

Adaptive dynamical networks

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DOI: <https://doi.org/10.3367/UFNe.2016.10.037902>

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Abstract. Dynamical networks are systems of active elements (nodes) interacting with each other through links. Examples are power grids, neural structures, coupled chemical oscillators, and communications networks, all of which are characterized by a networked structure and intrinsic dynamics of their interacting components. If the coupling structure of a dynamical network can change over time due to nodal dynamics, then such a system is called an adaptive dynamical network. The term ‘adaptive’ implies that the coupling topology can be rewired; the term ‘dynamical’ implies the presence of internal node and link dynamics. The main results of research on adaptive dynamical networks are reviewed. Key notions and definitions of the theory of complex networks are given, and major collective effects that emerge in adaptive dynamical networks are described.

Keywords: dynamical networks, adaptive links, synchronization, attractor, coupled oscillators

1. Introduction

In his famous paper [1] published in *Physics–Uspekhi*, Vitaly Lazarevich Ginzburg put nonlinear physics, including turbulence, solitons, chaos, and strange attractors, as number 11 on the list of “particularly important and interesting problems.” He wrote: “...The attention to nonlinear physics is ever increasing. To a significant degree, this stems from the fact that the use of modern computer facilities enables the analysis of problems one could only have dreamed of previously.” Complex dynamical networks can be fully regarded as belonging to this class of problems. Systems with a networked structure are widespread in Nature, technical applications, and society, and they attract the attention of researchers from very different branches of science (see, e.g.,

reviews [2–4]). Network-related issues have moved to the forefront in a wide range of disciplines—from social networks and the Internet to network structures in the brain and genomes.

Despite substantial differences in the origin of structures, and behavior of networks, they exhibit qualitatively similar traits and universal features that reflect the organizing principles specific to all networks. Any network consists of some finite, but commonly very large, number of nodes (vertices) which are connected to each other by links (edges). The structure (configuration) of links between the nodes, irregular as a rule, sets the topology (or architecture) of the network. Two nodes are called neighboring (or adjacent) if connected directly. An important network characteristic is the so-called nodal degree — the number of neighbors the node is connected to. In real networks, the connection topology is commonly irregular, and yet not random. Networks with such properties are conventionally referred to as complex.

The most typical task for complex networks lies in finding their structural properties and answering the question on the principles one topology or another forms and on the laws governing the network’s *evolution* — the change with time in its topology (for example, the birth or disappearance of nodes). This class of problems often includes network systems for which the characteristic time of changes in a network state by far exceeds the characteristic time of changes in topology, so that dynamical processes in nodes and links can be ignored. The overwhelming majority of publications concerning complex networks is devoted to research in this important area. However, this approach is insufficient for many other problems, and one needs to move to the next level of description when dynamical processes unfolding in nodes and links are taken into account. An example can be furnished by brain’s neural networks—here, one needs to analyze not only the structure of links but also the electrochemical activity of separate neurons and axonal connections among them. To explore such systems, learning only about their structure is insufficient and one needs to account for the network *state* — the values of dynamical variables that govern the behavior of nodes and links. A fundamental task in such an approach is to study the network dynamics — the changes in the state of its nodes and links with time. If the

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Received 24 August 2016, revised 3 October 2016
Uspekhi Fizicheskikh Nauk 187 (7) 745–756 (2017)
DOI: <https://doi.org/10.3367/UFNr.2016.10.037902>
Translated by S D Danilov; edited by A Radzig

characteristic time of the change in the state of a network is smaller than the characteristic time of change in its topology, then in the first approximation the topology can be treated as static, i.e., fixed in time, in the analysis of the influence of structural and dynamical parameters on complex network collective activity.

Until recently, publications concerned with complex networks could be broadly attributed to one of the two areas described above, which can be generalized as follows. The first one is the *evolution of nondynamical networks* when the network topology evolves with time according to certain rules, but the dynamics of nodes and links are lacking. The second one covers the *dynamics on static networks*. In the latter case, each node and each link are represented by dynamical systems, all being connected through some static topology. However, from the example of a neural network given above, it is already clear that in general the structure evolves simultaneously with the dynamics of the nodes and links in real networks, both processes being interrelated. The peculiarities of connection topology have a direct impact on the nodal collective dynamics, whereas changes in the state of nodes and links may lead to a change in the network structure, and the characteristic time scales for changing the dynamical state and topology of network are frequently of the same order. Thus, one arrives at the next level of network system description—that is, of so-called *adaptive dynamical networks* or *dynamical networks with varying topology*. The term ‘adaptive’ characterizes the possibility of changes in the structure of links, while ‘dynamical’ characterizes the presence of internal dynamics of nodes and links.

In a general form, the evolution and dynamics of adaptive networks can be viewed as a two-level process (see Refs [5, 6] for more details). On the one hand, the state of nodes and links varies with time according to some multidimensional dynamical system (a schematic of phase trajectories is given in Fig. 1 on the right). On the other hand, the connectivity pattern is ever changing, i.e., the network evolves (see the left-hand side of Fig. 1). The topology influences in an obvious way the character of the collective dynamics of network components, its nodes and links, but their dynamics also

feed back on topology rewiring, so that network evolution and dynamics permeate each other.

This review deals with the main results obtained for adaptive dynamical networks up to the present day. In Section 2, we collect basic definitions from the complex network theory needed for further exposition. Section 3 concentrates on concrete examples of adaptive dynamical systems, and the major findings are discussed and summarized in the Conclusion.

2. Basic concepts and definitions

The basic concepts of the theory of complex networks, valid for networks lacking dynamics of nodes and links, as well as for dynamical networks with static topology, are presented in a series of reviews [2–4, 7–11]. A network is commonly defined as an ensemble consisting of a finite number N of nodes (vertices) coupled through L directed or undirected links (edges). The topology of links is set by the *adjacency matrix* (or connectivity matrix) $A = \{a_{ij}\}$, $i, j = 1, \dots, N$, with elements $a_{ij} = 1$, if node j is connected to node i , and $a_{ij} = 0$ otherwise. The network can have both directed (oriented) and undirected links. The adjacency matrix for oriented networks is asymmetric in the general case, but it is symmetric for networks with undirected links: $a_{ij} = a_{ji}$. Notice that a network with undirected links can be viewed as an oriented network in which each pair of nodes is connected by two oppositely directed edges. In general, the links between the nodes may act in various ways (for example, in neural networks the synapses can be excitatory or inhibitory), and be also characterized by different weights (connection or coupling strengths) w_{ij} , which are the components of the weight matrix W .

Each node i is characterized by its *degree* k_i —the number of neighboring nodes it is connected with. For oriented networks, we introduce the outgoing node degree $k_i^{\text{out}} = \sum_{j=1}^N a_{ji}$ —the number of outgoing (efferent) links—and the incoming node degree $k_i^{\text{in}} = \sum_{j=1}^N a_{ij}$ —the number of incoming (afferent) links. The full degree of the node in this case is defined as $k_i = k_i^{\text{in}} + k_i^{\text{out}}$.

One of the most important characteristics of any complex network is the *distribution function of nodal degrees*, $P(k)$, which expresses the probability of an arbitrarily selected node having the definite number of links k . For example, in random networks, where a connection exists between any two nodes with some probability p (also called random graphs, or Erdős–Rényi networks), most of the nodes have the degree equal to the network mean value $\langle k \rangle = (1/N) \sum_{i=1}^N k_i$, and the function $P(k)$ represents the Poisson distribution, which passes a maximum at $k = \langle k \rangle$ and has an exponentially decaying tail.

