \dot{x})$ of the original equation, as well as any its function $\mathcal{H}(H(x, \dot{x}))$. Using relation (5), we can use the generalized energy to reconstruct the Lagrangian

$$L(x, \dot{x}) = \dot{x}\int^{\dot{x}} \frac{1}{v^2}\,\frac{\partial H(x, v)}{\partial v}\,\mathrm{d}v\,, \tag{7}$$

the canonical momentum

$$p(x, \dot{x}) = \frac{\partial L}{\partial \dot{x}} = \int^{\dot{x}} \frac{1}{v}\,\frac{\partial H(x, v)}{\partial v}\,\mathrm{d}v\,, \tag{8}$$

the effective mass

$$\mu(x, \dot{x}) = \frac{\partial^2 L}{\partial \dot{x}^2} = \frac{1}{\dot{x}}\,\frac{\partial}{\partial \dot{x}}\,H(x, \dot{x})\,, \tag{9}$$

and, having expressed the generalized energy as a function of the coordinate and momentum, the Hamiltonian

$$H(x, p) = H\big(x, \dot{x}(x, p)\big)\,.$$

It can readily be seen that in a stationary case, the equation for effective mass (4) is a consequence of Eqns (6) and (9). In this case, Eqn (6) for the integral of motion frequently proves to be more convenient than the equation for the effective mass, because it has a simpler solution.

The integral of motion for an autonomous second-order equation can be found with the help of a formal substitution $\dot{x} = y(x)$. In this case, the original equation $\ddot{x} = f$ reduces to the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x, y)}{y}\,. \tag{10}$$

If we can find the general solution $y = y(x, H)$ for Eqn (10), which depends on an arbitrary constant H, then, to construct the integral of motion, it suffices to express this constant though the coordinate and velocity

$$\dot{x} = y(x, H) \;\Rightarrow\; H = H(x, \dot{x})\,.$$

If the general solution of the original equation for $x(t)$ is known for all possible initial conditions, the integral of motion can be found without solving Eqn (10). Projecting all possible $x(t)$ on the two-dimensional plane $(x, \dot{x})$, we can build the system 'phase portrait'. Because the phase curves of autonomous equations can intersect only at special points, the possibility always exists of labeling all the trajectories by a continuous parameter H. This means that the equations for the entire family of phase trajectories can be represented in the form

$$x = x(t, H)\,, \qquad \dot{x} = \frac{\partial}{\partial t}\,x(t, H)\,.$$

Eliminating time from these relations, we conclude that the phase trajectories obey the equation $H(x, \dot{x}) = \text{const}$. Evidently, $H(x, \dot{x})$ found in this way, or any function of it, is the sought integral of motion.

We see that the possibility of describing an autonomous system on a plane with the help of a stationary Lagrange or Hamilton function is only related to the possibility of 'counting' phase trajectories. The statements given above remain valid for truly dissipative equations that include friction. Here, we face a contradiction with the conventional paradigm. Physicists are used to contrasting the properties of conservativeness and dissipation. In many courses of theoretical mechanics and the theory of oscillations, dissipation is defined as contributions (terms in the equations) that violate the conservation of the energy integral and hence the symmetry characteristic of conservative Hamilton's or Lagrange's equations. Reciprocally, the existence of a conserved energy integral is attributed to conservativeness [2–5, 31]. A formal way of resolving this contradiction lies in the definition of the energy integral, which does not need to be reducible to the total mechanical energy of the system or its analogs in nonmechanical systems in the presence of dissipative forces. The distinction between conservative and dissipative systems is blurred, at least in the one-dimensional case.

3. Linear oscillator with viscous friction

In keeping with the title of this paper, we describe a damped oscillator with viscous friction, obeying Eqn (2), in the framework of the fully conservative formalism. As mentioned, the possibility of introducing a stationary Lagrange function is related to the procedure whereby all phase trajectories are labeled.

[1] This statement is formulated as Theorem I.1 in Ref. [7], and kinetic equation (4) for the effective mass reduces to the statement of Theorem I.2.

