 © 2017 Uspekhi Fizicheskikh Nauk, Russian Academy of Sciences

METHODOLOGICAL NOTES

PACS numbers: **02.20.–a**, **03.65.–w**, 31.15.xh

On the spin-statistics theorem

E D Trifonov

DOI: https://doi.org/10.3367/UFNe.2017.02.038061

Contents

Abstract. **The possibility of proving the theorem on the connection between spin and statistics in the nonrelativistic quantum mechanical framework is examined.**

Keywords: identical particles, spin and statistics, Pauli exclusion principle, irreducible representations of the rotation group, spinor fields

"... the final 'truth' on the subject is still 'dwelling in the abyss..."

W Pauli [1]

1. Introduction

As is well known, Pauli's theorem on the connection between spin and statistics is proved in the theory of quantized fields involving invariant properties under the Lorentz group (including time, space and charge reversal [1–5]) and satisfying the positive energy requirement and the fulfilment of causality condition. On the other hand, the formulation of the theorem is quite concise: the wave function of a system of identical particles with integer (half-integer) spin is symmetric (antisymmetric) under simultaneous permutation of the particle coordinates and spin variables. This connection has found wide and effective use, in particular, in the nonrelativistic theory of many-electron systems and in molecular physics (see, for example, Refs [6, 7]). Because the way a wave function transforms under three-dimensional rotations is determined by spin, restrictions on permutation symmetry are naturally expected to be connected with rotation group symmetry. It should be noted that attempts are still ongoing to explain the connection between spin and statistics within the framework of nonrelativistic quantum mechanics (see Refs [8–11] and references cited therein).

2. Spin-statistics connection for systems of identical particles

According to Bethe's 'simple symmetry' principle, a unique choice can only be made for each class of identical particles between two possibilities: symmetry or antisymmetry (see 'arguments in favor of simple symmetry' in Ref. [12, p. 25]); a detailed analysis of this problem is carried out in Ref. [13] (see also Refs [14, 15]). To establish a connection between spin and statistics for a given class of identical particles, it suffices to prove that the theorem *necessarily* holds for any specific case. Because the symmetry property we consider should be independent of the number of particles in the system, the simplest case of two identical particles appears to be a natural starting point.

Let us consider a system of two particles with half-integer spins. The wave function of a particle with half-integer spin (for simplicity, we can confine ourselves to spin-1/2 particles with zero orbital moment) is a spinor field, i.e., spinor specified at each point in space, $\chi_m(\mathbf{r})$, $m = \pm 1/2$. Let both particles be on the x-axis, and let the z-axis be the spin projection quantization axis.

The transformation of a spinor field under three-dimensional spatial rotation g (i.e., a transformation to a new orthogonal reference frame) is given by (see, for example, Ref. [7])

$$\chi'_m(g^{-1}\mathbf{r}) = \sum_{m'=\pm 1/2} D^{(1/2)}_{m'm}(g)\,\chi_{m'}(\mathbf{r})\,, \tag{1}$$

where $D^{(1/2)}_{mm'}(g)$ is the matrix of the irreducible representation, with weight 1/2, of the rotation group. The primes denote the spinor components in the new reference frame obtained from the original one by a rotation g. Notice that the spinor components $\chi_m(\mathbf{r})$ are multiplied by $\exp(\mathrm{i}\pi m) = \pm \mathrm{i}$ under a rotation by π around the spin projection quantization z-axis.

The wave function of two such particles is a second-rank spinor field (a second-rank spinor depending on two vector arguments, the radius vectors of these particles). The transformation law for a second-rank spinor can be written out as

$$\chi'_{m_1 m_2}(g^{-1}\mathbf{r}_1, g^{-1}\mathbf{r}_2) = \sum_{m'_1, m'_2=\pm 1/2} D^{(1/2)}_{m'_1 m_1}(g) D^{(1/2)}_{m'_2 m_2}(g) \chi_{m'_1 m'_2}(\mathbf{r}_1, \mathbf{r}_2)\,. \tag{2}$$

The values of the spatial arguments can be permuted by a rotation through π about any axis which passes through the center of inertia of the two particles and which is orthogonal to the line segment connecting them: $g^{-1}\mathbf{r}_1 = \mathbf{r}_2$, $g^{-1}\mathbf{r}_2 = \mathbf{r}_1$. We choose the projections of the spin on this axis as the spin variables. Then, under this rotation, the spin variables retain

E D Trifonov Herzen State Pedagogical University of Russia,
Naberezhnaya reki Moiki 48, 191186 St. Petersburg, Russian Federation
E-mail: thphys@herzen.spb.ru

Received 11 February 2016, revised 13 January 2017
Uspekhi Fizicheskikh Nauk **187** (6) 667 – 668 (2017)
DOI: https://doi.org/10.3367/UFNr.2017.02.038061
Translated by E G Strel'chenko; edited by A Radzig

their values, whereas the one-particle wave functions (the components of the first-rank spinor) are multiplied by i or −i. *For identical values of the spin variables* (the diagonal components of the second-rank spinor), the two-particle wave function reverses sign:

$$\chi'_{m,m}(\mathbf{r}_2, \mathbf{r}_1) = -\chi_{m,m}(\mathbf{r}_1, \mathbf{r}_2)\,, \tag{3}$$

consistent with Eqn (2). Thus, when transforming into a new reference frame obtained by rotating the original one through π about the spin projection quantization axis passing through the center of inertia of the two particles and orthogonal to their connecting line, the spatial particle coordinates have their values permuted and the wave function has its sign reversed.

