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Quasilinear theory of plasma turbulence. Origins, ideas, and evolution of the method

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Contents

Abstract. The quasilinear method of describing weak plasma turbulence is one of the most important elements of current plasma physics research. Today, this method is not only a tool for solving individual problems but a full-fledged theory of general physical interest. The author's objective is to show how the early ideas of describing the wave–particle interactions in a plasma have evolved as a result of the rapid expansion of the research interests of turbulence and turbulent transport theorists.

Keywords: quasilinear theory, turbulent transport, diffusion coefficients, stochastic magnetic field, plasma

1. Introduction

The quasilinear theory of weak plasma turbulence is indisputably a key element of modern plasma physics [1–7]. Indeed, a voluminous book would be too small to cover all its aspects. Some issues concerning the quasilinear approach have been highlighted in numerous reviews and monographs, whereas others are either forgotten or stay on the sidelines amid the exponentially growing number of publications concerning plasma turbulence.

Ideas paramount to the creation of the quasilinear theory have not arisen in a vacuum. In fact, they emerged from the work of concrete researchers, among whom were mainly Soviet scientists, such as A A Vlasov, L D Landau, I A Akhiezer, Ya B Fainberg, B I Davydov, A B Migdal, V M Galitskii, B V Chirikov, and Yu L Klimontovich, to name but a few, and certainly Yu A Romanov and G F Filippov. Since a discussion of the studies reported by Vlasov and Landau has a long history, the present review is focused on a relatively small number of literature sources, allowing us to follow the formation of theoretical concepts used as a basis by A A Vedenov, E P Velikhov, and R Z Sagdeev for the development of the quasilinear method, not only as a tool for addressing specific problems, but also as a full-fledged theory of general physical interest.

For example, the concept of formation of a quasilinear 'plateau' immediately attracted the attention of theorists, as appears from V D Shafranov's preprint [8] issued in 1960, i.e., before the publication of fundamental articles [2–5] and only two months or so after a comprehensive discussion of the quasilinear theory at a LIPAN seminar.[1] The author already

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[1] LIPAN (Russian acronym) — Laboratory of Measurement Tools of the USSR Academy of Sciences, subsequently redesignated as the I V Kurchatov Institute of Atomic Energy (today's National Research Centre 'Kurchatov Institute').

represented the mechanism of plateau formation in the form of a 'mnemonic' rule allowing the state into which the system passes as a result of the development of Cherenkov type instability to be predicted. Actually, the quasilinear theory became for the next decade one of the few 'foci of comprehension' for plasma turbulence.

The elegance of considering interactions between resonant particles and a wave packet in the framework of the quasilinear approach accounts for the not infrequently formal attitude toward the notionally important findings of predecessors, whereas the quasilinear method itself is viewed now, after 50 years, as an almost 'literal' replication of the work of Vlasov and Landau [9–12]. However, it needs to be emphasized that the near 15-year time gap between Landau's article and the widely cited series of publications by Vedenov, Velikhov, and Sagdeev was a period of tremendous growth of plasma physics. It was therefore important for the purpose of the present review to identify publications dating back to the period from 1947 to the 1960s, crucial for establishing the basis of the quasilinear theory.

We start considering issues pertinent to the prehistory of the quasilinear theory from Davydov's article [13] and Galitskii's thesis [14] in the first place, because we are interested in the influence of the quasilinear method on the development of the theory of turbulent transport. It is a somewhat artificial limitation, because such an approach disregards earlier results obtained at the time when the Vlasov equation began to be extensively utilized. It is hardly possible to cover the whole knot of problems related to the theory of plasma waves and oscillations or describe the slowing-down of charged particles and beam propagation investigated by physicists during that period.

It is noteworthy that the first publication of the authors of Ref. [1] appeared as a preprint issued by the I V Kurchatov Institute of Atomic Energy in 1960 [in a year, they published a review under the same title in *Uspekhi Fizicheskikh Nauk* (*Physics–Uspekhi*)]. In 1958, volume 1 of the four-volume work having the general title *Plasma Physics and the Problem of Controlled Thermonuclear Fusion* came out. It contained material on declassified reports highlighting early thermonuclear studies, including the important article by B I Davydov [13] concerning the role of plasma oscillations in the context of diffusion and thermal conductivity research, actually written as early as 1951. In fact, it was the first article raising the question of the influence of the turbulence on transport processes.

On the other hand, the book of collected memories of A A Vedenov contains papers by Velikhov and Sagdeev describing the work of Galitskii [14] and Romanov and Filippov [2] as important sources of their own research. Notice that both the introduction to the thesis by Galitskii [14] and the article by Romanov and Filippov [2] include references to Davydov's pioneering work [13]. Unfortunately, the articles by Klimontovich [16] and Chirikov [17] published in 1958 and 1959 are rarely mentioned in connection with the quasilinear theory. Despite the absence of direct references to these articles, it is necessary to emphasize their notational and temporal relation to publications dealing with the quasilinear theory. Naturally, the list of predecessors could be extended by appending the authors of publications on the role of resonant particles [18, 19], beam instability [20, 21], etc. But for the purpose of the present review, it is quite sufficient to refer here to a few of the aforementioned studies [2, 13, 14, 16].

The quasilinear method is currently one of the most popular analytical tools for the study of weak plasma turbulence. Suffice it to say that all turbulent transport coefficients containing the quadratic dependence of pulsation amplitudes are traditionally called quasilinear coefficients. To date, however, i.e., more than half a century after the publication of Refs [1–7, 22–24], it is important not only to summarize the data obtained by the quasilinear method but also to characterize the most significant components of this approach. It is clear at first sight that the problem encompasses a few fundamental ideas, viz. the consideration of Cherenkov mechanisms, distinguishing between fastly and slowly evolving components of the distribution function, a description of the evolution of an excited plasma wave spectrum and diffusive evolution of the particle distribution function under weak turbulence conditions, nonlinear effects associated with the 'capture' of particles in potential wells, the formation of the stochastic layer in the vicinity of separatrices, and decorrelation mechanisms governed by stochastic instability.

Indeed, each of these components had been in some way or other investigated by predecessors of the authors of the quasilinear method. Nevertheless, comprehensive consideration of all the ideas forwarded by that time was certainly an important breakthrough. This is perfectly obvious now, even from the number of references to and applications of the quasilinear method.

There are a lot of valuable books in which the majority of the problems discussed in the present review are discussed [25–57]. However, most of them are devoted to special issues and intended largely for plasma physicists; moreover, the authors are limited by constraints imposed by the length of the publications and cannot give much attention to the discussion of the mutual influence of quasilinear and general physical concepts.

This review is focused on theoretical issues, bearing in mind that the authors of pioneering studies [1–5] and many later breakthrough studies were leading Soviet and foreign theorists. The choice of topics for the present article was far from random. The author's objective was to show how the early ideas of describing wave–particle interactions in a plasma have evolved as a result of the rapid expansion of the research interests of turbulence and turbulence transport theorists.

2. Influence of plasma oscillations on transport

Davydov's paper considered in this section is called "The influence of the oscillations of a plasma on its electrical and thermal conductivity" [13]. Actually, there had been no attempts to evaluate such effects before it came out. Transport coefficients were calculated from collisional processes making use of the particle mean free path as a key parameter. Limitations of the approach, taking into account only the interplay between electrons and ions at distances shorter than the Debye radius, gave Davydov reason to point to the importance of considering interactions between charged particles (electrons and ions) with reference to plasma waves. Therefore, he proposed, based on the standard structure of the kinetic equation, to introduce the effective mean free path l_{eff}:

$$\frac{1}{l_{\mathrm{eff}}} = \frac{1}{l_{\mathrm{turb}}} + \frac{1}{l_{\mathrm{e}}} + \frac{1}{l_{\mathrm{i}}} \,, \tag{1}$$

where l_{turb}, l_{e}, and l_{i} are the characteristic path scales under turbulence, electron collisions, and ion collisions. Then, the quasiclassical transport coefficients in a plasma assume the form

$$D = \frac{1}{3}\langle V\rangle l_{\text{eff}}\,, \quad \sigma = \frac{e^2 n}{k_{\text{B}} T}\,D\,, \quad \chi = 3\,\frac{n}{T}\,D\,. \tag{2}$$

Here, D is the diffusion coefficient of particles, σ is the electrical conduction coefficient of the plasma, χ is the thermal conduction coefficient of the plasma, and k_{B} is the Boltzmann constant. The author of Ref. [13] does not appear to have been interested in specific features imparted by the applied magnetic field, even though he was one of the first physicists involved through I V Kurchatov in the project on magnetic plasma confinement for the purpose of controlled thermonuclear fusion (CTF) [58]. According to A D Sakharov's recollections, L A Artsimovich and M A Leontovich discussed the ideas proposed by himself and I E Tamm at the end of 1950 and early in 1951 [59].

It is worthwhile to note that Davydov was a multidiscipline theorist [60] well-aware of analogies between specific plasma problems and the fundamentals of solid-state physics, in which interactions of electrons and lattice vibrations constitute a key mechanism [61, 62]. Comparing ion oscillations (ion-acoustic plasma waves) with acoustic vibrations of ionic crystals and of electron plasma (Langmuir) oscillations with optical vibrations in crystals makes it convenient to use the quantum-mechanical perturbation theory to calculate the scattering probabilities. Such calculations were made by Davydov and Shmushkevich for optical oscillations as early as 1940 [63], and by Landau and Kompaneets for acoustic vibrations in the 1930s [64].

It was shown that the probability of electron transition from the state with the wave vector $\mathbf{k} = m\mathbf{V}$ into the $\mathbf{k}+\mathbf{q}$ state upon absorption of a quantum with frequency ω and wave vector $\mathbf{q}$ is given by the formula

$$W_{\text{e}}^{+} = \frac{2ne^4}{m\omega q^2}\,N_{\omega}\,, \tag{3}$$

where m is the electron mass, e is the electron charge, n is the plasma density, and N_{ω} is the number of quanta per unit cell. Thus, the probability of the $\mathbf{k} \to \mathbf{k}-\mathbf{q}$ transition corresponding to the emission of a quantum is

$$W_{\text{e}}^{-} = \frac{2ne^4}{m\omega q^2}\,(N_{\omega}+1)\,. \tag{4}$$

Davydov argued that a plasma normally contains no ions capable of interacting with Langmuir oscillations having phase velocities much higher than the thermal velocities of ions. Getting ahead of the story, let us note that the above probabilities of emission and absorption of wave quanta were used later in the pioneering work of Romanov and Filippov [2] when calculating the quasilinear diffusion coefficient of electrons in the field of Langmuir oscillations, while the 'quantum' method itself became one of the main tools in plasma turbulence research. This is hardly a surprise, because the language of quantum mechanics was extensively used by the majority of physicists working at that time.

It should be noted that Davydov does not mention any relationship between his quantum method and the Vavilov–Cherenkov effect. However, he apparently emphasizes the similarity of the processes under consideration to hydrodynamic conceptions implying the impossibility of wave emission and absorption for bodies traveling in a liquid with velocities below the speed of sound.

Accordingly, for ion-acoustic waves, the following expressions hold (bearing in mind the equiphase conditions for electron and ion vibrations):

$$W_{\text{i}}^{+} = \frac{2ne^4}{M\Omega q^2}\,N_{\Omega}\,, \quad W_{\text{i}}^{-} = \frac{2ne^4}{M\Omega q^2}\,(N_{\Omega}+1)\,, \tag{5}$$

where M is the ion mass, Ω is the frequency of ion-acoustic vibrations, and N_{Ω} is their intensity (the number of quanta per unit volume).

The exchange between charged particles and the wave field by means of quantum emission and absorption is believed to be associated with minor changes in energy and momentum which, in turn, suggests the diffusive evolution of the distribution function. Being a prominent authority in physical kinetics [65], Davydov found no difficulty in deriving the Fokker–Planck equations best matched to the problem under consideration. An additional argument for using the diffusion approximation was the studies by Bohm and Gross [18, 19, 66–69] published one or two years before Davydov's article. These papers contained a detailed discussion of interactions between electrons and plasma waves, taking into consideration the estimates of energy transferred in 'collisions':

$$\Delta\varepsilon \approx mV_{\text{ph}}(V - V_{\text{ph}}) = m\,\frac{\omega}{k}\left(V - \frac{\omega}{k}\right). \tag{6}$$

Here, $V_{\text{ph}} = \omega/k$ is the phase velocity of a wave. Noteworthily, Landau did not discuss the qualitative picture of collisionless damping described in his original paper, while Bohm and Gross actually used the model to describe energy transport from 'light to heavy components', well known in plasma kinetics.

Davydov made use of such estimates at various times in the mid-1930s to derive the Fokker–Planck equations describing the distribution function of electrons in a weakly ionized plasma placed in an electric field (the Druyvesteyn distribution was obtained theoretically) [63, 65, 70]. Davydov distinctly formulated the necessity of taking into account diffusion effects in velocity space for the analysis of plasma kinetic problems. Many studies carried out by Davydov were concerned with diffusion models of various phenomena. Moreover, he was also well aware of the importance of 'nondiffusive' transport [71] for the description of turbulent diffusion.

The basic kinetic equation was chosen in paper [13] in the form of the following Fokker–Planck equation:

$$\frac{\partial F}{\partial t} = \frac{1}{V^2}\frac{\partial}{\partial V}\left[V^2\left(D_V\frac{\partial F}{\partial V} + G_V F\right)\right] + \frac{1}{V^2}\,D_{\vartheta}\left(\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\,\frac{\partial F}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2 F}{\partial\varphi^2}\right), \tag{7}$$

where coordinates (V, ϑ, φ) traditional for plasma kinetics are used, D_V and D_{ϑ} are the corresponding diffusion coefficients, and G_V is the drag coefficient. Naturally, the expression

$$F = F_0\cos\vartheta\exp\left(-\frac{t}{\tau}\right) \tag{8}$$

is used for the model solution, where $F_0(V, T)$ denotes the equilibrium distribution. In the absence of an external force and spatial inhomogeneity, we come to an equation describing Maxwellian type equilibrium. In the work under consideration, the divergent approach is employed as developed by Davydov to address problems of weakly ionized plasma kinetics. He considered a particle flux S_V in the phase space:

$$\frac{1}{V^2}\frac{\partial}{\partial V}(V^2 S_V) = \frac{1}{V^2}\frac{\partial}{\partial V}\left[V^2\left(D_V\frac{\partial F_0}{\partial V} + GF_0\right)\right] = 0\,, \quad (9)$$

and arrived at the condition for coupling the diffusion (D_V) and drag (G_V) coefficients, taking advantage of the absence of a singularity in the expression for the flux of particles S_V at zero:

$$S_V = D_V\frac{\partial F_0}{\partial V} + G_V F_0 = 0\,. \quad (10)$$

Since the equilibrium distribution function is given by the classical Maxwellian formula

$$F_0 = n\left(\frac{m}{2\pi\theta}\right)^{3/2}\exp\left(-\frac{mV^2}{2\theta}\right), \quad (11)$$

where the notation $\theta = k_{\mathrm{B}}T$ was introduced, it leads to the coupling condition for kinetic coefficients in the form

$$G_V = -\frac{mV}{\theta}\,D_V\,. \quad (12)$$

Then, estimation of characteristic relaxation time τ_* and characteristic spatial scale l_{turb} reduces to calculating the particle diffusion coefficient in the phase space.

Coefficients D_V and D_ϑ are given by the classical Einstein expressions

$$D_V = \frac{1}{2}\frac{\left\langle(\Delta V\cos\vartheta)^2\right\rangle}{\Delta t}\,, \quad (13)$$

$$D_\vartheta = \frac{1}{2}\frac{\left\langle(\Delta V\sin\vartheta\cos\varphi)^2\right\rangle}{\Delta t}\,, \quad (14)$$

each breaking into two items standing for quantum absorption (probability W_+) and emission (probability W_-), respectively, with a change in velocity $\Delta\mathbf{V} = \hbar\mathbf{q}/m$ in either case. The averaging procedure is actually reduced to taking the corresponding integrals over q, ϑ, and φ.

For example, the following expressions are obtained in the case of interaction between electrons and Langmuir waves:

$$D_V(V) \approx 2\pi\,\frac{ne^4}{m^2V}\,\frac{\theta}{mV^2}\,, \quad (15)$$

$$D_\vartheta(V) \approx 2\pi\,\frac{ne^4}{m^2V}\,. \quad (16)$$

Assuming that the effective mean free path is estimated from the relation $l_{\mathrm{turb}} \propto V\tau_*$, where

$$\tau_*(V) = \frac{V^2}{2D_\vartheta(V)} \propto \frac{m^2V^3}{ne^4}\,, \quad (17)$$

we arrive at Davydov's scaling for the particles' mean free path brought about by turbulent fluctuations:

$$l_{\mathrm{turb}}(V) \propto \left(\frac{ne^4}{m^2V^4}\right)^{-1} \propto V^4\,. \quad (18)$$

As a matter of fact, these results coincide with elementary kinetic (fluctuation-dissipative) estimates for Coulomb collisions:

$$D_V D_{\mathrm{R}} = \frac{(k_{\mathrm{B}}T)^2}{m}\,, \quad (19)$$

where diffusion in the configuration space is given by the formula $D_{\mathrm{R}} = l_{\mathrm{coll}}V$. The collisional range l_{coll} may be estimated from dimensional notions when considering Coulomb collisions:

$$\frac{e^2}{r_0} \approx \frac{mV^2}{2}\,, \quad (20)$$

where $l_{\mathrm{coll}}r_0^2 n \propto 1$. Here, the Coulomb collision cross section is assessed through the characteristic spatial scale in a purely dimensional manner: $\sigma \propto r_0^2$. As is well known, the rigorous kinetic theory of Coulomb collisions developed by Landau assigns an important role to collisions resulting in small-angle scattering of charged particles due to their long-range interaction in a plasma. Naturally, this enlarges the effective cross section $\Lambda = \ln(r_{\mathrm{D}}/r_0)$ times and consequently shortens the range l_{coll}. Here, r_{D} is the Debye radius. In the problems of interest, the Coulomb logarithm is usually amounted to $\Lambda \approx 10-20$ [72].

Similar results are obtained in considering interactions between electrons and ion-acoustic waves:

$$D_V \approx 2\pi\,\frac{ne^4}{m^2V}\,\frac{\theta}{mV^2}\,, \quad (21)$$

$$D_\vartheta \approx 2\pi\,\frac{ne^4}{m^2V}\,. \quad (22)$$

Based on these calculations, Davydov arrived at the conclusion that the contribution from the particle–wave interactions in a plasma is small in comparison with that of Coulomb collisions at a noise level commensurate with the thermal energy.

Today, it is clear that the choice of the N_ω and N_Ω values as indicators of plasma oscillation intensity corresponding to thermal equilibrium was at odds with the real role of turbulence in a plasma. Davydov himself pointed out that it provided only a lower-bound estimate of the role of oscillations. It required almost another 10 years to construct the first self-consistent model for evolution of the distribution function in the Langmuir oscillation field, taking into consideration the temporal evolution of oscillation energy [1–5]. The key point here was the employment of both the Landau damping mechanism and the diffusion type equation describing the electron distribution function.

3. Kinetic equation for waves and thermal conductivity of plasma

Theoretical studies on plasma turbulence initiated by Davydov were continued under the supervision of A B Migdal by his postgraduate student V M Galitskii, who investigated the influence of Langmuir oscillations on the transport processes in a plasma placed in magnetic traps. The volume of collected works by V M Galitskii came out only in the early 1980s; it included, inter alia, his declassified thesis for Candidate of physical and mathematical sciences, defended in 1954 [14]. Davydov was forced to stop his work in the field of plasma physics after he was dismissed from LIPAN at the request of bodies controling the regime of secrecy [60], while the undergraduate student T F Volkov and postgraduate student S I Braginskii conducting research under his guidance switched to investigations into the hydrodynamics and collisional kinetics of magnetized plasma [73, 74].

The last chapter in Galitskii's thesis was entitled "Kinetic equation for waves and plasma thermal conductivity." In the introduction, the author directly pointed out the motivational role of Davydov's work and stated that determining the thermal conductivity coefficient related to electron oscillations was the primary objective of his study. He understood that waves make only a negligible contribution under thermal equilibrium. Using the relevant qualitative estimations based on the classical Rayleigh–Jeans method for counting the number of states (waves), Galitskii came to the conclusion that the number of waves outside the Debye sphere under near-equilibrium conditions was equal to

$$N_W = \frac{\int_{k_{\min}}^{k_{\max}} \mathrm{d}^3\mathbf{k}}{\delta k^3} \approx \frac{L_0^3}{(2\pi)^3}\,\frac{3}{4}\,\pi k_{\mathrm{D}}^3 = \frac{1}{6\pi^2}\left(\frac{L_0}{r_{\mathrm{D}}}\right)^3. \tag{23}$$

Here, $\delta k \approx 2\pi/L_0$ is the minimal wave number, and L_0^3 is the plasma volume with the characteristic spatial scale L_0, and $k_{\max} \approx 2\pi/r_{\mathrm{D}} \gg k_{\min}$, where

$$r_{\mathrm{D}} = \sqrt{\frac{k_{\mathrm{B}}T}{4\pi e^2 n}}.$$

Notice that the estimate obtained by Galitskii is similar to Debye's calculated result for the heat capacity of solids. According to Debye, the maximum wave number allowing integral divergence to be avoided is related to the minimal characteristic scale coincident with the crystal lattice period. In the case of Langmuir oscillations, $k_{\max} \propto 2\pi/r_{\mathrm{D}}$ due to Landau damping on $r < r_{\mathrm{D}}$ scales.