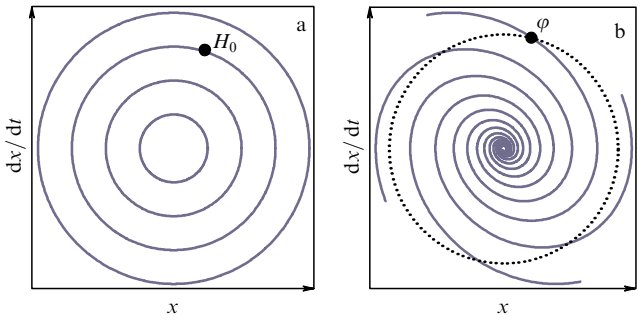

Figure 1. 'Center' equilibrium state of (a) a nondamped linear oscillator and (b) the 'focus' equilibrium state for a linear oscillator with viscous friction. Solutions with the 'node'-type equilibrium state are also possible for a damped oscillator.

Figure 1 displays the well-known phase portraits that illustrate the sets of such trajectories for undamped and damped oscillators. Here, it is important that these two cases are *topologically* different. For the undamped oscillator, the role of the label is played by the conventional 'mechanical' energy

$$H_0 = \frac{1}{2}\,\dot{x}^2 + \frac{1}{2}\,\omega_0^2 x^2\,.$$

The phase trajectory is defined as the line of constant H_0, and the oscillation phase determines only the position of an image point on this trajectory. For a damped oscillator, the situation is directly the opposite: the trajectory is uniquely specified by the phase, while the mechanical energy determines the position on the phase curve. Indeed, all trajectories pass through any given section $H_0 = \text{const}$ (depicted by dots in Fig. 1b) exactly once, because, for a damped oscillator, the derivative $\mathrm{d}H_0/\mathrm{d}t = -2\gamma\dot{x}^2$ preserves its sign in solutions. Thus, a trajectory is uniquely specified by its intersection point with some selected curve $H_0 = \text{const}$. We characterize this intersection point by some phase φ. The phase introduced in this way can play the role of a Hamiltonian for a linear oscillator with viscous friction. We note that the phase is a single-valued integral of motion everywhere except the equilibrium state and a ray emanating from there, where it jumps by 2π.

For definiteness, we consider a 'focus'-type equilibrium state, which corresponds to a real frequency $\omega = (\omega_0^2 - \gamma^2)^{1/2} > 0$. In this case, the solutions of Eqn (2) can be written as

$$x = a\exp(-\gamma t)\cos(\omega t + \varphi)\,,$$
$$\dot{x} + \gamma x = -a\omega\exp(-\gamma t)\sin(\omega t + \varphi)\,.$$

From these relations, we readily find equations for the instantaneous phase and time:

$$\theta(x,\dot{x}) \equiv \omega t + \varphi = -\arctan\frac{\dot{x}+\gamma x}{\omega x}\,,$$
$$a^2\omega^2\exp(-2\gamma t) = \omega^2 x^2 + (\dot{x}+\gamma x)^2\,.$$

We eliminate time, assuming without loss of generality that $a\omega = 1$ in the last equation (the coordinate in this case has the dimension of time). As a result, we obtain the initial phase expressed in terms of instantaneous values of coordinate and velocity:

$$\varphi(x,\dot{x}) = -\arctan\frac{\dot{x}+\gamma x}{\omega x} + \frac{\omega}{2\gamma}\ln\left(\omega^2 x^2 + (\dot{x}+\gamma x)^2\right).$$

By construction, $\varphi(x,\dot{x})$ is an integral of motion of Eqn (2), and hence an arbitrary function

$$H = \mathcal{H}(\varphi)$$

can be treated as the generalized energy. We note that the same result can be obtained directly from Eqn (10), whose implicit solution is in our case given by $\varphi(x,\dot{x}) = \text{const}$.

The instantaneous phase $\theta(x,\dot{x})$ is a monotonic function of velocity for a fixed coordinate; hence, it can be used instead of the velocity. The initial phase and velocity can be written as

$$\varphi = \theta + \frac{\omega}{\gamma}\ln\left|\frac{\omega x}{\cos\theta}\right|, \qquad \dot{x} = -\gamma x\,\frac{\partial\varphi}{\partial\theta}\,.$$

Such a representation is convenient for computing the generalized momentum by formula (8) because all additional factors that occur in using the chain rule to calculate the derivative drop out:

$$p = \int^{\dot{x}}\frac{1}{\upsilon}\,\frac{\partial H(x,\upsilon)}{\partial\upsilon}\,\mathrm{d}\upsilon = \int^{\theta}\frac{1}{\dot{x}}\,\frac{\partial\mathcal{H}}{\partial\varphi}\,\frac{\partial\varphi}{\partial\theta}\,\mathrm{d}\theta = -\frac{1}{\gamma x}\int^{\theta}\frac{\partial\mathcal{H}}{\partial\varphi}\,\mathrm{d}\theta\,. \tag{11}$$