For the majority of real large-scale networks, the degree distribution substantially deviates from the Poisson distribution. In particular, for many complex networks, including the Internet and some neural ensembles, the function $P(k)$ takes a power-law form: $P(k) \sim k^{-\gamma}$. Such networks are called *scale-invariant* (or *scale-free*): most of their nodes have a small number of neighbors, while a relatively small number of nodes (so-called *hubs*) are, in contrast, connected to a vast number of neighbors. It has been shown that the scale-free property arises, in particular, in growing networks, where new nodes appear, building connections with the already existing nodes, so that a new node forms with a higher probability a link with the node already having many

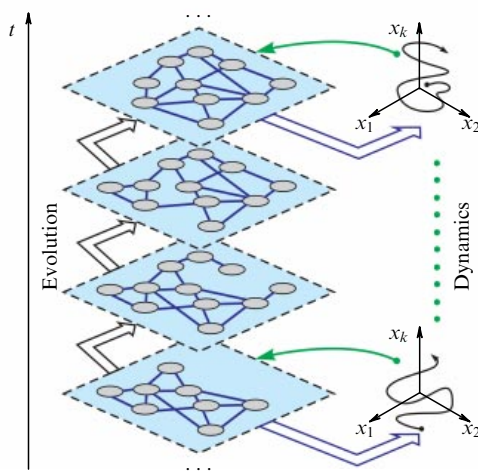


Figure 1. Schematic representation of adaptive dynamical network: the network evolution (on the left), accompanied by the topology rewiring, proceeds in parallel with the dynamics of nodes and links (on the right), related to some trajectory in phase space.

neighbors (see also Refs [12, 13]). The last property reflects the principle of preferential attachment (or the so-called rich-club phenomenon) (see Ref. [14]).

Most real networks have rather sparse links, i.e., only a relatively small fraction of all potentially possible links do actually exist. Two nodes, i and j , picked at random are disconnected with a high probability; however, there is a possibility of drawing a *path*—a sequence of intermediate links—also between the nonadjacent nodes. The number of these links defines the *length* of this path. The *distance* d_{ij} between nodes i and j is the length of the shortest path between them, i.e., the minimum number of links (edges) to be visited on moving from j to i . If there is the link $j \rightarrow i$, then $d_{ij} = 1$. If the path from j to i can only be built through intermediate node k by means of the transitions $j \rightarrow k \rightarrow i$, then $d_{ij} = 2$, and so on. If no path can connect i and j , then $d_{ij} = \infty$. The *average length l of the shortest path* (sometimes called the network diameter) is the network average distance d_{ij} over all node pairs i and j of the network:

$$l = \frac{1}{N(N-1)} \sum_{i,j \in N, i \neq j} d_{ij}. \quad (1)$$

The *clustering coefficient* C is the next topological characteristic of complex networks. Consider a node i which has k_i neighbors. If all these nodes were connected to each other, there would be $k_i(k_i - 1)/2$ links altogether. The ratio between the number E_i of actually present links and the maximum possible number $k_i(k_i - 1)/2$ gives the value of the clustering coefficient: $C_i = 2E_i/[k_i(k_i - 1)]$ for node i . The average taken over all individual C_i defines the network clustering coefficient C :

$$C = \frac{1}{N} \sum_{i \in N} C_i. \quad (2)$$

Notice that in a random graph, where the edge between two randomly selected nodes exists with the probability p , the clustering coefficient is $C = p$. However, for the majority of real networks the value of C is, as a rule, much higher than that for a random network with close numbers of nodes and edges. For example, in networks with the property of a *small world*, the clustering coefficient attains a large value, but the average path length stays relatively small.

In addition to the mentioned basic topological characteristics of complex networks, the following notions are frequently used. In a network of N nodes and L edges, the *density* of the network is the fraction of the existing links from their possible number: $\rho = L/[N(N - 1)]$. It should be noted that the shortest path between two nodes is commonly not unique; there can be several alternative shortest paths. To characterize the influence of individual nodes on the collective interaction in the network, we introduce the notion of *betweenness centrality* $C_{BC}(i)$ —the number of all shortest paths passing through node i :

$$C_{BC}(i) = \sum_{s \neq i, t \neq i}^N \frac{\sigma_{st}(i)}{\sigma_{st}},$$

where $\sigma_{st}(i)$ is the number of shortest paths starting at s , passing through i , and ending at t , and σ_{st} is the number of all shortest paths from s to t .

The scale invariance of connection topology is closely related to the phenomena of *assortative mixing*, in which

nodes-hubs are connected with the highest probability to other hubs, and *disassortative mixing*, when hubs connect with higher probability to nodes with low degrees. It turns out that many social groups (co-authors of publications, film actors, heads of companies) are characterized by high assortativeness, whereas biological (synaptic connections in the neural network of the nematode *C. elegans*, food webs in marine or lake populations) and technological (the network of autonomous systems in the Internet, the network of hyper-links between web sites in a given domain zone) networks are disassortative [15]. It is established that assortative networks are more stable against elimination of nodes-hubs, and the disassortative ones are more vulnerable. Thus, social communities are more trustworthy with respect to intrusions and attacks than technological ones.

One pressing problem in the theory of adaptive dynamical networks is the task of building intellectual power grids which would provide uninterrupted electric power supply from the generators to the consumers under conditions of possible failures or overloads, and also assuming heterogeneous and decentralized power production. From the standpoint of nonlinear dynamics, the working regime is that of global stability, when the balance of electric power is maintained between the power supplied to the network by generators and that extracted from the network by consumers. A possible method to ensure stable functioning of electric power networks under changing conditions lies in using adaptive principles of organization, when the network architecture is dynamically adjusted depending on instantaneous redistribution of power flows. As basic models of power grids, the ensembles of coupled phase oscillators are often used; a thorough description of them is offered in Section 3 (for more detail on power networks, see, for example, Refs [16–18]).

3. Collective effects in adaptive dynamical networks

Each node in adaptive dynamical systems is characterized by its own dynamics (regular or chaotic), whereas internodal links depend on these dynamics, and their behavior can also be governed by dynamical systems. Such interconnection between the nodes and links may lead to unexpected effects and complex collective behavior in the network. The network evolution, i.e., the change in its topology with time, can be described differently in various models. The simplest way is the change in the weight variables for the fixed connectivity structure. In this case, the link strength is changing, and, furthermore, the links can change their direction or action character (excitation/inhibition).

The next way to describe evolution is the rewiring of connectivity. In this case, we select a link at some time instant, which is then rewired: it is either fully eliminated in favor of a new link that connects some other nodes, or one of the link ends retains its node, while the other one is switched to a new node. In this rewiring, the total number of nodes and links is preserved with time. The best known example of a network which is formed through topology rewiring is the small-world network defined by the Watts–Strogatz model [19]. The initial network topology in this model appears as a regular lattice, then some fraction of links is randomly rewired. It turns out that even a small number of rewired links substantially modifies the network

topological and dynamical properties, in particular, the network diameter sharply decreases.

A more complex way to describe network evolution, as compared to the previous one, lies in creating new or removing old links, keeping the number of nodes, but varying the number of links with time.

Finally, the most complex form for describing topology evolution assumes the creation (removal) of nodes in (from) the network. In this case, the node removal modifies the links that were connected to this node (for example, they also disappear). Moreover, adding a new node entails the creation of new links to join the other nodes in the network. In this evolution scheme, the number of nodes, as well as that of links, vary. A typical example is the Barabási–Albert model [12], where the numbers of nodes and links in the network increase. The initial topology in the model is set by a small network of connected nodes. In what follows, we sequentially add a node after the node, and each new node has m links through which it is connected with higher probability to the nodes of higher degree (the principle of preferential attachment).

The most typical basic properties of adaptive dynamical networks established at present include the following:

- the formation of activity clusters, i.e., groups of synchronous oscillators of various configurations and high multi-stability of the resulting regimes. This also includes so-called chimera states—co-existing clusters of synchronous activity and groups of oscillators with asynchronous dynamics (their consideration is beyond the scope of this review);
- the emergence of complex topologies characterized by a nontrivial distribution of node degrees, in particular, by the appearance of various types of heterogeneity in an initially homogeneous network (for example, the formation of hierarchical structures or modules). In this case, the property of self-organized criticality—the network self-organizes into a critical state, which is accompanied by the appearance of power-law distributions in various network characteristics—is often observed;
- the presence of various transient or switching regimes accompanied by repeated rewiring in connection topology and alteration in collective activity regimes.

