# About recognition of mirror-like objects 

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#### Abstract

We discuss the problem of the visual recognition of mirror-like objects, i.e., bodies with an ideally smooth (analytic) surface. The visual perception of such objects is essentially dependent on their environment, in contrast to bodies with an irregular (rough) surface, whose visual images are independent of the environment.


Keywords: mirror-like object, wave field, scatterer, analytic continuation, singularities of analytic continuation of the wave field, carrier of secondary sources, crosscut, Riemann surface, imagery

## 1. Introduction

As is known, monochromatic waves (both electromagnetic and sound) can be described by a function $U^{1}(\mathbf{r}) \equiv U^{1}(x, y, z)$ that satisfies the Helmholtz equation

$$
\begin{equation*}
\Delta U^{1}+k^{2} U^{1}=0 \tag{1}
\end{equation*}
$$

where $\Delta \equiv \partial^{2} / \partial x^{2}+\partial^{2} / \partial y^{2}+\partial^{2} / \partial z^{2}$ is the Laplace operator, $k=2 \pi / \lambda$ is the wave number, and $\lambda$ is the wavelength of the simulated oscillations. The function $U^{1}$ is referred to as a wave field in what follows.

If a primary wave field $U^{0}(x, y, z)$ encounters an obstacle (scatterer), i.e., a body bounded by a surface S , as it propagates from a radiation source, then the wave field

[^0]reflected (scattered) by the obstacle is found from the relation (see, e.g., Refs [1, 2]):
\[

$$
\begin{equation*}
U^{1}(\mathbf{r})=\int_{\mathrm{S}}\left(U\left(\mathbf{r}^{\prime}\right) \frac{\partial G_{0}\left(\mathbf{r} ; \mathbf{r}^{\prime}\right)}{\partial n^{\prime}}-\frac{\partial U\left(\mathbf{r}^{\prime}\right)}{\partial n^{\prime}} G_{0}\left(\mathbf{r} ; \mathbf{r}^{\prime}\right)\right) \mathrm{d} s^{\prime} \tag{2}
\end{equation*}
$$

\]

where $U \equiv U^{0}+U^{1}$ is the total (incident plus scattered) field, $G_{0}$ is the fundamental solution of the Helmholtz equation (the free-space Green's function),
$G_{0}\left(\mathbf{r} ; \mathbf{r}^{\prime}\right)= \begin{cases}\frac{1}{4 \pi} \frac{\exp \left(-\mathrm{i} k\left|\mathbf{r}-\mathbf{r}^{\prime}\right|\right)}{\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} & \text { in three dimensions, } \\ \frac{1}{4 \mathrm{i}} H_{0}^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}^{\prime}\right|\right) & \text { in two dimensions, }{ }^{1}\end{cases}$
and $\partial / \partial n^{\prime}$ is the differentiation operator along the outer normal to S .

The function $G_{0}\left(\mathbf{r} ; \mathbf{r}^{\prime}\right)$ represents the field at a point $M(\mathbf{r})$ produced by a point-like source located at $N\left(\mathbf{r}^{\prime}\right)$. Thus, in accordance with Eqn (2), the field of primary (for example, light) sources results from the radiation of a large number of secondary sources located on its surface $S$. In other words, what we see is the result of the radiation of these sources.

We consider a situation where the object under study is mirror-like, i.e., polished to a sufficient extent to make its surface roughness features visually imperceptible. An example is a perfectly polished sphere like those familiarly used as Christmas tree decorations. If there are no objects in the vicinity of such a sphere, then, if illuminated in darkness by a source of light, the contours of the sphere are somewhat uncertain.

It is to be noted that the visualization of mirror objects is a frequent problem for artists, as exemplified by works of the famous Russian artist Petrov-Vodkin, notably by his 'Still Life with a Samovar', which we reproduce here.

Importantly for what follows, the function $U^{1}(\mathbf{r})$ that solves homogeneous Helmholtz equation (1) with $M(\mathbf{r}) \in \mathbb{R}^{3} \backslash \overline{\mathrm{D}}$, where D is a domain inside S and $\overline{\mathrm{D}} \equiv \mathrm{D} \cup \mathrm{S}$,
${ }^{1} H_{0}^{(2)}$ is the zeroth-order Hankel function of the second kind.


K S Petrov-Vodkin, "Still Life with a Samovar."
is analytic everywhere outside D [1] and can therefore be represented as a power series in the neighborhood of each of its points of analyticity (an example is the Atkinson-Wilcox series in $1 /(k r)[1,2])$. As is well known, such a series converges outside a sphere of a radius equal to the distance from the center of expansion to the most distant singularity point of $U^{1}(\mathbf{r})$, allowing the function $U^{1}(\mathbf{r})$ to be analytically continued beyond the domain of its original definition [2-4]. The function $U^{1}(\mathbf{r})$ necessarily has singular points (otherwise it would be everywhere zero [2]) and these clearly lie inside or on the boundary of the domain D .