In equilibrium, wave and particle energy densities are identical, and, therefore, the following estimate is valid:

$$W = \frac{N_W}{L_0^3}\,k_{\mathrm{B}}T = \frac{k_{\mathrm{B}}T}{4\pi^2 r_{\mathrm{D}}^3} \propto \frac{n k_{\mathrm{B}}T}{N_{\mathrm{D}}}, \tag{24}$$

where it was taken into account that $N_{\mathrm{D}} = n r_{\mathrm{D}}^3 = n[T/(4\pi e^2 n)]^{3/2}$, and n is the plasma density. To recall, $N_{\mathrm{D}} \gg 1$ and $\ln N_{\mathrm{D}} = \Lambda$. Galitskii assumed the wave energy transfer rate determined by group velocity $\mathrm{d}\omega/\mathrm{d}k$ and close to the particle velocity. Characteristic wave time τ_* introduced by Davydov is of the same order of magnitude as electron mean free path time τ_{ei} prior to collision with plasma ions. As a result, a comparison of wave thermal conductivity with the kinetic (collisional) one is akin to a comparison between wave number density N_W/L_0^3 and particle (electron) number density:

$$\frac{\chi_{\mathrm{turb}}}{\chi_{\mathrm{coll}}} \propto \frac{(6\pi^2 r_{\mathrm{D}}^3)^{-1}}{n} = \frac{e^2/r_{\mathrm{D}}}{k_{\mathrm{B}}T} \ll 1\,. \tag{25}$$

The choice of thermal conductivity coefficients instead of the diffusion coefficient to analyze the contribution from turbulent pulsations to the transport processes was dictated by the necessity to avoid additional difficulties related to ambipolarity effects. Clearly, these calculations confirmed the known inference deduced by Davydov suggesting a negligible influence of turbulence at a thermal noise level [13].

Galitskii's further studies were motivated by the conjecture that the transverse collisional transport should be suppressed in a strong magnetic field. Indeed, a Larmor radius inversely proportional to the magnetic field amplitude serves in this case as the characteristic spatial correlation scale. Under these conditions typical of magnetic plasma confinement problems, the contribution from interactions of particles with turbulent fluctuations becomes an essential factor. At an earlier stage of CTF research, much attention was given to the stabilizing effect of strong magnetic fields. But even at that time, theorists began to think about the possible role of kinetic instabilities.

A novelty element of importance for all the subsequent analysis of plasma turbulence, as compared to Davydov's approach, is the consideration of the equation for wave energy spectral density W_k. In this case, energy flux density is given by the formula

$$\mathbf{S}(\mathbf{r}, t) = \int \frac{\partial \omega}{\partial k}\;W_k(t, \mathbf{r}, \mathbf{k})\,\mathrm{d}\mathbf{k}\,. \tag{26}$$

Here, spectral wave energy density is assumed to depend on both wave vector $\mathbf{k}$ and coordinate $\mathbf{r}$. Galitskii presented the appropriate equation for field evolution in the form

$$\frac{\partial W(t,\mathbf{r},\mathbf{k})}{\partial t} + \frac{\partial\omega}{\partial\mathbf{k}}\,\frac{\partial W(t,\mathbf{r},\mathbf{k})}{\partial\mathbf{r}} - \frac{\partial\omega}{\partial\mathbf{r}}\,\frac{\partial W(t,\mathbf{r},\mathbf{k})}{\partial\mathbf{k}} = \frac{\mathrm{d}W}{\mathrm{d}t} \tag{27}$$

by using the geometric acoustic approximation

$$\frac{\partial\mathbf{r}}{\partial t} = \frac{\partial\omega}{\partial\mathbf{k}}\,, \tag{28}$$

$$\frac{\partial\mathbf{k}}{\partial t} = -\frac{\partial\omega}{\partial\mathbf{r}}\,. \tag{29}$$

Davydov did not consider wave field evolution, assuming noise amplitudes to be given. However, Galitskii failed to propose a scheme for matching this equation and Vlasov's equation for the particle distribution function. At that time, solid-state physics theorists had a clear notion of phonon kinetics, but they had to deal with near-equilibrium states. Later on, kinetic equations for waves in plasma became the most important tool for the analysis of nonlinear problems in the theory of turbulence extensively discussed in the literature [26, 34, 36, 37, 40, 47].

It is worthwhile to note in the context of this review that E P Velikhov mentioned in his memoirs on the work on quasilinear theory the Galitskii thesis as a stimulus that made him turn to research in a new area [15]. "When I thought about a subject for my own thesis, Ya A Smorodinskii advised me: "Look at what is going on with Galitskii." At that time Viktor Mikhailovich worked on his thesis. It proved to be a classified theoretical study in which the wave–particle interaction in plasma was considered for the first time, continuing in some ways the famous classical work of Landau. I looked through the Galitskii thesis and was greatly impressed by his ideas.

Plasma phenomena, i.e., the interplay between two gases (particle-gas and wave-gas), caught my imagination, and I decided to seek the advice of Sagdeev and Vedenov. We began to work on the quasilinear theory of turbulence, which was completed shortly before the *First Nuclear Fusion Congress* held in Salzburg.

It was high time to report our results, because William Drummond had come by then to similar conclusions, although his was a botched job compared with our elegant

Таблица 1

Тип взаимодействия	Плазменный процесс (продольные волны)	Электромагнитный процесс (поперечные волны)
Взаимодействие одной частицы с волнами	Уменьшение энергии одной частицы и возникновение волны	Черенковское излучение
	Увеличение энергии одной частицы и уничтожение волны	Черенковское поглощение
Взаимодействие двух частиц с волнами	Возникновение волны при столкновении двух частиц	Тормозное излучение
	Уничтожение волны с передачей энергии двум сталкивающимся частицам	Фотоэффект в непрерывном спектре

Table 1

Interaction	Plasma process (longitudinal waves)	Electromagnetic process (transverse waves)
Interaction of a single particle with waves	Energy loss by a single particle and wave formation	Cherenkov emission
	Energy gain by a single particle and annihilation of a wave	Cherenkov absorption
Interaction of two particles with waves	Origin of a wave in a two-particle collision	Bremsstrahlung emission
	Destruction of a wave with energy transfer to two colliding particles	Photoeffect in a continuous spectrum

Figure 1. Table from Galitskii's thesis for Cand. phys.-math. sci. (1954) [14].

theory. Both versions were delivered at the congress; preference was given to our model and it was recognized as the standard theory.''

The notion of wave-gas undoubtedly gives evidence of understanding the importance of the correlation between wave and particle kinetics by theorists. Thereafter, Romanov and Filippov [2], as well as Vedenov, Velikhov, and Sagdeev [1, 3, 4], used the equation for spectral density of wave energy W_k in a reduced form, preserving only effects related to Landau collisionless damping as one of the main relations in quasilinear theory:

$$\frac{\partial W_k(t)}{\partial t} = 2\gamma_{\mathrm{L}} W_k(t)\,, \tag{30}$$

$$\gamma_{\mathrm{L}} = \frac{\pi\omega_{\mathrm{L}}^3}{2k^2}\,\frac{\partial f}{\partial V}\bigg|_{V=\omega/k}\,. \tag{31}$$

The traditional notation for oscillation energy density $W(t) = \int W(\mathbf{k}, t)\, \mathrm{d}^3\mathbf{k}$ is adopted here.

A few bibliographic remarks are in order. To begin with, Galitskii in all his calculations considered Landau damping as interpreted by Bohm and Gross (with reference to the relevant publications). Strange as it may seem, he made no reference to the classical work of Akhiezer and Fainberg [20, 21] but mentions the later paper by Akhiezer and Sitenko [75]. In this context, a characteristic quotation from a *Personalia* published in *Physics–Uspekhi* in commemoration of the 60th anniversary of the birthday of A I Akhiezer [76] may be of interest. It reads: ``The classical work of A I Akhiezer and Ya B Fainberg dated to 1949 predicted the beam instability effect, namely, the exponential growth of fluctuations in a plasma traversed by an electron beam. This work, together with those of A A Vlasov and L D Landau, provided a basis for investigations of collective interactions in a plasma.''

Notice that Galitskii in 1954 extensively exploited the notion of Landau damping as the Cherenkov effect (Fig. 1) in the context of description of particle–wave interactions in a plasma (Tamm, Cherenkov, and Frank were awarded the Nobel Prize in Physics only in 1958). Also, Galitskii attended to effects related to the passage of a beam of charged particles through a plasma by repeating calculations of his predecessors with the employment of a somewhat different method. For example, the formula derived by Vlasov [12] in 1945 to estimate energy losses by particles, viz.

$$-\frac{\mathrm{d}W}{\mathrm{d}l} \approx \frac{\omega_0^2}{V^2}\, e^2 \ln \frac{k_{\max} V}{\omega_{\mathrm{L}}}\,, \tag{32}$$

gave rise to hopes for a possible qualitative evaluation of transport effects from dimensional considerations making use of the quantity

$$\frac{\mathrm{d}W}{\mathrm{d}l} \propto \frac{e^2}{r_{\mathrm{D}}}\,\frac{1}{r_{\mathrm{D}}} \propto e^2 \left(\frac{\omega_{\mathrm{L}}}{V}\right)^2 \tag{33}$$

representing the reciprocal of plasma frequency $\omega_{\mathrm{L}} = \sqrt{4\pi e^2 n/m}$. Here, a 'unified' estimate is used for the characteristic spatial scale in the form of the Debye radius, $r_{\mathrm{D}} \propto V_T/\omega_{\mathrm{L}}$, which allows adequately characterizing the inverse dependence of energy loss by a particle on its velocity. Such an approach is in excellent agreement with the classical collisional description of the plasma. Indeed, the mean free path markedly increases with particle velocity, $l_{\mathrm{coll}} \propto V^4$, i.e., the losses have to decrease as velocity grows. Nevertheless, the characteristic frequency of Langmuir oscillations is absent in Galitskii's estimations of turbulent transport disregarding the influence of the magnetic field.

It can be noted by way of an intermediate summary that Davydov's work and Galitskii's thesis provided a basis for the subsequent rigorous analysis of effects associated with plasma turbulence. However, it proved necessary to move from a phenomenological description of particle diffusion effects to a systematic analysis of Vlasov's equation in order to construct the consistent quasilinear theory.

4. Phenomenological equation for the plasma wave potential

From the standpoint of an analysis of key theoretical research preceding the development of the quasilinear theory, it is worthwhile to mention the paper of Klimontovich [16], dating back to 1959, which summarizes the most important data on the interaction between electron beams and plasma available at that time. They were included in the well-known review article by Klimontovich and Silin [77] (see Section 7) published in *Physics–Uspekhi* in 1960.

At the beginning of Ref. [16], its author draws attention to the inapplicability of the known analytical formulas describing the slowing-down of individual charged particles in a plasma to the description of electron beam scattering. The unusually fast scattering noted long ago by Langmuir and beam instability were repeatedly investigated by many theorists (e.g., Vlasov, Bohm, Akhiezer); the excitation of plasma oscillations associated with beam propagation was already observed in experiment [78, 79]. Thus, calculations based on Coulomb collisions showed that in a plasma of density 10^{13} cm^{-3} the electron range must be of order 10^6 m at a beam energy of 2 keV. However, real beams tend to be attenuated on laboratory scales. Klimontovich proposed to construct a rigorous theory with the use of the Bogolyubov hierarchy [80] for the derivation of a kinetic equation taking account of plasma wave excitations. His calculations brought him to the collision integral in the form

$$\mathrm{St} = n\,\frac{\partial}{\partial \mathbf{r}} \int U(\mathbf{r}-\mathbf{r}')\,\frac{\partial G}{\partial \mathbf{p}'}\,\mathrm{d}\mathbf{r}'\,\mathrm{d}\mathbf{p}'\,, \tag{34}$$

where the correlation function is expressed as

$$G = \frac{\partial}{\partial \mathbf{r}} \int_0^\infty U\left(\left|\mathbf{r}-\mathbf{r}'-\frac{\mathbf{p}-\mathbf{p}'}{m}\,\tau\right|\right)\mathrm{d}\tau \left\{ f_1\,\frac{\partial f_1}{\partial \mathbf{p}} - \frac{\partial f_1}{\partial \mathbf{p}'}\,f_1 \right\}. \tag{35}$$

Noting that this equation is analogous to the Landau kinetic equation with Coulomb collisions of charged particles [42–44, 72], Klimontovich linearized it following the traditional method and arrived at a classical Fokker–Planck equation. However, he obtained the diffusion coefficient for particles in velocity space, taking account of plasma oscillations:

$$D_V \approx \frac{e^2 k_\mathrm{B} T}{2\pi} \int \delta(\omega_\mathrm{L} - \mathbf{k}\mathbf{V})\,\mathrm{d}^3\mathbf{k}\,, \tag{36}$$

in a form substantially differing from the aforementioned result of Davydov. Here, delta function $\delta(\omega_\mathrm{L} - \mathbf{k}\mathbf{V})$ makes it possible to take account in explicit form of the resonant character of electron–Langmuir wave interactions. The qualitative estimate can be written out in terms of Langmuir oscillation frequency ω_L:

$$D_V \propto \frac{e^2}{V^3}\,k_\mathrm{B} T \omega_\mathrm{L}^2 \propto \frac{ne^4}{V}\,\frac{k_\mathrm{B} T}{mV^2}\,, \tag{37}$$

though it actually coincides with Davydov's scaling.

Klimontovich was not satisfied with such 'duality' and attempted, in the same article, to describe the evolution of the beam distribution function based on direct solution of the Vlasov equation by the perturbation method representing the distribution function in the form of a fundamental harmonic and perturbation: $f = f_0 + f_1$. In fact, Klimontovich postulated his own system of equations. He chose one of them as the equation for the perturbed part f_1 of the electron distribution function. It was obtained from the classical Vlasov equation:

$$\frac{\partial f_1}{\partial t} + v\,\frac{\partial f_1}{\partial x} + \frac{e}{m}\,\frac{\partial \varphi}{\partial x}\,\frac{\partial f_1}{\partial v} = 0\,. \tag{38}$$

The other, being a phenomenological equation, describes the evolution of the plasma wave potential amplitude by virtue of 'phenomenological' modification of the classical Poisson equation:

$$\frac{\partial^2 \varphi}{\partial x^2} + 2\,\frac{\gamma_\mathrm{L}}{V_\mathrm{ph}}\,\frac{\partial \varphi}{\partial x} - \frac{1}{V_\mathrm{ph}^2}\,\frac{\partial^2 \varphi}{\partial t^2} = 4\pi e\left(\int f_1\,\mathrm{d}V - n\right). \tag{39}$$

Because Klimontovich aimed to describe beam instability, he considered the simplest 'two-hump' distribution function, the positive derivative of which on the left-hand side-hump corresponding to beam particles initiates the development of Cherenkov type instability (Fig. 2). Here, the Landau increment is given by the classical formula

$$\gamma_\mathrm{L} = \frac{\pi \omega_\mathrm{L}^3}{2k^2}\,\frac{\partial f}{\partial V}\bigg|_{V=\omega/k} \approx \frac{\pi}{4}\,\omega_\mathrm{L}\,\frac{n_\mathrm{b}}{n}\left(\frac{V_\mathrm{b}}{\Delta V}\right)^2, \tag{40}$$

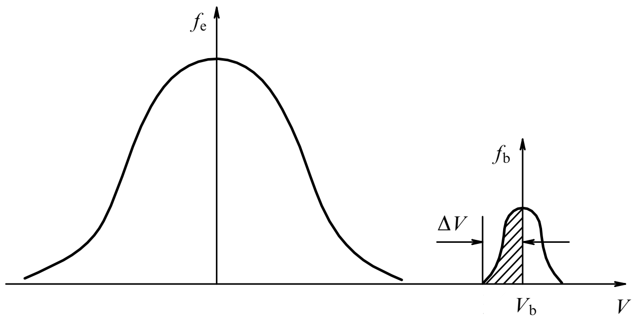

Figure 2. Beam instability.

where n_b is the particle density in the beam, V_b is the characteristic beam velocity, and ΔV is the velocity spread.

According to the author's intention the term $(1/V_\mathrm{ph}^2)\,\partial^2\varphi/\partial t^2$ would allow shifting from the classical Poisson equation to a wave equation, while the term $2(\gamma_\mathrm{L}/V_\mathrm{ph})\,\partial\varphi/\partial x$ accounted for effects associated with the Landau resonance mechanism and thereby made possible the self-consistent description of the evolution of plasma oscillations under turbulence conditions.

This aspect of Klimontovich's work deserves special attention, because it resembles Davydov's phenomenological approach to the description of turbulent diffusion in the atmosphere [71]. Thus, the classical diffusion equation for admixture (scalar) transport takes the form

$$\frac{\partial n(x,t)}{\partial t} = D_0\,\frac{\partial^2 n(x,t)}{\partial x^2}\,. \tag{41}$$

Here, n is the density of admixture particles, and D_0 is the molecular diffusion coefficient of admixture in the medium. To take into account nonlocal effects arising from turbulent convection leading to the considerable acceleration of the transport processes, Davydov included in this equation an additional term $\partial^2 n/\partial t^2$ converting the parabolic transport equation into the hyperbolic type wave equation

$$\tau_*\,\frac{\partial^2 n}{\partial t^2} + \frac{\partial n}{\partial t} = D\,\frac{\partial^2 n}{\partial x^2}\,. \tag{42}$$

Here, τ_* is the characteristic time.

Such 'body-checking strategy' used by Davydov had seemed like a sort of revolutionary approach in the mid-1930s but found wide application just in the 1950s to describe nonlocal effects in connection with the expansion of research into atmospheric turbulence and air pollution [81–84]. To recall, Klimontovich did not refer to Davydov's early work [71] but included his article on the influence of plasma oscillations on electron transport [13] in the list of cited literature. Interestingly, Klimontovich appealed to such phenomenology several more times afterwards to renormalize the kinetic equation for the particle distribution function [85].

It is clear now that the small perturbations of which Klimontovich spoke cannot generate a finite-amplitude wave. The artificial character of Klimontovich's approach is quite obvious today when the evolution of the electric potential in the classical quasilinear theory of a weak plasma turbulence is described in spectral terms. The effectiveness of this approach has already been confirmed in the theory of vortex hydrodynamic turbulence [86–92]. The use of spectral terms to describe wave packets in plasma problems also looks more natural.

The main elements of the Klimontovich theoretical model are, in fact, similar to those formulated later in the quasilinear theory. They include, inter alia, the employment of the Vlasov equation for the electron distribution function, the delta function in the expression for the particle diffusion coefficient in the phase space, taking account of the resonant wave–particle interaction, and the inclusion of classical Landau damping (with the increment borrowed from the linear theory). It will be shown in Section 5 that the ideas Klimontovich put forward in 1959 were further developed already in 1960–1961 when the first quasilinear equations were derived.

5. Coefficient of diffusion in the wave random phase approximation

The comprehensive article by Romanov and Filippov [2] published in 1961 (although known to theorists since 1960) summarized results of the efforts undertaken during a preceding decade in search of an adequate model for one of the most important problems of plasma turbulence. The authors of Ref. [2] criticized attempts to explain abnormally fast beam scattering by voltage fluctuations at the boundary and singled out the well-known beam instability related to Landau damping reversal as the main factor responsible for turbulent fluctuations in the presence of a spectral region with $\partial f/\partial V > 0$. Another essential point is understanding the necessity to consider fluctuations in terms of spectral energy density W_k of plasma waves:

$$W_k(t) = \frac{\delta W(\mathbf{k}, t)}{\delta \mathbf{k}^3} \,. \tag{43}$$

To recall, the turbulence theory developed by Kolmogorov and Obukhov away back in 1941 for the spectral distribution of energy became by the 1960s a classical tool for specialists engaged in research on atmospheric and oceanic turbulence. Galitskii [14] also proposed to use spectral representation, but his thesis could hardly be known to the authors of Ref. [2]. At the same time, Romanov and Filippov were among the participants in the Soviet Atomic project [59], and the analysis of results of many atmospheric nuclear tests inevitably implied investigations into atmospheric turbulence.

On page 2 of their article [2], Romanov and Filippov recognize the influence of Klimontovich's work [16] but point out its main drawback arising from the unjustified assumption of the formation of a steady wave in the plasma. They proposed an equation describing the evolution of plasma wave energy density $W_k(\mathbf{k}, \mathbf{r}, t)$ in the form

$$\frac{\partial W_k}{\partial t} + \frac{\partial \omega}{\partial k_i}\frac{\partial W_k}{\partial x_i} = A + 2\gamma_{\mathrm{L}} W_k \,, \tag{44}$$

where A is the term taking account of spontaneous plasmon emission and γ_{L} is the Landau increment. Here, the Galitskii collision integral acquired a more concrete form, because it took into account, in a 'self-consistent manner', both excitation and attenuation of plasma waves under the effect of Cherenkov mechanisms.

Romanov and Filippov argue that diffusion effects in the case of interaction between a beam of fast electrons and longitudinal plasma waves can be calculated by Davydov's method with the involvement of quantum concepts. Specifically, these effects account for appearing the probabilities of emission and absorption of the quantum with frequency ω and wave vector $\mathbf{k}$ by an electron in the form

$$w^+ = \frac{\omega_{\mathrm{L}}^2}{2\pi\omega k^2}(N_\omega + 1) \,, \tag{45}$$

$$w^- = \frac{\omega_{\mathrm{L}}^2}{2\pi\omega k^2}\, N_\omega \,, \tag{46}$$

where $\omega_{\mathrm{L}}^2 = 4\pi e^2 n/m$. Importantly, probabilities w^+ and w^- presented by Romanov and Filippov precisely match probabilities used by Davydov in the study on electron scattering by Langmuir waves.