Inserting the known dependence $\theta(x,\dot{x})$, we obtain the Lagrange function

$$L(x,\dot{x}) = \dot{x}\,p(x,\dot{x}) - \mathcal{H}\left(\theta(x,\dot{x}) + \frac{\omega}{\gamma}\ln\left|\omega x\sec\theta(x,\dot{x})\right|\right).$$

And if we succeed in expressing the instantaneous phase as a function of the coordinate and generalized momentum from relation (11), we obtain the Hamilton function

$$H(x,p) = \mathcal{H}\left(\theta(x,p) + \frac{\omega}{\gamma}\ln\left|\omega x\sec\theta(x,p)\right|\right).$$

Thus, the problem that is dissipative from the physical standpoint is described by a fully conservative Hamiltonian. We do not even need to redefine the time and coordinate: the job of taking dissipation into account is entirely done by the generalized momentum.

We consider the simplest case, which is realized when the initial phase is taken as the energy integral:

$$\mathcal{H} = \varphi\,, \qquad \frac{\partial\mathcal{H}}{\partial\varphi} = 1\,.$$

In this case, $\theta(x,p) = -\gamma x p$. The corresponding Hamilton function

$$H = -\gamma p x + \frac{\omega}{\gamma}\ln\left|\frac{\omega x}{\cos(\gamma p x)}\right| \tag{12}$$

leads to Hamilton's equations

$$\begin{cases}\dot{x} = -x\left[\gamma - \omega\tan(\gamma p x)\right],\\ \dot{p} = -\dfrac{\omega}{\gamma x} + p\left[\gamma - \omega\tan(\gamma p x)\right].\end{cases} \tag{13}$$

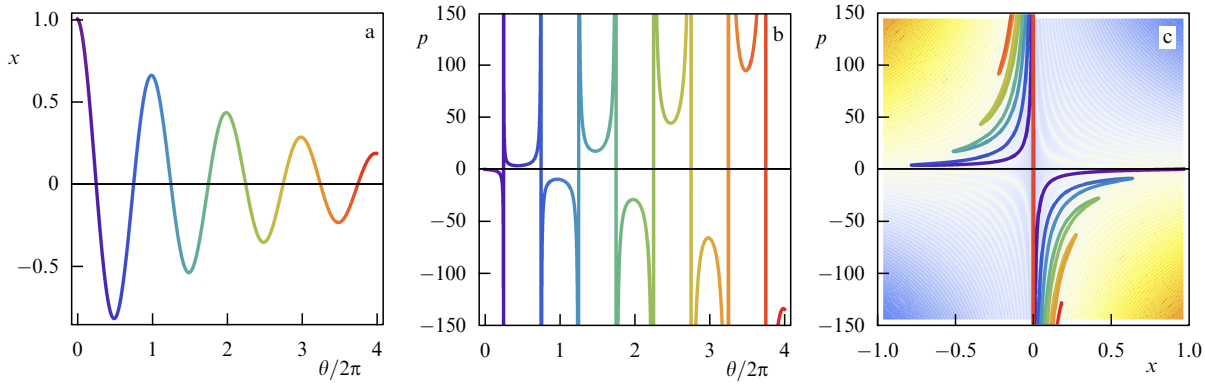

Figure 2. (Color online.) (a, b) Solution of Eqns (13) for $\gamma = 1$ and $\omega = 15$. (c) The same trajectory in the phase space (x, p); the background color indicates the value of Hamiltonian (12): the phase trajectory lies in a valley and corresponds to a constant value of the Hamiltonian.

It is easy to see that these equations, after eliminating p and $\dot{p}$, are indeed reduced to Eqn (2) describing an oscillator with viscous friction. We note that by virtue of Hamilton's equations,

$$\dot{x}p + x\dot{p} = -\frac{\omega}{\gamma}\,,$$

which means that γpx plays the role of an instantaneous phase (it varies linearly with time). Hence, it immediately follows that the conservation of Hamiltonian (12) is related to the coordinate that varies by a decaying harmonic law.

