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Killing vector fields and a homogeneous isotropic universe

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Contents

Abstract. Some basic theorems on Killing vector fields are reviewed. In particular, the topic of a constant-curvature space is examined. A detailed proof is given for a theorem describing the most general form of the metric of a homogeneous isotropic space–time. Although this theorem can be considered to be commonly known, its complete proof is difficult to find in the literature. An example metric is presented such that all its spatial cross sections correspond to constant-curvature spaces, but it is not homogeneous and isotropic as a whole. An equivalent definition of a homogeneous isotropic space–time in geometric terms of embedded manifolds is also given.

Keywords: Killing vector field, homogeneous universe, isotropic universe, Friedmann metric

1. Introduction

We first introduce our conventions and definitions, and then formulate the theorem.

Definition. *A space–time is a pair* $(\mathbb{M}, g)$, *where* $\mathbb{M}$ *is a four-dimensional differential manifold and* g *is a metric of the Lorentzian signature* $(+ - - -)$ *defined on it.*

We assume that both the manifold $\mathbb{M}$ and the metric g are sufficiently smooth. Furthermore, we assume that the manifold is geodesically complete, i.e., any geodesic can be extended in both directions beyond any value of the canonical parameter. In general relativity, a space–time manifold is commonly geodesically incomplete due to singularities of solutions of the Einstein equations. In this paper, we do not consider any dynamical equations, focusing on the kinematical aspect only. Therefore, our assumption on the geodesic completeness is quite natural and, for example, does not allow considering only a part of the whole sphere.

If the null coordinate line in local coordinates x^α, $\alpha = 0, 1, 2, 3$, is time-like, $(\partial_0, \partial_0) = g_{00} > 0$, where the parentheses denote the scalar product, then the coordinate $x^0 := t$ is called time. Spatial indices are denoted by Greek letters from the middle of the alphabet: $\mu, \nu, \ldots = 1, 2, 3$. Then $\{x^\alpha\} = \{x^0, x^\mu\}$.

The modern observational data indicate that our Universe is homogeneous and isotropic (the cosmological principle), at least in the first approximation. Most cosmological models rely on the following statement.

Theorem 1.1. *Let a four-dimensional manifold be the topological product* $\mathbb{M} = \mathbb{R} \times \mathbb{S}$, *where* $t \in \mathbb{R}$ *is the time coordinate and* $x \in \mathbb{S}$ *is a three-dimensional space of constant curvature. We suppose that* $\mathbb{M}$ *is endowed with a sufficiently smooth metric of the Lorentzian signature. It follows that if the space–time is homogeneous and isotropic, then a coordinate system* t, x^μ, $\mu = 1, 2, 3$ *exists in the neighborhood of each point such that the metric takes the form*

$$ds^2 = dt^2 + a^2 \overset{\circ}{g}_{\mu\nu} dx^\mu dx^\nu, \tag{1}$$

where $a(t) > 0$ *is an arbitrary function of time* (*the scale factor*) *and* $\overset{\circ}{g}_{\mu\nu}(x)$ *is a negative-definite constant-curvature metric on* $\mathbb{S}$ *depending on the spatial coordinates* $x \in \mathbb{S}$ *only*.

Thus, the most general metric of a homogeneous and isotropic universe has form (1) up to a coordinate transformation.

The theorem is independent of the dimension of the manifold $\mathbb{M}$ and the signature of the metric g. The first condition of the theorem can be replaced by the following: "Let any constant-time slice $t \in \mathbb{R}$ of the space–time $\mathbb{M}$ be a space of constant curvature." An exact definition of a homogeneous and isotropic universe is given in Section 3.

Theorem 1.1 is fundamental in relativistic cosmology and is therefore very important. Metric (1) was originally considered in [1–11]. We make a few comments on those parts of the original papers that are related to the form of the metric.

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Friedmann pioneered the use of metric (1) in cosmological models of general relativity [1, 2]. He did not write about a homogeneous and isotropic universe, but simply required that all constant-time spatial cross sections $t = \text{const}$ be constant-curvature spaces, and assumed the metric to have form (1). In his first and second papers, Friedmann considered spatial cross sections of positive and negative curvatures.

Lemaître analyzed solutions of the Einstein equations that describe a closed universe [3]. However, he did not formulate Theorem 1.1. A more general class of cosmological models was considered in [4], but nevertheless the theorem was not yet formulated.

Robertson gave the theorem in [5, 6], but did not provide a proof. Instead, he cited papers [12, 13]. The proof consists of two parts. The first (Theorem 5.1 in Section 5) was proved in general by Hilbert [12]. The second part (Theorem 5.2 in Section 5) was proved in one direction by Fubini [13] (see also [14], Chapter IV, Exercise 3). Namely, Fubini showed that metric (1) is homogeneous and isotropic, but the converse proposition that *any* homogeneous and isotropic metric has this form was not proved. In [7], metric (1) was derived in a different way by considering observers possessing certain properties. By construction, the resulting metric was homogeneous and isotropic. However, Robertson proposed (see the discussion above Eqn (2.1) in [7]) that the spatial part of the metric describes a constant-curvature space where the curvature can take only the discrete values $\pm 1, 0$, and therefore the general form of metric (1) was not shown. Metric (49) below fits the construction but is not of form (1).

Tolman obtained linear element (1) on a different basis [8–10]. In particular, he assumed that there is a spherical symmetry and that the time-coordinate lines are geodesics. In addition, he required that the Einstein equations be satisfied. Homogeneity and isotropy were not discussed in his papers.

In [11, Section 10], Walker proved Theorem 1.1 in one direction: metric (1) is homogeneous and isotropic. However, he did not prove that *any* homogeneous and isotropic metric is of this form. In fact, metric (49) given in Section 6 satisfies Eqn (52) in [11], but is not of form (1).