Let us show that the 'permutation antisymmetry' property can also be extended to other states of the system under consideration (with $m_1 \neq m_2$). To do this, consider a rotation through π about the y-axis orthogonal both to the connecting line between the particles and to the spin quantization axis. A rotation by π about the y-axis results in permutation of the spinor components:

$$\chi'_{1/2} = \mathrm{i}\chi_{-1/2}\,, \qquad \chi'_{-1/2} = \mathrm{i}\chi_{1/2}\,. \tag{4}$$

For a second-rank spinor we obtain

$$\chi'_{m_2,m_1}(\mathbf{r}_2, \mathbf{r}_1) = -\chi_{m_1,m_2}(\mathbf{r}_1, \mathbf{r}_2) \tag{5}$$

according to Eqn (2). Equation (5) still does not imply, however, that the two-particle wave function with equal values of the spin variables should be antisymmetric under the permutation of the spatial coordinates in a specific (one and the same) reference frame. The wave function will be antisymmetric in the specified (original) reference frame if we require *additionally* that the following relationship holds:

$$\chi'_{m_1,m_2}(\mathbf{r}_1, \mathbf{r}_2) = \chi_{m_1,m_2}(\mathbf{r}_1, \mathbf{r}_2)\,. \tag{6}$$

Relying on — and treating as an accepted fact — the antisymmetry property of the wave function of half-integer spin particles, we can indeed draw this conclusion, i.e., Eqn (6) can be viewed as a reverse theorem following from the wave function antisymmetry with respect to permutation of half-integer spin particles. Relation (6) can be interpreted as a quantum mechanical expression of the indistinguishability of identical particles. In our present discussion, this equality is regarded as a *hypothesis*.

Thus, the antisymmetry property of the wave function of two identical particles with spin $1/2$ is fully determined by the half-integer value of the spin. If this fact is taken to be established, then the symmetry properties of the wave functions of systems of identical particles with other spin values are most easily determined by treating them as 'composite particles' made up of an odd or even number of spin-$1/2$ particles. It is known that the permutation symmetry properties of 'composite particles' (e.g., such as atoms) are determined by the value of the total spin (for an atom, the sum of the spins of the nucleus and electrons). Not only is this suggested by the logic of theoretical analysis, but it also has a long history of experimental confirmation (the superconductivity of helium and Bose condensation of dilute laser-cooled gases).

3. Conclusion

In the present methodical article, the connection between spin and statistics is proved using an additional hypothesis, Eqn (6) — the relationship which we suggest should be considered a formal expression of the indistinguishability of identical particles.

After the first version of this paper was submitted to the *Physics–Uspekhi* Editorial Board, we found that similar analyses had already been proposed [8, 9], which, however, came under criticism — in particular, from Ref. [11] — presumably for the logical flaw our hypothesis made up for.

The treatment presented in this paper cannot be considered — and this is what the epigraph is about — as a complete proof of the theorem on the connection between spin and statistics, the reason being the use of an additional hypothesis. However, the connection which we found to exist between the transformation properties of the wave functions for a system of identical particles is of methodical interest and will possibly stimulate a deeper analysis of the problem in the framework of nonrelativistic quantum mechanics.

Sincere thanks for discussions go to M G Benedikt, M M Mestechkin, L N Labzovsky, and A V Tulub. The work was supported by the RFBR, project 15-02-08369-A.

References

1. Pauli W "Exclusion principle, Lorentz group and reflection of space-time and charge", in *Niels Bohr and the Development of Physics* (Ed. W Pauli) (New York: McGraw-Hill, 1955) p. 30; Translated into Russian: "Printsip zapreta, gruppa Lorentsa, otrazhenie prostranstva, vremeni i zaryada", in *Nil's Bor i Razvitie Fiziki* (Ed. W Pauli) (Moscow: IL, 1958)
2. Fierz M *Helv. Phys. Acta* **12** 3 (1939)
3. Pauli W *Phys. Rev.* **58** 716 (1940)
4. Rumer Yu B, Fet A I *Teoriya Grupp i Kvantovannye Polya* (The Theory of Groups and Quantized Fields) (Moscow: Nauka, 1977)
5. Bogoliubov N N, Shirkov D V *Introduction to the Theory of Quantized Fields* (New York: John Wiley, 1980); Translated from Russian: *Vvedenie v Teoriyu Kvantovannykh Polei* (Moscow: Nauka, 1984)
6. Landau L D, Lifshitz E M *Quantum Mechanics. Non-Relativistic Theory* (Oxford: Pergamon Press, 1977); Translated from Russian: *Kvantovaya Mekhanika. Nerelyativistskaya Teoriya* (Moscow: Fizmatlit, 2004)
7. Petrashen M I, Trifonov E D *Applications of Group Theory in Quantum Mechanics* (Mineola, NY: Dover Publ., 2009); Translated from Russian: *Primenenie Teorii Grupp v Kvantovoi Mekhanike* (Moscow: URSS, 2014)
8. Broyles A A *Am. J. Phys.* **44** 340 (1976)
9. Bacry H *Am. J. Phys.* **63** 297 (1995)
10. Duck I, Sudarshan E C G *Pauli and the Spin-Statistics Theorem* (Singapore: World Scientific, 1997)
11. Duck I, Sudarshan E C G *Am. J. Phys.* **66** 284 (1998)
12. Jabs A *Found. Phys.* **40** 776 (2010); *Found. Phys.* **40** 793 (2010)
13. Bethe H A *Intermediate Quantum Mechanics* (New York: W.A. Benjamin, 1964); Translated into Russian: *Kvantovaya Mekhanika* (Moscow: Mir, 1965)
14. Kaplan I G *Sov. Phys. Usp.* **18** 988 (1975); *Usp. Fiz. Nauk* **117** 691 (1975)
15. Kaplan I G *Found. Phys.* **43** 1233 (2013)