Coefficients entering the Fokker–Planck equation were calculated by considering the mean and root-mean-square (rms) acceleration imparted to electrons by the turbulent electric field. For friction, one obtains

$$\langle \Delta V \rangle = \frac{e}{m}\int_0^{\Delta t} \big\langle \mathbf{E}\big(\mathbf{r}(t), t\big)\big\rangle \, \mathrm{d}t \,, \tag{47}$$

and for diffusion in the velocity space, the result reads as follows:

$$\langle \Delta V_\alpha \Delta V_\beta \rangle = \frac{e^2}{m^2}\int_0^{\Delta t} \mathrm{d}t \int_0^{\Delta t} \Big\langle \mathbf{E}_\alpha\big(\mathbf{r}(t), t\big)\mathbf{E}_\beta\big(\mathbf{r}(t'), t'\big)\Big\rangle \, \mathrm{d}t' . \tag{48}$$

In these calculations, it was taken into consideration that both the width of the Langmuir wave spectrum Δk and Landau damping satisfy the condition

$$\frac{1}{\gamma_{\mathrm{L}}} \gg \Delta t \gg \frac{1}{V\Delta k} \approx \tau_{\mathrm{ph}} \,. \tag{49}$$

This is actually the phase chaoticity condition for the waves present in the packet under consideration; it is needed to ensure the randomness of particle bouncing and, therefore, the possibility of applying the diffusion model of the distribution function evolution. It is also important that this criterion includes the Landau increment as one of the parameters of the problem. The following expression holds for the fast electron flow:

$$\frac{\Delta k}{k} \gg \frac{n}{n_0}\left(\frac{V}{\Delta V}\right)^2 . \tag{50}$$

The relevance of the condition for disregarding higher-order terms in series expansion of the distribution function has the form

$$W = \int W_k \, \mathrm{d}^3\mathbf{k} \ll \left(\frac{\Delta k}{k}\right)^4 n_0 m V^2 \,. \tag{51}$$

On the one hand, it is the weak turbulence condition; on the other hand, it offers a wide enough wave packet. Now, the use of spectral terms looks quite natural in the description of wave packets. The importance of conditions for the applicability of quasilinear expressions will be demonstrated in the next sections.

Another important issue concerns the derivation of the equation describing diffusion of electrons as a result of their interaction with the field of turbulent pulsations of the electric potential:

$$D_{ik} = \frac{4\pi e^2 \omega_0^2}{m^2} \int \frac{k_i k_k}{k^2} \frac{W_k}{\omega^2} \, \delta(\mathbf{k}\mathbf{v} - \omega) \, \mathrm{d}^3 k \,. \tag{52}$$

This formula is remarkable for several reasons. To begin with, it is the first expression for the electron diffusion coefficient in the velocity space containing, in a correct manner, the Langmuir oscillation frequency $\omega = \omega(k)$. Moreover, even a superficial qualitative analysis permits relating this expression to the classical definition of turbulent diffusion proposed by Taylor for the admixture transport under hydrodynamic turbulence conditions:

$$D_T = \int \left\langle V(0) V(t) \right\rangle \mathrm{d}t \propto V_{\mathrm{p}}^2 \tau_{\mathrm{cor}} \,, \tag{53}$$

where $\langle V(0) V(t) \rangle$ is the autocorrelation function of the velocity of Lagrangian liquid particles. It is an example of the quadratic dependence of the transport coefficient in the usual configuration space on the amplitude V_{p} of velocity pulsations. In our kinetic case, we have to deal with a velocity space in which pulsations are related to accelerations: $A \propto eE/m$, and scaling can be expected in the form

$$D_V = A^2 \tau \approx \left(\frac{eE}{m} \right)^2 \frac{1}{\omega_{\mathrm{L}}} \,. \tag{54}$$

It is to such a dimensional estimate that the Romanov–Filippov formula leads.

There is one more peculiarity arising from the almost parallel development of hydrodynamic and plasma turbulence theories. In 1960, I D Howells published in the *Journal of Fluid Mechanics* the now classical article in which the coefficient of turbulent diffusion of the admixture was expressed via the spectral energy density $E_{\mathrm{p}}(k)$ of hydrodynamic pulsations [93–95]:

$$D_{\mathrm{p}}^2 = \int_k^\infty \frac{E_{\mathrm{p}}(k)}{k^2} \, \mathrm{d}k \,. \tag{55}$$

Here, normalization of the energy spectral density to the square of the characteristic amplitude of pulsations of the liquid particle velocity, V_{p}^2, was used:

$$\frac{V_{\mathrm{p}}^2}{2} = \int_0^\infty E(k) \, \mathrm{d}k \,. \tag{56}$$

The equations thus obtained make it possible to estimate the quasilinear relaxation time of an electron beam. Romanov and Filippov derived the following expression containing as a multiplier the logarithm of the ratio between electrostatic noise energy densities at the beginning and end of the process:

$$\tau_{\mathrm{rel}} \approx \frac{1}{\gamma_{\mathrm{L}}} \ln \frac{W}{W_0} \approx \frac{1}{\omega_{\mathrm{L}}} \frac{n_{\mathrm{b}}}{n} \ln \frac{nmV_{\mathrm{b}}^2}{k_{\mathrm{B}} T / r_{\mathrm{D}}^3} \approx \frac{1}{\omega_{\mathrm{L}}} \ln \Lambda \,. \tag{57}$$

It should be noted by way of an intermediate summary that Romanov and Filippov [2] integrated the main ideas of their predecessors, viz.

— 'reversal' of the Landau damping effect (Akhiezer–Fainberg, Bohm–Gross);

— diffusive evolution of the distribution function (Davydov);

— spectral representation of turbulent fluctuations (Galitskii), and

— consideration of 'resonant' particle–wave interaction in the expression for the coefficient of electron diffusion in the phase space (Klimontovich).

Setting priorities is beyond the scope of this review confined to the analysis of the evolution of theoretical models designed in describing the weakly turbulent state of the plasma. Nevertheless, the fact that three articles by independent groups of authors appeared almost simultaneously cannot be overlooked. Certainly, Romanov and Filippov were the first to publish their paper [2]. They refer to the work by Klimontovich [16] and the report by Davydov [13, 60] but appear to be unaware of the Galitskii thesis [14].

It is easily seen that the equations used by Romanov and Filippov are defined as quasilinear in modern terminology. References to their article can be found in the very first work by Vedenov, Velikhov, and Sagdeev [1] on this subject. Moreover, Sagdeev writes [15]: "The quasilinear theory has become an illustration of how a nonlinear theory should be constructed. One idea was to apply a method analogous to that proposed by another outstanding physics theorist Bogolyubov, who was interested, as a young researcher, in devising the methods concerned with nonlinear oscillations of systems simpler than plasma. An impetus to use Bogolyubov's method was given by the participation of Yuri Romanov and my fellow student Gennady Filippov in our seminar. They presented a report in which they insisted on the necessity to consider quantum and semiquantum recoil effects for an electron experiencing interactions with waves. It was an elegant work!

However, we made use of the Bogolyubov method, i.e., averaging over minor oscillations. This gave rise to the quasilinear theory. We extended it over other types of waves known to occur in a plasma, especially those propagating in a plasma placed in a magnetic field."

6. Quasilinear method and 'plateau' formation

We discussed in Section 5 the quasilinear diffusion coefficient reported in the article by Romanov and Filippov [2]. Another widely known quasilinear method for the description of weak turbulence is based on the work initiated by Vedenov, Velikhov, and Sagdeev [1, 3, 4]. Their quasilinear method for the analysis of Vlasov equations became prototypical and found especially wide application. It is fairly well described in many review articles (including those of the authors themselves) and monographs [28–52]. Therefore, the relevant calculations are only schematically described below.

The authors of classical work [2] considered the interaction of particles with wave packets based on the Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{V} \frac{\partial f}{\partial \mathbf{x}} + \mathbf{E} \frac{\partial f}{\partial \mathbf{V}} = 0 \,, \tag{58}$$

$$\operatorname{div} \mathbf{E} = 4\pi n e \int f \, \mathrm{d}\mathbf{V} \,. \tag{59}$$

Here, f is the velocity distribution function, $\mathbf{E}$ is electrical field strength, and n is the plasma density. Such an approach appears to be quite natural in the context of turbulent plasma theory. Romanov and Filippov showed that the Landau linear mechanism is sufficient to account for abnormal beam scattering. On the other hand, the energy exchange must result in the phase mixing of electrons, and the equation for the averaged particle distribution function in the resonance region must take the diffusive form. Indeed, Fourier components of the field independently acted on electrons in the linear approximation used by Landau in the collisionless damping problem. Turning to the nonlinear analysis, one encounters a situation in which even a weak field may exert a considerable influence on resonant particles and thereby appreciably alter their distribution function.

Vedenov, Velikhov, and Sagdeev considered the problem of the evolution of the electron distribution function in the velocity space on the assumption that in the one-dimensional case in the absence of spatial inhomogeneity it can be presented in the form

$$f = f_0(V,t) + \tilde{f}_1(V,x,t)\,, \tag{60}$$

where f_0 and $\tilde{f}$ are slow and fast oscillating functions of time, respectively. Accordingly, the electric field is represented as

$$E = E_0(t) + \tilde{E}(x,t)\,, \tag{61}$$

free from the mean electric field, $E_0 \equiv 0$. Here, the spatial homogeneity of the smoothed distribution function $f_0(V,t)$ follows from the assumption of the absence of electron 'capture' by individual harmonics of the electric field having the shape of a wide enough wave packet:

$$\Delta V = \frac{\omega}{k}\bigg|_{\max} - \frac{\omega}{k}\bigg|_{\min} \gg \sqrt{\frac{e\Phi_0}{m}}\,, \tag{62}$$

where $\Phi_0^2 = \int \mathrm{d}k\,|E_k|^2/k^2$ is the characteristic amplitude of the electric field potential. In fact, electrons migrate 'between packet harmonics' (in the reference frame moving with a phase velocity of waves, the packet forms an 'undulating landscape'), and they can no longer stay in the potential well of one of the waves longer than the time of flight over this region.

The substitution then leads to the expression for the Vlasov equation modified in accordance with the perturbation method:

$$\frac{\partial}{\partial t}(f_0 + \tilde{f}_1) + V\frac{\partial}{\partial x}(f_0 + \tilde{f}_1) - \frac{e}{m}\,\tilde{E}\,\frac{\partial}{\partial V}(f_0 + \tilde{f}_1) = 0\,. \tag{63}$$

Averaging over fast oscillations and subtracting the averaged equation from the complete one yield two equations

$$\frac{\partial f_0}{\partial t} - \frac{e}{m}\left\langle \tilde{E}\,\frac{\partial \tilde{f}_1}{\partial V}\right\rangle = 0\,, \tag{64}$$

$$\frac{\partial \tilde{f}_1}{\partial t} + V\frac{\partial \tilde{f}_1}{\partial x} - \frac{e\tilde{E}}{m}\frac{\partial f_0}{\partial V} = 0\,. \tag{65}$$

This set of equations is called quasilinear. An argument for the choice of such a name is the conservation of the nonlinear term in the equation for the averaged particle distribution function and the employment of the linear equation for its perturbations. This approach is widely applied not only to solve problems related to Vlasov's equation but also to construct various passive scalar transport models [96–100], the stochastic magnetic field evolution [101–103], etc.

The diffusive evolution of the averaged distribution function $f_0(V,t)$ is quite apparent even at this early stage. The use of the equation for $\tilde{f}_1$ in a simplified form ignoring the spatial derivative $\partial\tilde{f}_1/\partial x$ gives

$$\tilde{f}_1 = \int \frac{e}{m}\,\tilde{E}\,\frac{\partial f_0}{\partial V}\,\mathrm{d}t\,. \tag{66}$$

The substitution into the evolution equation for the averaged distribution function yields the sought diffusion relation

$$\frac{\partial f_0}{\partial t} = \left[\left(\frac{e}{m}\right)^2 \int \langle E^2\rangle\,\mathrm{d}t\right]\frac{\partial^2 f_0}{\partial V^2}\,, \tag{67}$$

in which the diffusion coefficient has, as expected, the characteristic 'Taylorian' form: $D_V \approx [(e/m)E_0]^2\tau$, with the amplitude E_0 of fluctuations of the electric field strength.

The formal method for the purpose is based on the use of the full equation for the fast oscillating part of the distribution function, which coincides with the Landau approach

$$\tilde{f} = \mathrm{i}\,\frac{e}{m}\,\frac{\tilde{E}}{\omega - kV}\,\frac{\partial f_0}{\partial V}\,. \tag{68}$$

At this stage, additional assumptions about the character of the wave field are needed. Assuming that the electric field can be represented as the superposition of independent harmonics (letting the Fourier component amplitude to be small and therefore disregarding interharmonic interactions):

$$\tilde{E}(x,t) = \int_{-\infty}^{\infty} E_k \exp\left[-\mathrm{i}(\omega_k t - kx)\right]\mathrm{d}k\,. \tag{69}$$

To recall, $\omega_k = \tilde{\omega}_k + \mathrm{i}\gamma_k$ and, because the electric field must be real-valued, $E_{-k} = E_k^*, \tilde{\omega}_k = -\tilde{\omega}_{-k}, \gamma_k = -\gamma_{-k}$. In the case of weak damping (small damping increments γ), the standard method for transformation of the denominator in the expression for the fast oscillating part of the distribution function $\tilde{f}$:

$$\frac{1}{\omega - kV} = \mathrm{P}\left(\frac{1}{\omega - kV}\right) - \mathrm{i}\pi\delta(\omega - kV)\,, \tag{70}$$

is used, which reflects the adiabaticity assumption of switching on the electric field in an infinitely remote past: $\tilde{f}_1(V,x,t \to -\infty) = 0$. This leads to a frequency renormalization consisting in the appearance of an infinitesimal positive addition: $\omega \to \omega + \mathrm{i}\varepsilon$. Here, the traditional symbol of the principal value is used:

$$\mathrm{P}\left(\frac{1}{x}\right) = \begin{cases} \dfrac{1}{x}\,, & x \neq 0\,,\\ 0\,, & x = 0\,. \end{cases} \tag{71}$$

It is now possible to evaluate the nonlinear term in the equation for the averaged particle distribution function f_0:

$$\langle \tilde{E}\tilde{f}\rangle = \left[\mathrm{i}\,\frac{e}{m}\int_{-\infty}^{\infty} \frac{E_k E_{-k}}{\tilde{\omega}_k - kV}\exp(-2\gamma_k t)\,\mathrm{d}k + \frac{e\pi}{m}\int E_k^2 \exp(-2\gamma_k t)\delta(\omega_k - kV)\,\mathrm{d}k\right]\frac{\partial f_0}{\partial V}\,. \tag{72}$$

Clearly, consideration of only the resonant term in this expression leads to the sought diffusion equation in the phase space:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial V}\left(D_V\,\frac{\partial f_0}{\partial V}\right), \tag{73}$$

with the quasilinear diffusion coefficient

$$D_V = \left(\frac{e}{m}\right)^2 \int \frac{|E_k|^2}{\omega(k) - kV} \, \mathrm{d}k \,. \tag{74}$$

These equations are supplemented by the relation describing the evolution of amplitudes of the Fourier harmonics of the electric field governed by Cherenkov mechanisms of wave emission and absorption:

$$\frac{\mathrm{d}}{\mathrm{d}t} |E_k|^2 = 2\gamma_k |E_k|^2 \,, \tag{75}$$

where the Landau increment γ_k is defined as

$$\gamma_k = 2\pi e^2 \omega \int \frac{\partial f_0}{\partial V} \, \delta(\omega - kV) \, \mathrm{d}V \,. \tag{76}$$

Here, $\omega(k)$ describes the dependence of frequency on the wave number k, while $|E_k|^2$ is the spectral function of the electric field. This closed system of equations for the averaged distribution function f_0 and the square of Fourier harmonics $|E_k|^2$ is the simplest form of quasilinear equations allowing the interaction between resonant particles and a wave packet to be described.

In the case of the one-dimensional formulation of the problem, the Cherenkov resonance condition $\omega_{\mathrm{L}} - k_z V_z = 0$ accounts for the one-to-one coupling of the wave vector and velocity: $k_z = \omega_{\mathrm{pe}}/V_z$. As a result, equations of the quasilinear theory written in terms of energy density $W(V_z, t)$ assume the form

$$\begin{gathered} \frac{\partial f_0}{\partial t} = \frac{\partial}{\partial V_z} \left(\frac{\pi e^2 V_z}{m^2 \omega_{\mathrm{L}}} \, |E_V|^2 \, \frac{\partial f_0}{\partial V_z} \right) , \\ \frac{\partial |E_V|^2}{\partial t} = 2 \, \frac{\pi V_z^2 \omega_{\mathrm{L}}}{2n} \, V_z^2 \, \frac{\partial f_0}{\partial V_z} \, |E_V|^2 \,. \end{gathered} \tag{77}$$

Distinguishing the common structural element $|E_V|^2 \, \partial f_0 / \partial V_z$ in equations (77) and using the substitution readily yield the quasilinear integral

$$\frac{\partial}{\partial t} \left[f_0 - \frac{e^2 n}{m \omega_{\mathrm{L}}^2} \frac{\partial}{\partial V_z} \left(\frac{|E_V|^2}{V_z} \right) \right] = 0 \,. \tag{78}$$

This means that we come to the equation relating the final and initial states:

$$f_\infty - \frac{e^2 n}{m\omega_{\mathrm{L}}^2} \frac{\partial}{\partial V_z} \left(\frac{|E_V(\infty)|^2}{V_z} \right) = f_0 - \frac{e^2 n}{m\omega_{\mathrm{L}}^2} \frac{\partial}{\partial V_z} \left(\frac{|E_V(0)|^2}{V_z} \right) . \tag{79}$$

If the initial fluctuations are thermal, $W_T \approx k_{\mathrm{B}} T (1/r_{\mathrm{D}}^3) \approx n k_{\mathrm{B}} T / N_{\mathrm{D}} \ll n k_{\mathrm{B}} T$, one can assume that $|E_k(t=0)|^2 = 0$; hence, the self-consistent picture of 'plateau' formation in the distribution function and of electrostatic noise enhancement (Fig. 3):

$$|E_V|^2 = 4\pi m V_z \int_0^{V_z} \big(f_0(\infty) - f_0(0) \big) \, \mathrm{d}V_z \,. \tag{80}$$

This result for the quasilinear diffusion coefficient of electrons in the velocity space proved to coincide with the result obtained by Romanov and Filippov. A few references to the problems pertinent to the setting of priority as regards a quasilinear description of the particle–wave packet interaction can be found in the volume dedicated to the memory of A A Vedenov. For information, here is an excerpt from Yu A Romanov's "Review of A A Vedenov's Thesis for Doct. Phys.-Math. Sci." [15]:

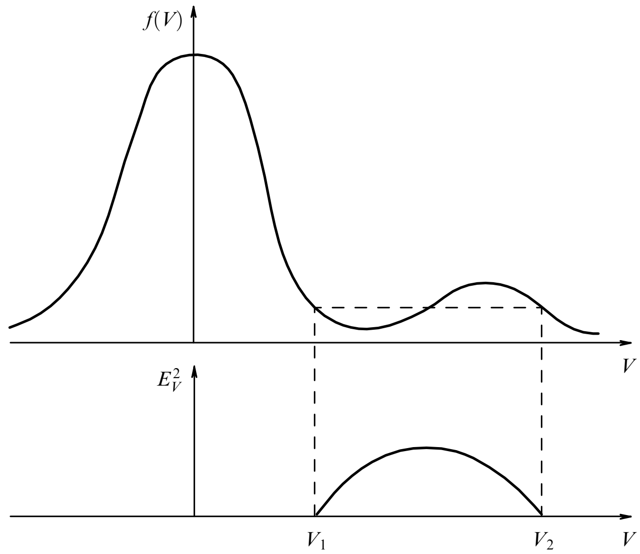

Figure 3. Formation of the quasilinear plateau in the distribution function.

"A A Vedenov's thesis presents an analysis of various phenomena in a plasma that occur when the energy density of plasma oscillations is much higher than the thermal noise energy density. These phenomena are considered based on the quasilinear equations derived earlier by other authors (Yu A Romanov and P F Filippov).

The central idea of the thesis embraces the existence of a quasilinear state to which plasma comes after the development of perturbations. In this state, the electron distribution function in a certain part of the phase space turns out to be constant, while plasma oscillation energy density becomes rather high. It should be emphasized that the quasilinear state is realized only when the electron distribution function can be regarded as one-dimensional."

On the other hand, Vedenov argued in his first review article on the quasilinear theory published in *Voprosy Teorii Plasmy* (*Reviews of Plasma Physics*) [6] that Romanov and Filippov "postulated their equations." This opinion is also justified: it will be shown in the following sections that Vedenov–Velikhov–Sagdeev's scheme gives an advantage in considering such issues as turbulent diffusion of admixtures (passive scalar), chaoticity of magnetic lines, and turbulent convection.

Formally speaking, it is no stretch to believe that Romanov and Filippov realized the potential of the quantum-mechanical approach proposed by Davydov for the case of Langmuir oscillations, which enabled them to derive the necessary equations, whereas Vedenov, Velikhov, and Sagdeev developed an efficacious perturbation method for the solution of Vlasov's equation and formulated the concept of 'plateau' formation in the averaged particle distribution function. Clearly, the quasilinear character of the theory accounts for the relatively small amplitudes, and these estimates need to be elaborated in greater detail by an in-depth consideration of both the wave packet structure and the decorrelation mechanisms responsible for the diffusive character of the distribution function evolution.

7. Stochasticity in a wave packet and correlation scales

The quasilinear description of a weakly turbulent plasma is based on the concept of the absence of particle capture by individual harmonics and the randomness of wave phases in the 'packet'; such an approach makes it possible to implement the decorrelation mechanism behind the diffusive wandering of particles in the velocity space (Fig. 4). Here is a quotation from Ref. [1]: "Let us assume the simultaneous presence of many waves with different vectors and randomly distributed phases in a plasma. Then, the wave packets are wide enough and the capture of particles by 'potential wells' of individual harmonics can be ignored... ." To recall, Drummond and Pines [5] did not even discuss this important problem.