Furthermore, I browsed more than 30 monographs on the general theory of relativity, including my favorite books [15–19] and found a proof of Theorem 1.1 only in [19].

The aim of this paper is to explicitly review the proof of Theorem 1.1. In Sections 2–4, we briefly review the general properties of the Killing vector fields and necessary features of constant-curvature Riemannian (pseudo-Riemannian) spaces. The proof of Theorem 1.1 is given by two theorems, 5.1 and 5.2. The main idea behind the proof is borrowed from [16], but the details are different. In particular, the proof of Theorem 5.1 given in [14] is simpler. In Section 6, we describe an example metric all of whose spatial cross sections are constant-curvature spaces, but nevertheless the total metric is not homogeneous and isotropic. Also, a new equivalent definition of a homogeneous and isotropic metric is given in this section.

We hope that the reader is familiar with the basic concepts of differential geometry, which can be found, e.g., in [20, 21].

2. Killing vector fields

We consider an n-dimensional Riemannian (pseudo-Riemannian) manifold $(\mathbb{M}, g)$ endowed with a metric $g(x) = g_{\alpha\beta}(x)\,dx^\alpha \otimes dx^\beta$, $\alpha, \beta = 0, 1, \ldots, n-1$ and the corresponding Levi-Civita connection Γ.

Definition. *A diffeomorphism*

$$\imath : \quad \mathbb{M} \ni \quad x \mapsto x' = \imath(x) \quad \in \mathbb{M}$$

is called an isometry of a Riemannian (pseudo-Riemannian) manifold $(\mathbb{M}, g)$ *if the metric remains invariant,*

$$g(x) = \imath^* g(x') \,, \tag{2}$$

where $\imath^*$ *is the induced map of differential forms.*

Because an isometry leaves the metric invariant, it follows that all other structures expressed in terms of the metric, like the Levi-Civita connection, the geodesics, and the curvature tensor, are also invariant.

Map (2) can be represented in coordinate form. Let points x and x' lie in one coordinate neighborhood, and have coordinates x^α and x'^α. Then the isometry $\imath$ of the form

$$g_{\alpha\beta}(x) = \frac{\partial x'^\gamma}{\partial x^\alpha} \frac{\partial x'^\delta}{\partial x^\beta}\, g_{\gamma\delta}(x') \tag{3}$$

relates the metric components at different points of the manifold.

Proposition 2.1. *All isometries of a given Riemannian (pseudo-Riemannian) manifold* $(\mathbb{M}, g)$ *form an isometry group denoted by* $\mathbb{I}(\mathbb{M}) \ni \imath$.

Proof. The composition of two isometries is an isometry. The product of isometries is associative. The identical map of $\mathbb{M}$ is an isometry identified with the group unit. Every isometry has an inverse, which is also an isometry.

For a given metric, Eqn (3) defines functions $x'(x)$ that give an isometry. In general, this equation has no solutions and the corresponding manifold has no nontrivial isometries. In this case, the unit is a single element of the isometry group. The larger the isometry group is, the smaller the class of Riemannian (pseudo-Riemannian) manifolds.

Example 2.1. Euclidean space $\mathbb{R}^n$ endowed with the Euclidean metric $\delta_{\alpha\beta}$ has an isometry group that is the inhomogeneous rotation group $\mathbb{IO}(n, \mathbb{R})$, with $\dim \mathbb{IO}(n, \mathbb{R}) = n(n+1)/2$, consisting of rotations, translations, and reflections.

The isometry group $\mathbb{I}(\mathbb{M})$ can be either a discrete group or a Lie group.

Definition. *If the isometry group* $\mathbb{I}(\mathbb{M})$ *is a Lie group, we can consider infinitesimal transformations. In this case, we are dealing with infinitesimal isometries*:

$$x^\alpha \mapsto x'^\alpha = x^\alpha + \epsilon K^\alpha \,, \qquad \epsilon \ll 1 \,. \tag{4}$$

Any infinitesimal isometry is generated by a sufficiently smooth vector field $K(x) = K^\alpha(x)\partial_\alpha$, *which is called a Killing vector field.*

Let $K = K^\alpha \partial_\alpha$ be a Killing vector field. Then invariance condition (4) takes the infinitesimal form

$$\mathsf{L}_K g = 0 \,, \tag{5}$$

where L_K is the Lie derivative along the vector field K. The coordinate form is given by

$$\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0 \,, \tag{6}$$

where $K_\alpha := K^\beta g_{\beta\alpha}$ is a Killing 1-form, and the covariant derivative

$$\nabla_\alpha K_\beta := \partial_\alpha K_\beta - \Gamma_{\alpha\beta}{}^\gamma K_\gamma$$

is defined by the Christoffel symbols $\Gamma_{\alpha\beta}{}^{\gamma}$ (the Levi-Civita connection).

Definition. *Equation* (6) *is called the Killing equation and integral curves of a Killing vector field $K = K^{\alpha}\partial_{\alpha}$ are called Killing trajectories. Any Killing vector field is uniquely associated with the* 1*-form $K = \mathrm{d}x^{\alpha}K_{\alpha}$, where $K_{\alpha} := K^{\beta}g_{\beta\alpha}$, which is called a Killing form.*

For any Riemannian (pseudo-Riemannian) manifold $(\mathbb{M}, g)$, Killing equation (5) always has the trivial solution $K = 0$. If the equation has the zero solution only, there are no nontrivial continuous symmetries.