Figure 2 gives an example of a phase trajectory for Hamiltonian (12). Evidently, this trajectory corresponds to a constant value of the Hamiltonian, and therefore any finite motion must be strictly periodic. In the dissipative case, motion is infinite: staying finite with respect to the coordinate, it is characterized by the generalized momentum going to infinity in a finite time. This happens twice per period when the coordinate passes through zero. We see that the solution for the damped oscillator is glued from pieces, each corresponding to a strictly conservative problem, whereas all the fragments correspond to the same energy integral. *Such a bundle is unavoidable when we describe motion in a plane, finite and nonperiodic from the physical standpoint, with the help of the conservative formalism*. For this reason, the singularity in momentum cannot be eliminated by the choice of the energy integral. The 'strange' behavior of the generalized momentum is a signature of the dissipative nature of the original problem.

These singularities are formally absent in the Lagrangian description, because it leads immediately to a second-order equation for the coordinate. Indeed, using the general formula, we find that the generalized energy $H = \varphi(x, \dot{x})$ corresponds to the Lagrange function

$$L = \frac{\dot{x} + \gamma x}{\gamma x} \arctan \frac{\dot{x} + \gamma x}{\omega x} - \frac{\omega}{2\gamma} \ln\left[\omega^2 x^2 + (\dot{x} + \gamma x)^2\right].$$

This function leads to Lagrange's equation that coincides with the equation for an oscillator with friction up to multiplication by the effective mass:

$$\mu = \frac{\omega}{\gamma}\left(\omega^2 x^2 + (\dot{x} + \gamma x)^2\right)^{-1} > 0\,.$$

The positive-definite effective mass does not contain singularities related to dissipation, except at the trivial point $x = \dot{x} = 0$; in the Lagrange formalism, they appear only in the integral for the generalized energy.

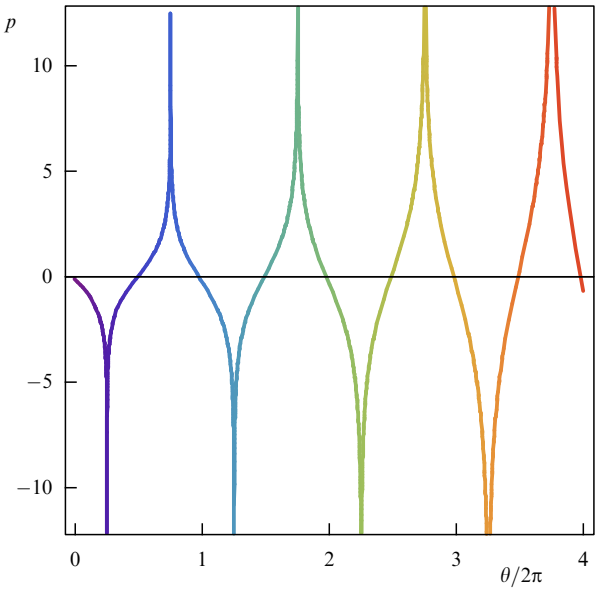

Figure 3. (Color online.) Dependence $p(\theta)$ for $\mathcal{H} = \exp(\gamma\varphi/\omega)$ with $\gamma = 1$ and $\omega = 15$.

We consider one more class of conservative equations generated by the integral of motion

$$\mathcal{H} = \exp\frac{\gamma\varphi}{\omega}\,. \tag{14}$$

In this case, generalized momentum (11) depends only on the instantaneous phase and does not depend on the coordinate:

$$p(\theta) = -\int^{\theta} \exp\left(\frac{\gamma\theta'}{\omega}\right)\frac{\mathrm{d}\theta'}{\cos\theta'}\,.$$

This integral can be expressed in terms of the hypergeometric function ${}_2F_1$:

$$p(\theta) = -\mathrm{Im}\left\{\frac{2\omega}{\omega - \mathrm{i}\gamma}\exp\left[\left(\mathrm{i} + \frac{\gamma}{\omega}\right)\theta\right] \times {}_2F_1\left(1, 1 - \frac{\mathrm{i}\gamma}{\omega}, \frac{3}{2} - \frac{\mathrm{i}\gamma}{2\omega}, -\exp(2\mathrm{i}\theta)\right)\right\}.$$

Figure 3 shows the dependence $p(\theta)$ for the same set of parameters as in Fig. 2. Clearly, because of the cosine in the

denominator, the generalized momentum becomes infinite twice per period. The corresponding Hamiltonian has the form

$$H = \frac{\omega x}{\cos\theta(p)} \exp\left(\frac{\gamma}{\omega}\,\theta(p)\right), \tag{15}$$

and Hamilton's equations are written as

$$\begin{cases} x = -x\bigl(\gamma + \omega\tan\theta(p)\bigr), \\ \dot p = -\dfrac{H}{x}. \end{cases}$$

From these equations, it can be readily seen that the phase varies linearly with time, $\dot\theta = \dot p\,\partial\theta/\partial p = \omega$, and hence the conservation condition for the Hamiltonian immediately implies that the coordinate must satisfy the decaying harmonic law.