These features are explained in detail in Sections 3.1–3.3.

3.1 Formation of activity clusters

One of the frequent effects occurring in adaptive dynamical networks is the formation of cluster states. A cluster comprises a group of synchronous network nodes; the cluster state exhibits the presence of one or more clusters of a certain configuration that coexist in the network.

References [20–22] consider networks of phase oscillators described by the system of equations

$$\dot{\varphi}_i = \omega_i + \frac{1}{N} \sum_j^N k_{ij} \Gamma_{ij}(\varphi_i - \varphi_j), \quad (3)$$

where the dynamics of weights k_{ij} is governed by the equations

$$\dot{k}_{ij} = \varepsilon A(\varphi_i - \varphi_j), \quad |k_{ij}| \leq 1, \quad (4)$$

where ε is the global link strength, and $A(\varphi)$ is the 2π -periodic function of plasticity, which defines how weights are modified in response to phase differences. The condition $|k_{ij}| \leq 1$ eliminates unbounded growth of the weights

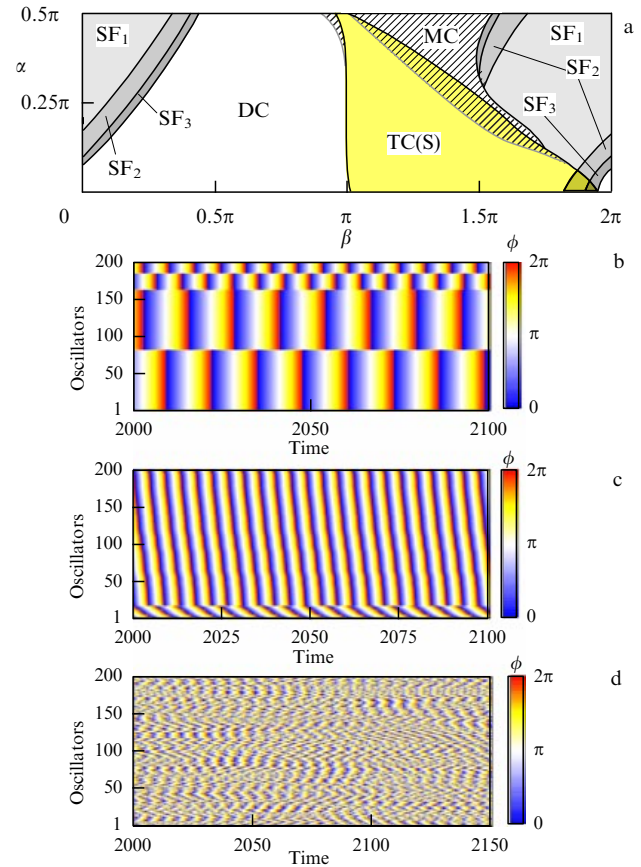


Figure 2. (a) Diagram of the collective activity regimes for system (5) in the plane of parameters (β, α) for $N = 200$ interacting oscillators. The dynamical states formed in the network conform: the regime of global synchronization S; synchronous states with a fixed phase relationship SF_i , where i indicates the number of frequency groups formed in the network; asynchronous states DC; two-cluster state TC, and multicenter states MC. The parameter $\varepsilon = 0.01$. (b–d) The dynamics of oscillator phases in different regimes for $\alpha = 1$: (b) four-cluster state, $\beta = 4.9$; (c) synchronous state with a fixed phase relationship for the case of two frequency groups, $\beta = 0.65$, and (d) asynchronous state, $\beta = 1.9$. The indices of oscillators are in ascending order of their frequencies and phases at the time instant $t = 1000$. (Data taken from Ref. [24].)

outside the interval $[-1, 1]$. The link function is given in the form $\Gamma_{ij}(\varphi) = -\sin(\varphi + \alpha)$, where α is the phase difference caused by the interaction delay. The plasticity function is defined as $A(\varphi) = -\sin(\varphi + \beta)$, where β controls the character of weight change depending on the phase difference. The initial values of variables are set in a random way, and the connection topology is selected as ‘all-to-all’. It is found that, depending on the values of parameters α and β , the network can evolve to one of three asymptotic regimes: a two-cluster state, a synchronous state with a fixed relationship between the phases, and asynchronous behavior. In the two-cluster state, the network splits into two groups, with synchronous oscillations inside each of them, whereas the group activity makes itself evident in antiphase (see Fig. 2, which illustrates a schematic of an analogous network). In the regime with a fixed phase relationship, the frequencies of all network oscillators are identical, and relative phase shifts between different oscillators take some fixed values that do not vary with time. The same regimes can also be observed in a network with a scale-free topology [22].

The modification of the plasticity rule which affects the connection strength leads to the appearance of new collective activity regimes. Thus, Refs [23, 24] consider a network of phase oscillators with variable connections:

$$\begin{aligned}\dot{\varphi}_i &= 1 - \frac{1}{N} \sum_{j=1, j \neq i}^N \kappa_{ij} \sin(\varphi_i - \varphi_j + \alpha), \\ \dot{\kappa}_{ij} &= -\varepsilon [\sin(\varphi_i - \varphi_j + \beta) + \kappa_{ij}].\end{aligned}\quad (5)$$

Because of the more diverse structure of connections formed in the network as a result of the joint dynamics of nodes and links, new types of collective activity become possible in such a network, when part of the internodal links acquires values close to zero. In addition to the globally synchronous, two-cluster, and asynchronous states, multicluster regimes have been discovered when several groups of synchronous oscillators are forming in the network. Among these groups, the pairs can be identified in which oscillators are in antiphase with each other. The frequencies of oscillators belonging to different pairs of synchronous groups are also different (see Fig. 2).

Reference [25] considers a network of phase oscillators

$$\dot{\theta}_m = \omega_m + \gamma \sum_{k=1}^N w_{mk} \sin[2(\theta_k - \theta_m)], \quad (6)$$

where $m = 1, \dots, N$, ω_m is the eigenfrequency of the m th oscillator, and w_{mk} are the coupling weights, with their dynamics governed by the equations

$$\dot{w}_{mk} = \varepsilon w_{mk} \exp\left(-2\left|\Delta\theta_{mk} - \frac{\pi}{2}\right|\right) \sin(2\Delta\theta_{mk}). \quad (7)$$

The oscillators split into dynamical groups where they are in phase, while the groups as a whole are in antiphase. In this case, the network becomes self-organized, so that its connection strengths are distributed according to a power law evolving from an initially random distribution, and modules may also form. If the resources needed to form the links are limited, the network becomes split into two dynamical groups: inside each of them, the oscillators are synchronized in phase, whereas oscillators of different groups are in antiphase. The connection strength inside the modules increases, and it decreases outside them, i.e., the modules become practically disconnected. If there is no limitation on resources, the two groups coalesce in a single one, as a consequence of the fact that the coupling strength between the oscillators with initially close phases increases, and it decreases for large phase differences.