The Killing trajectories $\{x^{\alpha}(\tau)\} \in \mathbb{M}$ with $\tau \in \mathbb{R}$ are defined by the system of ordinary differential equations

$$\dot{x}^{\alpha} = K^{\alpha}\,, \tag{7}$$

which has a unique solution passing through a point $p = \{p^{\alpha}\} \in \mathbb{M}$ for any differentiable Killing vector field. For small $\tau \ll 1$, the trajectory has the form

$$x^{\alpha}(t) = p^{\alpha} + \tau K^{\alpha}(p) + o(\tau)\,, \tag{8}$$

where the integration constant is chosen such that the trajectory passes through the point p at $\tau = 0$.

Any Killing vector field generates a one-parameter subgroup of the isometry group. If a Killing vector field vanishes at some point, this point is stationary under the action of the isometry group generated by the vector field. Killing vector fields are called complete if Killing trajectories are defined for all $\tau \in \mathbb{R}$. They must have this property because the isometries form a group.

A given Killing vector field defines not only an infinitesimal symmetry but also the whole one-parameter isometry subgroup in $\mathbb{I}(\mathbb{M})$. For this, we have to find integral curves (Killing trajectories) passing through any point $p \in \mathbb{M}$. If $x(0) = p$, then there is a diffeomorphism

$$\imath: \quad \mathbb{M} \ni \quad p \mapsto x(t) \quad \in \mathbb{M}\,,$$

corresponding to each value $\tau \in \mathbb{R}$.

The contravariant component form of Killing equation (6) is given by

$$g_{\alpha\gamma}\partial_{\beta}K^{\gamma} + g_{\beta\gamma}\partial_{\alpha}K^{\gamma} + K^{\gamma}\partial_{\gamma}g_{\alpha\beta} = 0\,. \tag{9}$$

This equation is linear in both the Killing vector and the metric. It follows that any two metrics differing by a prefactor have the same Killing trajectories. Moreover, Killing vector fields are defined modulo an arbitrary nonzero constant prefactor. In particular, if K is a Killing vector, then $-K$ is also a Killing vector. If there are several Killing vector fields, their linear combination is again a Killing vector field. In other words, Killing vector fields form a vector space over real numbers, which is a subspace of the vector space of all vector fields $\mathcal{X}(\mathbb{M})$ on the manifold $\mathbb{M}$. This space is endowed with a bilinear form. It is easy to show that the commutator of two Killing vector fields K_1 and K_2 is another Killing vector field:

$$\mathsf{L}_{[K_1, K_2]}g = \mathsf{L}_{K_1} \circ \mathsf{L}_{K_2}g - \mathsf{L}_{K_2} \circ \mathsf{L}_{K_1}g = 0\,.$$

It follows that Killing vector fields form a Lie algebra $\mathfrak{i}(\mathbb{M})$ over real numbers, which is a subalgebra in the infinite-dimensional Lie algebra of all vector fields, $\mathfrak{i}(\mathbb{M}) \subset \mathcal{X}(\mathbb{M})$. This is the Lie algebra of the isometry Lie group $\mathbb{I}(\mathbb{M})$.

Proposition 2.2. *Let a Riemannian* (*pseudo-Riemannian*) *manifold* $(\mathbb{M}, g)$ *have $N \leqslant \dim\mathbb{M}$ nonvanishing commuting and linearly independent Killing vector fields K_i, $i = 1, \ldots, N$. A coordinate system then exists such that the metric is independent of N coordinates corresponding to Killing trajectories. The converse statement is that if the metric is independent of N coordinates in some coordinate system, then the metric has at least N nonvanishing commuting Killing vector fields.*

Proof. For any nonvanishing Killing vector field, there is a coordinate system where the field has components $(1, 0, \ldots, 0)$. For a set of independent commuting vector fields K_i, this implies that there is a coordinate system $x^1, \ldots, x^n$ such that each Killing vector field has just one nonvanishing component $K_i = \partial_i$. In this coordinate system, Killing equation (9) is particularly simple:

$$\partial_i g_{\alpha\beta} = 0\,, \qquad i = 1, \ldots, N \leqslant \dim\mathbb{M}\,. \tag{10}$$

This implies that the metric components are independent of the coordinates x^i.

In this coordinate system, the Killing trajectories are given by the equations

$$\dot{x}^i = 1\,, \qquad \dot{x}^{\mu} = 0, \qquad \mu \neq i\,.$$

We see that the coordinate lines x^i are Killing trajectories.

As regards the converse statement, if the metric components are independent of N coordinates, then Eqns (10) are satisfied. These are the Killing equations for commuting vector fields $K_i := \partial_i$.

It follows that in the limit case where the number of commuting Killing vectors is equal to the dimension of the manifold $N = n$, a coordinate system exists such that the corresponding metric components are constant.

Example 2.2. In the Euclidean space $\mathbb{R}^n$, the metric in Cartesian coordinates x^{α}, $\alpha = 1, \ldots, n$ has constant components $g_{\alpha\beta} = \delta_{\alpha\beta}$. This metric has n commuting Killing vectors $K_{\alpha} := \partial_{\alpha}$, which generate translations. All coordinate lines are Killing trajectories.

If a Riemannian manifold $(\mathbb{M}, g)$ has two or more noncommuting Killing vectors, this does not mean that there is a coordinate system such that the metric components are independent of two or more coordinates.

Example 2.3. We consider the two-dimensional sphere $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$ embedded into a three-dimensional Euclidean space in the standard manner. Let the metric g on the sphere be the induced metric. Then the Riemannian manifold $(\mathbb{S}^2, g)$ has three noncommuting Killing vector fields corresponding to the rotation group $\mathbb{SO}(3)$. It is obvious that there is no coordinate system where the metric components are independent of two coordinates. Indeed, in such a coordinate system, the metric components are constant and therefore the curvature is zero. But this is impossible because the curvature of the sphere is nonzero.

In general relativity, we suppose that the space–time is a pseudo-Riemannian manifold $(\mathbb{M}, g)$ endowed with a metric of the Lorentzian signature. Using the notion of a Killing vector field, we can give the following invariant definition.