What are the above-mentioned singular points of the function $U^{1}(\mathbf{r})$ ? We consider a simple example where a source of light is placed in front of a plane mirror. When looking in the mirror, we see the source of light at a point symmetric relative to S located behind the mirror (Fig. 1). We see this imaginary source (image) by straight-line continuation of the rays reflected from the mirror. The image of the source is exactly the result of the mirrorreflected field being analytically continued into the region behind the mirror.

The image obtained is perceived by us as an additional light source located on the other side of the mirror (Alice's


Figure 1. Mapping in a plane mirror.
looking-glass). If we start moving the source away from the mirror, its wonderland counterpart moves symmetrically arbitrarily far away.

A totally different situation occurs for reflection from a nonplanar surface. We imagine that we start bending a planar mirror by lifting its edges. The space behind the mirror starts to 'shrink' to form a 'fold', in which process the possibility exists for the image to disappear, in part or as whole - a phenomenon familiar to everyone who has happened to view curved mirror images in a funhouse. This phenomenon occurs because in the hypothetical behind-the-mirror medium, the above-mentioned 'folds' form: these are regions in which two (or more) images should appear simultaneously, with the disappearing part of the image 'hiding in the fold' (see Ref. [4] for more details).

## 2. Localization of the singularities of the analytically continued diffraction field

We now discuss the localization of singularities of the function $U^{1}(\mathbf{r})$ in more detail. We consider representation (2) that and limit ourselves to the case of two dimensions. We suppose the scatterer boundary $S$ (which we consider to be star-shaped) is specified by the equation $r=\rho(\varphi)$ and introduce the complex variable

$$
\begin{equation*}
\zeta=\rho(\varphi) \exp (\mathrm{i} \varphi) . \tag{3}
\end{equation*}
$$

If the quantity $\varphi$ in Eqn (3) is real, the contour $C$ described by $\zeta$ in the plane $z=r \exp (\mathrm{i} \varphi)$ is geometrically coincident with S. However, setting $\varphi=\varphi_{1}+\mathrm{i} \varphi_{2}$ causes the contour C to be deformed (in particular, for positive $\varphi_{2}$, to be compressed).

Such a deformation is possible until mapping (3) remains bijective. Also, because the boundary values $\left.U(\mathbf{r})\right|_{\mathrm{S}}$ and $\partial U(\mathbf{r}) /\left.\partial n\right|_{\mathrm{S}}$ involve the values $\left.U^{0}(\mathbf{r})\right|_{\mathrm{S}}$ and $\partial U^{0}(\mathbf{r}) /\left.\partial n\right|_{\mathrm{S}}$, it follows that the above deformation is bounded by singular points of the function $\left.U^{0}(\mathbf{r})\right|_{\mathrm{S}}$ continued to the domain of complex $\varphi$.

Clearly, the bijectivity of mapping (3) is violated at a point for which

$$
\begin{equation*}
\zeta^{\prime}(\varphi) \equiv\left[\rho^{\prime}(\varphi)+\mathrm{i} \rho(\varphi)\right] \exp (\mathrm{i} \varphi)=0 . \tag{4}
\end{equation*}
$$

It is important to note that Eqn (4) is equivalent to the system of two equations

$$
\left[\begin{array}{l}
{\left[\rho^{\prime}(\varphi)+\mathrm{i} \rho(\varphi)\right]=0,}  \tag{4a}\\
\exp (\mathrm{i} \varphi)=0,
\end{array}\right.
$$

where the equation $\exp (\mathrm{i} \varphi)=0$ cannot be dropped because ignoring its roots results in losing some (for certain shapes, very essential) part of the singularities.

The singularities of the function $\left.U^{0}(\mathbf{r})\right|_{\mathrm{S}}$ are located at the 'image points' of the primary field source, which they reach by following the complex characteristics and whose coordinates are found using the Riemann-Schwarz symmetry principle [4, 5].

We now turn to considering another way of defining the coordinates of the image of a point-like source. Let the pointlike source field be of the form

$$
U^{0}(\mathbf{r})=\frac{1}{4 \mathrm{i}} H_{0}^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}_{0}\right|\right)
$$

In coordinates $(\rho, \varphi)$, we have

$$
\begin{aligned}
& \left.\left|\mathbf{r}-\mathbf{r}_{0}\right|^{2}\right|_{\mathrm{S}}=\rho^{2}(\varphi)+r_{0}^{2}-2 r_{0} \rho(\varphi) \cos \left(\varphi-\varphi_{0}\right) \\
& \quad=\left[\rho(\varphi) \exp (\mathrm{i} \varphi)-r_{0} \exp \left(\mathrm{i} \varphi_{0}\right)\right] \\
& \quad \times\left[\rho(\varphi) \exp (-\mathrm{i} \varphi)-r_{0} \exp \left(-\mathrm{i} \varphi_{0}\right)\right] .
\end{aligned}
$$