References [26, 27] explore the networks of logistic maps

$$x_{n+1}^i = f(x_n^i) + \frac{\varepsilon}{m} \sum_{j=1}^N A_n^{ij} [f(x_n^j) - f(x_n^i)], \quad (8)$$

where $f(x) = 4x(1-x)$, and A_n is the adjacency matrix at the instant n . Each node can have up to $\sum_{j=1}^N A_n^{ij} = N-1$ incoming links. The topology evolution obeys the principle of preferential attachment in the following way. At each step, the so-called matrix of *distances* is computed, namely

$$D_n^{ij} = \begin{cases} |f(x_n^i) - f(x_n^j)| & \text{at } A_{n-1}^{ij} \neq 0, \\ 0 & \text{at } A_{n-1}^{ij} = 0, \end{cases} \quad (9)$$

which is used to compute the average distance of node i from its neighbors, $\langle D \rangle_n^i = (1/a_n^i) \sum_{j=1}^N D_n^{ij}$, where a_n^i is the number of neighbors of the i th node at time instant n . The set of ‘bad’ nodes \mathcal{B}_n^i is determined among the neighboring nodes of node i , i.e. those nodes the distance to which exceeds the mean $\langle D \rangle_n^i$: $j \in \mathcal{B}_n^i$ if $D_n^{ij} > \langle D \rangle_n^i$; all others are considered ‘good’ ones, $\mathcal{G}_n^i = \{1, \dots, N\} \setminus (\mathcal{B}_n^i \cup \{i\})$. The links coming from the nodes in the set \mathcal{B}_n^i are then disrupted and undergo rewiring to some arbitrarily selected nodes from \mathcal{G}_n^i . The initial topology consists of symmetric links ‘all-to-all’, so that each node is connected to all possible $N-1$ neighbors and $A_0^{ii} = 0$. It is shown that for an arbitrary magnitude of the coupling strength ε the network asymptotically evolves into a stationary state, and the topology rewiring ceases. The final topology is characterized by hierarchical properties, and the dynamics of nodes within some parameter domain is polysynchronous: the nodes can be separated into groups of nodes synchronous in different ways; however, the nodes within individual groups are not necessarily connected with each other.

Reference [28] finds that the connection plasticity leads to the appearance of multistable states — desynchronization and formation of a two-cluster state in which the clusters are in antiphase with each other. The multistable states occur for the random distribution of initial phases, and only a desynchronized state evolves if the initial phases are distributed uniformly.

A model of neural network is considered in Ref. [29]:

$$\dot{\theta}_i = \omega_i - \frac{1}{N} \sum_{j=1}^N K_{ij} \sin(\theta_i - \theta_j), \quad i = 1, \dots, N, \quad (10)$$

where the coupling coefficients vary according to the law reflecting the spike-timing dependent plasticity (STDP) of synapses:

$$\dot{K}_{ij} = \varepsilon \begin{cases} (\alpha - K_{ij}) \exp \frac{\theta_i - \theta_j}{\tau_p}, & (\theta_i - \theta_j) \in [-\pi, 0], \\ -K_{ij} \exp \frac{\theta_i - \theta_j}{\tau_d}, & (\theta_i - \theta_j) \in [0, \pi], \end{cases} \quad (11)$$

where α is the maximum coupling strength, τ_p and τ_d are the constants of synaptic potentiation and depression, respectively, and ε is a small parameter. This plasticity rule is asymmetric, because the constants are different: $\tau_d > \tau_p$. It is shown that this asymmetry in the plasticity law leads to the phenomenon of network multistability. Depending on the parameter α , a fully synchronous regime, a state of desynchronization, and diverse multicluster regimes may occur. In some intervals of α stable regimes coexist, each characterized by a different number and size of clusters — groups of oscillators with the same frequency.

Reference [30] studies the dynamics of Kuramoto phase oscillators with a slowly varying adjacency matrix:

$$\dot{\varphi}_i = \omega_i - \frac{1}{N} \sum_{j=1}^N K_{ij} \sin(\varphi_i - \varphi_j), \quad (12)$$

where $i = 1, \dots, N$, ω_i is the eigenfrequency of the i th oscillator, and K_{ij} are the coupling coefficients whose dynamics are governed by the equations

$$\dot{K}_{ij} = \varepsilon [\alpha \cos(\varphi_i - \varphi_j) - K_{ij}]. \quad (13)$$

The parameter ε is small here; therefore, the coupling coefficients are slowly varying functions of phase differences, growing faster for the oscillators in phase and decaying faster for those in antiphase—that is, the generalized Hebb rule works here. The slow dynamics lead to multistability: stabilization of synchronous clusters with different sizes and phase relationships between the oscillators takes place. The difference in phases between the oscillators can be used to store and retrieve information in the system. One natural constraint of the reduced phase description exists—the assumption that all the oscillators reside in an excited state.

Reference [31] explores the dynamics of scale-free networks of chaotic oscillators in which the coupling strength changes adaptively, depending on the synchronization among the nodes. A network of N connected identical chaotic oscillators is considered:

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sum_{j=1}^N G_{ij}(\mathbf{H}(\mathbf{x}_j) - \mathbf{H}(\mathbf{x}_i)), \quad (14)$$

where the function $\mathbf{F}(\mathbf{x})$, describing the dynamics of an individual oscillator $\mathbf{x} = (x, y, z)$, is given either by the Rössler system

$$\mathbf{F}(\mathbf{x}) = (-0.97x - z, 0.97x + 0.15y, x(z - 8.5) + 0.4)$$

or the food web model:

$$\mathbf{F}(\mathbf{x}) = (x - 0.2g(x, y), -0.7y + 0.2g(x, y) - xz, -10x(z - 0.006) + xz),$$

where $g(x, y) = xy/(1 + 0.05x)$. The function $\mathbf{H}(\mathbf{x})$ describes a linear coupling between the oscillators [for the first system $\mathbf{H}(\mathbf{x}) = (x, 0, 0)$, and for the second $\mathbf{H}(\mathbf{x}) = (0, y, z)$]. G_{ij} is the weighted connectivity matrix, $G_{ij} = A_{ij}W_{ij}$, where A_{ij} is the binary adjacency matrix, W_{ij} is the connection strength from node j to node i . The strength of connection of the i th node with its neighbors k_i grows uniformly over all k_i edges, whereas the difference Δ_i between the average activity of neighbors decreases:

$$W_{ij}(t) = V_i(t), \quad \dot{V}_i = \frac{\gamma \Delta_i}{1 + \Delta_i}, \quad (15)$$

where $\gamma > 0$ is the adaptation parameter, and Δ_i is the quantity governed by the average activity of neighbors, which decreases with time:

$$\Delta_i = \left| \mathbf{H}(\mathbf{x}_i) - \frac{1}{k_i} \sum_j A_{ij} \mathbf{H}(\mathbf{x}_j) \right|.$$

The initial connection strength is a small random quantity, and the connection topology is scale-free.

The law of connection change (15) assumes that the strength of links among the nodes in different states increases. Thus, with time the network evolves into a more homogeneous state with higher synchronization. The authors of Ref. [31] showed that the transition to synchronization occurs in a hierarchic way: first, the nodes with the largest degree k_i are synchronized, and then the nodes with smaller degrees. Thus, Δ_i in Eqn (15) decreases with an increase in k_i , and the nodes with the larger k_i have a smaller rate of increase $V_i(t)$. The structure of weight distribution becomes practically stationary past a relatively short time interval, after

which the transition to synchronization begins, resembling such in static networks. The adaptive rule essentially enhances the tendency to network synchrony, which finds important applications in the control of dynamical networks.

Thus, the adaptive networks in which the state of links depends on node dynamics, and vice versa, demonstrate multistable dynamics. This manifests itself in the existence of a variety of collective activity regimes related to the appearance of clusters—the groups of synchronous oscillators. The number of clusters and the relations of oscillation phases between them are governed by the law describing how the coupling coefficients evolve and by the control parameters.