It is convenient to consider the correlation-related aspects of quasilinear diffusion in the framework of the Chirikov concept suggesting the appearance of a stochastic layer near the physical pendulum separatrix in the presence of modulating perturbation [17, 45, 46]. In this case, the use of Chirikov's criterion for the stochastic layer overlap (Fig. 5) allows us to obtain useful estimates of key parameters of the quasilinear model. For example, the amplitude of electric field perturbations is given in the spectral representation by the expression

$$\delta E \propto |E_k|^2 \, \delta k \, . \tag{81}$$

Here, $|E_k|^2$ is the spectral distribution, and δk is the wave packet width in the wave number space. The width of the stochastic layer can be estimated from the particle velocity δV_{S} acquired in the field of fluctuations of the electric potential $\delta\varphi$:

$$\frac{m(\delta V_{\mathrm{S}})^2}{2} \propto e\,\delta\varphi \, . \tag{82}$$

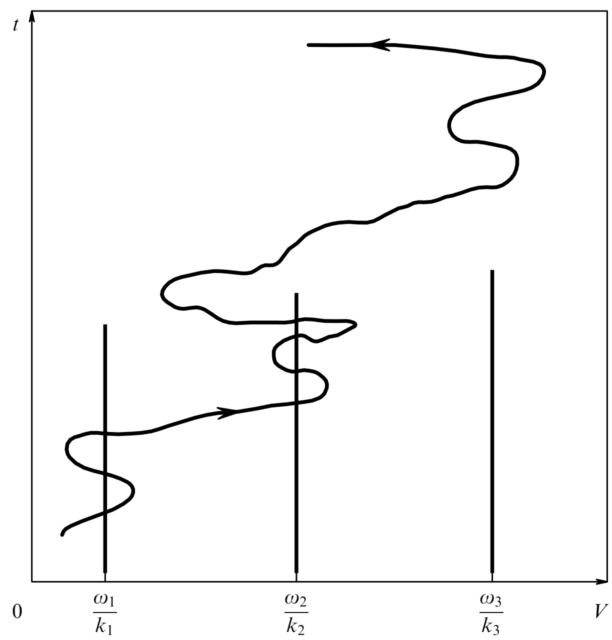

Figure 4. Wandering of electrons between wave packet harmonics.

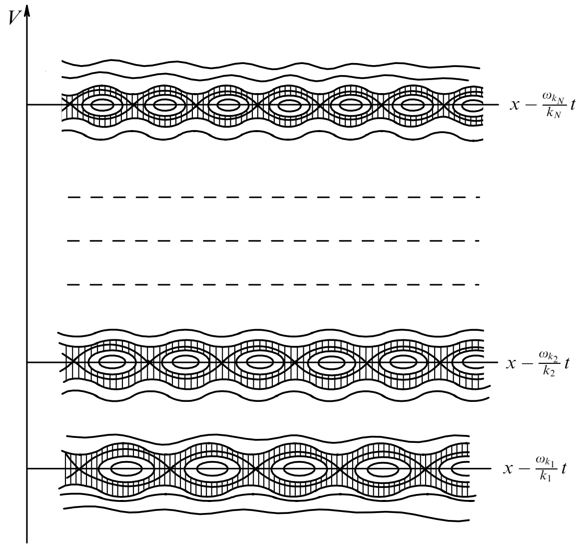

Figure 5. Overlapping island 'chains' create conditions for particle diffusion in the velocity space.

The use of $\delta E \propto \nabla\varphi \propto \delta\varphi\, k$ representation leads to the expression for the characteristic amplitude of electric fluctuations $\delta\varphi$ created by wave packet harmonics in the form $k^2\,\delta\varphi^2 \propto E_k^2\,\delta k$. Substituting the expression for $\delta\varphi$ into the equation for the stochastic layer width δV_{S} yields

$$\delta V_{\mathrm{S}} \propto \left(\frac{e^2}{m^2}\,\frac{E_k^2}{k^2}\,\delta k\right)^{1/4} . \tag{83}$$

The distance between harmonics of the wave packet of interest in the velocity space is given then by the formula

$$\delta\left(\frac{\omega}{k}\right) = \delta V = \frac{\partial(\omega/k)}{\partial k}\,\delta k = \left(\frac{1}{k}\,\frac{\partial\omega}{\partial k} - \frac{1}{k}\,\frac{\omega}{k}\right)\delta k$$
$$= \frac{\delta k}{k}\,(V_{\mathrm{g}} - V_{\phi}) \, . \tag{84}$$

In the case under consideration, it is supposed that the waves composing the packet have amplitudes of the same order, $V_{\phi} \ll V_{\mathrm{g}}$, which gives the simple estimate for power-law dependences $\omega(k)$:

$$\delta V(\delta k) \propto \frac{\omega(k)}{k^2}\,\delta k \, . \tag{85}$$

The resonance overlapping condition $\delta(\omega/k) \leqslant \delta V_{\mathrm{S}}$ is useful at once in two ways. First, it allows determining the lower boundary of plasma oscillation noise level in the quasilinear approximation:

$$\left(\frac{e^2}{m^2}\,\frac{E_k^2}{k^2}\,\delta k\right)^{1/4} \geqslant \frac{\omega}{k^2}\,\delta k \, . \tag{86}$$

Second, δk actually makes up a model parameter that can be deduced from the 'moderate' Chirikov condition for resonance overlapping that takes, in this case, the form of the algebraic equation for δk:

$$\left(\frac{e^2}{m^2}\,\frac{E_k^2}{k^2}\,\delta k_*\right)^{1/4} = \frac{\omega}{k^2}\,\delta k_* \, . \tag{87}$$

Simple calculations permit estimating the packet width involving key parameters of the wave field:

$$\delta k_* = \left(\frac{e^2}{m^2}\, E_k^2\right)^{1/3} \frac{k^2}{\omega^{4/3}}\,, \tag{88}$$

where the condition $kV_\phi = \omega$ is fulfilled.

Not only the possibility but also the randomness of transitions must be provided in the framework of the random walk model. It yields the stochasticity condition for wave phases. Assuming that the wave packet 'misphasing' takes a characteristic time $\tau_{\rm ph}$ allows us to write out the following estimates

$$\Delta\theta \propto \Delta(\omega t) \propto \Delta V k \tau_{\rm ph} \approx 1\,. \tag{89}$$

Here, ΔV is the characteristic width of the wave packet in the velocity space. Therefore, the phase mixing time may be estimated as follows:

$$\tau_{\rm ph} \propto \frac{1}{k\Delta V} \propto \frac{1}{k}\,\frac{1}{\delta V N}\,, \tag{90}$$

where N is the number of waves in the packet. Evidently, the mixing time $\tau_{\rm ph}$ is much shorter than correlation time $\tau \approx 1/(k\delta V)$, as far as a wide packet having $N \gg 1$ harmonics is concerned.

The use of correlation scales $\delta V_* \approx \delta V_{\rm S}(\delta k_*)$ and $\tau_* \approx 1/(k\delta V_*)$ makes it possible to write out the condition for the maximum permissible packet energy density, which is actually a criterion for rapid 'stirring' of wave phases:

$$\tau_{\rm ph} \ll \tau_{\rm rel} \approx \frac{(\Delta V)^2}{D_V}\,, \tag{91}$$

where $D_V \propto \delta V_*^2/(\tau_*(k))$ is the diffusion coefficient in the velocity space. For the 'jump' amplitude in the velocity space one obtains

$$\delta V_* \approx \delta V_{\rm S} \propto \left(\frac{e^2}{m^2}\, E_k^2\, \frac{1}{\omega}\right)^{1/3}. \tag{92}$$

The dimensional estimate $\tau_*(k) \propto 1/(k\delta V_*)$ needs to be utilized as the characteristic correlation time, because $\delta V_* \approx \delta V_{\rm S} \geqslant \delta V$. Substitution yields the following quasiclassical result for the diffusion coefficient:

$$D_V \propto \frac{\delta V_*^2}{\tau_*(k)} \propto \left(\frac{e}{m}\right)^2 \frac{E_k^2}{V_\phi}\bigg|_{V_\phi=\omega/k}. \tag{93}$$

It was shown earlier that this formula is in line with Taylor notions of turbulent diffusion. But we are dealing here with the velocity space, instead of a merely ordinary one. The analysis with the use of the stochastic layer overlap model allowed important features of the decorrelation mechanism to be elucidated. For example, the Landau increment in the quasilinear method behaves externally only as a characteristic determining the evolution of the wave packet harmonics, whereas the mechanism of Cherenkov interaction creates a wave modulation effect underlying stochastization and 'overlapping' of separatrices; as a consequence, the particle motion in the packet field acquires a diffusive character. Formally, it is possible to calculate the width of the stochastic layer by the Mel'nikov integration method [104–108], but it yields only the linear dependence on the perturbation amplitude $\delta V_* \propto \Phi_*$, whereas parametric dependences in the expression for correlation scales will be lost.

It should be noted that the introduction of the frequently used formal criterion for stochasticity K, viz.

$$K = \frac{\delta V_{\rm S}}{\delta V} > 1\,, \tag{94}$$

allows representing the applicability conditions of the quasilinear approximation $\tau_{\rm ph} < \tau_* < \tau_{\rm rel}$ in the form

$$\frac{1}{k\delta V}\,\frac{1}{N} < \frac{1}{k\,\delta V_{\rm S}} < \frac{(N\delta V)^2}{D} \approx N^2\,\frac{(\delta V)^2}{(\delta V_{\rm S})^3 k}\,, \tag{95}$$

which leads to $K < N$ or, in fact, to the boundedness condition for the stochastic layer width: $\delta V_{\rm S} < \Delta V$.

The question of characteristic times goes beyond the $\tau_{\rm ph} < \tau_* < \tau_{\rm rel}$ hierarchy: τ_* can be readily redefined in terms of the quasilinear diffusion coefficient:

$$\tau_* \approx \frac{1}{k\delta V_*} \propto \frac{1}{(k^2 D)^{1/3}}\,. \tag{96}$$

This characteristic scale is of the same order of magnitude as the time scale τ_K characterizing stochastic instability and as the 'bounce-period' $\tau_B \approx (k\sqrt{e\Phi_*/m}\,)^{-1}$. The time scale hierarchy can now be represented as

$$\tau_{\rm ph} < \tau_* \approx \tau_K \approx \tau_B < \tau_{\rm rel}\,. \tag{97}$$

Bass, Fainberg, and Shapiro [109] appear to have been the first to pay attention in 1965 to the relationship between the quasilinear approach and Chirikov's ideas and to emphasize the importance of taking into account the finiteness of correlation splitting time. It is worthwhile to mention that Chirikov published his article on stochasticity in dynamical systems [17] as early as 1959. Moreover, he worked at LIPAN under the supervision of G I Budker, doing research on plasma confinement in magnetic traps from 1954 till 1960, i.e., in the same period as the authors of the quasilinear method (at the Theoretical Sector headed by M A Leontovich) [1]. For all that, calculations of correlation effects and stochastic layer width began to be systematically considered in the framework of the Hamiltonian formalism only in a series of publications by Zaslavskii, Sagdeev, and Filonenko in 1966–1968 [110–112].

8. Self-similarity and 'fronts' in velocity space

This section opens with a quotation from the article by Gurevich, Pariiskaya, and Pitaevskii [113] published in 1966 that reads as follows: "The system of equations describing collisionless plasma is very complicated, which makes it difficult to formulate a nonstationary nonlinear problem that would have a clear physical sense, the solution of which could be brought to completeness." But it is exactly such a problem that was solved in the paper by Ivanov and Rudakov dated 1966 [114]. It was the time of tremendous upgrowth of the kinetic plasma theory and exponential development of its analytical methods [115]. The concepts of diffusion and source–sink in the velocity space became commonplace. It was time to import one more beautiful hydrodynamic idea related to 'front' propagation [114]. The 'inner side-hump' of the electron distribution function is known to generate

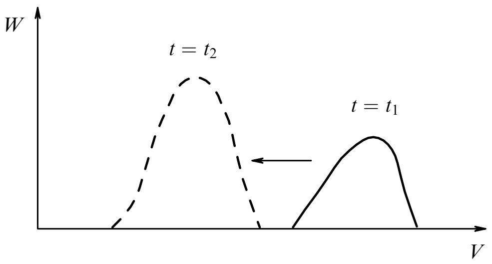

Figure 6. Quasi-linear front movement.

Cherenkov radiation by virtue of $\partial f/\partial V > 0$ and thereby tends toward the region of lower velocities due to beam electron energy loss. Importantly, the side-hump becomes increasingly steeper, because more energetic particles lose proportionally more energy. It is actually the front movement in the velocity space (Fig. 6).

The analytical representation of the front propagation process was obtained by introducing self-similarity variables that were already used at that time in collisionless plasma kinetics (see the aforementioned studies by Gurevich and Pitaevskii [113, 116]), but in the context of the spatio-temporal problem. Ivanov and Rudakov [117] reduced the set of quasilinear equations to the well-known class of nonlinear diffusion (thermal conductivity) equations. It is convenient to move from the squares of Fourier harmonics to energy density W_V as a function of velocity V_z. In this case, $W_V \omega_L (\mathrm{d}V_z/V_z^2)$ is the energy density of oscillations with phase velocity in a range from V_z to $V_z + \mathrm{d}V_z$. As a result, the equations of the one-dimensional quasilinear theory in terms of energy density $W_V(V_z, t)$ will take the form

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{\pi}{\omega_{\mathrm{p}}} \left(\dfrac{e}{m}\right)^2 \dfrac{\partial}{\partial V} \left(V W_V \dfrac{\partial f}{\partial V} \right), \\ \dfrac{\partial W_V}{\partial t} = \dfrac{\pi \omega_{\mathrm{p}}}{n} V^2 W_V \dfrac{\partial f}{\partial V}. \end{cases} \tag{98}$$

This system can be simplified by introducing a new 'renormalized' noise density function

$$\psi = W_V V^{\alpha}. \tag{99}$$

Then, trying to get identical 'combinations' in both equations, the following expressions are obtained with the self-similarity parameter $\alpha = -1$:

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{\pi}{\omega_{\mathrm{p}}} \left(\dfrac{e}{m}\right)^2 \dfrac{\partial}{\partial V} \left(V^2 \psi \dfrac{\partial f}{\partial V} \right), \\ \dfrac{\partial \psi}{\partial t} = \pi \dfrac{\omega_{\mathrm{p}}}{n} V^2 \psi \dfrac{\partial f}{\partial V}. \end{cases} \tag{100}$$

The algebraic substitution of the second equation into the first one and subsequent integration over time yield a system in which the equation for noise density ψ allows an independent solution:

$$\frac{\partial \psi}{\partial t} = \frac{\pi}{\omega_{\mathrm{p}}} \left(\frac{e}{m}\right)^2 V^2 \psi \frac{\partial^2}{\partial V^2} (\psi - \psi_0) + \pi \frac{\omega_{\mathrm{p}}}{n} V^2 \psi \frac{\partial f_0}{\partial V}. \tag{101}$$

It is easily seen that the equation for the 'renormalized' noise density function $\psi = W_V/V$ is a nonlinear diffusion equation (power-law nonlinearity) with a source. Here, $f_0(V)$ and $\psi_0(V)$ are initial distributions of the renormalized particle and noise distribution functions, respectively.

Self-similarityl variables provide an efficient tool for analytical studies of such problems [117]. However, the presence of a $\partial f_0/\partial V$-related source permits additionally singling out characteristic stages of the beam relaxation process. For example, the initial stage of plateau formation and noise level enhancement is mainly due to the influence of term $\pi(\omega_{\mathrm{p}}/n) V^2 \psi \, \partial f_0/\partial V$ under small nonlinearity. Consequent 'weakening' of the electron distribution function profile permits neglecting the term with the source ($\partial f_0/\partial V \to 0$) and thereby making use of self-similarity variables for the solution of the reduced equation in noise density:

$$\frac{\partial \psi}{\partial t} = \frac{\pi}{\omega_{\mathrm{p}}} \left(\frac{e}{m}\right)^2 V^2 \psi \frac{\partial^2}{\partial V^2} (\psi - \psi_0). \tag{102}$$

It is this self-similarity regime that provides an approach to the description of both the propagation velocity and the width of the perturbation front for ψ and f in a form analogous to the thermal wave front.

Because the traditional scaling method for nonlinear equations in thermal conductivity does not yield a fundamentally new invariance relation, Ivanov and Rudakov [114] introduced the self-similarity variable in the form

$$\xi(U, \tau) = 1 - \frac{U}{\sqrt{\tau}}, \tag{103}$$

analogous to the variable in the classical spatial diffusion problem where the spatial and temporal scales are related by the expression $\xi = l/\sqrt{t}$. In addition, dimensionless quantities for the key parameters of the model were introduced:

$$\tilde{F} = \frac{V_{\mathrm{b}}}{n_{\mathrm{b}}} f(V), \quad w = W_k \frac{\omega_{\mathrm{p}}}{2\pi m n_{\mathrm{b}} V_{\mathrm{b}}^3}, \tag{104}$$

$$\tau = \pi \omega_{\mathrm{p}} \left(\frac{n_{\mathrm{b}}}{n_0}\right) t, \quad U = \frac{V}{V_{\mathrm{b}}}. \tag{105}$$

Transformation then yields an ordinary differential equation for the function $\varphi = w(U, \tau) - w_0(U)$ being sought:

$$-\frac{\xi}{2} \frac{\partial \varphi}{\partial \xi} = (\varphi + w_0) \frac{\partial^2 \varphi}{\partial \xi^2}. \tag{106}$$

Here, there are two boundary conditions, $\varphi(0) = 1$ and $\varphi(\infty) = 0$. The authors of Ref. [114] found an approximate solution to this equation for the case of increasing noises ($\ln(\varphi/w_0) \gg 1$):

$$\varphi = \frac{1}{2} \xi_0 (\xi_0 - \xi) \ln \frac{1}{w_0}, \tag{107}$$

where the constant ξ_0 is given by the condition $\varphi(0) = 1$; evidently, it determines position of the front traveling in the velocity space: $\xi_0 = \sqrt{2/\ln(1/w_0)}$. Turning back to dimensional quantities, one can use the definition of the self-similarity variable

$$\xi \propto \frac{1}{V_{\mathrm{b}}} \frac{V_{\mathrm{b}} - V}{\sqrt{\pi \omega_{\mathrm{p}} (n_{\mathrm{b}}/n_0) t}} \tag{108}$$

in order to determine beam relaxation time in the velocity interval $\Delta V = V_{\rm b} - V(\Delta t)$:

$$\Delta t \approx \frac{1}{2}\left(\frac{\Delta V}{\Delta V_{\rm b}}\right)^2 \frac{1}{\pi\omega_{\rm p}}\frac{n_0}{n_{\rm b}}\ln\frac{1}{w_0}\,. \qquad (109)$$

Because the Landau increment is estimated by the expression

$$\gamma_{\rm L} \approx \omega_{\rm p}\frac{n_{\rm b}}{n}\left(\frac{V_{\rm b}}{\Delta V}\right)^2,$$

it is easy to obtain the relaxation time

$$\tau_{\rm rel} \approx \frac{1}{2\gamma_{\rm L}}\ln\frac{1}{w_0} \gg \frac{1}{\gamma_{\rm L}}\,. \qquad (110)$$

The logarithmic factor is comparable to the Coulomb logarithm, since it is related to the noise amplitude ratio at the beginning and end of the relaxation process.

A qualitative assessment of the front velocity is given by the ratio of the characteristic 'displacement' $\delta V_{\rm F} \propto \Delta V/\ln(1/w_0)$ to the time it needs to occur (inverse increment $1/\gamma_{\rm L}$). Ivanov and Rudakov also investigated in detail the structure of the front by simplifying the key self-similar equation. In doing so, the evaluation of the width of the transition zone, traditional for thermal wave propagation problems, is impossible, because the spatial scale 'drops out' of the model equation.

This elegant solution proposed by Ivanov and Rudakov demonstrates the effectiveness of the simultaneous solution of the diffusion equation for the averaged distribution function nonlinearly related to the equation for noise spectrum evolution. Indeed, it opens up a new level of understanding the Vlasov approach in which the kinetic equation is regarded as 'self-consistent' with the Maxwell equations. Since then, the self-similarity solutions of kinetic equations have greatly contributed to the analysis of three-dimensional beam relaxation models [41], ion-acoustic turbulence [118], and strongly nonequilibrium suprathermal electron distribution functions [40, 52, 119–121].

9. Wave scattering effects and sectoral plateau

The quasilinear approach makes it possible to analyze not only one-dimensional problems. Specifically, the 'plateau-formation' phenomenology can be employed to consider electron scattering by an ion-acoustic wave packet. This mechanism was postulated by Kadomtsev and Sagdeev as being of primary importance for explaining anomalous plasma resistance to an electric current passing through it [26, 122, 123]. Marked non-one-dimensionality (anisotropy) in the development of instability of ion-acoustic waves in a current-carrying plasma was demonstrated in the numerical experiments of Field and Fried [122]. It was shown that the role of waves directed at a large enough angle to the electric current vector becomes important at long times. Moreover, electron diffusion develops in the velocity space along the azimuthal angle.

The manifestation of anisotropy effects can be accounted for in terms of the mechanism underlying resonant interactions between electrons and ion-acoustic waves. The latter become unstable in a nonisothermal plasma, where electron temperature is much higher than ion temperature ($T_{\rm e} \gg T_{\rm i}$) under conditions when the directed velocity of electrons U exceeds a certain critical value: $U > U_* \approx c_{\rm s}$ [26, 47]. It occurs, for example, in the presence of an external electric field. Electron scattering by ion-acoustic waves is related to the Cherenkov mechanism of interaction, for which the resonance condition has the form

$$\omega = \mathbf{k}\mathbf{V}_{\rm e} = kV_{\rm e}\cos\theta_*\,, \qquad (111)$$

where θ_* is the angle between electron velocity direction $\mathbf{V}_{\rm e}$ and wave vector $\mathbf{k}$ of ion sound. At small wave numbers (the long wavelength limit $\omega/k = \sqrt{T_{\rm e}/m_{\rm i}}$, $k \ll 1/r_{\rm De}$), this condition is fulfilled only when $\cos\theta_* \approx \omega/(kV_{\rm e}) \approx (m_{\rm e}/m_{\rm i})^{1/2}$. This means that only those electrons that travel toward a wave at an almost right angle interact with it, while incremental velocity $\Delta V_{\rm e}/V_{\rm e} \approx \cos\theta_*$, being parallel to the wave vector $\mathbf{k}$, turns out perpendicular to $\mathbf{V}_{\rm e}$. Indeed, a change in momentum in each quantum emission–absorption act, $\Delta\mathbf{p} = m\Delta\mathbf{V} = \hbar\mathbf{k}$, leads to a change in energy $\Delta\varepsilon = m\mathbf{V}\Delta\mathbf{V} = \hbar\mathbf{k}\mathbf{V} = \hbar\omega$. Formal estimations yield

$$\frac{\Delta p}{p} \propto \frac{\hbar\omega}{mV^2/2}\frac{kV}{\omega} \propto \frac{\Delta\varepsilon}{\varepsilon}\sqrt{\frac{m_{\rm i}}{m_{\rm e}}}\,, \qquad (112)$$

which allows us to use with confidence the elastic scattering model.