Definition. *A space–time* $(\mathbb{M}, g)$ *or its domain is called stationary if there is a time-like Killing vector field.*

Killing vector fields have a number of remarkable features. We consider the simplest of them.

Proposition 2.3. *The length of a Killing vector along the Killing trajectory is constant*:

$$\mathsf{L}_K K^2 = \nabla_K K^2 = K^{\alpha}\,\partial_{\alpha}K^2 = 0\,. \tag{11}$$

Proof. Contracting Eqns (6) with $K^\alpha K^\beta$ yields the equalities

$$2K^\alpha K^\beta \nabla_\alpha K_\beta = K^\alpha \nabla_\alpha K^2 = K^\alpha \partial_\alpha K^2 = 0\,.$$

Corollary. *Killing vector fields on a Lorentzian manifold are oriented time-like, light-like, or space-like.*

A metric on the manifold defines two particular types of curves: geodesics (extremals) and Killing trajectories, if they exist.

Proposition 2.4. *Let* $(\mathbb{M}, g)$ *be a Riemannian* (*pseudo-Riemannian*) *manifold with a Killing vector field K. A Killing trajectory is geodesic if and only if the length of the Killing vector is constant on* $\mathbb{M}$: $K^2 = \text{const}$ *for all* $x \in \mathbb{M}$.

Proof. A Killing trajectory $x^\alpha(\tau)$ is given by the system of equations

$$\dot{x}^\alpha = K^\alpha\,. \tag{12}$$

The length of the infinitesimal interval of the Killing trajectory

$$\mathrm{d}s^2 = g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta\,\mathrm{d}\tau^2 = K^2\,\mathrm{d}\tau^2$$

is constant along the trajectory, i.e., the parameter τ is proportional to the length of the trajectory and is therefore canonical. Differentiating equation (12) with respect to the canonical parameter τ yields the relation

$$\ddot{x}^\alpha = \partial_\beta K^\alpha \dot{x}^\beta = (\nabla_\beta K^\alpha - \Gamma_{\beta\gamma}{}^\alpha K^\gamma)\dot{x}^\beta\,,$$

which can be represented as

$$\ddot{x}^\alpha = K^\beta \nabla_\beta K^\alpha - \Gamma_{\beta\gamma}{}^\alpha \dot{x}^\beta \dot{x}^\gamma\,. \tag{13}$$

Using the Killing equations, we can rewrite the first term in the right-hand side as

$$K^\beta \nabla_\beta K^\alpha = -\frac{1}{2}\,g^{\alpha\beta}\partial_\beta K^2\,.$$

Then Eqn (13) takes the form

$$\ddot{x}^\alpha = -\frac{1}{2}\,g^{\alpha\beta}\partial_\beta K^2 - \Gamma_{\beta\gamma}{}^\alpha \dot{x}^\beta \dot{x}^\gamma\,.$$

The last equation coincides with the geodesic equation if and only if $K^2 = \text{const}$.

It follows that Killing trajectories differ from the geodesics in general.

Example 2.4. We consider the Euclidean plane $\mathbb{R}^2$ endowed with the Euclidean metric. This metric is invariant under the three-parameter inhomogeneous rotation group $\mathbb{IO}(2)$. We let x, y and r, φ denote Cartesian and polar coordinates on the plane. Then the Killing vector fields corresponding to rotations and translations are $K_1 = \partial_\varphi$ and $K_2 = \partial_x$, $K_3 = \partial_y$. The squared vector norms are

$$K_1^2 = r^2\,, \qquad K_2^2 = K_3^2 = 1\,.$$

The Killing vector fields K_2 and K_3 have a constant length on the whole plane. Their trajectories are straight lines, which are geodesics. This agrees with Proposition 2.4. The Killing trajectories corresponding to rotations K_1 are concentric circles around the origin. In accordance with Proposition 2.3, the length of the Killing vector K_1 is constant along the circles, but nonconstant on the whole plane $\mathbb{R}^2$. The corresponding Killing trajectories are circles, which are not geodesics.

Example 2.5. We consider a semisimple Lie group $\mathbb{G}$ as a Riemannian (pseudo-Riemannian) manifold endowed with the Cartan–Killing form as an invariant metric. Then the left-invariant and right-invariant vector fields on $\mathbb{G}$ generate right and left group actions. Both left and right group actions leave the metric invariant. Therefore, the left- and right-invariant vector fields are Killing vector fields. Their length equals ± 1. Hence, the corresponding Killing trajectories are geodesics.

Contracting Killing equation (6) with the metric shows that the divergence of a Killing vector field is zero:

$$\nabla_\alpha K^\alpha = 0\,. \tag{14}$$

The covariant derivative ∇^β of Killing equation (6) takes the form

$$\nabla^\beta(\nabla_\beta K_\alpha + \nabla_\alpha K_\beta) = \Delta K_\alpha + (\nabla^\beta \nabla_\alpha - \nabla_\alpha \nabla^\beta)K_\beta = 0\,,$$

where we used relation (14), and where $\Delta := \nabla^\beta \nabla_\beta$ is the Laplace–Beltrami operator on the manifold $\mathbb{M}$. Using the equality

$$[\nabla_\alpha, \nabla_\beta]K_\gamma = -R_{\alpha\beta\gamma}{}^\delta K_\delta$$

for the commutator of covariant derivatives, we arrive at the following equation for the Killing vector:

$$\Delta K_\alpha = R_{\alpha\beta} K^\beta\,, \tag{15}$$

where $R_{\alpha\beta} := R_{\alpha\gamma\beta}{}^\gamma$ is the Ricci tensor.