3.2 Formation of modular topology

One of the most widespread effects occurring in adaptive dynamical networks is the formation of modular topology in an initially homogeneous network. Such a topology is characterized by the presence of modules—subnetworks with a higher density of links inside than among the nodes belonging to different subnetworks. For example, Refs [32, 33] consider the networks of phase oscillators with randomly selected initial topology, directed links, and weight evolution governed by the dynamics of oscillators. The adaptation rule relies on two key principles which compete with each other: (1) the links between synchronous nodes are enhanced, and (2) the resources available to maintain the node links to other network nodes are limited. The first principle is in essence the Hebb rule, well known in neural dynamics, whereas the second principle reflects the capability of maintaining dynamical equilibrium, i.e., homeostasis. The model represents an ensemble of N phase oscillators in which each node i is characterized by the phase φ_i and interacts with M randomly selected neighboring nodes forming the set \mathcal{N}_i . The dynamics of network nodes are governed by the equations

$$\dot{\varphi}_i = \omega_i + \lambda \sum_{j \in \mathcal{N}_i} w_{ij} \sin(\varphi_j - \varphi_i), \quad (16)$$

where ω_i are the eigenfrequencies of oscillators randomly selected from the interval $[-\pi, \pi]$ and uniformly distributed there, w_{ij} is the weight of the link between nodes i and j , and λ is the global coupling strength. Initially, the phases are randomly chosen from the interval $[-\pi, \pi]$, and all the weights are equal to $1/M$. The dynamics (evolution) of the weights w_{ij} in Eqn (16) obey the relationship

$$\dot{w}_{ij} = p_{ij} - \left(\sum_{j \in \mathcal{N}_i} p_{ij} \right) w_{ij}, \quad (17)$$

where p_{ij} is the mean phase correlation between the i th and j th oscillators for the characteristic time T :

$$p_{ij} = \frac{1}{T} \left| \int_{-\infty}^t \exp\left(-\frac{t-t'}{T}\right) \exp[i(\varphi_i(t') - \varphi_j(t'))] dt' \right|. \quad (18)$$

From equations (17) it follows that during the entire time T the normalization condition $\sum_{j \in \mathcal{N}_i} w_{ij} = 1$ is observed, i.e., the sum of weights of all incoming links remains constant for each node.

It should be noted that the first and second terms on the right-hand side of equation (17) relate to the Hebb rule and homeostasis, respectively. The level of global synchronization

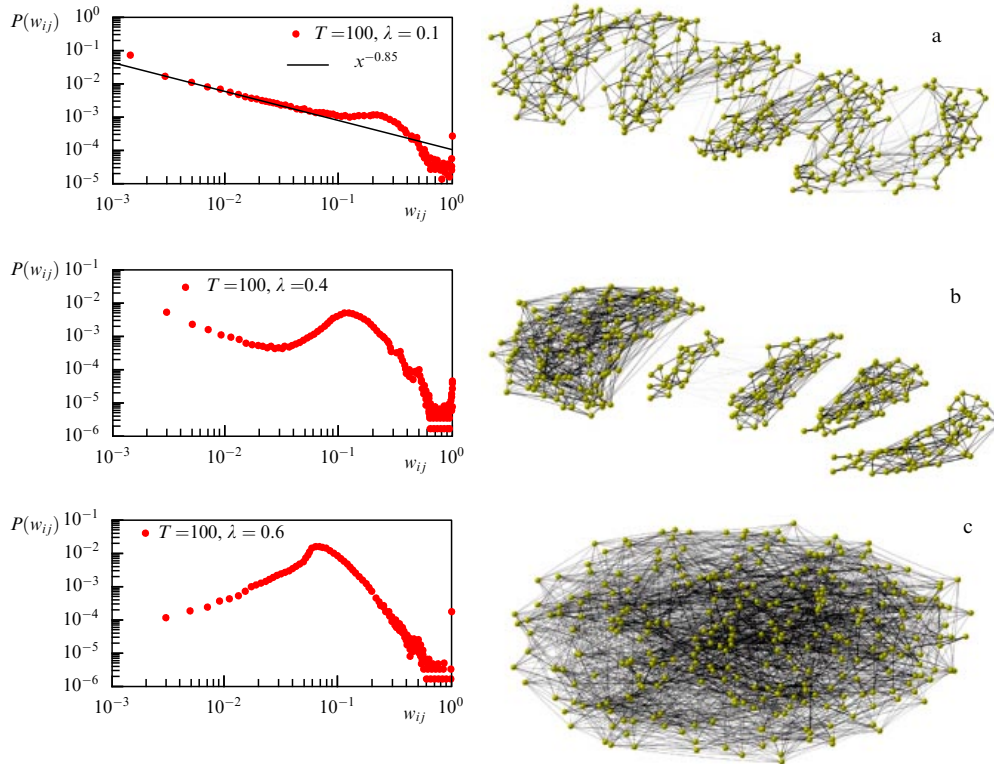


Figure 3. Network structure and respective weight distributions for the system described by equations (16)–(18). The left part of the figure plots the weight distributions $P(w_{ij})$ for $T = 100$ and various values of λ indicated in the figure. Notice that for small λ a network is formed with a power-law weight distribution $P(w_{ij}) = w_{ij}^{-\alpha}$. On the right-hand part of the figure, the formation of respective modules is shown: with λ increasing from $\lambda = 0.1$ (panel a) the modules become more and more isolated, down to the states when they become fully disconnected at $\lambda = 0.4$ (panel b). With a further increase in λ to 0.6, the weight distribution becomes progressively more and more uniform and in the end the network becomes a single module (panel c). (Data taken from Ref. [33].)

in the network of phase oscillators can be conveniently estimated with the help of the order parameter $r(t) = 1/N |\sum_{i=1}^N \exp(i\varphi_i(t))|$ [34]. The system arrives at some asymptotic state past some time t_s , in which the weights w_{ij} take definite values with small stationary fluctuations. As a measure of synchronism, the mean order parameter is computed in this case giving

$$r = \lim_{\Delta t \rightarrow \infty} \frac{1}{\Delta t N} \int_{t_s}^{t_s + \Delta t} \left| \sum_{i=1}^N \exp(i\varphi_i(t')) \right| dt', \quad (19)$$

where Δt is a sufficiently large time interval. Additionally, denoting the mean degree of synchronization between the i th and j th oscillators as

$$r_{ij} = \lim_{\Delta t \rightarrow \infty} \frac{1}{\Delta t} \left| \int_{t_s}^{t_s + \Delta t} \exp[i(\varphi_i(t') - \varphi_j(t'))] dt' \right|,$$

we determine the measure of local synchronization [35]:

$$r_{\text{link}} = \frac{1}{N} \sum_{i=1}^N \sum_{j \in \mathcal{N}_i} w_{ij} r_{ij}. \quad (20)$$

In the case of a weak (strong) link, both quantities r and r_{link} are close to zero (unity), but in intermediate cases interesting dynamical regimes are observed, in which the measure r_{link} of local synchronization is high, whereas global synchronization has not been yet achieved. With an increase in the connection strength λ , the network gradually evolves into several interconnected modules which grow in size, while the links become concentrated inside the modules. Under these

conditions, the weight distribution bears a clearly identifiable power-law character. A further increase in the strength leads to the network's disintegration into several independent modules, and the power-law weight distribution becomes violated: it acquires a clearly expressed maximum (Fig. 3). Similar effects are also seen in a more complex — multi-layer — network of phase oscillators if the above-described principles of adaptation act there too (see Ref. [36]).

Different dynamics of nodes and links relying on similar adaptation principles are considered in Ref. [37], which explores the network of logistic maps with alternate links:

$$x_{n+1}^i = f \left[(1 - c)x_n^i + c \sum_{j=1}^N w_n^{ij} x_n^j \right], \quad (21)$$

where x_n^i are the variables describing the state of the i th node ($i = 1, \dots, N$) at time instant n . The coefficient c characterizes the link strength of the network nodes with the given node ($0 < c < 1$), and w_n^{ij} are the alternate weights of directed links from the j th to i th node at time instant n . The function $f(x) = ax(1 - x)$ defines the logistic map at each node. The weights of links vary according to the maps:

$$w_{n+1}^{ij} = \frac{(1 + \delta g(x_n^i, x_n^j)) w_n^{ij}}{\sum_{j=1}^N (1 + \delta g(x_n^i, x_n^j)) w_n^{ij}}, \quad (22)$$

where $\delta = 0.1$ is the plasticity parameter, and $g(x_n^i, x_n^j) = 1 - 2|x_n^i - x_n^j|$ is a monotone decaying function of the absolute value of the difference between its arguments.