The singularities of $\left.U^{0}(\mathbf{r})\right|_{\mathrm{S}}$ are then located at

$$
\rho(\varphi) \exp ( \pm \mathrm{i} \varphi)=r_{0} \exp \left( \pm \mathrm{i} \varphi_{0}\right)
$$

We also note that the function $U^{0}(\mathbf{r})$ has a singularity at infinity. The root of the equation $\rho(\varphi) \exp (\mathrm{i} \varphi)=r_{0} \exp \left(\mathrm{i} \varphi_{0}\right)$ corresponds to the singularity of $U^{0}(\mathbf{r})$ at the position of the source, $r_{0} \exp \left(\mathrm{i} \varphi_{0}\right)=x_{0}+\mathrm{i} y_{0}$.

We now consider the equation

$$
\begin{equation*}
\rho(\hat{\varphi}) \exp (-\mathrm{i} \hat{\varphi})=r_{0} \exp \left(-\mathrm{i} \varphi_{0}\right) \equiv z^{-} \tag{5}
\end{equation*}
$$

Assuming a real $\hat{\varphi}$ in Eqn (5), we obtain the same point as in the preceding case. Therefore, we solve Eqn (5) by assuming that $\hat{\varphi}=\varphi_{1}+\mathrm{i} \varphi_{2}$ is a complex quantity. From Eqns (3) and (5), we have

$$
\zeta_{\text {sing }} \exp \left(-\mathrm{i} 2 \hat{\varphi}\left(z^{-}\right)\right)=z^{-}
$$

that is,

$$
\begin{equation*}
\zeta_{\text {sing }}=z^{-} \exp \left(\mathrm{i} 2 \hat{\varphi}\left(z^{-}\right)\right) \tag{6}
\end{equation*}
$$

where $\zeta_{\text {sing }} \equiv \zeta\left(\hat{\varphi}\left(z^{-}\right)\right)$. It is this equation that gives the coordinates of the 'image' of the source. Additionally, it is necessary to find the image of the infinitely remote point [2].

The above techniques for localizing singularities apply unchanged to the three-dimensional case where the scatterer is a body of revolution. It goes without saying that Eqns (4) and (6) give the coordinates of singularities in the axial cross section of the scatterer.

In the general three-dimensional case, an analytic deformation of the boundary $S$ can be implemented by introducing the complex variable

$$
\begin{equation*}
\zeta=\rho(\theta, \varphi) \exp (\mathrm{i} \theta) \tag{7}
\end{equation*}
$$

(where $r=\rho(\theta, \varphi)$ is the equation for the boundary S in spherical coordinates), assuming the angle $\theta$ to be complex and taking the angle $\varphi$ as a parameter [2].

Having found all singularities of the analytic continuation of the wave field to the interior of the scattering body (i.e., domain D ), we can use the principle of equivalence to represent the field $U^{1}(\mathbf{r})$ scattered by this object by using sources located on a certain surface $\Sigma$ that envelopes the set of singularities. In other words, Eqn (2) can then be replaced by [2]

$$
\begin{equation*}
U^{1}(\mathbf{r})=\int_{\Sigma}\left(U\left(\mathbf{r}^{\prime}\right) \frac{\partial G_{0}\left(\mathbf{r} ; \mathbf{r}^{\prime}\right)}{\partial n^{\prime}}-\frac{\partial U\left(\mathbf{r}^{\prime}\right)}{\partial n^{\prime}} G_{0}\left(\mathbf{r} ; \mathbf{r}^{\prime}\right)\right) \mathrm{d} \sigma^{\prime} \tag{8}
\end{equation*}
$$

Thus, the image of an object is formed by sources located on a certain support $\Sigma$ inside the object. In some cases, as we see in Section 3, the support of secondary sources $\Sigma$ can be located very deep inside the scatterer.

To see how deep inside the scatterer the support $\Sigma$ of the scattered field sources can be located, we consider some
examples of the localization of singularities of the analytic continuation of the wave field.