In the case of a constant-curvature space, the Ricci tensor is proportional to the scalar curvature [see Eqn (26) below], and therefore Eqn (15) is simplified to

$$\Delta K_\alpha = \frac{R}{n}\,K_\alpha\,, \qquad R = \text{const}\,.$$

In other words, each component of the Killing vector is an eigenfunction of the Laplace–Beltrami operator.

Proposition 2.5. *Let* $X, Y \in \mathcal{X}(\mathbb{M})$ *be two arbitrary vector fields on a Riemannian* (*pseudo-Riemannian*) *manifold* $(\mathbb{M}, g)$ *and K be a Killing vector. Then the following equality holds*:

$$g\big((\mathsf{L}_K - \nabla_K)X, Y\big) + g\big(X, (\mathsf{L}_K - \nabla_K)Y\big) = 0\,,$$

where $\mathsf{L}_K X = [K, X]$ *is the Lie derivative and* $\nabla_K X = K^\alpha(\partial_\alpha X^\beta + \Gamma_{\alpha\gamma}{}^\beta X^\gamma)\partial_\beta$ *is the covariant derivative of a vector field X along the Killing vector field K.*

Proof. Direct verification using the Christoffel symbols and Killing equations (6).

3. Homogeneous and isotropic spaces

Killing equation (6) imposes severe restrictions on Killing vector fields, which we have to discuss. Using the formula for the commutator of covariant derivatives, we find the relation

$$\nabla_\alpha \nabla_\beta K_\gamma - \nabla_\beta \nabla_\alpha K_\gamma = -R_{\alpha\beta\gamma}{}^\delta K_\delta\,. \tag{16}$$

Then, using the identity

$$R_{\alpha\beta\gamma}{}^\delta + R_{\beta\gamma\alpha}{}^\delta + R_{\gamma\alpha\beta}{}^\delta = 0$$

for the curvature tensor and Killing equation (6), we find the equality

$$\nabla_\alpha\nabla_\beta K_\gamma + \nabla_\beta\nabla_\gamma K_\alpha + \nabla_\gamma\nabla_\alpha K_\beta = 0\,,$$

where the terms are related by cyclic permutations. Using this equality, we can represent (16) as

$$\nabla_\gamma\nabla_\alpha K_\beta = R_{\alpha\beta\gamma}{}^\delta K_\delta\,. \tag{17}$$

Contracting the indices γ and α, we obtain exactly equality (15) from Section 2.

Equation (17) follows from the Killing equations. However, they are not equivalent. Nevertheless, Eqn (17) has important consequences. We assume that Killing vector fields are real analytic functions, i.e., their components can be represented as Taylor series converging in some neighborhood $\mathbb{U}_p$ of a point $p \in \mathbb{M}$. We suppose that all components of the Killing 1-form $K_\alpha(p)$ and their first derivatives $\partial_\beta K_\alpha(p)$ are given at some fixed point $p \in \mathbb{M}$. Then the second partial derivatives of the Killing 1-form $\partial^2_{\beta\gamma}K_\alpha$ can be found from Eqn (17). Now, we evaluate the covariant derivative of Eqn (17), thereby obtaining some relation for the third derivatives, and so on, up to infinity. It is important that all the relations are linear in the Killing vector components and their derivatives. It follows that the Killing 1-form components in some neighborhood $\mathbb{U}_p$ are of the form

$$K_\alpha(x,p) = A_\alpha{}^\beta(x,p)K_\beta(p) + B_\alpha{}^{\beta\gamma}(x,p)\big[\partial_\beta K_\gamma(p) - \partial_\gamma K_\beta(p)\big]\,, \tag{18}$$

where $A_\alpha{}^\beta(x,p)$ and $B_\alpha{}^{\beta\gamma}(x,p)$ are some functions. The antisymmetry in the indices β and γ in the last term is achieved by expressing the symmetrized partial derivative in terms of the Killing vector components by means of Killing equation (6). Therefore, Killing 1-form components in some neighborhood $\mathbb{U}_p$ are linear combinations of the Killing form and their exterior derivative components at the point p.

The Killing form $K_\alpha(x,p)$ depends on two variables. The second variable p shows that the form has properties specified at the point $p \in \mathbb{M}$. By assumption, representation (18) holds at any point $p \in \mathbb{M}$: it is just necessary to know the values $K(p)$ and $\mathrm{d}K(p)$. We suppose that the functions $K_\alpha(x,p)$ are real analytic in both variables x and p.

It is assumed that the Killing form components can be expanded in Taylor series near any point $p \in \mathbb{M}$. Let $\mathbb{U}_p$ be a neighborhood of the point p where representation (18) holds and is invertible, i.e., the variables x and p can be replaced with some new functions A and B. We consider a point q outside $\mathbb{U}_p$. For this point, an invertible representation like (18) also holds in some neighborhood $\mathbb{U}_q$. We suppose that a point q lies close enough to $\mathbb{U}_p$ such that the neighborhoods overlap, $\mathbb{U}_p \cap \mathbb{U}_q \neq \emptyset$. Then, for any point belonging to the intersection $x \in \mathbb{U}_p \cap \mathbb{U}_q$, representation (18) holds with respect to the components of $K(p)$ and $K(q)$ and their exterior derivatives. We see that the Killing form and its exterior derivative at q can be linearly expressed in terms of their values at p. Therefore, representation (18) holds in the union $\mathbb{U}_p \cup \mathbb{U}_q$. This construction can be extended to the whole manifold $\mathbb{M}$. As a result, representation (18) holds for all points $x, p \in \mathbb{M}$.