Equation (22) governs the generalized Hebb rule, i.e., the links are strengthened for nodes i and j with close values of

variables x_n^i and x_n^j . Furthermore, map (22) accounts for the normalization of links, which reflects another important property of real networks—the boundedness of resources needed to form the links. The initial values of weights are taken to be identical for all links: $w_0^{ii} = 0$ and $w_0^{ij} = 1/(N-1)$ for $i \neq j$. The initial values of variables are taken as random and uniformly distributed in the interval $(0, 1)$. Reference [37] reported that the structure of this network undergoes self-organization: ordering emerges even in the absence of synchronization between the interacting nodes. Two groups of nodes can be identified in the network. The nodes of the first (central) group interact with each other owing to the large number of links and send many links to the nodes of the second (peripheral) group, strongly influencing through that the dynamics of its nodes. The nodes of the second group interact among themselves only weakly and send only a very limited number of links to the nodes of the central group.

Reference [38] considers a model based on coupled neural masses (32 or 64 masses) each simulating a large population of interacting excitatory and inhibitory neurons and generating an EEG/MEG-like signal (EEG stands for electroencephalography, and MEG for magnetoencephalography). It is shown how the interaction between the dynamics and evolution lead to the emergence of complex network properties, in particular, of module topology.

The network evolution is modelled in two ways: owing to plastic links depending on synchronization or to the plasticity depending on the network growth. In the first case, the links between the masses strengthen when the masses are synchronous, and weaken otherwise. In the second case, a homeostatic process takes place, with a random formation, depending on the distance, of new links between the neural masses. The first rule, as applied to random networks, leads to clustering, but without pronounced modules. The second one leads to a stable network with a link strength that depends on the distance, with small-world properties, but without correlation between nodal degrees and only with a weak modularity. The well-expressed modularity appears only when both rules operate, with the number and size of modules depending on the relative strength of both processes and also on the network size. In such a network, the properties of a small world (high clustering and small mean path length), assortative mixing and modular structure also manifest themselves. Network damage (elimination of some links between the adjacent nodes) after the stable state is established leads to a transient disturbance in the network structure followed by the recovery of the modular architecture.

Reference [39] considers a network composed of phase oscillators

$$\dot{\varphi}_i = \omega_i + \frac{r}{M_i} \sum_j^N W_{ij} \sin(\varphi_j - \varphi_i), \quad (23)$$

where $i = 1, \dots, N$, ω_i is the eigenfrequency of the i th oscillator, r is the coupling strength, M_i is the number of neighbors of the i th oscillator, and W_{ij} is the symmetric adjacency matrix. The evolution algorithm works as follows. For each time interval T , the mean oscillation frequency is computed for each oscillator: $\Omega_i = (1/T) \int_{t_i}^{t_i+T} \dot{\varphi}_i(t') dt'$, then some oscillator i is picked up at random and quantities $\delta_{ij} = |\Omega_i - \Omega_j|$ are computed for all $j \neq i$. Then such an oscillator j_1 is sought for which δ_{ij_1} is minimum among all δ_{ij} . Further, an oscillator j_2 is sought among the neighbors of

oscillator i , for which the value of δ_{ij_2} is at a maximum. If the j_1 th node is the neighboring one for the i th node, the topology is kept without changes. Otherwise, the link between the nodes i and j_2 is disrupted and re-established between the nodes i and j_1 . After this topology rearrangement, a new interval of duration T is started and the process is repeated.

References [40, 41] consider an ensemble of neurons (threshold integrators) interacting through an exchange of spikes (short pulses). The state of neuron $i = 1, \dots, N$ is described by a variable $h_i \geq 0$ characterizing the membrane potential, which varies according to the equation

$$\dot{h}_i = \delta_{i,\zeta(t)} I + \frac{1}{N} \sum_{j=1}^N u_{ij}(t_{sp}) J_{ij}(t_{sp}) \delta(t - t_{sp}^j - \tau_d), \quad (24)$$

where $\delta_{i,\zeta(t)}$ is the Kronecker symbol ($\delta_{i,\zeta} = 1$ if $i = \zeta$, and $\delta_{i,\zeta} = 0$ if $i \neq \zeta$), ζ stands for a random process, I is the magnitude of input stimulus, and t_{sp} is the moment of spike generation. The amount of transmitter entering the synaptic gap is characterized by the variable J_{ij} , and the available fraction of transmitter at a given moment is $u_{ij} \in [0, 1]$, where the subscripts i and j are related to pre- and postsynaptic neurons, respectively. The coupling strength is given by the product $u_{ij} J_{ij}$. A spike from neuron j acts on neuron i with the time delay τ_d , which is accounted for in equation (24) by the Dirac delta-function $\delta(t)$.

If the membrane potential $h_i \geq 0$ exceeds some threshold value θ at the time instant t_{sp}^i , neuron i generates a spike. The potential is then readjusted by the threshold value: $h_i(t_{sp}^i) = h_i(t_{sp}) - \theta$. The synaptic activity also leads to a reduction in the amount of transmitter, so that J_{ij} decreases immediately after the spike ceasing. In the time intervals between the instants when spikes are generated, the resources are replenished and J_{ij} reaches the unperturbed state α/u_0 through the characteristic time τ_J :

$$\dot{J}_{ij} = \frac{1}{\tau_J} \left(\frac{\alpha}{u_0} - J_{ij} \right) - u_{ij} J_{ij} \delta(t - t_{sp}^j). \quad (25)$$

It is found that such a network possesses the property of self-organized criticality—the presence of a threshold of avalanche neuron excitation. The adaptivity of links leads to the appearance of new effects: a hysteresis between the states of critical and noncritical dynamics, switching the dynamics between different states, depending on incoming stimuli, and the first- and second-order phase transitions.

Reference [42] considers a network of N integrate-and-fire neurons, such that the membrane potential $V_i(t) \in [0, 1]$ of each i th neuron varies according to the equation

$$\dot{V}_i(t) = a - V_i(t) + I_i(t), \quad i = 1, \dots, N. \quad (26)$$

On reaching the threshold $V_i = 1$, the neuron generates in such a network the α -pulse $p_\alpha(t) = \alpha^2 t \exp(-\alpha t)$ which instantaneously propagates to all other neurons, whereas the value of V_i drops to zero. Here, $a > 1$ is the constant current above the threshold value, $I_i = gE_i$ is the synaptic current, and $g > 0$ is the coupling strength. The field E_i represents a linear superposition of pulses received by the i th neuron, and its dynamics are governed by the second-order differential equation

$$\ddot{E}_i(t) + 2\alpha \dot{E}_i(t) + \alpha^2 E_i(t) = \frac{\alpha^2}{N-1} \sum_{n|t_n < t, j} w_{ij} \delta(t - t_n). \quad (27)$$

In the case of the postsynaptic (presynaptic) spike emitted by neuron $i(j)$ at the time instant t , the weights w_{ij} increase (decrease) according to the relationship $w_{ij}(t^+) = w_{ij}(t^-) + \Gamma_{ij}(t)$, where

$$\Gamma_{ij}(t) = \begin{cases} p(w_{\max} - w_{ij}(t^-)) \exp\left(-\frac{\delta_{ij}}{\tau_+}\right), & \text{if } \delta_{ij} > 0, \\ -dw_{ij}(t^-) \exp\left(\frac{\delta_{ij}}{\tau_-}\right), & \text{if } \delta_{ij} < 0. \end{cases} \quad (28)$$

Here, p is the coefficient of excitatory connection, d is the coefficient of inhibitory connection, w_{\max} is the maximum coupling strength, $0 \leq w_{ij} \leq w_{\max}$, τ_+ (τ_-) is the duration of time intervals over which the action of postsynaptic (presynaptic) spikes leads to potentiation (depression), and $\delta_{ij} = t - t^{(j)} > 0$ ($\delta_{ij} = t^{(i)} - t < 0$) is the time difference relative to the spike generation instant.