## 3. Examples of the localization of singularities of the analytically continued diffraction field

### 3.1 Singularities of the scatterer boundary equation mapped on the complex plane

The singularities of mapping (3) can be found by solving Eqns (4a). We give examples of solutions of these equations for various geometries.
3.1.1 Elliptic cylinder. As the first example, we consider the diffraction of a plane wave on an elliptic cylinder. The equation for the cross section contour S has in this case the form

$$
\rho(\varphi)=\frac{b}{\sqrt{1-\varepsilon^{2} \cos ^{2} \varphi}}
$$

where $b$ is the minor semiaxis, $a=b / \sqrt{1-\varepsilon^{2}}$ is the major semiaxis, and $\varepsilon=\sqrt{1-b^{2} / a^{2}}$ is the ellipse eccentricity. After elementary manipulations, we see that the complex root $\varphi_{0}$ of the first of Eqns (4a) satisfies

$$
\cos ^{2} \varphi_{0}-\mathrm{i} \cos \varphi_{0} \sin \varphi_{0}-\frac{1}{\varepsilon^{2}}=0
$$

whence

$$
\exp \left(\mathrm{i} \varphi_{0}\right)= \pm \frac{\varepsilon}{\sqrt{2-\varepsilon^{2}}}
$$

and therefore

$$
\zeta_{01} \equiv \zeta\left(\varphi_{0}\right)= \pm \frac{b \varepsilon}{\sqrt{1-\varepsilon^{2}}}= \pm a \varepsilon= \pm f
$$

(where $2 f$ is the interfocal distance), implying that the wave field singularities inside the ellipse lie at its focuses. A singularity at the origin corresponds to the second of Eqns (4a). It can be shown (see Refs $[2,3]$ ) that the wave field has second-order branch points at the focuses of the ellipse. To define a single-valued branch, a cut must be made connecting the points $\pm f$.

In three dimensions, with the spheroid-shape scatterer, the singularities lie at the focuses of the axial cross section of the spheroid.
3.1.2 Cassini oval. As the second example, let $S$ be a Cassini oval,

$$
\rho(\varphi)=a \sqrt{1+\varepsilon^{2} \cos ^{2} \varphi}, \quad \varepsilon<1
$$

The first of Eqns (4a) can then be written as

$$
\varepsilon^{2} \cos ^{2} \varphi_{0}+\mathrm{i} \varepsilon^{2} \cos \varphi_{0} \sin \varphi_{0}+1=0
$$

solving which yields

$$
\exp \left(\mathrm{i} \varphi_{0}\right)= \pm \mathrm{i} \frac{\sqrt{2+\varepsilon^{2}}}{\varepsilon}
$$

whence

$$
\zeta_{02}= \pm \mathrm{i} \frac{a}{\varepsilon} \sqrt{1+\varepsilon^{2}}
$$



Figure 2. Cassini oval.
and therefore

$$
\left|\zeta_{02}\right|=\frac{a}{\varepsilon} \sqrt{1+\varepsilon^{2}}>\rho\left( \pm \frac{\pi}{2}\right)=a, \quad \arg \zeta_{02}= \pm \frac{\pi}{2}
$$

Hence, the points $\zeta_{02}$ lie outside the boundary S (Fig. 2).
Solving the second of Eqns (4a) yields

$$
\zeta\left(\varphi_{0}\right)= \pm \frac{a \varepsilon}{2}=\zeta_{01}
$$

These singular points lie inside S because $\left|\zeta_{01}\right|=$ $a \varepsilon / 2<\rho(0)=\rho(\pi)=a \sqrt{1+\varepsilon^{2}}$.
3.1.3 Multifoil. We now consider the case where the boundary S is a multifoil whose equation is

$$
\begin{equation*}
\rho(\varphi)=a(1+\tau \cos q \varphi), \quad 0 \leqslant \tau<1, \quad q=1,2,3, \ldots \tag{9}
\end{equation*}
$$

I. The case $q=1$ has to be treated separately. Curve (9) is in this case called Pascal's limaçon. As in Section 3.1.2, both Eqns (4a) should be considered. Solving the second of these gives

$$
\zeta\left(\varphi_{0}\right)=\frac{a \tau}{2}=\zeta_{01} .
$$

This point is readily seen to lie inside $S$ (Fig. 3).
The first of Eqns (4a) now becomes

$$
\rho^{\prime}\left(\varphi_{0}\right)+\mathrm{i} \rho\left(\varphi_{0}\right)=-a \tau \sin \varphi_{0}+\mathrm{i} a+\mathrm{i} a \tau \cos \varphi_{0}=0,
$$



Figure 3. Pascal's limaçon.
and has the solution $\exp \left(\mathrm{i} \varphi_{0}\right)=-1 / \tau$, whence

$$
\begin{aligned}
\zeta_{02} & \equiv \rho\left(\varphi_{0}\right) \exp \left(\mathrm{i} \varphi_{0}\right) \\
& =a\left\{1+\frac{\tau}{2}\left[\exp \left(\mathrm{i} \varphi_{0}\right)+\exp \left(-\mathrm{i} \varphi_{0}\right)\right]\right\} \exp \left(\mathrm{i} \varphi_{0}\right) \\
& =-\frac{a}{2 \tau}\left(1-\tau^{2}\right) .
\end{aligned}
$$