Once the instability increment of ion-acoustic waves excited in a plasma through which electricity flows is small, $U \gg c_{\rm s}$, a quasilinear plateau corresponding to a concrete wave vector direction forms in the electron distribution function (Fig. 7). Here, U is the drift velocity of current-carrying electrons. Kadomtsev [26] maintained that if wave vectors fill a certain sector in the current flow direction, the plateau formation region (electron distribution function isotropization region) broadens appreciably, because the distribution function must be constant at the intersections of different resonance belts. Evidently, such 'non-one-dimensionality' significantly complicates theoretical analysis in comparison with that of a one-dimensional quasilinear model [1–7].

It was noted in the foregoing discussion that Davydov examined the influence of ion-acoustic waves on the electron distribution function $F_{\rm e}$ for the case of thermal noise ($T_{\rm e} \gg T_{\rm i}$) using the model Fokker–Planck equation [13]. He

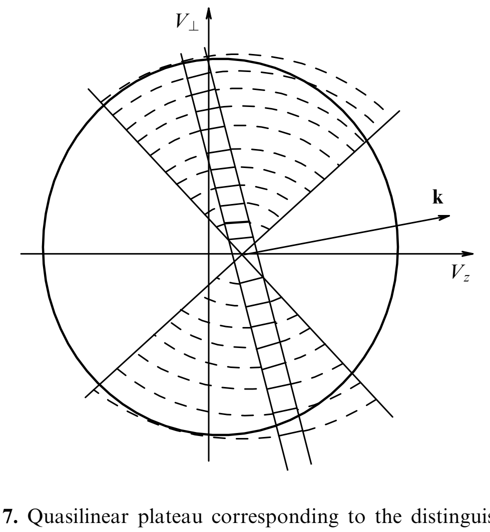

Figure 7. Quasilinear plateau corresponding to the distinguished wave vector direction.

also described the scattering effect by invoking the collision integral analogous to the integral of electron collisions with infinitely heavy ions (the Lorentz approximation):

$$\mathrm{St}\,[F_{\mathrm{e}}] = \frac{1}{\sin\theta'}\,\frac{\partial}{\partial\theta'}\left(v_{\mathrm{eff}}\sin\theta'\,\frac{\partial F_{\mathrm{e}}}{\partial\theta'}\right). \tag{113}$$

Here, $v_{\mathrm{eff}} \approx \omega_{\mathrm{pe}} W/(nTe)$ is the effective collision frequency, and θ' is the azimuthal angle of the electron velocity vector. Traditional normalization in terms of noise energy density was employed:

$$W = 2\pi\int_0^{\infty} k^2\,\mathrm{d}k\int_0^{\pi} W(k,\theta')\sin\theta'\,\mathrm{d}\theta'. \tag{114}$$

The 'sectoral plateau formation' concept and the kinetic (Lorentz) model of electron scattering by noises actually provided a basis for the development of the ion-acoustic turbulence theory in Refs [124–128]. Rudakov and Korablev [125] were the first to thoroughly investigate a quasistationary state in which the dynamic friction force prevents the majority of the electrons from passing into the free acceleration regime. The simplified kinetic equation for the electron distribution function is traditionally written out with the use of spherical coordinates V, θ', φ' for particle velocity. This equation lacks the dependence on φ' due to a problem symmetry, and contains the variable $\xi = \cos\theta'$:

$$F_{\mathrm{e}}(V,\xi) = f_0(V) + f_1(V,\xi)\,, \tag{115}$$

$$F_{\mathrm{e}}(V,\xi) = f_0(V) + f_1(V,\xi) - \frac{eE}{m_{\mathrm{e}}}\,\xi\,\frac{\partial f_0}{\partial V} = \frac{V_T^3}{V^3}\,\frac{\partial}{\partial\xi}\left[(1-\xi^2)\,v_{\mathrm{eff}}(\xi)\,\frac{\partial f_1}{\partial\xi}\right]. \tag{116}$$

Here, the effective collision frequency $v_{\mathrm{eff}}(\xi)$ appears as a result of calculating the quasilinear collisional term

$$\mathrm{St}_{\mathrm{QL}} = \frac{e^2}{m_{\mathrm{e}}^2}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^3}\,\mathbf{k}\,\frac{\partial}{\partial\mathbf{V}}\,|\varphi_k|^2\pi\delta(\omega_k - \mathbf{kV})\left(\mathbf{k}\,\frac{\partial F}{\partial\mathbf{V}}\right). \tag{117}$$

The expression for v_{eff} is written through the definition of the spectral oscillation density

$$N(\mathbf{k}) = \frac{k^2|\varphi_k|^2}{8\pi}\,\frac{\partial}{\partial\omega}\left[\omega\varepsilon(\omega,\mathbf{k})\right] \approx \left(\frac{\omega_{\mathrm{pi}}}{\omega}\right)^2\mathbf{k}^2\,\frac{|\varphi_k|^2}{4\pi}\,, \tag{118}$$

and has the following form

$$v_{\mathrm{eff}}(\xi) = \frac{V_T}{n_0 T_{\mathrm{e}}}\int_0^{\infty}\frac{k^3\,\mathrm{d}k}{(2\pi)^2}\int_{-\sqrt{1-\xi^2}}^{\sqrt{1-\xi^2}}\frac{N(k,x)}{\sqrt{1-\xi^2-x^2}}\left(\frac{\omega}{kc_{\mathrm{s}}}\right)^2\frac{x^2}{1-\xi^2}\,\mathrm{d}x\,, \tag{119}$$

where the spherical coordinate system k, θ, φ for the wave vector and notation $x = \cos\theta$ are used.

The central point in the development of the quasilinear method concerns the consideration by Rudakov and Korablev of the stationary equation for spectral energy density of ion-acoustic oscillations:

$$\gamma_{\Sigma}(k,\theta)\,N(k,\theta) = 0\,. \tag{120}$$

Here, γ_{Σ} is the sum of increments responsible for excitation and damping of ion-acoustic waves. In the simplest formulation [125], the increment of excitation of ion-acoustic waves with the aid of the nonequilibrium electron distribution function was taken into account, namely

$$\gamma_{\mathrm{e}} = \frac{2\pi^2 e^2}{m_{\mathrm{e}} k^2}\,\frac{\omega}{\omega_{\mathrm{pi}}^2}\int\mathrm{d}\mathbf{V}\,\delta(\omega - \mathbf{kV})\left(\mathbf{k}\,\frac{\partial f_1}{\partial\mathbf{V}}\right), \tag{121}$$

as was the wave damping on ions by virtue of Cherenkov interaction (Landau damping):

$$\gamma_{\mathrm{i}} = 2\pi^2 k\left(\frac{\omega}{k}\right)^4 F_{\mathrm{i}}\left(\frac{\omega}{k}\right). \tag{122}$$

Here, the authors of paper [125] used for the first time the separation-of-variables approximation $N(k,\theta) = N(k)\Phi(\cos\theta_k)$, where k is the modulus of an acoustic wave vector, and θ is the angle between the wave vector and the electron drift velocity. The 'peaked' delta-shaped distribution was chosen for $N(k)$:

$$N(k) = \frac{(2\pi)^3}{\omega}\,\frac{\delta(k-k_0)}{k_0^2}\,, \tag{123}$$

where k_0 was found from the increment extremality condition (there is an analogous condition in Ref. [124]):

$$\frac{\partial}{\partial k}\,\gamma_{\Sigma}(k_0,\theta)\Big|_{\theta\to 0} = 0\,, \qquad \gamma_{\Sigma}(k_0,\theta)\Big|_{\theta\to 0} = 0\,. \tag{124}$$

Importantly, physical concepts of the mechanism behind ion-acoustic wave excitation make it possible to obtain the equation describing the angular dependence of wave distribution function $\Phi(\cos\theta)$. This, it is natural to suppose that waves are excited in the region shaped like a cone broadening in the current flow direction. As a follow-up to Kadomtsev's inference, the authors of Ref. [125] suggested that the apex angle of the cone $\theta_0 < \pi/2$; therefore, $\Phi(\cos\theta)$ differs from zero only in the region $1 \geqslant \cos\theta \equiv x \geqslant \cos\theta_0 \equiv x_0$.

We can safely say that theorists had here a piece of good luck, because the stationary equation for spectral wave density reduces in this case to the Abel integral equation

$$g(x) = \int_0^x \frac{\Phi(z)}{(x-z)^{\alpha}}\,\mathrm{d}z\,, \qquad 0 < \alpha < 1\,. \tag{125}$$

It is a special case of the Volterra integral equation of the first kind having the analytical solution

$$\Phi(x) = \frac{\sin(\pi\alpha)}{\pi}\,\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (x-t)^{\alpha-1} g(t)\,\mathrm{d}t\,. \tag{126}$$

Omitting the cumbersome calculations, it suffices to note here that in the case considered by Rudakov and Korablev, $\alpha = 1/2$.

However, the angular distribution of turbulent pulsations obtained by solving the Abel equation leads to divergence as $\theta \to 0$, suggesting the infinity of the wave packet energy. This difficulty was overcome by Kovrizhnykh [126], who included pair ion collisions in the consideration and added a relevant increment

$$\gamma_{\mathrm{st}} = \frac{1}{2\sqrt{\pi}}\left(\frac{m_{\mathrm{e}}}{m_{\mathrm{i}}}\,\frac{T_{\mathrm{e}}}{T_{\mathrm{i}}}\right)^{1/2}\frac{c_{\mathrm{s}}^2 k^2}{\omega}\,v_{\mathrm{ei}}\,, \tag{127}$$

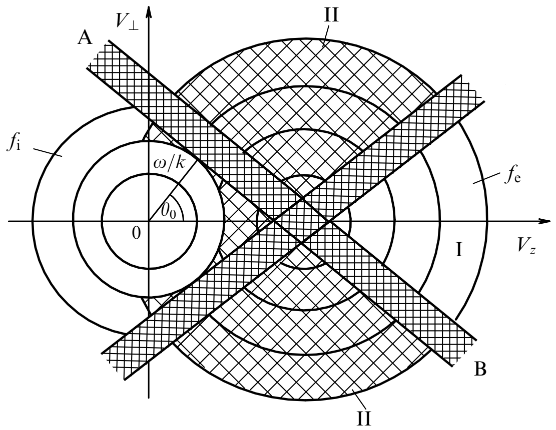

Figure 8. The quasilinear diffusion region in the Rudakov–Korablev model.

where

$$\nu_{ei} = \frac{4\pi e^4 n}{m_e^2 V^3} \ln \Lambda \,, \tag{128}$$

in the expression for γ_Σ. As a result, he obtained the following angular distribution:

$$\Phi(\cos\theta) \equiv \Phi(x) = c_0 \frac{\lambda}{x(1-\lambda x)} \Big\{ p\big[p + 3x^2(1-\lambda x)\big] - \frac{3}{4}\lambda x_0^2 \ln \frac{x + p^{1/2}}{x_0} \Big\} \,, \tag{129}$$

serving in this place as an illustration of the potential of the analytical approach. Here, $x^2 - x_0^2 = p$, c_0 is a constant, and λ is the parameter differing from unity that takes into account the contribution from ion–ion collisions. The absence of collisions in Ref. [125] corresponds to $\lambda = 1$ and, as a consequence, to the $\Phi(x)$ divergence as $x \to 1$, $\Phi(x) \propto 1/[x(1-x)]$. The Kadomtsev sectoral diagram conforms to a small excess of drift velocity U over phase velocity c_s of acoustic waves. An illustration for the general case is presented in Fig. 8, where the resonance condition is shown schematically.

The analysis of quasilinear results revealed serious problems associated with the model theory. For example, a rise in temperature leads to a greater number of runaway electrons and a system escape from the stationary state, meaning that the applied electric field must be much weaker than the critical one that makes electrons escape, i.e., the Dreicer field E_D rapidly decreasing with temperature:

$$E_D \propto \frac{e}{\lambda_e^2(T_e)} \ln \Lambda \propto \frac{1}{T_e^2} \,, \tag{130}$$

where λ_e is the electron range, and $\ln \Lambda$ is the Coulomb logarithm. Indeed, elementary estimates give $n\sigma\lambda_e \approx 1$, $e/\sigma \approx k_B T_e$, where σ is the cross section. On the other hand, the effects of ion distribution function anisotropy, resonant wave absorption by ions, and nonlinear wave-to-wave interaction (induced scattering, etc.) are disregarded here.

The solution of kinetic equations for plasmons, taking account of nonlinear effects, became a natural area of further studies [127–130]. In 1956, Sizonenko and Stepanov [124] noted that the level of ion-acoustic oscillations must be correlated with nonlinear processes. It is this modified equation for the number of quanta that was used by Silin and co-workers [131, 132], who disregarded the model delta-shaped $N(k)$ profile but retained the variable separation method and the Abel integral, to obtain the self-consistent analytical solution to the problem of quasistationary ion-acoustic turbulence. Much as we would like to further elaborate on this theme, we have to confine ourselves to this brief comment. To conclude this section, it is worthwhile to note that the description of ion-acoustic turbulence (IAT) is a nonacademic problem, and many of its important aspects remain to be clarified [132–134].

10. Increment balance and suprathermal particles

The quasilinear approach to the analysis of collective effects associated with oscillations excited in the plasma considerably expanded the circle of solvable problems in the kinetic plasma theory. For example, it proved possible to describe, in the framework of the Lorentz approximation and with the use of the quasilinear diffusion coefficient, anisotropic distortions of the particle velocity distribution function related to oscillations under the influence of an external electric field. Here, we shall consider the phenomenological method for the solution of quasilinear equations based on the balance of characteristic times for the problem. Indeed, the expression for the Landau damping increment includes a derivative of the particle distribution function. Therefore, the comparison of this increment with other characteristic inverse times describing oscillation decay (e.g., due to particle–particle collisions) allows us to obtain a simple differential equation for the distribution function. In the simplest case, it is a mere balance of two increments.

An interesting model of this type was proposed by Ryutov [135], who considered the evolution of Langmuir noises arising in the problem of runaway electrons. The author was interested in the case of weak fields in which an electric field E is lower than the critical Dreicer field $E_D \propto (mV_T/e)\nu_{ei}(V_T)$, where $\nu_{ei}(V_T)$ is the frequency of electron–ion collisions at particle velocities of the order of thermal ones, V_T. Under these conditions, only a small fraction of particles can undergo acceleration and give rise to the formation of a second maximum in the function of electron distribution over velocities. Because fast electrons are virtually uninvolved in pair collisions, $\lambda_e \propto V_e^4$, the collective processes associated with the buildup of Langmuir oscillations began to play an important role. The quasilinear equation for spectrum evolution assumes the form

$$\frac{\partial W(k)}{\partial t} = 2\big[\gamma_L(k) - \nu_{ei}(V_T)\big]\, W(k) \,, \tag{131}$$

where

$$\gamma_L(k) = \pi\omega_{pe} \frac{n'}{n} \left(V^2 \frac{\partial f}{\partial V} \right)\Bigg|_{V=\omega_{pe}/k} \,, \tag{132}$$

with n' being the characteristic fast electron beam density scale. In a one-dimensional case close to a stationary one (the absence of an electric field in the first approximation), it is expected that the increment balance in the interval of wave numbers, $k_1 < k < k_2$, can be written in the form

$$\gamma_L(k) - \nu_{ei}(V_T) = 0 \,. \tag{133}$$

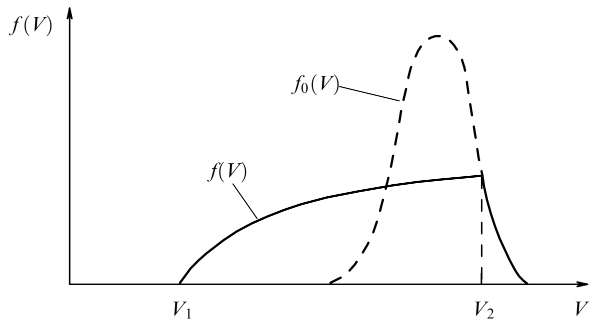

Figure 9. Particle distribution function in the Ryutov model.

This equation looks simpler than the Rudakov–Korablev stationary balance, although it is easy to see that the former is equally efficient. Suffice it to say that substituting the increment and the frequency matched to the problem under consideration results in the following simple form of the sought equation for the distribution function:

$$\frac{\partial f(V)}{\partial V} = \frac{\alpha_{\mathrm{R}}}{V^2}, \tag{134}$$

where $\alpha_{\mathrm{R}} = (n/n')\nu_{\mathrm{ei}}(V_T)/(\pi\omega_{\mathrm{pe}})$. Integration yields

$$f(V) = \alpha_{\mathrm{R}}\left(\frac{1}{V_1} - \frac{1}{V}\right), \tag{135}$$

where

$$\frac{\omega_{\mathrm{pe}}}{k_2} = V_1 < V < V_2 = \frac{\omega_{\mathrm{pe}}}{k_1}. \tag{136}$$

Outside this velocity range, the distribution function remains unaltered and retains its initial form $f_0(V)$ (Fig. 9). In such a formulation, the parameters are V_1 and V_2 velocities found from two conditions. One is the law of conservation of the total number of particles:

$$\alpha_{\mathrm{R}}\left(\frac{V_2}{V_1} - \ln\frac{V_2}{V_1} - 1\right) + \int_{V_2}^{\infty} f_0(V)\,\mathrm{d}V = 1. \tag{137}$$

The other ensures continuous distribution of particles over velocities:

$$\alpha_{\mathrm{R}}\left(\frac{1}{V_1} - \frac{1}{V_2}\right) = f_0(V_2). \tag{138}$$

The author of Ref. [135] proposed that a weak electric field be taken into account by considering points V_1 and V_2 as mobile and representing the distribution function for $V > V_2$ in the form

$$f(V, t) = f_0\left(V - \frac{e}{m}\,Et\right). \tag{139}$$

This naturally leads to redefining the characteristic scales of V_1 and V_2 velocities due to the inclusion of Et dependence.

Moreover, it becomes possible to find the spectrum of Langmuir oscillations, the energy of which vanishes in the absence of an electric field. Thus, the classical one-dimensional quasilinear equation

$$\frac{\partial f}{\partial t} + \frac{e}{m}\,E\,\frac{\partial f}{\partial V} = \frac{4\pi^2 e^2}{m^2}\,\frac{\partial}{\partial V}\left[\frac{W(\omega_{\mathrm{pe}}/V)}{V}\,\frac{\partial f}{\partial V}\right] \tag{140}$$

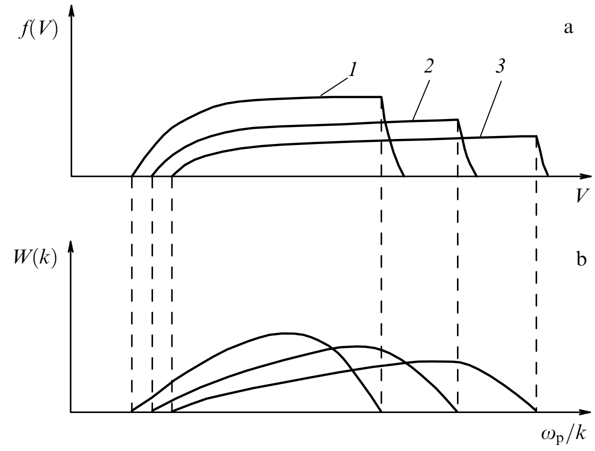

Figure 10. Evolution of the spectral function in the Ryutov model.

takes (after simple modifications) the form

$$-\frac{1}{V_1^2}\,\frac{\mathrm{d}V_1}{\mathrm{d}t}\left(\frac{\omega_{\mathrm{pe}}}{k}\right) - \frac{eEk}{m\omega_{\mathrm{pe}}} = \frac{4\pi^2 e^2}{m^2}\,\frac{k^3}{\omega_{\mathrm{pe}}}\,W(k) + \mathrm{const}. \tag{141}$$

Because $\omega(k_1) = 0$, one obtains

$$\mathrm{const} = -\frac{\omega_{\mathrm{pe}}}{k_1 V_1^2}\,\frac{\mathrm{d}V_1}{\mathrm{d}t} - \frac{eEk_1}{m\omega_{\mathrm{p}}}. \tag{142}$$

As a result, the evolving spectral function is given by the expression (Fig. 10)

$$W(k) = \frac{neE}{\pi k^4}(k - k_1)(k - k_2). \tag{143}$$

This rather simple model demonstrates the efficiency of considering the balance between characteristic increments (times). The quantities characterizing such important nonlinear processes as induced scattering or various decorrelation mechanisms can be used as the increments. Taken together, the data that can be obtained by such a simple method depict the qualitative picture of the phenomenon, without which it is impossible to construct a rigorous theory.

11. Force line diffusion and 'island' structures

Consideration of electron–wave packet interaction in quasilinear terms was based on the concept of particle wandering inside a wave packet. Such a mechanism was visualized in the phase plane as particle transition between the waves due to overlapping of stochastization regions in the vicinity of separatrices. Investigations into the structure of magnetic surfaces in high-temperature plasma confinement devices revealed 'island' structures delimited by separatrices close to the resonant surfaces [136–141], analogous to the phase portraits of wave groups (Fig. 11). Such a similarity is not purely coincidental, the analogy being due to the Hamiltonian structure of the equations describing magnetic field lines trajectories.

Rosenbluth et al. [142] made use of this property to construct a quasilinear model of stochastic magnetic field line wandering in plasma traps with destroyed magnetic surfaces based on the phenomenological definition of the

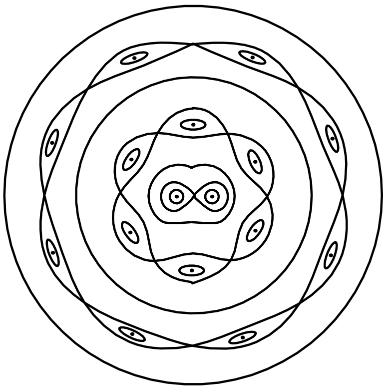

Figure 11. Cross section of toroidal magnetic surfaces and island structures.