We now assume that a Riemannian (pseudo-Riemannian) manifold $(\mathbb{M}, g)$ has several Killing vector fields K_i, $i = 1, \ldots, N$. Then representation (18) holds for each Killing vector:

$$K_{i\alpha}(x,p) = A_\alpha{}^\beta(x,p)K_{i\beta}(p) + B_\alpha{}^{\beta\gamma}(x,p)\big[\partial_\beta K_{i\gamma}(p) - \partial_\gamma K_{i\beta}(p)\big]\,. \tag{19}$$

The functions $A_\alpha{}^\beta(x,p)$ and $B_\alpha{}^{\beta\gamma}(x,p)$ are the same for any Killing form, because they are defined by relation (17), which is linear in the Killing form components and their derivatives. They are uniquely defined by the metric, the curvature, and its covariant derivatives. It is supposed that in the resulting representation, the point $p \in \mathbb{M}$ is arbitrary but fixed, while the point $x \in \mathbb{M}$ ranges the whole manifold $\mathbb{M}$.

Equality (17) is a system of partial differential equations for the Killing form components and has nontrivial integrability conditions. One of them has the covariant form

$$[\nabla_\gamma, \nabla_\delta]\nabla_\alpha K_\beta = -R_{\gamma\delta\alpha}{}^\epsilon\nabla_\epsilon K_\beta - R_{\gamma\delta\beta}{}^\epsilon\nabla_\alpha K_\epsilon\,,$$

where the square brackets denote the commutator of covariant derivatives. Substituting the initial equation (17) for the second derivatives in the left-hand side of this equation, by straightforward computation we find that

$$\begin{aligned}\big(R_{\alpha\beta\gamma}{}^\epsilon\delta^\zeta_\delta - R_{\alpha\beta\delta}{}^\epsilon\delta^\zeta_\gamma + R_{\gamma\delta\alpha}{}^\epsilon\delta^\zeta_\beta - R_{\gamma\delta\beta}{}^\epsilon\delta^\zeta_\alpha\big)\nabla_\zeta K_\epsilon \\ = (\nabla_\gamma R_{\alpha\beta\delta}{}^\epsilon - \nabla_\delta R_{\alpha\beta\gamma}{}^\epsilon)K_\epsilon\,.\end{aligned} \tag{20}$$

When the curvature is nontrivial, this equation is a linear relation between components of the Killing form K_α and their covariant derivatives $\nabla_\beta K_\alpha$. Conversely, if we know some properties of the Killing form, the resulting equality can determine the structure of the curvature tensor. In Theorem 3.1 in what follows, Eqn (20) is used to prove the statement that a homogeneous and isotropic manifold is a constant-curvature space.

Definition. *A Riemannian (pseudo-Riemannian) manifold* $(\mathbb{M}, g)$ *of dimension* $\dim\mathbb{M} = n$ *is called homogeneous at a point* $p \in \mathbb{M}$ *if there are infinitesimal isometries mapping this point to any other point in some neighborhood* $\mathbb{U}_p$ *of* p*. In other words, the metric should have Killing vector fields with arbitrary directions at* p*. Because Killing vectors form a linear space, it is necessary and sufficient to have a set of* n *Killing forms in the dual space* $K^{(\gamma)} = \mathrm{d}x^\alpha K_\alpha{}^{(\gamma)}(x,p)$*, where the index* γ *in parenthesis labels Killing forms, such that the following relations are satisfied*:

$$K_\alpha{}^{(\gamma)}(p,p) = \delta^\gamma_\alpha\,. \tag{21}$$

If a Riemannian (pseudo-Riemannian) space $(\mathbb{M}, g)$ *is homogeneous at any point* $x \in \mathbb{M}$*, it is called homogeneous. In other words, the isometry group acts on* $\mathbb{M}$ *transitively.*

A Riemannian (pseudo-Riemannian) manifold $(\mathbb{M}, g)$ *is called isotropic at a point* $p \in \mathbb{M}$ *if there are infinitesimal isometries with Killing forms* $K(x,p)$ *such that the given point is stable, i.e.,* $K(p,p) = 0$*, and the exterior derivative* $\mathrm{d}K(x,p)$ *at* p *takes all possible values in the space of 2-forms* $\Lambda_2(\mathbb{M})\big|_p$ *at* p*. This happens if and only if there is a set of* $n(n-1)/2$ *Killing forms* $K^{[\gamma\delta]} = -K^{[\delta\gamma]} = \mathrm{d}x^\alpha K_\alpha{}^{[\gamma\delta]}(x,p)$*, where the indices* γ*,* δ *label Killing forms, such that the following relations are satisfied*:

$$\begin{aligned}&K_\alpha{}^{[\gamma\delta]}(p,p) = 0\,,\\ &\left.\frac{\partial K_\beta{}^{[\gamma\delta]}(x,p)}{\partial x^\alpha}\right|_{x=p} = \delta^{\gamma\delta}_{\alpha\beta} - \delta^{\delta\gamma}_{\alpha\beta}\,.\end{aligned} \tag{22}$$

If a Riemannian (pseudo-Riemannian) manifold $(\mathbb{M}, g)$ *is isotropic at any point, it is called isotropic*.

By continuity, it follows that the forms $K^{(\gamma)}$ and $K^{[\gamma\delta]}$ are linearly independent in some neighborhood of p.

Proposition 3.1. *Any isotropic Riemannian (pseudo-Riemannian) manifold* $(\mathbb{M}, g)$ *is also homogeneous*.