It is found that in such a network irregularly repeating transitions take place between strongly and weakly synchronous states. The mechanism of such transitions is set by the so-called Sisyphus effect — a continuous feedback between synaptic changes and coherence of neuron spike activity. This is accompanied by periodic variability of weights: small (large) synaptic weights bring the network into a strongly (weakly) synchronous state; the induced activity of neurons, in turn, leads to an increase (decrease) in weights.

Reference [43] considers the dynamics of a network of N elements, each characterized by the variable $x_i(t)$, called the resource by the authors of Ref. [43]. The resource dynamics at the network nodes is described by the equations

$$\begin{aligned} x_i(t+1) &= x_i(t) + F(x_i(t)) \\ &+ D \sum_{j \in \mathcal{N}} \left(\frac{w_{ij}(t)}{s_j(t)} x_j(t) - \frac{w_{ji}(t)}{s_i(t)} x_i(t) \right), \end{aligned} \quad (29)$$

where $F(x) = -\kappa(x-1)$ reflects the resource dissipation at the node, and the weights $w_{ij}(t)$ control the process of diffusion. It can be represented in terms of random walk, assuming that a random transition from node i to node j occurs within one time step with the probability $Dw_{ji}(t)/s_i(t)$. Here, $s_i(t)$ is the node strength, expressed as $s_i(t) \equiv \sum_{j \in \mathcal{N}} w_{ji}(t) = \sum_j a_{ij} w_{ij}(t)$, where \mathcal{N}_i is the set of neighbors of node i . The dynamics of weights is described by the equations

$$w_{ij}(t+1) = w_{ij}(t) + \varepsilon [x_i(t)x_j(t) - w_{ij}(t)], \quad (30)$$

where ε is the relaxation parameter. The joint dynamics of nodes and links leads asymptotically to the buildup of power-law distributions of the resource (at the nodes) and weights, even if the topology of links is not scale-invariant. Furthermore, it is shown that in the presence of dissipation ($\kappa \neq 0$), even though the distributions become stationary, the values of variables at an individual node or link can vary with time.

Thus, the more complex rules of dynamical network adaptation than those considered in Section 3.1 lead to new effects which affect not only the dynamics, but also the structure of networks. In particular, the competition between the two principles — the boundedness of resources needed to form the links and the synchronization of oscillators with close values of dynamical variables — leads to the appearance of modules. Owing to the joint dynamics of nodes and links, the network acquires qualitatively new

structural properties expressed in the presence of weakly interacting subgroups of tightly coupled oscillators.

3.3 Switching dynamics of adaptive networks

The property of adaptivity, which consists in interaction between the network dynamics and evolution, can manifest itself in finite times, as well as in the long-time limit. In the first case, the adaptive network system arrives at an attractor on which the network topology ceases to vary with time, whereas the dynamics of nodes and links may continue to unfold. In the second case, the system may arrive at an attractor where the varying topology and dynamical network state will continue to influence each other. On the route to these attractors, the network state and topology may experience successive changes over long time intervals, which are more interesting than the final state of the network. In this case, the subject of research is the transient dynamics of the network system, which consists of a chain of switched states. Even though the topology and dynamical variables do not lead to fixed states, some average characteristics (the number of nodes and their mean degree) can reach stable values.

Reference [44] pays considerable attention to the heterogeneity of nodes as a driving force of the network structure evolution. The network evolution follows the principle of finding among all configurations a structure with the minimum number of links, which guarantees frequency synchronization among the nodes. The dynamics of the nodes is described by the phase equations

$$\dot{\varphi}_i = \omega_i + \frac{1}{N} \sum_{j=1}^N w_{ij} \sin(\varphi_j - \varphi_i), \quad (31)$$

in which the coupling evolution is governed by the second-order equations with a potential having two wells, $V(w_{ij}) = bw_{ij}^2(w_{ij}-1)^2$, where the parameter b specifies the barrier height:

$$\ddot{w}_{ij} + d\dot{w}_{ij} + \frac{\partial V(w_{ij})}{\partial w_{ij}} = h(\|\varphi_j - \varphi_i\|). \quad (32)$$

Here, d is the decay coefficient, and $h(\|\varphi_j - \varphi_i\|) = \alpha[1 - (1/2)|\exp(i\varphi_i) + \exp(i\varphi_j)|]$. Owing to the shape of the potential $V(w_{ij})$, the weights w_{ij} with the initial conditions $w_{ij}(0) = 0$ and $\dot{w}_{ij}(0) = 0$ converge with time to zero (no connection) or to unity (the connection is activated), in relation to an external stimulus defined by the function $h(\|\varphi_j - \varphi_i\|)$. The frequencies ω_i of oscillators are different and are taken according to the Gaussian distribution with zero mean and standard deviation σ , i.e., σ is the measure of heterogeneity of the nodes in the network.

The action of the evolution operator includes two stages reflecting such properties of the natural evolution as variability and selection. The realization of the variability mechanism assumes the generation of a set of unweighed networks with the aid of equations (31) and (32), each time for different initial values of variables. A selection of n_S initial conditions is used, which are randomly taken from the interval $\varphi_i(0) \in [0, 2\pi)$, $i = 1, 2, \dots, N$. After this procedure, for each link we compute the probability of its appearance p_{ij} as the ratio of the number n_{ij} of generated networks, in which a given link is available, to the total number n_S of realizations, i.e., $p_{ij} = n_{ij}/n_S$. The elements of the obtained stochastic matrix P of the size $N \times N$ are the probabilities of activation of any possible connection between the nodes.

The selection mechanism consists in taking only those links whose activation probability exceeds some critical threshold value p^* , i.e., such that $p_{ij} > p^*$. The p^* value is chosen so that the resulting network be connected and have the minimum number of links.

An increase in nodal heterogeneity entails structural changes in the network: its maximum and minimum nodal degrees increase. The links are activated with higher probability if they involve nodes with the largest distinction in frequencies, and those nodes whose frequencies deviate the most from the mean frequency with a higher probability become the hubs—nodes having a higher degree relative to the degrees of the remaining nodes. The dynamics of weights depends on the dynamics of nodes and the resulting (unweighted) network emerges as the result of the joint evolution of the structure and nodal dynamics. The strategy relies on the distributed adaptive nonlinear system and therefore presents a rather general decentralized mechanism.