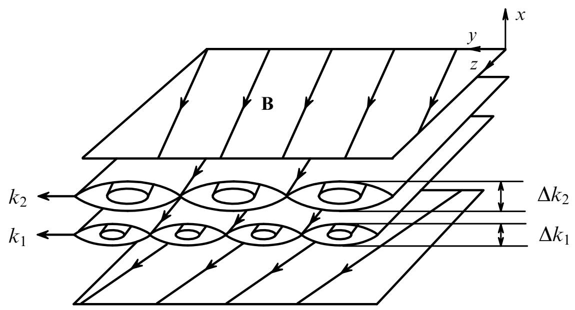

Figure 12. System of magnetic islands.

magnetic field line diffusion coefficient D_{m}:

$$D_{\mathrm{m}} = \frac{\left\langle \Delta \mathbf{r}_{\perp}^2(l) \right\rangle}{l} \,. \tag{144}$$

This formula has been derived by calculating the transverse displacement $\Delta \mathbf{r}_{\perp}(l)$ of the magnetic field line as seen by an observer moving along it at a distance l.

Let us consider a plane section through magnetic surfaces stretched along the z-axis, in which the island structure is formed by several chains (Fig. 12). The similarity with the phase portrait of a longitudinal wave packet is very obvious. In this case, the z-axis is analogous to the time axis, x stands for velocities, and y is particle displacement in the wave field. Then, the analog of the oscillator wave phase $\varphi = kx - \omega t$ is the magnetic field perturbation phase $\varphi_{\mathrm{m}} = k_y y - k_z z$, and the individual perturbation harmonic assumes the form

$$B_x = B_{\perp} \cos(k_y y - k_z z) \,. \tag{145}$$

The equations of 'motion' of the magnetic field line have the traditional form

$$\frac{\mathrm{d}x}{\mathrm{d}l} = \frac{B_x}{B_0} = b_x \,, \tag{146}$$

$$\frac{\mathrm{d}y}{\mathrm{d}l} = \frac{B_y}{B_0} = b_y \,, \tag{147}$$

where B_0 is the unperturbed magnetic field along the z-axis, and b_x and b_y are the characteristic velocities of motion of the magnetic field line in the transverse direction.

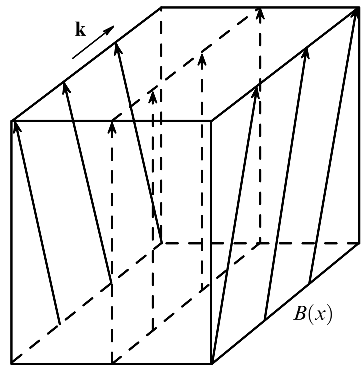

Figure 13. Sheared magnetic field model.

For the complete understanding of the above analogy, the simplest equation for the evolution of resonant particle velocity in the wave field should be written out taking advantage of the Cherenkov resonance condition $\omega = kV$ and presenting the equation of particle motion in the frame of reference moving together with the wave:

$$\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} = \frac{e}{m} E_0 \cos(k\tilde{x}) \,, \tag{148}$$

where $x = (\omega/k)t + \tilde{x}$. Representing the particle displacement $\tilde{x}$ in the integral form, viz.

$$\tilde{x} = \int \tilde{V}(t) \, \mathrm{d}t \,, \tag{149}$$

yields the equation describing the interaction between a resonant particle and an individual harmonic of the electric field:

$$\frac{\mathrm{d}\tilde{V}}{\mathrm{d}t} = \frac{e}{m} E_0 \cos\left(k \int \tilde{V} \, \mathrm{d}t\right) . \tag{150}$$

The resonance condition can be just as well formulated for magnetic field harmonics using the derivative $\mathrm{d}y/\mathrm{d}l = B_y/B_0$ as an analog of velocity. The phase steadiness condition, $k_y y + k_z z$, together with the adequately chosen dependence $B_y(x)$ allows us to obtain the condition for the resonant surface x_0. Using the above analogy, dependence $B_y(x)$ must be linear in x. Such dependence is provided by the simplest sheared field approximation (Fig. 13)

$$B_y(x) = \frac{\partial B_y}{\partial x} \, x = \frac{B_0}{L_{\mathrm{s}}} \, x \,, \tag{151}$$

where L_{s} is the characteristic spatial scale (shear parameter). Integrating the equation of motion of the magnetic field line yields

$$y = \frac{x_0 l}{L_{\mathrm{s}}} + \frac{1}{L_{\mathrm{s}}} \int_{x_0}^{x} x \, \mathrm{d}l \,. \tag{152}$$

It follows from the resonance condition at $x = x_0$ that the expression relating the main parameters of the problem has the form

$$k_z = -k_y \, \frac{x_0}{L_{\mathrm{s}}} \,. \tag{153}$$

Indeed, the resonance condition $k_z = -k_y(y/z)$ should be compared with the equation for the unperturbed component $z/B_0 = y/B_y^0$ or $B_y^0 = (B_0/L_s)x_0$. Hence, we have the equation for the displacement velocity of a magnetic line toward the x-axis:

$$\frac{\mathrm{d}x}{\mathrm{d}l} = \frac{B_\perp}{B_0} \cos\left(\frac{k_y}{L_s}\int_{x_0}^{x} x\,\mathrm{d}l\right), \tag{154}$$

which is a close analog of the above equation of particle motion in the electric field of an individual harmonic. It establishes a direct correspondence between parameters of the problem of particle–wave packet interaction and that of force line wandering in the system of magnetic islands:

$$\frac{B_\perp}{B_0} \leftrightarrow \frac{e}{m} E_0\,, \qquad \frac{k_y}{L_s} \leftrightarrow k\,. \tag{155}$$

Consideration of the ensemble of magnetic field harmonics in the form of Fourier expansion

$$B_x = \sum_{k_y k_z} B_k \exp\left(\mathrm{i}k_z z + \mathrm{i}k_y y\right) \tag{156}$$

allows generalizing the quasilinear diffusion coefficient for resonant particles in the phase space:

$$D_V = \sum_k \frac{\pi e^2 E_k^2}{m^2}\,\delta(\omega_k - kV)\,. \tag{157}$$

Here, the electric field is defined by a set of Fourier harmonics:

$$E = \sum_k E_k \exp\left[-\mathrm{i}(\omega t - kx)\right]. \tag{158}$$

The quasilinear diffusion coefficient of stochastic magnetic field lines acquires the analogous form

$$D_\mathrm{m} = \sum_{k_y k_z} \pi\,\frac{|B_k|^2}{B_0^2}\,\delta\left(k_z + k_y\,\frac{x}{L_s}\right). \tag{159}$$

To study the transport of particles in a stochastic magnetic field, additional arguments concerning decorrelation mechanisms are needed [143–150]. In the simplest case of ballistic (collisionless) motion of 'magnetized' particles along the magnetic lines, the transport depends only on the decorrelation properties of the magnetic field:

$$D_T = \frac{\Delta r_\perp^2(\tau)}{\tau} = \frac{\Delta r_\perp^2(\tau)}{l_\mathrm{cor}}\,\frac{l_\mathrm{cor}}{\tau} = V_\parallel D_\mathrm{m}\,, \tag{160}$$

where characteristic correlation time $\tau = l_\mathrm{cor}/V_\parallel$, $V_\parallel$ is the particles' longitudinal velocity, and l_cor is the longitudinal correlation spatial scale.

Interesting possibilities arise from the application of magnetic line diffusion coefficients to the problems of fast electron propagation in different stochastization regions of a magnetic field. The model kinetic equation describing distortions of the tail of the distribution function of suprathermal electrons with the collision integral in the Fokker–Planck form looks like

$$-|\mu| v D_\mathrm{m}\,\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\,\frac{\partial}{\partial v}\left[v^2 \nu_\mathrm{e}(v)\left(vf + \frac{T_\mathrm{e}(x)}{m}\,\frac{\partial f}{\partial v}\right)\right] + \nu_\mathrm{e}(v)\,\frac{\partial}{\partial\mu}\left[(1-\mu^2)\,\frac{\partial f}{\partial\mu}\right]. \tag{161}$$

Here, ν_e is the frequency of electron collisions, $\nu_\mathrm{e}(v, x) = 4\pi e^4 \Lambda n_\mathrm{e}(x)/(m^2 v^3)$, $\mu = \cos\theta$, $f(x, v, \mu)$ is the electron distribution function, D_m is the anomalous diffusion coefficient, T_e is the bulk electron temperature, n_e is the electron number density, θ is the pitch-angle of the electron velocity, and v is the electron velocity modulus. The term related to the influence of the electric field E can be disregarded [120], because the disturbances it causes become apparent only at high enough electron energies, $\varepsilon > \varepsilon_\mathrm{D} \approx 4\pi e^3 n_\mathrm{e}\Lambda/E$. The use of the self-similarity variables permits the description of a spatially nonuniform medium to be reduced to an expression similar to that describing runaway electrons in a homogeneous plasma [150–152]. The solution of such equations was technically well developed by Gurevich and Lebedev. For example, the distribution function in a given concrete problem shows different behavior in the regions delimited by the values of the velocity $v_1(D_\mathrm{m})$ and $v_2(D_\mathrm{m})$. These theoretically computed values can be used to determine D_m in experiment from results of X-ray measurements [119, 153].

It is noteworthy that Ref. [142] appeared only in 1966, whereas L S Solov'ev, interested in the splitting of magnetic surfaces in toroidal plasma traps, had referred in his Ref. [137] to the review by Vedenov, Velikhov, and Sagdeev published as early as 1961. The gap can be attributed to the fact that the relationship between the quasilinear method and the Chirikov criterion for resonance overlapping had not been established in due time. Nevertheless, the combination of quasilinear ideas, Chirikov's approach, and Hamiltonian models for a system of nested magnetic surfaces gave a powerful impetus to the development of the dynamic chaos theory.

12. Transport of admixture and correlation effects

One of the lines along which the quasilinear approach developed was its application to describing the turbulent diffusion of passive admixtures (scalars) in hydrodynamic flows. By passive admixtures are meant particles introduced into a hydrodynamic flow and having no effect on its character. Under conditions in which the temperature of each liquid particle remains unaltered, i.e., 'frozen' in the medium, it can also serve as a scalar.

Notice that the idea of averaging the transport equation was effectively implemented shortly before the appearance of the quasilinear method for the description of anomalous scalar diffusion by Taylor [154], who considered the scalar transport in the laminar flow of a fluid with the nonmonotonic velocity profile:

$$\frac{\partial n}{\partial t} + V_x(y, z)\,\frac{\partial n}{\partial x} = D_0 \nabla^2 n\,. \tag{162}$$

Here, n is the particle number density, V_x is the longitudinal (along the x-axis) velocity, and D_0 is the seed (molecular) diffusion coefficient. Also considered was the effect of occasional (due to infusion) inflow of particles into a nonuniform longitudinal velocity field, creating longitudinal diffusion in addition to the molecular one. Here, we analyze a slab model by representing the longitudinal flow profile in the form $V(y) = (V_0/L^2) \times (L^2 - y^2)$, where V_0 is the characteristic velocity, and L is the characteristic spatial scale. The density and velocity distributions are represented as the average plus a correction:

$$n = \frac{1}{2L}\int_{-L}^{L} n(x,y,t)\,\mathrm{d}y + n_1(x,y,t) = n_0 + n_1(x,y,t)\,, \quad (163)$$

$$V = \frac{1}{2L}\int_{-L}^{L} V(y)\,\mathrm{d}y + V_1(y) \equiv \frac{2}{3}\,V_0 + V_0\left[\frac{1}{3} - \left(\frac{y}{L}\right)^2\right]. \quad (164)$$

Let us consider a stationary case. Substituting expressions for n and V into the initial equation and averaging yield the expressions for the evolution of mean density $n_0(x)$ and perturbation $n_1(x,y)$:

$$V_0\,\frac{\partial}{\partial x}\,n_0 + \left\langle V_1\,\frac{\partial}{\partial x}\,n_1\right\rangle = D_0\,\frac{\partial^2 n_0}{\partial x^2}\,, \quad (165)$$

$$V_1\,\frac{\partial n_0}{\partial x} + V_0\,\frac{\partial n_1}{\partial x} + V_1\,\frac{\partial n_1}{\partial x} - \left\langle V_1\,\frac{\partial n_1}{\partial x}\right\rangle = D_0\left(\frac{\partial^2 n_1}{\partial x^2} + \frac{\partial^2 n_1}{\partial y^2}\right). \quad (166)$$

Let us further distinguish the terms responsible for the expected effect in the equation for perturbation of scalar density n_1 and take account of the smallness of $\partial n_1/\partial x$ and $\partial^2 n_1/\partial x^2$ in comparison with meaningful $\partial n_0/\partial x$ and $\partial^2 n_1/\partial y^2$:

$$V_1\,\frac{\partial n_0}{\partial x} = D_0\,\frac{\partial^2 n_1}{\partial y^2}\,. \quad (167)$$

Here, the nontrivial dependence $n_1 \propto V_0/D_0$ is readily discernible. Hence, the easy solution for n_1 can be found:

$$n_1 = \frac{\partial n_0}{\partial x}\,\frac{V_0}{3D_0}\left(\frac{y^2}{2} - \frac{y^4}{4L^2}\right) + \mathrm{const}\,. \quad (168)$$

Condition $\langle n_1\rangle = 0$ also gives

$$\mathrm{const} = \frac{\partial n_0}{\partial x}\,\frac{V_0}{3D_0}\left(-\frac{7}{60}\,L^2\right).$$

The expression $\langle V_1\,\partial n_1/\partial x\rangle$ defines the additional contribution to the longitudinal diffusive transport:

$$\left\langle V_1\,\frac{\partial n_1}{\partial x}\right\rangle = -D_1\,\frac{\partial^2 n_0}{\partial x^2} = -\frac{8}{945}\,\frac{(V_0 L)^2}{D_0}\,\frac{\partial^2 n_0}{\partial x^2}\,. \quad (169)$$

The resulting dependence of the additional longitudinal diffusion coefficient retains the 'quasilinear' (quadratic) character in terms of velocity amplitude, but proves anomalous from the standpoint of molecular diffusion contribution: $D_1 \propto 1/D_0$, and

$$D_{\mathrm{eff}} = D_0 + \frac{8}{945}\,\frac{V_0^2 L^2}{D_0}\,. \quad (170)$$

Similar effects are known to occur in turbulent flows. Naturally, the new diffusion mechanism will manifest itself a long way downstream, because the equation thus obtained holds true only for times $t \gg \tau_{\mathrm{D}} \approx L^2/D$. On the other hand, the condition of smallness was used in comparison of the transverse spatial scale L with the longitudinal scale l: $L \ll l$.

It is a striking example of the method suitable for obtaining a linear (convenient for solution) perturbation equation by averaging the initial equation. In Section 13 dealing with renormalization of quasilinear equations, it will be a diffusion type equation. It is noteworthy that the quasilinear approach to describing turbulent admixture (scalar) transport was utilized after it had found wide application in plasma physics. Specifically, one can consider, instead of Vlasov's equation, the continuity equation for particle number density of a passive scalar involved in an incompressible flow:

$$\frac{\partial n}{\partial t} + V\,\frac{\partial n}{\partial x} = 0\,. \quad (171)$$

Here, $n(x,t)$ is the spatial density of the passive scalar, and $V(t)$ is the random velocity field. Let us apply averaging over the realization ensemble for the continuity equation on the assumption that the density field can be represented as the sum $n = n_0 + n_1$ of mean density n_0 and the fluctuation part $n_1 = n - \langle n\rangle$, and that $\langle n_1\rangle = 0$ and $v = v_0 + v_1$, where $v_0 = \mathrm{const}$, $\langle v_1\rangle = 0$. Simple calculations yield two equations

$$\frac{\partial n_0}{\partial t} + v_0\,\frac{\partial n_0}{\partial x} + \left\langle v_1\,\frac{\partial n_1}{\partial x}\right\rangle = 0\,, \quad (172)$$

$$\frac{\partial n_1}{\partial t} + v_0\,\frac{\partial n_1}{\partial x} + v_1\,\frac{\partial n_0}{\partial x} + v_1\,\frac{\partial n_1}{\partial x} - \left\langle v_1\,\frac{\partial n_1}{\partial x}\right\rangle = 0\,. \quad (173)$$

Suppose that fluctuations n_1 and v_1 have the order of smallness ε in comparison with the mean field n_0. The 'quasilinearity' of the assumption consists in retention of the nonlinear term of order ε^2 in the equation for n_0, whereas only terms of first order in ε are conserved in the equation for n_1:

$$\frac{\partial n_1}{\partial t} + v_0\,\frac{\partial n_1}{\partial x} = -v_1\,\frac{\partial n_0}{\partial x}\,. \quad (174)$$

The structure of the set of equations describing the evolution of scalar density proves analogous to that of equations for the mean and perturbed parts of the distribution function in the quasilinear plasma theory.

The solution of the density perturbation equation is obtained by the Green's function method, regarding it as a first-order linear hyperbolic equation with source $I(x,t) = -v_1(\partial n_0/\partial x)$ and homogeneous initial condition $n_1(x,0) = 0$. Here, $\partial n_0/\partial x$ is a parameter of the equation. Let us consider the equation for Green's function G:

$$\frac{\partial G}{\partial t} + v_0\,\frac{\partial G}{\partial x} = \delta(x - x_1)\,\delta(t - t_1)\,. \quad (175)$$

The solution of this equation is easy to find using the Laplace transform in time t and Fourier transform in spatial coordinate x:

$$\tilde{\tilde{G}}_{k,s} = \frac{\exp(-t_1 s)}{s + \mathrm{i}kv_0}\,\exp(\mathrm{i}kx)\,. \quad (176)$$

Hereinafter, a tilde mark $\sim$ made above a letter denotes either the Fourier or Laplace transform. The solution has a clear physical sense of perturbation propagation along characteristic $z = x - v_0(t - t_1)$:

$$G(x,t,x_1,t_1) = \delta(x - x_1 - v_0(t - t_1))\,\Theta(t - t_1)\,,$$

where Θ is the Heaviside function. Then, the solution for $n_1(x,t)$ takes the form

$$n_1(x,t) = -\int_0^t v_1(t_1)\,\frac{\partial n_0(z,t_1)}{\partial z}\,\mathrm{d}t_1\,. \quad (177)$$

Substituting Eqn (177) for n_1 into the mean density equation and simple transformations yield

$$\frac{\partial n_0}{\partial t} + v_0 \frac{\partial n_0}{\partial x} = \int_0^t \langle v_1(t) v_1(t_1) \rangle \frac{\partial^2 n_0(z, t_1)}{\partial z \, \partial x} \, \mathrm{d}t_1 \,. \tag{178}$$

The integral form of this equation reflects the Lagrangian character of the relationship between derivatives of $n_0(x, t)$, which makes it essentially different from the 'fundamentally local' classical diffusion equation. Characteristic $z = z(x, t)$ in our consideration relates the derivatives determined at different time moments. For example, the left-hand side of equation (178) contains partial derivatives with respect to x and t, while the right-hand side is the sum of $\partial^2 n_0 / \partial x^2$ calculated along the characteristic with weight $C(t, t_1) = \langle v_1(t) v_1(t_1) \rangle$, which is an autocorrelation function of velocity pulsations. In the case of a stationary random process, the function $C(t, t_1) \approx C(t - t_1)$ in this equation plays the role of a 'memory' function. The final form of the transport equation in the framework of the quasilinear approach depends on the approximation of the correlation function.

The simplest meaningful case is a reduction of the above equation to the classical diffusion equation

$$\frac{\partial n_0}{\partial t} + v_0 \frac{\partial n_0}{\partial x} = D_T \frac{\partial^2 n_0(x, t)}{\partial x^2} \,. \tag{179}$$

The reduction is possible if the main contribution to the integral on the right-hand side of the integral equation comes from the small interval $(t - t_0; t)$, where $t_0 \ll t$. Given that the second derivative within this small interval does not change appreciably, one arrives at

$$\int_0^t \langle v(t) v(t_1) \rangle \frac{\partial^2 n_0(z, t_1)}{\partial z \, \partial x} \, \mathrm{d}t_1 \approx \frac{\partial^2 n_0(x, t)}{\partial x^2} \int_{t - t_0}^t C(t - t_1) \, \mathrm{d}t_1$$
$$\approx \frac{\partial^2 n_0(x, t)}{\partial x^2} \int_0^\infty C(\tau) \, \mathrm{d}\tau \,. \tag{180}$$

We actually assumed a rapid decay of correlations ('short' correlations) and arrived at Taylor's turbulent diffusion coefficient [155].

In this example, we disregarded the influence of molecular diffusion, responsible, for instance, for nontrivial effects of particle 'return' and, as a result, for the striking difference between the correlation functions and the 'convenient' exponential approximation [156–158]. Nor were spectral properties of the turbulent field taken into consideration. Unavoidable difficulties were overcome by both rigorous and phenomenological renormalization methods, e.g., making use of seed diffusion to couple Lagrangian and Euler correlations [159]. It is significant that such renormalizations led to the substitution of the quadratic dependence of diffusion coefficients on fluctuation amplitudes by the linear dependence that was observed with increasing frequency in high turbulence experiments [160–165]. The same issues continued to be developed in plasma physics, where they were employed to renormalize quasilinear equations.

13. Phase mixing and renormalization

We can speak with confidence that attempts to improve the quasilinear model of weak plasma turbulence date back to the time of its birth. In the classical review published in 1964, Kadomtsev [26] considered various aspects of transition to the strong turbulence problem with reference to the interaction between waves making up a wave packet. Reference [109] is worth mentioning too, since it not only discusses the possibility of transition from δ-correlated fields but also cites for the first time the Chirikov results in relation to the quasilinear approach [17, 166]. Nevertheless, the problem of renormalization of quasilinear equations is usually associated with the classical work of Dupree [167] that appeared in 1966.