Proof. If a manifold is isotropic, the Killing forms $K^{[\gamma,\delta]}(x,p)$ and $K^{[\gamma,\delta]}(x,p+\mathrm{d}p)$ satisfy Eqns (22) in some neighborhoods of the respective points p and $p+\mathrm{d}p$. Their arbitrary linear combination and therefore arbitrary linear combination of derivatives

$$c^\alpha \frac{\partial K_\beta{}^{[\gamma\delta]}(x,p)}{\partial p^\alpha} := c^\alpha \lim_{\mathrm{d}p^\alpha \to 0} \frac{K_\beta{}^{[\gamma,\delta]}(x,p+\mathrm{d}p) - K_\beta{}^{[\gamma,\delta]}(x,p)}{\mathrm{d}p^\alpha}$$

are Killing forms for arbitrary constants c^α. We differentiate the Killing form $K^{[\gamma\delta]}$ with respect to x at the point p. From the first relation in (22), it follows that

$$\frac{\partial}{\partial p^\alpha} K_\beta{}^{[\gamma\delta]}(p,p) = \left.\frac{\partial K_\beta{}^{[\gamma\delta]}(x,p)}{\partial x^\alpha}\right|_{x=p} + \left.\frac{\partial K_\beta{}^{[\gamma\delta]}(x,p)}{\partial p^\alpha}\right|_{x=p} = 0\,.$$

Using the second condition in (22), we obtain the equality

$$\left.\frac{\partial K_\beta{}^{[\gamma\delta]}(x,p)}{\partial p^\alpha}\right|_{x=p} = -\delta^{\gamma\delta}_{\alpha\beta} + \delta^{\delta\gamma}_{\alpha\beta}\,.$$

Now, from $K^{[\gamma\delta]}$ we can build Killing forms that take arbitrary values $\mathrm{d}x^\alpha a_\alpha$ at the point p, where $a_\alpha \in \mathbb{R}$. For this, it is sufficient to assume that

$$K_\alpha := \frac{a_\gamma}{n-1} \frac{\partial K_\alpha{}^{[\gamma\delta]}(x,p)}{\partial p^\delta}\,.$$

By choosing appropriate constants a_γ, we find a set of Killing forms satisfying equalities (21).

Thanks to the theorem, it suffices to use the term 'isotropic universe'. However, we prefer to call it 'homogeneous and isotropic', because this name emphasizes important physical properties.

Theorem 3.1. *The Lie algebra of infinitesimal isometries* $\mathfrak{i}(\mathbb{M})$ *of a connected Riemannian (pseudo-Riemannian) manifold* $\mathbb{M}$ *has the dimension not exceeding* $n(n+1)/2$, *where* $n := \dim \mathbb{M}$. *If the dimension is maximal,* $\dim \mathfrak{i}(\mathbb{M}) = n(n+1)/2$, *then the manifold* $\mathbb{M}$ *is homogeneous and isotropic, being a constant-curvature space*.

Proof. The dimension of the Lie algebra $\mathfrak{i}(\mathbb{M})$ is equal to the maximal number of linearly independent Killing vector fields on the manifold $\mathbb{M}$. From Eqn (19), it follows that the number N of linearly independent Killing vectors cannot exceed the number of independent components of Killing forms $\{K_\alpha(p)\}$ and their exterior derivatives $\{\partial_\beta K_\alpha(p) - \partial_\alpha K_\beta(p)\}$ at a fixed point $p \in \mathbb{M}$. The number of independent components of an arbitrary 1-form does not exceed n, and the number of independent components of its exterior derivative cannot exceed $n(n-1)/2$. Thus, we find a restriction on the dimension of the Lie algebra of isometries generated by Killing vector fields:

$$\dim \mathfrak{i}(\mathbb{M}) \leqslant n + \frac{1}{2}\, n(n-1) = \frac{1}{2}\, n(n+1)\,.$$

This proves the first statement of the theorem. The real analyticity is important here because it was used to obtain representation (19).

The connectedness of $\mathbb{M}$ guarantees that a number of independent Killing vector fields is defined unambiguously. If $\mathbb{M}$ has several connected components, the number of independent Killing vector fields may depend on a particular component.

There are at most $n(n+1)/2$ independent Killing vector fields on a homogeneous and isotropic manifold. By Eqn (19), they define all possible Killing vector fields on the manifold $\mathbb{M}$. Consequently, if a manifold has the maximal number of independent Killing fields, it is necessarily homogeneous and isotropic.

We now prove that any homogeneous and isotropic manifold is a constant-curvature space. If a manifold is homogeneous and isotropic, for any point $x \in \mathbb{M}$ there are Killing forms such that $K_\alpha(x) = 0$, while their derivatives $\nabla_\beta K_\alpha(x)$ can be arranged into an antisymmetric matrix. As a consequence, an antisymmetric coefficient at $\nabla_\zeta K_\epsilon$ in Eqn (20) must be zero. It follows that

$$\begin{aligned} &R_{\alpha\beta\gamma}{}^\epsilon \delta^\zeta_\delta - R_{\alpha\beta\delta}{}^\epsilon \delta^\zeta_\gamma + R_{\gamma\delta\alpha}{}^\epsilon \delta^\zeta_\beta - R_{\gamma\delta\beta}{}^\epsilon \delta^\zeta_\alpha \\ &\quad = R_{\alpha\beta\gamma}{}^\zeta \delta^\epsilon_\delta - R_{\alpha\beta\delta}{}^\zeta \delta^\epsilon_\gamma + R_{\gamma\delta\alpha}{}^\zeta \delta^\epsilon_\beta - R_{\gamma\delta\beta}{}^\zeta \delta^\epsilon_\alpha\,. \end{aligned} \tag{23}$$

If the space is homogeneous and isotropic, then for any point $x \in \mathbb{M}$ there are Killing forms taking arbitrary values at this point. From Eqns (20) and (23), it follows that

$$\nabla_\gamma R_{\alpha\beta\delta}{}^\epsilon = \nabla_\delta R_{\alpha\beta\gamma}{}^\epsilon\,. \tag{24}$$

In Eqn (23), we contract the indices δ and ζ and then lower the upper index. As a result, we express the curvature tensor in terms of the Ricci tensor and the metric:

$$(n-1) R_{\alpha\beta\gamma\delta} = R_{\beta\delta} g_{\alpha\gamma} - R_{\alpha\delta} g_{\beta\gamma}\,. \tag{25}$$