Reference [45] builds a model of the network of the synaptically coupled Morris–Lecar neurons whose dynamics are described by the following system of equations:

$$\begin{aligned} C \frac{dv_i}{dt} &= -g_L(v_i - v_L) - g_{Ca} M_{\text{inf}}(v_i) [v_i - v_{Ca}] \\ &\quad - g_K n_i(v_i - v_K) + I_i^{\text{ext}} - s_i g_{\text{syn}}(v_i - v_{\text{rev}}), \\ \frac{dn_i}{dt} &= \frac{n_{\text{inf}}(v_i) - n_i}{\tau_n(v_i)}, \\ \frac{dr_i}{dt} &= f_1(r_i) - s_i - k_1, \\ \frac{ds_i}{dt} &= \varepsilon \left[f_2(r_i) - s_i - k_2 - \mu \sum_{j=1, j \neq i}^N H(v_j - \theta_{ji}) \right], \end{aligned} \quad (33)$$

where the subscripts i and j enumerate the network elements. The first two equations in set (33) describe the neuron dynamics, and the two remaining ones describe synaptic connections between them. The variables v_i and n_i characterize the membrane potential and the activation of the potassium ion channel of the i th neuron, C is the membrane electric capacitance, and the terms on the right-hand side in equations (33) describe currents traversing the neuron cellular membrane. The first three terms describe, respectively, the leakage current and calcium and potassium currents passing through the membrane, I_i^{ext} describes the external currents, g_L , g_{Ca} , and g_K are the maximum conductivities, and v_L , v_{Ca} , and v_K are the equilibrium potentials (reversion potentials) for the respective ion channels; $M_{\text{inf}}(v)$, $n_{\text{inf}}(v)$, and $\tau_n(v)$ are the stationary states of the activation levels and the characteristic relaxation time, depending on the membrane potential according to the sigmoid law:

$$\begin{aligned} M_{\text{inf}}(v) &= \frac{1}{2} \left(1 + \tanh \frac{v - v_1}{v_2} \right), \\ n_{\text{inf}}(v) &= \frac{1}{2} \left(1 + \tanh \frac{v - v_3}{v_4} \right), \\ \tau_n(v) &= \left(\varepsilon \tanh \frac{v - v_3}{2v_4} \right)^{-1}. \end{aligned}$$

The last term in the first equation of set (33) defines the synaptic current arising due to the action of other neurons on neuron i through chemical synapses. The parameter g_{syn}

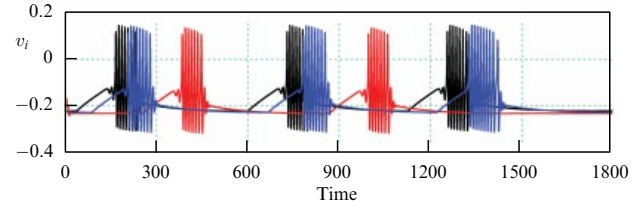


Figure 4. An example of the transient dynamics of system (33) consisting of 3 neurons. As a reaction to an external stimulus, transient dynamics take place in the network: the neurons generate series of spikes in a certain order, and then the system returns to equilibrium. (Data taken from Ref. [45].)

describes the maximum synaptic conductivity, v_{rev} is the reversal potential for the synaptic current, and the variable s_i characterizes the dependence of the i th neuron membrane conductivity on the presynaptic potentials v_j ($j \neq i$) of neurons acting on it. The functions in the synaptic connection model (33) are taken as

$$\begin{aligned} f_1(r) &= r - \frac{r^3}{3}, \quad f_2(r) = \begin{cases} \alpha r, & r < 0, \\ \beta r, & r \geq 0, \end{cases} \\ H(x) &= \left[1 + \exp \left(-\frac{x}{\kappa} \right) \right]^{-1}, \end{aligned}$$

where $\alpha = 0.5$, $\beta = 2$, and $\kappa = 0.001$ are the model constants. It is found that system (33) demonstrates diverse structurally stable transient dynamics. In response to the action of an external stimulus, a certain sequence of metastable oscillatory states of activity evolves in an ensemble initially staying at rest (the life span of such states is substantially longer than the action duration of a single pulse generated by an individual neuron). Depending on the form of the adjacency matrix $\{\theta_{ji}\}$, a sequential activation of neurons can be observed in the network, which generate one after the other a series of spikes or more complex patterns (Fig. 4).

An increase in the number of neurons in the network extends essentially the selection of possible routes of transient dynamics and can make their spatial configuration substantially more complex. In particular, spatially chaotic transient sequences become possible in large-scale neural networks. Notice that the trajectory corresponding to complex switching dynamics of neurons, which evolves in a multidimensional phase space, asymptotically approaches a stable equilibrium state and, as a consequence of the robustness (structural stability) of this equilibrium, is also robust.

Reference [46] considers a network of five chaotic maps which qualitatively describe the activity of a neuron:

$$\begin{aligned} x_{i,n+1} &= x_{i,n} + F_{\theta}(x_{i,n}) - y_{i,n} + I_{i,n}, \\ y_{i,n+1} &= y_{i,n} + \varepsilon(x_{i,n} - J_i), \quad i = 1, \dots, N, \end{aligned} \quad (34)$$

where $i = 1, \dots, 5$, $n = 0, 1, 2, \dots$, $F_{\theta}(x) = x(x-a)(1-x) - \beta\theta(x-d)$, and $\theta(x)$ is the Heaviside function; the parameters a , β , and d control the shape of the generated signal. The connection between the nodes is set by the term $I_{i,n}$, which describes the influence of other nodes on the i th node:

$$I_{i,n} = -g \sum_{j=1, j \neq i}^N G_{ij,n}(x_{i,n} - v) \theta(x_{j,n} - \theta), \quad (35)$$

where the parameter g is the coupling strength, v is the reversal potential, and θ is the presynaptic threshold. The

adjacency matrix $G_{ij,n}$ sets the topology: $G_{ij,n} = 1$ if the j th node influences node i at the time instant n , and $G_{ij,n} = 0$ otherwise.

The rearrangement of topology follows the integrate-and-fire principle. The integration is described by the variable q , which depends on the network mean field:

$$q_{n+1} = \begin{cases} q_n + \mu X_n & \text{at } q_n < 1, \\ 0 & \text{at } q_n \geq 1, \end{cases} \quad (36)$$

where $\mu = 0.01$ and

$$X_n = \frac{1}{N} \sum_{i=1}^N x_{i,n}. \quad (37)$$

On reaching the threshold $q = 1$ at the instant $n = n^*$, the topology becomes rearranged and the value of q is reset to zero.

The topology rewiring at time instant $n = n^*$ takes place under the action of the operator P_{kl} :

$$G_{ij,n+1} = P_{kl} G_{ij,n}, \quad (38)$$

where P_{kl} permutes rows k and l in $G_{ij,n}$, and then columns k and l in the matrix obtained. The numbers k and l are chosen based on the condition establishing which nodes are active at the instant of rearrangement and which were active before it; namely a pair is selected with a minimum distance computed on a cyclically ordered set of enumerated nodes $1, \dots, i, \dots, N$. It is shown that owing to the mutual influence of node dynamics and structure evolution, such networks are capable of generating switching sequences of cluster states. Each such state represents a cyclic change of clusters of synchronous activity of various configurations. It is found that an external signal acting on one of the nodes and forming the initial topology causes a response in the network in the form of a sequence of cluster states, which is stable to small noise, a change in initial conditions, and parameter detuning.

Thus, a more complex adaptation mechanism leads to the appearance of nontrivial transient processes in complex networks. Sequences of various regimes of synchronous activity replacing one after another emerge. The set of all possible regimes can be presented as a graph of dynamical network states, the edges of which correspond to the allowed transitions between different regimes. The action of external stimulus gives rise to a structurally stable response of the adaptive network in the form of a switching sequence which corresponds to a certain path along the graph.

4. Conclusion

Adaptive dynamical networks offer the most relevant model to describe numerous real systems in Nature, technical applications, and society. Examples can be provided by networks of interacting lasers and Josephson junctions, the immune and nervous systems, competing species populations and food webs, coupled chemical reactions, networks of roads and flight connections, power grids, wireless communication networks, the Internet, virtual social networks, networks for online dating, scientific communication, and many others.

This review presents the main model systems describing adaptive dynamical networks. It is shown that, the differences between concrete models notwithstanding, common phenom-

ena evolve in such networks, which include the emergence of cluster synchronization regimes, the appearance of topological modules, power-laws in the distributions of nodal degrees and connection weights, self-organized criticality, and multi-stable switching dynamics. One of the key problems in the theory of adaptive dynamical networks encompasses the questions of how nontrivial global topologies emerge in networks where one type of local interaction or another operates between the nodes, and which topological properties arise owing to a given set of evolution rules.

The work was carried out with the financial support of the Russian Science Foundation (grant 16-42-01043).

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