For the purpose of the present article, it is worthwhile to use 'intuitive' arguments, because a detailed discussion of 'pulsation' mechanisms would greatly increase the size of this section. The interested reader can find all necessary arguments and calculations in the relevant literature. Here, only the relationship (not fixed in the available literature) between the Dupree approach and the Corrsin's [168] and Taylor's [154] studies needs to be emphasized. These authors, rarely mentioned in the modern literature, greatly contributed to understanding decorrelation mechanisms related to seed diffusion under conditions of spatial flow anisotropy. Both these studies were united by the common idea of describing longitudinal displacements of particles in terms of decorrelation time $\tau_\perp$ determined by diffusion wanderings in the transverse direction. Specifically, there is the Taylor scaling $D_{\mathrm{eff}} \propto V_0^2 L^2 / D_0$ for the above problem. This type of simple estimate can be obtained taking account of the 'pulsation' character of the anomalous contribution to the efficient transport:

$$D_* \propto V_0^2 \tau_\perp \,, \quad \text{where} \quad \tau_\perp(D_0) \propto \frac{L^2}{D_0} \,. \tag{181}$$

Here, as before, V_0 is the characteristic longitudinal velocity, L is the characteristic spatial scale, and D_0 is the seed (molecular) diffusion.

On the other hand, in 1959, Corrsin [159] applied the diffusion approximation to describe the 'clouds' of Lagrangian trajectories in the mature turbulent flow (Fig. 14). Thus, he proposed coupling Lagrangian and Euler velocity correlation functions by the phenomenological relationship

$$\big\langle V\big(x(t), t\big) V\big(x(t + \tau), t + \tau\big) \big\rangle$$
$$= \int_{-\infty}^\infty \big\langle U(x, t) U(x + \varDelta, t + \tau) \big\rangle \, \rho_D(\varDelta, \tau) \, \mathrm{d}\varDelta \,, \tag{182}$$

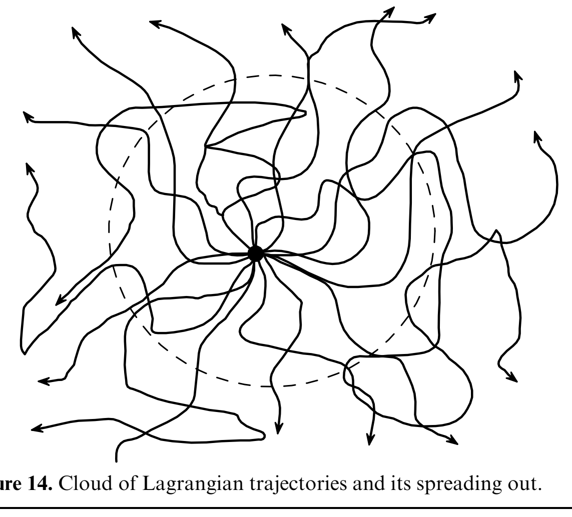

Figure 14. Cloud of Lagrangian trajectories and its spreading out.

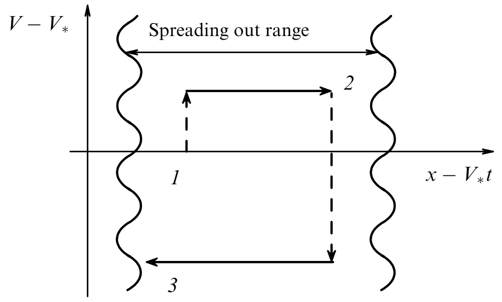

Figure 15. Velocity pulsations in the configuration space.

where the Gaussian distribution function was used:

$$\rho_D(\Delta, \tau) = \frac{1}{(4\pi D_0 \tau)^{3/2}} \exp\left(-\frac{\Delta^2}{4D_0\tau}\right). \tag{183}$$

It is worthwhile to note, when moving to the quasilinear problem, that electron trajectories were assumed to be unperturbed by electric field fluctuations, as follows from the equation for the disturbed part of the distribution function. However, diffusion in the velocity space inevitably causes 'spreading out' (diffusion) of particle trajectories. The analogy with the Corrsin and Taylor problems becomes increasingly clear if the necessity of taking into account the interdependence between configuration and velocity spaces is understood. In terms of assessments (Fig. 15), one obtains

$$(\delta x)^2 \propto (\delta V \tau)^2 \propto D_V \tau^3 . \tag{184}$$

The system of quasilinear equations makes it possible to elegantly perform such diffusion renormalization by including an additional term $D\,\partial^2 f_1/\partial V^2$ (taking into consideration diffusion in the velocity space) in the equation for the perturbed part of the particle distribution function. Then, the equations assume the form [167]

$$\frac{\partial f_0}{\partial t} - \frac{e}{m}\left\langle E\,\frac{\partial f_1}{\partial V}\right\rangle = 0 , \tag{185}$$

$$\frac{\partial f_1}{\partial t} + V\,\frac{\partial f_1}{\partial x} = D_V\,\frac{\partial^2 f_1}{\partial V^2} + \frac{eE}{m}\,\frac{\partial f_0}{\partial V} = 0 . \tag{186}$$

The inclusion of this term looks justified from the formal standpoint as well, because it approximates the 'small difference between two second-order terms', $V\partial f_1/\partial x - \langle V\partial f_1/\partial x\rangle$. Also important is the remaining possibility of applying the Fourier analysis to solve the perturbation equation due to the linear character of the renormalized equation

$$\mathrm{i}(kV-\omega)\,\tilde{f}_k - D_V\,\frac{\partial^2 \tilde{f}_k}{\partial V^2} = \frac{e}{m}\,\tilde{E}_k\,\frac{\partial f_0}{\partial V} . \tag{187}$$

For example, simple transformations lead to a new expression for the Fourier transform of the perturbed distribution function:

$$\tilde{f}_k = \frac{e}{m}\,\frac{\tilde{E}_k}{\mathrm{i}(kV-\omega) + (k^2 D_V/3)^{1/3}}\,\frac{\partial f_0}{\partial V} . \tag{188}$$

The term $(k^2 D_V/3)^{1/3}$ is responsible for the broadening of resonance inherent in the classical approach and allows the respective characteristic correlation time to be determined:

$$\tau_{\mathrm{P}} = \left(\frac{k^2 D_V}{3}\right)^{-1/3} \approx \tau_{\mathrm{K}} . \tag{189}$$

The expression for the diffusion coefficient acquires the nonquasilinear form:

$$D_V = \left(\frac{e}{m}\right)^2 \sum_k \int_0^\infty |E_k|^2 \exp\left[\mathrm{i}(kV-\omega_k\tau) - \frac{1}{3}\,k^2 D_V \tau^3\right] \mathrm{d}\tau . \tag{190}$$

Renormalization of the quasilinear diffusion coefficient can be considered in terms of the acceleration autocorrelation function:

$$C_a(t) = \left(\frac{e}{m}\right)^2 \big\langle E\big(x(t), t\big) E\big(x(0), 0\big)\big\rangle . \tag{191}$$

The traditional representation of the electric field as a totality of many independent Fourier transforms, $E(x,t) = \sum_k E_k \exp\left[\mathrm{i}(kx - \omega_k t)\right]$, yields the formula for the correlation function:

$$C(t) = \left(\frac{e}{m}\right)^2 \sum_{kk'} \big\langle E_k \exp\left[\mathrm{i}\big(kx(t) - \omega_k t\big)\right] E_{k'} \exp\left[\mathrm{i}k' x(0)\right]\big\rangle . \tag{192}$$

Then, by analogy with the Corrsin method for diffusion approximation of 'Lagrangian trajectory spreading', the independence hypothesis

$$C(t) = \left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \big\langle \exp\left[\mathrm{i}\big(kx(t) - \omega_k t\big) + \mathrm{i}k\Delta x(t)\right]\big\rangle \tag{193}$$

and the assumption of Gaussian statistics $\langle \exp A\rangle = \exp\left(\langle A^2\rangle/2\right)$ can be utilized. In this case, the expression for the correlation function takes the form

$$C(t) = \left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \exp\left[\mathrm{i}(kx - \omega_k t) - \frac{k^2 \big\langle \Delta x^2(t)\big\rangle}{2}\right] . \tag{194}$$

The use of diffusive estimate $\mathrm{d}\langle \Delta V^2(t)\rangle/\mathrm{d}t \approx 2D_V$ readily leads to the expression for the mean square displacement $\langle \Delta x^2(t)\rangle \approx (2/3) D_V t^3$. Under conditions of one-dimensional electrostatic turbulence (when $t \to \infty$), the diffusion coefficient is then given by the Dupree formula [167]

$$D_V = \int_0^t C(\tau)\,\mathrm{d}\tau = \left(\frac{e}{m}\right)^2 \sum_k \int_0^\infty |E_k|^2 \exp\left[\mathrm{i}(kV - \omega_k \tau) - \frac{1}{3}\,k^2 D_V \tau^3\right] \mathrm{d}\tau . \tag{195}$$

After the introduction of variable $J = (k^2 D_V/3)^{1/3}\tau$, convenient for integration, this expression gives a scaling substantially different from the quasilinear one:

$$D_V(E_k) \propto |E_k|^{3/2} . \tag{196}$$

The quasilinear result implies a 'steeper' dependence $D_{\mathrm{QL}}(E_k) \propto |E_k|^2$. The renormalized scaling for the diffusion coefficient obtained by Dupree was verified many times in numerical experiments reported in Refs [169–176]. Ambiguities in the results thus obtained precluded a definitive conclusion. The numerically found diffusion coefficient proved much smaller than the theoretically predicted one. O Ishihara and A Hirose confirmed these results. Moreover, they recalculated diffusion effect using the A Salat method and concluded that D_V can depend not only on amplitude fluctuations but also on time. It is appropriate at this point to cite a review article by Kadomtsev [54] published in 1972: "A crude quasilinear theory is in some respects even more preferable as inherently more approximate and therefore less pretentious. The development of a more exact theory encounters difficulties arising from nonanaliticity in the next approximations. Evidently, the following step will bring us to new concepts and approaches close to strong turbulence phenomena rather than to terms of a higher order of smallness."

Despite the disadvantages of such a phenomenological method, the Dupree approach permits us to visualize correlation effects dismissed in the quasilinear approach and opens up new possibilities for evaluating transport processes by both renormalization of the density perturbation equation and the diffusion approximation of correlation effects [159].

14. Stochastic magnetic field and strong turbulence

The estimation of amplitude dependence of the diffusion coefficient in Dupree's approach remained insufficiently smooth in the context of strong turbulence problems. It became conclusively clear at that time that quasilinear scalings failed to reflect the specificity of transport processes in strong plasma and hydrodynamic turbulence problems. In the meantime, the Bohm linear scaling that served as a benchmark for theorists was already discernible in renormalization models of scalar turbulent transport by the hydrodynamic turbulence field [93, 94, 177]. Specifically, it could be constructed by dimensional modeling based on spectral energy density [89, 93]:

$$D_T \propto \sqrt{\int_{k_{\min}}^{k_{\max}} \frac{E(k)}{k^2}\,\mathrm{d}k}\,. \tag{197}$$

Kadomtsev and Pogutse [178] managed to implement such an approach by integrating key ideas borrowed from different sources. Thus, before their work the application of the passive scalar model for the description of a stochastic magnetic field yielded only a quasilinear result. The authors of paper [178] considered a magnetic field with a rather specific configuration, but the quasilinearity of their result gives evidence of the possibility of employing a more general method based on invoking the continuity equation for magnetic line density n_B.

We are considering here a specific case of the magnetic line wandering problem, assuming the presence of a strong constant magnetic field with superimposed small random transverse perturbations (Fig. 16). It is possible to totally reproduce calculations for the case of a scalar transport taking advantage of the prominence of the longitudinal direction and the possibility of matching it with the time axis:

$$\frac{\partial n_B}{\partial l} + \mathbf{b}_\perp \nabla_\perp n_B = 0\,. \tag{198}$$

Here, $\mathbf{b}_\perp(l, r_\perp)$ is the perturbations of the magnetic field in the direction orthogonal to the force line, $n_B = n_0 + n_1$, where

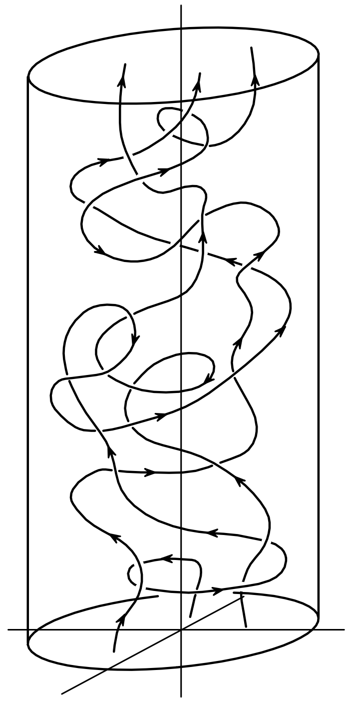

Figure 16. Stochastic magnetic field lines.

$n_0 = \langle n_B \rangle$, with the result remaining 'purely quasilinear':

$$D_{\mathrm{m}}(b_0) = \frac{1}{4}\int_{-\infty}^{\infty} \big\langle b(l,0)b(0,0)\big\rangle\,\mathrm{d}l \propto b_0^2 L_\parallel\,, \tag{199}$$

where b_0 is the characteristic amplitude of transverse fluctuations, and $L_\parallel$ is the characteristic longitudinal correlation scale.

Consideration of this scaling reveals the limitations of the method in which a single correlation scale is utilized. Kadomtsev and Pogutse broke apart characteristic spatial scales in this essentially anisotropic problem, bearing in mind that force lines are apt to decorrelate much earlier when transverse fluctuations are large. A criterion allowing the identification of these regimes is the condition $b_0 L_\parallel \geqslant \lambda_\perp$ or, in terms of the dimensionless parameter (the Kubo number Ku) introduced by the authors, the condition

$$\mathrm{Ku}_{\mathrm{m}} = \frac{b_0 L_\parallel}{\lambda_\perp} > 1\,. \tag{200}$$

Here, transverse displacement $\lambda_\perp$ in the Taylor expression for turbulent diffusion coefficients $\mathbf{b}(z, \lambda_\perp) \approx \mathbf{b}(z, 0)$ cannot be disregarded.

Based on the ideas of Dupree, Corrsin, and Howells, Kadomtsev and Pogutse proposed 'renormalizing' the quasilinear equations describing scalar transport:

$$\frac{\partial n_0}{\partial z} + \nabla_\perp \langle \mathbf{b} n_1 \rangle = 0\,, \tag{201}$$

$$\frac{\partial n_1}{\partial z} + \mathbf{b}\nabla_\perp n_0 = b_1\,\frac{\partial n_1}{\partial x} - \left\langle b_1\,\frac{\partial n_1}{\partial x}\right\rangle, \tag{202}$$

substituting the formerly discarded terms $b\,\partial n_1/\partial x - \langle b\,\partial n_1/\partial x\rangle$ by the diffusion term describing transverse 'spreading out' of correlations:

$$\frac{\partial n_1}{\partial z} + \mathbf{b}\nabla_\perp n_0 = D_{\mathrm{m}} \nabla_\perp^2 n_1\,. \tag{203}$$

Following Dupree [167], they used the effective diffusion coefficient D_{m} of magnetic field lines, which distinguishes their model from the scalar transport models of Corrsin [159] and Dreizin and Dykhne [179]. It preserves the linearity of the equation despite its conversion from hyperbolic to parabolic. Its formal solution is found by the Green's function method:

$$\frac{\partial G}{\partial z} - D_{\mathrm{m}} \nabla_{\perp}^2 G = \delta(\mathbf{r} - \mathbf{r}') \,. \tag{204}$$

The substitution yields the sought mean density equation

$$\frac{\partial n_0(z, \mathbf{r})}{\partial z} = \left(\frac{1}{2} \int \frac{b^2(\mathbf{k})}{\mathrm{i}k_z + k_{\perp}^2 D_{\mathrm{m}}} \, \mathrm{d}\mathbf{k} \right) \nabla_{\perp}^2 n_0 \,, \tag{205}$$

where the magnetic diffusion coefficient and the Fourier spectrum of perturbed amplitudes are given by the formulas

$$D_{\mathrm{m}} = \frac{1}{2} \int \frac{b^2(\mathbf{k})}{\mathrm{i}k_z + k_{\perp}^2 D_{\mathrm{m}}} \, \mathrm{d}\mathbf{k} \,, \tag{206}$$

$$b^2(\mathbf{k}) = \frac{1}{(2\pi)^2} \int \langle b(0) b(r) \rangle \exp(-\mathrm{i}\mathbf{kr}) \, \mathrm{d}\mathbf{r} \,. \tag{207}$$

Kadomtsev and Pogutse's solution allows using the notions of characteristic scales elaborated for vortical turbulence. For example, it is possible to distinguish, based on dimensional estimates, between strong and weak turbulences by comparing longitudinal and transverse scales. In the case of $\Delta k_z > k_{\perp}^2 D_{\mathrm{m}}$, the classical quasilinear expression was derived:

$$D_{\mathrm{m}} = \frac{\pi}{2} \int \mathrm{d}\mathbf{k} \, b^2(\mathbf{k}) \delta(k_z) \propto b_0^2 \lambda_z \propto \mathrm{Ku}_{\mathrm{m}}^2 \,, \tag{208}$$

where λ_z is the longitudinal correlation scale. In case of strong transverse correlations, $\Delta k_z < k_{\perp}^2 D_{\mathrm{m}}$, the Bohm (linear) type scaling is obtained (Fig. 17):

$$D_{\mathrm{m}} = \sqrt{\frac{1}{2} \int \frac{b^2(\mathbf{k})}{k_{\perp}^2} \, \mathrm{d}\mathbf{k}} \propto b_0 \lambda_{\perp} \propto \mathrm{Ku}_{\mathrm{m}} \,. \tag{209}$$

The analysis of turbulent transport problems in terms of the Kubo number is currently an indispensable research component of strong turbulence states due to the possibility of concentrating attention on the key decorrelation mechanisms in low-frequency regimes and comparing theoretical scalings with experimental findings.

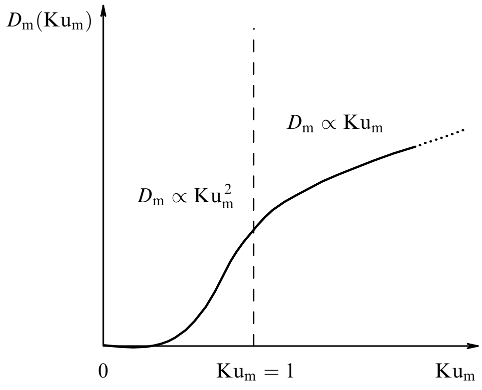

Figure 17. Magnetic field dependence of the force line diffusion coefficient.

15. Drift turbulence and vortex structures

The problem of renormalization of quasilinear equations was equally acute in the context of research on low-frequency drift instability of a plasma. These 'universal' modes were investigated in classical studies of the early 1960s [26, 34, 180, 181]. Large turbulent pulsation amplitudes observed in experiments suggested the inapplicability of the formal quasilinear theory. The development of a rigorous theory of strong turbulence in a magnetized plasma encountered the same difficulties as in the hydrodynamic case, since the importance of vortex structures (Fig. 18) and the necessity of taking into account energy transfer from small to large scales were becoming increasingly clear.

The qualitative picture of the anomalous transport processes was represented in the framework of Bohm's estimates [182]. Thus, the expression for the drift velocity of charged plasma particles in electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) crossed fields has the form

$$\mathbf{V}_E = c \, \frac{\mathbf{B} \times \mathbf{E}}{B^2} \propto \frac{\nabla \varphi}{B_0} \,. \tag{210}$$

Here, $\mathbf{V}_E$ is the drift velocity, and φ is the electric potential. For electric field fluctuations lower than the ion cyclotron frequency (low-frequency limit), the motion of particles in the plasma can be represented as the superposition of the rotation around a magnetic field line and the drift of the driving center with velocity $\mathbf{V}_E$. Dimensional considerations permit evaluating the characteristic decorrelation time τ_{cor} of particle transport via electric fluctuation field:

$$\delta V_E \approx \frac{c}{B_0} \, \delta E_{\mathrm{p}} \approx \frac{c}{B_0} \frac{\delta \varphi}{L_B} \tag{211}$$

with the characteristic spatial scale L_B:

$$\tau_{\mathrm{cor}} \approx \frac{L_B}{\delta V_E} \approx \frac{L_B^2}{c \delta \varphi} \, B_0 \,. \tag{212}$$

Here, $\delta\varphi \approx \delta E_{\mathrm{p}} L_B$ is the potential perturbation on vortex scales, and δE_{p} is the corresponding perturbation of the electric field strength. Fluctuations in the electric field are

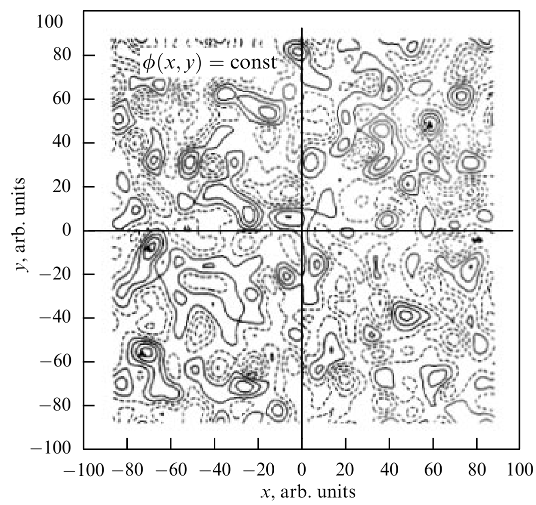

Figure 18. Current lines (equipotentials) depicting random two-dimensional vortex flow.

easy to relate to plasma temperature T_p by the expression $e\delta\varphi \approx e\delta E_p L_B \approx T_p$; hence, the Bohm scaling for anomalous particle diffusion in a turbulent magnetized plasma takes the form

$$D_B \propto \frac{L_B^2}{\tau_{cor}} \approx L_B \delta V_E \approx \frac{c}{B_0}\,\delta\varphi \approx \frac{cT_p}{eB_0}\left\langle\left(\frac{e\delta\varphi}{T_p}\right)^2\right\rangle^{1/2}. \quad (213)$$

The last expression does not contain the characteristic spatial scale L_B of vortex structures introduced at the initial stage of calculations. On the one hand, it may be attributed to the universal character of the resultant estimate. However, such a formulation does not provide an opportunity of analyzing the spatial correlation scales needed, as shown by Kadomtsev and Pogutse [178, 183], for the construction of a more comprehensive model of anomalous transport.