Thus, $\left|\zeta_{02}\right|=(a / 2 \tau)\left(1-\tau^{2}\right), \arg \zeta_{02}=\pi$, and hence the point $\zeta_{02}$ lies outside S (see Fig. 3).
II. Turning now to the cases $q=2,3, \ldots$, the first of Eqns (4a) becomes

$$
\rho^{\prime}\left(\varphi_{0}\right)+\mathrm{i} \rho\left(\varphi_{0}\right)=a\left[-q \tau \sin \left(q \varphi_{0}\right)+\mathrm{i}+\mathrm{i} \tau \cos \left(q \varphi_{0}\right)\right]=0 .
$$

Setting $\exp \left(\mathrm{i} \varphi_{0}\right)=t$, we have the following set of solutions (Ref. [3], Ch. II):
$t_{1 m}=\left[\frac{-1+\sqrt{1+\tau^{2}\left(q^{2}-1\right)}}{\tau(q+1)}\right]^{1 / q} \exp \left(\mathrm{i} \frac{2 m \pi}{q}\right)$,
$t_{2 m}=\left[\frac{1+\sqrt{1+\tau^{2}\left(q^{2}-1\right)}}{\tau(q+1)}\right]^{1 / q} \exp \left(\mathrm{i} \frac{(2 m+1) \pi}{q}\right)$,

$$
\begin{equation*}
m=0,1, \ldots, q-1 \tag{11}
\end{equation*}
$$

Now, noting that $\zeta=\rho(\varphi) \exp (\mathrm{i} \varphi)$ [see Eqn (3)] and using solutions (10) and (11), we find that the singularities of the analytic continuation of the wave field into the multifoil are located at points that are at the distance

$$
\begin{align*}
\left|\zeta_{01}\right| & =a \frac{q\left[q+\sqrt{1+\tau^{2}\left(q^{2}-1\right)}\right]}{q^{2}-1} \\
& \times\left[\frac{-1+\sqrt{1+\tau^{2}\left(q^{2}-1\right)}}{\tau(q+1)}\right]^{1 / q} \tag{12}
\end{align*}
$$

from the origin on rays drawn at the angles

$$
\begin{equation*}
\arg \zeta_{01}=\frac{2 m \pi}{q}, \quad m=0,1, \ldots, q-1 \tag{13}
\end{equation*}
$$

(see Fig. 4 in which $\zeta_{02}$ denotes the singularities due to the continuation of the inner field to the outer region). The roots


Figure 4. Trifolium.
of the equation $\exp (\mathrm{i} \varphi)=0$ are mapped to the infinity in the $z$ plane. We note that, for example, at singularities inside the scatterer [see Eqns (12) and (13)], the field has a $q$ th-order analytic branching [3, Ch. II]. More precisely, the character of the singularity in the neighborhood of the $m$ th singular point ( $m=0,1, \ldots, q-1$ ) has the form

$$
\begin{equation*}
\frac{1}{\left[r \exp (\mathrm{i} \varphi)-r_{0 m} \exp \left(\mathrm{i} \varphi_{0 m}\right)\right]^{1-1 / q}}, \tag{14}
\end{equation*}
$$

where $r$ and $\varphi$ are polar coordinates of a point in the neighborhood of the $m$ th singular point with coordinates $r_{0 m}=\left|\zeta_{01}\right|$ and $\varphi_{0 m}=\arg \zeta_{01}$.

It follows from Eqn (14) that, in particular, as $q \rightarrow \infty$, the character of the singularity becomes close to the singularity at a first-order pole. It is also seen [see Eqn (12)] that as $\tau \rightarrow 0$, the singular points tend to the origin. Thus, passing to a circle (in three dimensions, to a sphere), i.e., simultaneously letting $q$ tend to infinity, $q \rightarrow \infty$, and $\tau$ to zero, $\tau \rightarrow 0$, gives rise to a single singularity at the center, a singularity which is an infinite-order pole and hence an essential singularity.

In other words, the field scattered by a circular cylinder (sphere) can be regarded as being produced by a number of sources occupying an arbitrarily small volume containing the origin within it (see also representation (8) in light of the above) or a single source - an infinite-order multipole - at the center. The question naturally arises as to why, for example, we still see a sphere rather than a luminous point at its center. An answer to this question can be attempted only after considering the second set of singularities, the maps of the outer sources (singular points).