It is of interest to trace the limit $\gamma \to 0$ to the oscillator without dissipation. This can be done most simply with the help of representation (14), (15). Indeed, having set $\gamma = 0$, we obtain $\tanh p = -\sin\theta$, whence we find Hamilton's equations

$$H = \pm\omega_0 x\cosh p \;\Rightarrow\; \begin{cases} \dot x = \pm\omega_0 x\sinh p, \\ \dot p = \mp\omega_0\cosh p. \end{cases}$$

Eliminating the momentum, we obtain the correct equation for the harmonic oscillator; however, the equations we started from describe continuous motion only during half a period, because the momentum tends to infinity at $\theta = \pi/2 + n\pi$. These singularities reflect the fact that we are dealing all the time with the integral of motion in the form of phase, and phase trajectories in the vicinity of 'center'-type nondissipative equilibrium cannot be parameterized by the phase!

We therefore see that the 'focus'-type dissipative equilibrium state can be described in the framework of the conservative formalism, sacrificing the uniqueness of the generalized energy integral at a single point. Indeed, the phase jump by 2π is not of a fundamental nature and can be eliminated, for example, by choosing $\mathcal{H} = \sin(2\varphi)$. At the same time, the singularity related to contraction of all trajectories to a single equilibrium point is an important and irreducible property of dissipative systems. Even though the system arrives at the equilibrium state in infinite time, this point defines the topology of the phase portrait as a whole.

One more type of dissipative equilibrium on the plane, a node, can be considered similarly. If we require the uniqueness of the energy integral on the entire phase space, as, for example, is done in monograph [32], then the traditional criterion of conservativeness is recovered. Based on geometric considerations, it can be proved that in that case only conservative equilibrium states — saddles and centers — can exist on the phase plane.

4. Conclusions

The possibility of describing an autonomous system on a plane with the help of stationary Lagrange or Hamilton functions is related only to the possibility of enumerating the phase trajectories with a real-valued index. From a mathematical perspective, this can always be done, and in an infinite number of ways, irrespective of whether the system is conservative in the conventional sense. However, for a two-dimensional phase space, finite nonperiodic motions are impossible, and therefore the description of physical dissipative problems leads unavoidably to infinite solutions in generalized coordinates and momenta. In Lagrange's formalism, this can be 'masked' by selecting a representation with a bounded coordinate, because the canonically conjugate momentum does not appear in Lagrange's equations. In Hamilton's formalism, the momentum is an independent variable, and the appearance of infinite trajectories is unavoidable. A global solution is obtained by gluing a series of local, but strictly conservative solutions, defined on a limited time interval. On a local level, there is indeed no difference between dissipative and conservative systems, and there is similarly no difference from the topological standpoint between the families of ellipses and spirals plotted in Fig. 1 in any simply connected region in phase space excluding the equilibrium point.

To recover the conventional division of autonomous physical systems into conservative and dissipative, it is apparently sufficient to require the existence of a single-valued energy integral for all admissible points in the phase space.

We note that the general discussion above does not extend directly to multidimensional systems, in which the Hamiltonian character of the system is a more complex property than the existence of an integral of motion [33, 34]. The conditions for the reduction of a multidimensional system are more naturally described in the framework of the Euler–Lagrange formalism [6–8]. There are simple examples where the Lagrange function is absent. In particular, the equations of two linearly coupled harmonic oscillators

$$\begin{cases} \ddot x + 2\gamma_1\dot x + \omega_1^2 x = k_1 y, \\ \ddot y + 2\gamma_2\dot y + \omega_2^2 y = k_2 x, \end{cases}$$

can be derived from the variational principle if and only if the oscillators are either uncoupled ($k_1 = k_2 = 0$) or coupled, but have different decay decrements ($k_1k_2 \neq 0$ and $\gamma_1 \neq \gamma_2$). This statement follows from the general Theorem VI formulated in Ref. [7] (see also problem no. 13.113 in [30]). For example, if one of the coupling constants is zero and the other is different from zero, the system cannot be reduced to a Lagrangian system for any selection of the remaining parameters [a necessary condition in Ref. [7], (II.2), is violated].

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