Unlike the Bohm phenomenology, the formal quasilinear particle transport model was based, as usual, on the Cherenkov electron–drift fluctuation interaction [26, 34]. The expression for the diffusion coefficient obtained after averaging over random field phases,

$$E_\perp = \sum_k E_{\perp k} \exp\left(\mathrm{i}k_\parallel x - \mathrm{i}\omega_k t\right), \quad (214)$$

has a traditional integral form

$$D_\perp = \int |V_k|^2 \pi\delta(\omega - kV_\parallel)\,\frac{F_0}{n_0}\,\mathrm{d}\mathbf{V}\,\mathrm{d}\mathbf{k}, \quad (215)$$

where $\omega/k_\parallel \propto V_i$ is the ion velocity, and $|V_k|^2 = (cE_\perp/B_0)_k^2$ is the spectral function of velocity $V_\perp$.The corresponding estimate of diffusion by virtue of temperature-drift instability, namely

$$D_\perp \propto \int \frac{|V_k|^2}{V_e k_\parallel}\,\mathrm{d}\mathbf{k} \propto \chi\,\frac{V_i}{V_e} \propto \chi\sqrt{\frac{m_e}{m_i}}, \quad (216)$$

gives values in agreement with the classical kinetic model of Coulomb collisions, but at variance with the observed anomalous estimates in which diffusion and thermal conductivity χ are quantities of the same order of magnitude. Estimating nonlinear effects of electron capture by drift waves and investigating individual convection cells failed to improve the results, as expected. Here is the conclusion of review [183] dated 1967: "The smallness of the diffusion coefficient is due to the fact that electrons tend to obey the Boltzmann distribution and it is difficult to displace them from magnetic surfaces. If, however, the magnetic surfaces are destroyed the particle diffusion occurs simultaneously with heat transfer, and coefficient $D_\perp$ has an order of χ."

In the context of overcoming difficulties, it is worthwhile to mention the work by Dupree [184], published in the same 1967, that reports the expression for the renormalized diffusion coefficient

$$D_\perp \propto \sum_{k,\omega} \frac{c^2|\tilde{E}_{k\omega}|^2}{B_0^2}\,\frac{k^2 D_\perp}{\omega^2 + k^4 D_\perp^2}. \quad (217)$$

The Bohm linear result holds for large pulsation amplitudes in the low-frequency approximation, $\omega\tau_D \ll 1$, where $\tau_D \propto 1/(k^2 D_\perp)$. The idea of such diffusion renormalization was further elaborated in many publications [185–191] with special emphasis placed on the effects in the magnetized plasma caused by the ensemble of vortex structures. The authors of these studies used the equations of motion of the guiding centers in the simplest form (Fig. 19):

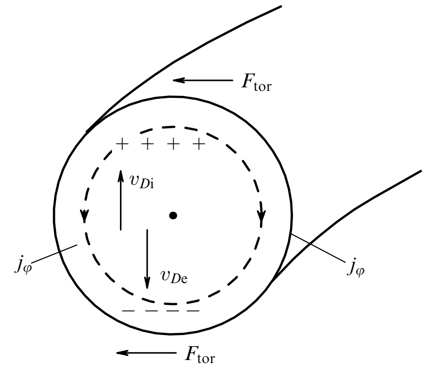

Figure 19. Motion of particles in a tokamak.

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = V_\parallel \mathbf{e}_z + \mathbf{V}_\perp = V_\parallel \mathbf{e}_z + c\,\frac{\mathbf{B}\times\nabla\varphi(\mathbf{r}_\perp, z, t)}{B^2}. \quad (218)$$

In the limit where the collision frequency is lower than the characteristic frequency of oscillations, longitudinal velocities can be regarded as constant and the electric potential presented in the simplest form

$$\varphi(\mathbf{r}, t) = \varphi(x, y, z_0 + V_\parallel t, t), \quad (219)$$

where z_0 is the initial particle coordinate. The corresponding Hamiltonian is given by the expression

$$\Psi(x, y, t) = -\frac{c}{B}\,\varphi(x, y, z_0 + V_\parallel t, t). \quad (220)$$

To avoid a detailed breakdown in the analysis of fluctuation spectra, the new version contained the results for the case $\omega \to 0$ presented as a combination of quasilinear scaling $D_\perp \propto V_0^2 \tau_D$ and the diffusion expression for correlation time $\tau_D \propto 1/(k_\perp^2 D_\perp)$. It is easy to see that the result obtained, $D_\perp \propto V_0 k_\perp$, is analogous to that obtained by Kadomtsev and Pogutse, $D_m \propto b_0 k_\perp$. Taking account of peculiar spectral features results in the appearance of an additional logarithmic factor [185]. Moreover, flat scaling regimes were found in which the dependence on the field amplitude was virtually absent.

The year 1978 was a significant watershed for transition to nonlinear models. To begin with, an explicit interpretation of numerical experiments was proposed in Ref. [191] based on the parametric decay of drift waves as the main mechanism of convective cell generation [192]. Sagdeev, Shapiro, and Shevchenko [192] distinguished a mode with $k_\parallel = 0$ and $\omega = 0$ in equations for drift wave interactions. The possibility of a quasistationary state was demonstrated with a Bohm-type transport by virtue of convective cells.

The second important step was reduced to consideration of the anomalous transport problem concerned with studies on the system of lines at the 'random' hilly landscape level and based on statistical topography methods [178, 193] (Fig. 20). Kadomtsev and Pogutse proposed this model to elucidate the role of mechanisms responsible for Bohm diffusion. By way of example, transport in the random two-dimensional flow of an incompressible fluid is described by a set of equations

$$V_x(x, y, t) = -\frac{\partial\Psi(x, y, t)}{\partial y}, \quad (221)$$

$$V_y(x, y, t) = \frac{\partial\Psi(x, y, t)}{\partial x}. \quad (222)$$

Figure 20. Random landscape as a generator of equipotentials.

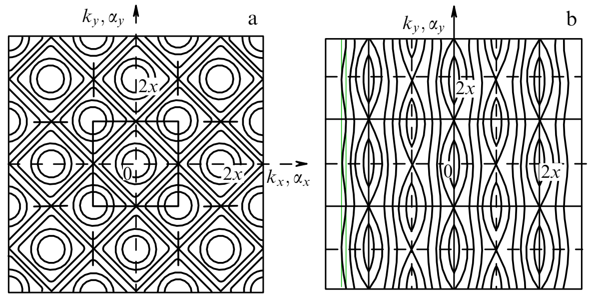

Figure 21. Regular vortex structure (a), and its rearrangement due to harmonic amplitude mismatch (b).

We are actually dealing with a Hamiltonian system having $1^1/_2$ degrees of freedom [45, 46]. The stochastic layers in the near-separatrix region of such systems were the subjects of extensive studies by both mathematicians and plasma physicists with reference to stochastization of magnetic field lines in the vicinity of resonant surfaces in magnetic confinement devices [194–197]. Kadomtsev and Pogutse proposed to obtain such a closed streamlined bunch that circumflows a large proportion of vortices in the flow and forms a mixing scale much greater than the size of a single vortex.

In this context, the important articles by Zaslavskii, Filonenko, and Sagdeev [110–112] should be mentioned, where the authors derived formulas for the width of stochastic layers formed as a result of separatrix perturbations. The authors presented two expressions for high and low perturbation frequencies:

$$\varDelta \approx \begin{cases} \lambda \exp\left(-\pi \dfrac{\varOmega_{\mathrm{p}}}{\omega_0}\right) & \text{for} \quad \varOmega_{\mathrm{p}} \gg \omega_0 \,, \qquad (223) \\ \lambda\varepsilon \dfrac{\varOmega_{\mathrm{p}}}{\omega_0} & \text{for} \quad \varOmega_{\mathrm{p}} \ll \omega_0 \,. \qquad (224) \end{cases}$$

Here, λ is the characteristic spatial scale, ε is the perturbation smallness parameter, $\varOmega_{\mathrm{p}}$ is the perturbation frequency, and ω_0 is the fundamental frequency. Notice that the width of the streamline bunch making the main contribution to transport in statistical topography-based models [193, 198, 199] can be estimated as the width of the stochastic layer $\varDelta$. This permits concentrating attention on the generation of realistic systems of equipotentials. Clearly, the problem of describing anomalous transport was moving further and further from the traditional quasilinear approach in which the pivotal role was ascribed to resonance (Cherenkov) mechanisms of wave–particle interplay.

16. Transport in convective cells

Switching in the late 1970s from the description of wave–particle interactions to transport models assuming the leading role of vortices had a dramatic impact on modern research. V I Petviashvili formulated the new concept in the following way: "Enhanced transport in the plasma is due to accumulation of vortices facilitating convective mixing with a very large characteristic spatial scale" (see, for instance, review [200]).

At the first stage, it was necessary to find a simple model making possible the analysis, at the new qualitative level, of the influence of the ensemble of vortex structures on the turbulent transport. It turned out that such a model is a 'close relative' of the quasilinear problem, while the suitable relief containing a regular system of vortex structures deliminated by separatrices had been considered by I M Lifshits [201] as early as 1956. This system of regular convective cells is described by the stream function given by the superposition of two oblique waves (Fig. 21):

$$\psi(x, y) = \frac{\psi_0}{2}\left[\cos \pi\left(\frac{x}{a} - \frac{y}{b}\right) - \cos \pi\left(\frac{x}{a} + \frac{y}{b}\right)\right]$$
$$= \psi_0 \sin\left(\frac{\pi}{a}\, x\right) \sin\left(\frac{\pi}{b}\, y\right). \qquad (225)$$

Such a stream function was later investigated in connection with the simplified model of *ABC*-flow [202]. In the presence of a minor secondary harmonic $\psi_1(x, y, t)$ of a similar type, streamline stochastization occurs in the vicinity of separatrices, which gives rise to a regular network of convective transport channels. The system of two drift waves considered by Hirshman and Horton reduces to the same system [203, 204]:

$$\psi(x, y) = ux + \psi_0 \sin x \cos y$$
$$+ \psi_1 \sin(k_x x + \alpha_x) \cos(k_y y + \alpha_y) \,. \qquad (226)$$

Importantly, the stochastization criterion in these models coincides with the condition of overlapping the Chirikov resonances. Moreover, here, as in the quasilinear theory, attention is not focused on wave packet evolution; rather, only the density distribution of scalar admixture particles is investigated in the velocity field given by function ψ.

The results of a numerical experiment on transport processes in a convective cell system [205] published in 1984 gave evidence of the dependence of the diffusion coefficient on the velocity pulsation amplitude, $D \propto \sqrt{V_0}$, regarded as nontrivial at that time. This seemingly simple problem nevertheless requires a subtle analysis to describe the particle interaction with such a flow structure. On the one hand, trapping effects related to capture by individual vortices cannot be ignored here. On the other hand, convective transport associated with the presence of stochastic layers in the vicinity of separatrices delimiting vortices begins to operate at long times. In this case, the transport is accompanied by scattering from the saddle points of the stream function, forming a square lattice. Notice that the effect of such scattering was studied by I M Lifshits [206] in 1961, a landmark year in the history of the quasilinear theory (Fig. 22).

Let us consider the simplest case, choosing the unified cell size λ and the characteristic velocity V_0 of the convective flow as parameters. Bearing in mind the importance of the near-separatrix layer, the effective diffusion coefficient can be

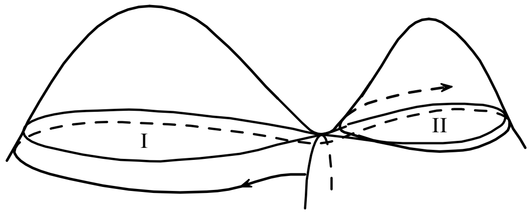

Figure 22. Random scattering of stream lines at the saddle point.

presented in the form

$$D_{\rm eff} \approx D_T P_\infty \,, \tag{227}$$

where $D_T \approx V_0^2 \tau$, P_∞ is the part of the space where the convection along separatrices occurs. In the case of convective cells, the value of P_∞ is readily estimated as $P_\infty \approx \lambda \Delta/\lambda^2 \approx \Delta(V_0)/\lambda$, where Δ is the width of the stochastic layer.

The problem reduces to the choice of a characteristic time. Formally, there are two options. One arises from the notion of diffusive escape from the stochastic layer. Then, $\tau \approx \Delta^2/D_0$ and, therefore, one has

$$D_{\rm eff} = \frac{V_0^2}{D_0}\left(\frac{\Delta^3}{\lambda}\right). \tag{228}$$

Here, D_0 is the seed (molecular) diffusion coefficient determined, for example, by collisions or diffusion due to stochastization of plasma particle trajectories near separatrices partitioning vortices. In a low-frequency streamlined perturbation regime, the use of the estimate $\Delta \propto \varepsilon \Omega_{\rm p}$ leads to the dependence $D_{\rm eff} \propto \Omega_{\rm p}^3$ [207, 208].

The other choice of correlation time arises from the consideration of the particle ballistic motion in the stochastic layer: $\tau \approx \lambda/V_0$; it becomes important when the velocity amplitude increases. Then, the expression for the turbulent diffusion coefficient coincides with the renormalized quasilinear Kadomtsev–Pogutse estimate discussed in connection with force line diffusion [178]:

$$D_{\rm eff}(V_0) \propto \lambda V_0 \,\frac{\Delta}{\lambda} = V_0 \Delta(V_0)\,. \tag{229}$$

The knowledge of balance between decorrelation mechanisms underlying particle balance in the layer allows comparing by an order of magnitude the characteristic time of particle escape from the boundary layer, $\tau \approx \Delta^2/D_0$, and the time of ballistic motion along the cell boundary, $\tau \approx \lambda/V_0$. As a result, here is an important estimate of the boundary layer width:

$$\Delta(V_0) = \left(\frac{D_0\lambda}{V_0}\right)^{1/2} \propto \frac{1}{\sqrt{V_0}}\,. \tag{230}$$

The final formula for the effective diffusion coefficient in the convective cell system [209] is written out as

$$D_{\rm eff} = {\rm const}\,\sqrt{D_0 V_0 \lambda} \propto V_0^{1/2}\,, \tag{231}$$

in agreement with modelled results. The strict solution to the problem of scalar transport by the stationary field of the 'vortex lattice' in the presence of seed diffusion by the method of multiple scales gives the same scaling [210]. There is a much smoother dependence than the quasilinear, $D_{\rm eff} \propto V_0^2$, or Bohm, $D_{\rm eff} \propto V_0$, one. These qualitative estimates of transport in the regular convective cell system made it possible to move from quasilinear transport models to nonlinear problems.

In general, the single-scale two-dimensional vortex flow is formed by the superposition of a large number of harmonics having the same wavelength λ, but different amplitudes, phases, and directions of wave vector $\mathbf{k}$:

$$\Psi(x,y) = \sum_j^N \psi_j \cos\left(\mathbf{k}_j \mathbf{r} + \varphi_j\right), \quad N \gg 1\,. \tag{232}$$

It allows simulating a random vortex field of 'general position' to which the aforementioned percolation ideas of Kadomtsev and Pogutse are applicable and thereby obtaining experimentally verified scalings for turbulent transport coefficients [210–217]. However, the computing techniques are no longer related to quasilinear notions.

17. Conclusions

Meticulous scrutinizing of classical work on turbulent plasma theory shows that the development of this research area proceeded in a far from haphazard fashion. Suffice it to mention the parallel discussions of Vedenov–Velikhov–Sagdeev and Drummond–Pines reports presented at the Salzburg conference in 1961. Although many pioneering studies were classified as secret by governments having different geopolitical interests, theorists were fairly well aware of the research carried out by their foreign colleagues. By that time, the fundamental role of Cherenkov mechanisms in wave emission and absorption was universally recognized.

B B Kadomtsev [30] pointed out that it had taken more than 10 years to understand and appreciate Landau's ideas concerning collisionless damping. The quasilinear method of Vedenov, Velikhov, and Sagdeev [3] provided one of the first efficient tools for the systematic application of the Landau increment to the solution to the complicated physical problem. In a sense, the success of the authors was predetermined by the fact that they were all graduates of the Faculty of Physics, Lomonosov Moscow State University, Vlasov's alma mater; moreover, two of them were Landau's students. The analysis of the range of problems pertaining to the origin and evolution of the quasilinear theory provides indisputable evidence of the importance of continuity in science.

The nonlinear theory developed side by side with the solutions to concrete quasilinear problems [26, 29, 34, 37, 41, 44]. The scope of applicability of the quasilinear theory of electron beam relaxation in a plasma is limited, because nonlinear interactions are essential for Langmuir waves even at low wave energy densities. Therefore, solving kinetic equations for plasmons, taking account of nonlinear effects, became a natural area of research. For example, Silin and coworkers [131, 132] arrived at the self-consistent analytical solution to the problem of quasistationary ion-acoustic turbulence by modifying the equation for the number of quanta without prejudice to the general idea of a kinetic description. In retrospect, the year 1967 appears to have been a conceptual watershed in the history of the quasilinear theory. On the one hand, Ivanov and Rudakov, Rudakov and Korablev, and Rosenbluth, Sagdeev, and Taylor published their elegant solutions in 1966; on the other hand, the article by Dupree demonstrated the necessity of developing renormalization techniques [184]. In this sense, the rational quasilinear theory had to face the sad fact that long-range correlations present in the system need to be taken into consideration.

The theory of plasma turbulence is now more than half a century old, and many motivational theoretical studies from the 1960s–1970s tend to be rarely and rather formally mentioned by modern authors. This is a deplorable and highly counterproductive practice, because basic ideas always stay relevant. It would be of special interest to consider from a single viewpoint both the sources of theoretical models of turbulent transport in a plasma and modern applications of quasilinear ideas for the investigation of anomalous transport under strong turbulence conditions. We confined ourselves to a single subject to demonstrate the synthesis of many challenging ideas in the problems related to the description of turbulent transport under conditions of well-developed two-dimensional structural turbulence with special reference to transport in a convective cell system, which has attracted the attention of researchers for many decades. It is worthwhile to note that many important aspects of this nonacademic problem await elucidation [56, 160].

During preparation of this article for publication, it occurred to the author that the 'quasihistorical' aspect unexpectedly emerges at 'small' temporal scales. This becomes evident, even when looking through the list of references, which brings to light a number of fun facts. Remarkably, the article by Davydov [13] does not contain references to the work by Akhiezer and Fainberg [20] or Bohm and Gross [18] published in widely circulating journals in 1949. Although Bohm's articles translated into Russian appeared in the journal *Problemy Sovremennoi Fiziki*, Davydov just mentioned in passing the nonequilibrium character of the plasma systems under consideration and repeated several times that "any ordered flows in the plasma provoke build-up of oscillations." Moreover, the Vlasov equation that later provided a basis for the derivation of the quasilinear approximation was used by Davydov only to define a criterion for the existence of ion-acoustic oscillations: $T_e \gg T_i$. It is traditionally accepted that this result was obtained by G V Gordeev in 1954 [218]. However, Davydov unambiguously asserted that it had been reported by T F Volkov in his graduation thesis in 1951 but could not be published in open access for reasons of secrecy.

Even more surprising is the absence of references to Davydov's paper in the classical study by Vedenov, Velikhov, and Sagdeev [3], which was discussed at length at the seminar led by Leontovich (and previously published as a preprint by the Institute of Atomic Energy [1] in 1960). Leontovich could hardly miss an opportunity to the draw attention of the authors to the paper of their predecessor, who had recently worked in the same department. All these inconsistencies may be due to security measures enacted at that time or to Davydov's suspension from participating in the controlled thermonuclear fusion program. On the other hand, Romanov's and Filippov's articles contain references to Davydov's paper of 1958. Today, few authors cite Davydov's publication, whereas references to the paper by Klimontovich are virtually nonexistent.

The National Research Centre 'Kurchatov Institute' has recently declassified Velikhov's Thesis for Cand. Phys.-Math. Sci. [219], in which a special section is devoted to the quasilinear approach. The archive of the National Centre stores reports by Davydov, Braginskii, Volkov, Galitskii, and other authors. Unfortunately, historians of science have not yet appreciated the value of plasma research nor have they taken full advantage of the documents collected in the archive of the Kurchatov Institute. True, the collected works of Galitskii, Ivanov, Kadomtsev, Larkin, and Leontovich have been published. This tradition should be continued to make accessible to the physical community the studies by Vedenov, Galeev, Dykhne, Mikhailovskii, Moiseev, Petviashvili, Shafranov, and other researchers. The author hopes that the present review will promote this process, if only modestly, by attracting attention to the beauty and depth of the work by Soviet theorists.

The author acknowledges the helpful comments by and discussions with Yu N Dnestrovskii, I O Zoteeva, N S Erokhin, S V Konovalov, L K Kuznetsova, A M Popov, V D Pustovitov, A A Rukhadze, V P Silin, and E I Yurchenko.

Editorial note

When consideration of this review article by O G Bakunin, "Quasilinear theory of plasma turbulence. Origins, ideas, and evolution of the method" by the editors of *Physics–Uspekhi* was underway, we learnt the tragic news of the sudden death of the author. Oleg Gennad'evich Bakunin was only 54 year old (08.09.1962–17.03.2017).

He was not merely an author but also a friend of our journal. We had ambitious plans to resume publication of the series of classical volumes of *Problems of Plasma Theory* both in Russian and in English (as *Reviews of Plasma Physics*), to complete the collection of materials for a memoir book about Boris Borisovich Kadomtsev and publish it, and to write new review articles for *Physics–Uspekhi*. These plans will be difficult to fully implement without the indomitable energy of Oleg Gennad'evich Bakunin. We shall always cherish his memory.

The obituary in memory of Oleg Gennad'evich Bakunin is published in the journal *Voprosy Atomnoy Nauki i Tekhniki* (*VANT*). Ser. Termoyaderny sintez, 2017, Vol. 40, Issue 2, pp. 92–93, http://vant.iterru.ru/vant_2017_2/ ogbakunin.pdf

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