### 3.2 Singularities of source images

We can use relations (5) and (6) to find the coordinates of these singularities. We again consider examples of their localization for various scatterer geometries.
3.2.1 Ellipse. Equation (5) here becomes

$$
\frac{b \exp (-\mathrm{i} \hat{\varphi})}{\sqrt{1-\varepsilon^{2} \cos ^{2} \hat{\varphi}}}=r_{0} \exp \left(-\mathrm{i} \varphi_{0}\right) \equiv z^{-}
$$

i.e.,
$4 b^{2} \exp (-\mathrm{i} 2 \hat{\varphi})=\left(z^{-}\right)^{2}\left\{4-\varepsilon^{2}[\exp (\mathrm{i} 2 \hat{\varphi})+\exp (-\mathrm{i} 2 \hat{\varphi})+2]\right\}$.
Solving this equation, we find

$$
\begin{aligned}
\exp (-\mathrm{i} 2 \hat{\varphi}) & =\left[\left(2-\varepsilon^{2}\right)+2 \sqrt{\frac{b^{2}}{\left(z^{-}\right)^{2}}-\frac{b^{2}}{r_{0}^{2}} \varepsilon^{2} \exp \left(\mathrm{i} 2 \varphi_{0}\right)}\right] \\
& \times\left[\varepsilon^{2}+4 \frac{b^{2}}{r_{0}^{2}} \exp \left(\mathrm{i} 2 \varphi_{0}\right)\right]^{-1} .
\end{aligned}
$$

Hence,

$$
\begin{align*}
\zeta_{\text {sing }} & =r_{0} \exp \left(-\mathrm{i} \varphi_{0}\right) \exp (\mathrm{i} 2 \hat{\varphi}) \\
& =r_{0}\left[\varepsilon^{2} \exp \left(-\mathrm{i} \varphi_{0}\right)+\left(\frac{2 b}{r_{0}}\right)^{2} \exp \left(\mathrm{i} \varphi_{0}\right)\right] \\
& \times\left[\left(2-\varepsilon^{2}\right)+2 \sqrt{\frac{b^{2}}{a^{2}}-\frac{b^{2}}{r_{0}^{2}} \varepsilon^{2} \exp \left(\mathrm{i} 2 \varphi_{0}\right)}\right]^{-1} . \tag{15}
\end{align*}
$$



Figure 5. Image in a circle and in a sphere.

As seen from Eqn (15), the point at infinity maps to the origin.
A. Special case $\varepsilon=0$ (circle or sphere). Here, it follows from Eqn (15)

$$
\zeta_{\text {sing }}=\frac{b^{2}}{r_{0}} \exp \left(\mathrm{i} \varphi_{0}\right)
$$

the map is at the point of inversion, with the product of the distances from the center of the circle (sphere) to the source and to the image equal to the squared radius of this circle (Fig. 5).
B. We let $r_{0}=b+\delta, \varphi_{0}=\pi / 2$, and $\delta \ll b$, assuming that the source is close to the boundary. Keeping terms of the order $\delta$ and neglecting terms of the order $\delta^{2}$, we obtain

$$
\begin{aligned}
\zeta_{\text {sing }} & \approx \mathrm{i} r_{0} \frac{4(1-2 \delta / b)-\varepsilon^{2}}{\left(2-\varepsilon^{2}\right)+2 \sqrt{(b / a)^{2}+\varepsilon^{2}(1-2 \delta / b)}} \\
& \approx \mathrm{i} r_{0} \frac{4-\varepsilon^{2}-8 \delta / b}{4-\varepsilon^{2}-2 \varepsilon^{2} \delta / b} \approx \mathrm{i}\left(r_{0}-2 \delta\right)
\end{aligned}
$$

Hence, approximately [up to $(\delta / b)^{2}$ ], the image is near the 'mirror' point, as it should be in accordance with what our visual experience suggests. As the source moves away along the straight line $\varphi_{0}=\pi / 2$, its image, as seen from Eqn (15), moves toward the interfocal cut until, at $r_{0}=2 a / \varepsilon$, it reaches the point $r=0$ on the cut. As $r_{0}$ is increased further, the source image intersects the cut (because it appears at $\varphi_{0}=-\pi / 2$ ), and thus appears on the nonphysical sheet of the Riemann surface. Hence, it turns out that for $r_{0}>2 a / \varepsilon$, as for the plane-wave excitation of an ellipse, the set of diffraction field singularities is represented only by the interfocal segment.

It is important to note that relations (5) and (6) allow finding not only the coordinates of the point source image but also the coordinates of the singular points 'generated' by the wave field singularities inside a nearby body, for example, in the situation shown in Fig. 6.
3.2.2 Pascal's limaçon. We recall the equation for Pascal's limaçon, $\rho(\varphi)=a+b \cos \varphi, 0 \leqslant b<a$. In this case, Eqn (5) takes the form

$$
(a+b \cos \hat{\varphi}) \exp (-\mathrm{i} \hat{\varphi})=r_{0} \exp \left(-\mathrm{i} \varphi_{0}\right) \equiv z^{-}
$$

or
$\exp (-\mathrm{i} 2 \hat{\varphi})+\frac{2 a}{b} \exp (-\mathrm{i} \hat{\varphi})+\left(1-\frac{2 z^{-}}{b}\right)=0$,


Figure 6. Superellipse near an ellipse.
whence

$$
\exp (-\mathrm{i} \hat{\varphi})=\frac{-a \pm \sqrt{a^{2}-b^{2}+2 b z^{-}}}{b}
$$

i.e.,

$$
\exp (\mathrm{i} \hat{\varphi})=\frac{b}{-a+a \sqrt{1-\left(b^{2}-2 b z^{-}\right) / a^{2}}}
$$

Here, we ignore the solution with the minus sign in front of the square root because it does not pass into the solution for a circle at $b=0: \exp (\mathrm{i} \hat{\varphi})=\left(a / r_{0}\right) \exp \left(\mathrm{i} \varphi_{0}\right)$.

Thus,

$$
\begin{aligned}
\zeta_{\text {sing }} & \equiv r_{0} \exp \left(-\mathrm{i} \varphi_{0}\right) \exp (\mathrm{i} 2 \hat{\varphi}) \\
& =r_{0} b^{2} \exp \left(-\mathrm{i} \varphi_{0}\right)\left[2 a^{2}-b^{2}+2 r_{0} b \exp \left(-\mathrm{i} \varphi_{0}\right)\right. \\
& \left.-2 a \sqrt{a^{2}-b^{2}+2 r_{0} b \exp \left(-\mathrm{i} \varphi_{0}\right)}\right]^{-1}
\end{aligned}
$$

To verify the solution, we set $r_{0}=a+b+\delta, \varphi_{0}=0$, and $\delta \ll a+b$ and again keep terms of the order $\delta$ and neglect terms of the order $\delta^{2}$. We obtain

$$
\begin{aligned}
\zeta_{\text {sing }} & \approx \frac{b^{2}(a+b+\delta)}{a^{2}+(a+b)^{2}+2 b \delta-2 a(a+b)\left(1+b \delta /(a+b)^{2}\right)} \\
& \approx(a+b) \frac{a+b+\delta}{a+b+2 \delta}
\end{aligned}
$$

or finally

$$
\zeta_{\mathrm{sing}} \approx a+b-\delta
$$

As in the case of an ellipse, the image lies at the mirror point.
From the examples given, we see that the image of a point source (singular point) can be sought using a small parameter expansion.
3.2.3 Parabola. As the last example, we consider the case where a parabolic cylinder mirror is excited by a point-like source (more precisely, a current filament) $U^{0}(\mathbf{r})=$ $(1 / 4 \mathrm{i}) H_{0}^{(2)}\left(k\left|\mathbf{r}-\mathbf{r}_{0}\right|\right)$ (Ref. [3], Ch. II). This problem is of applied interest in the theory of antennas, etc.

We assume that the point-like source field is incident on the parabola

$$
\begin{equation*}
y=a-b x^{2} . \tag{16}
\end{equation*}
$$

The problem is conveniently treated in Cartesian coordinates. Introducing the complex variable

$$
\begin{equation*}
\zeta=x+\mathrm{i} y(x), \tag{17}
\end{equation*}
$$

we can find the points where the bijectivity of mapping (17) is violated from the equation

$$
\begin{equation*}
\zeta^{\prime}(x) \equiv 1+\mathrm{i} y^{\prime}(x)=0 . \tag{18}
\end{equation*}
$$

In the case of the parabola in Eqn (16), Eqn (18) becomes

$$
1-\mathrm{i} 2 b x=0
$$

and has the solution

$$
x_{0}=\frac{1}{2 b \mathrm{i}} .
$$

In the complex-z plane, the singular point

$$
\zeta_{0}=\mathrm{i}\left(a-\frac{1}{4 b}\right) .
$$

corresponds to this solution. From this point, we should draw a cut to the point at infinity. The point with coordinates

$$
\begin{equation*}
x_{0}=0, \quad y_{0}=a-\frac{1}{4 b} \tag{19}
\end{equation*}
$$

corresponds to the singular point $\zeta_{0}$ in the real $x y$ plane, which is the focus of the parabola.

We now find the coordinates of the point source image. The discussion is entirely similar to the one above.

We again consider the expression (now in Cartesian coordinates)

$$
\begin{aligned}
\left.\left|\mathbf{r}-\mathbf{r}_{0}\right|^{2}\right|_{\mathrm{S}} & =\left[(x+\mathrm{i} y(x))-\left(x_{0}+\mathrm{i} y_{0}\right)\right] \\
& \times\left[(x-\mathrm{i} y(x))-\left(x_{0}-\mathrm{i} y_{0}\right)\right] .
\end{aligned}
$$

It follows that the singularities of $U^{0}(\mathbf{r})$ occur at

$$
\hat{x} \pm \mathrm{i} y(\hat{x})=x_{0} \pm \mathrm{i} y_{0} \equiv z_{0}^{ \pm} .
$$

The point $z_{0}=x_{0}+\mathrm{i} y_{0}$, which is the point of location of the source and hence is of no interest, corresponds to the equation $\zeta \equiv \hat{x}+\mathrm{i} y(\hat{x})=z_{0}^{+}$in the $z=x+\mathrm{i} y$ plane. We consider the point for which

$$
\begin{equation*}
\hat{x}-\mathrm{i} y(\hat{x})=z_{0}^{-}, \tag{20}
\end{equation*}
$$

where $\hat{x}=x_{1}+\mathrm{i} x_{2}$ is a complex quantity. Equation (20) is equivalent to

$$
\begin{equation*}
2 \hat{x}\left(\zeta_{\text {sing }}\right)-\zeta_{\text {sing }}=z_{0}^{-}, \tag{21}
\end{equation*}
$$

where $\hat{x}(\zeta)$ is the solution of the equation

$$
\begin{equation*}
\hat{x}+\mathrm{i} y(\hat{x})=\zeta, \tag{22}
\end{equation*}
$$

defined everywhere except the points where Eqn (18) holds. Equation (20) can be written as

$$
\hat{x}-\mathrm{i} a+\mathrm{i} b \hat{x}^{2}=z_{0}^{-},
$$

and its solution has the form

$$
\hat{x}\left(z_{0}^{-}\right)=\frac{\mathrm{i} \pm \mathrm{i} \sqrt{1-4 a b+4 \mathrm{i} b z_{0}^{-}}}{2 b}
$$

where the minus-sign solution is redundant because it becomes wrong at $b=0$. Thus [see Eqn (21)],

$$
\begin{align*}
\zeta_{\text {sing }} & \equiv 2 \hat{x}\left(z_{0}^{-}\right)-z_{0}^{-}=\frac{\mathrm{i}-\mathrm{i} \sqrt{1-4 a b+4 \mathrm{i} b z_{0}^{-}}}{b}-z_{0}^{-} \\
& =\frac{\mathrm{i}-b\left(x_{0}-\mathrm{i} y_{0}\right)-\mathrm{i} \sqrt{1-4 a b+4 \mathrm{i} b\left(x_{0}-\mathrm{i} y_{0}\right)}}{b} \tag{23}
\end{align*}
$$

Taking the real and imaginary parts of relation (23), we obtain the coordinates of the point source image in a paraboloidal mirror.

We now investigate some limit cases (Ref. [3], Ch. II). First, let $|a b| \ll 1$ and $\left|b z_{0}\right| \ll 1$ (a weakly curved surface). We then see from Eqn (23) that $\zeta_{\text {sing }}=\left(x_{0}-\mathrm{i} y_{0}\right)+\mathrm{i} 2 a$. Thus, in the $x y$ plane, the source image is at the point with the coordinates

$$
\begin{equation*}
x_{\mathrm{sing}}=x_{0}, \quad y_{\mathrm{sing}}=-y_{0}+2 a \tag{24}
\end{equation*}
$$

i.e., at the same place as for reflection from the $y=a$ plane. Similarly, for $\left|b x_{0}\right| \ll 1$ and $\left|b y_{0}\right| \ll 1$ (i.e., in the case of a near-surface source), the image is again at the point with coordinates (24).

By comparing this result with those described above when considering compact scatterers, it is possible to formulate the following locality principle: if the source is located near a convex curved surface at a distance that is small compared to the curvature radius of the surface at the point considered, then its image is located at (approximately) the same point as in the case of reflection from the tangent surface.

## 4. Conclusion

We can now attempt to answer the question raised in Section 3.1: why do we see a sphere rather than a luminous point at the center of the sphere? From the examples we have given in Section 2 and 3, it is clear that if a perfectly polished sphere (polished, of course, to a scale exceeding the resolving capability of the human eye) is placed in the field of a plane wave (for example, if it is illuminated by a laser beam) in a room with absorbing walls, we indeed see nothing but a luminous point at the center of the sphere. The brightness of the point is determined by the properties of the sphere surface; for example, an ideally polished sphere made of an absorbing material would be perceived as a dim point. In a usual setting, for example, we see the same sphere surrounded by many other objects that are imaged in it and, by the totality of these images, form a visual image that we perceive as a sphere. If the sphere surface is rough, this means, in fact, that we are dealing with an object of a nearly spherical shape with many singular points of the surface that serve as sources of scattered field.

By extending the above discussion to nonspherical objects, it is possible to understand in general terms why is we see what we see. Indeed, for a mirror surface body, the sources of the light it scatters are inside it. This is the reason why such an object is recognized the better, the more other objects are close to it and imaged in it. Conversely, if the surface of the object is rough, the sources of the scattered field lie in the immediate neighborhood of the surface, dramati-
cally simplifying the problem of recognizing the shape of the object.

Thus, we arrive at the conclusion that for a scatterer to have the property of invisibility, i.e., to be poorly recognizable, its shape should be as close as possible to a perfect analytic surface and be free of irregularities (angles, bulges, etc.), and its surface should have absorbing properties [2, 4].

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