Because the right-hand side of (25) has to be antisymmetric in δ and γ, there is the additional restriction

$$R_{\beta\delta} g_{\alpha\gamma} - R_{\alpha\delta} g_{\beta\gamma} = -R_{\beta\gamma} g_{\alpha\delta} + R_{\alpha\gamma} g_{\beta\delta}\,.$$

Contracting the indices β and γ yields a relation between the Ricci tensor and the scalar curvature,

$$R_{\alpha\delta} = \frac{1}{n} R g_{\alpha\delta}\,, \tag{26}$$

where $R := g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature. Substituting the above relation in equality (25) results in the following expression for the full curvature tensor:

$$R_{\alpha\beta\gamma\delta} = \frac{R}{n(n-1)} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})\,. \tag{27}$$

Now, to complete the proof, we have to show that the scalar curvature R of a homogeneous and isotropic space is constant. For this, we use the contracted Bianchi identity

$$2\nabla_\beta R_\alpha{}^\beta - \nabla_\alpha R = 0\,.$$

Substituting formula (26) for the Ricci tensor in this identity yields the equation

$$\left(\frac{2}{n} - 1\right) \partial_\alpha R = 0\,.$$

For $n \geqslant 3$, it follows that $R = \mathrm{const}$.

The case $n = 2$ is to be considered separately. Contracting the indices β and ϵ in Eqn (25) yields the equality

$$\nabla_\gamma R_{\alpha\delta} - \nabla_\delta R_{\alpha\gamma} = 0 .$$

Then, contracting with $g^{\alpha\delta}$ and using relation (26) yields the equation $\partial_\gamma R = 0$, and hence $R = \text{const}$ also in the case $n = 2$.

Thus, the scalar curvature in (27) has to be constant, $R = \text{const}$, and therefore a maximally symmetric Riemannian (pseudo-Riemannian) manifold is a constant-curvature space.

Example 3.1. We consider the Euclidean space $\mathbb{R}^n$ with a zero-curvature metric, i.e., $R_{\alpha\beta\gamma\delta} = 0$. This space obviously has a constant curvature. It follows that there is a coordinate system x^α, $\alpha = 1, \ldots, n$ such that all metric components are constant. The Christoffel symbols in this coordinate system are zero. Equation (17) for Killing vector fields takes the simple form

$$\partial^2_{\beta\gamma} K_\alpha = 0 .$$

The general solution of this equation is linear in coordinates:

$$K_\alpha(x) = a_\alpha + b_{\alpha\beta} x^\beta ,$$

where a_α and $b_{\alpha\beta}$ are some constants. It follows from Killing equation (6) that this expression defines the Killing form if and only if the matrix $b_{\alpha\beta}$ is antisymmetric, i.e., $b_{\alpha\beta} = -b_{\beta\alpha}$. Therefore, we can define $n(n+1)/2$ linearly independent Killing forms:

$$K_\alpha{}^{(\gamma)}(x) = \delta^\gamma_\alpha ,$$
$$K_\alpha{}^{[\gamma\delta]}(x) = \delta^\delta_\alpha x^\gamma - \delta^\gamma_\alpha x^\delta .$$

Hence, an arbitrary Killing form is the linear combination

$$K_\alpha = a_\gamma K_\alpha{}^{(\gamma)} + \frac{1}{2}\, b_{\delta\gamma} K_\alpha{}^{[\gamma\delta]} .$$

Here, the n Killing vectors $K^{(\gamma)}$ generate translations along coordinate axes in $\mathbb{R}^n$, while the $n(n-1)/2$ Killing vectors $K^{[\gamma\delta]}$ generate rotations around the origin. Thus, a zero-curvature metric has the maximal number $n(n+1)/2$ of Killing vectors, and therefore the space is homogeneous and isotropic.

It is known that the metric can be diagonalized by linear coordinate transformations such that the main diagonal elements are ± 1, depending on the metric signature. If the metric is Riemannian (positive definite), it can be mapped into the the standard form $g_{\alpha\beta} = \delta_{\alpha\beta}$. This metric is invariant under the inhomogeneous rotation group $\mathbb{IO}(n)$.

We have proved that a homogeneous and isotropic space has constant curvature. The converse is also true. This can be formulated in several steps.

Theorem 3.2. *Let* $(\mathbb{M}, g)$ *be a Riemannian (pseudo-Riemannian) space of constant curvature with the curvature tensor given by* (27), *where* $R = \text{const}$ *is the scalar curvature. We assume that the metric signature is* (p, q). *Then, in some neighborhood of a point* $x \in \mathbb{M}$, *there is a coordinate system (stereographic coordinates) such that the metric is given by*

$$\mathrm{d}s^2 = \frac{\eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta}{(1 - Rx^2/8)^2} , \tag{28}$$

where

$$\eta := \operatorname{diag}(\underbrace{+\ldots+}_{p}\underbrace{-\ldots-}_{q}) , \qquad x^2 := \eta_{\alpha\beta} x^\alpha x^\beta .$$

Proof. (See, e.g., Theorem 2.4.12 in [22].)

If $R = 0$, then the full curvature tensor (27) is also zero. It follows that a zero-curvature space is locally isomorphic to the Euclidean (pseudo-Euclidean) space $\mathbb{R}^{p,q}$, and formula (28) holds.

We consider the case $R \neq 0$. Metric (28) is the induced metric on the sphere $\mathbb{S}^{p+q}$ or the hyperboloid $\mathbb{H}^{p+q}$ embedded into the higher-dimensional pseudo-Euclidean space $\mathbb{R}^{p+1,q}$. Indeed, let u, x^α be Cartesian coordinates in $\mathbb{R}^{p+1,q}$. Then the metric takes the form

$$\mathrm{d}s^2 := \mathrm{d}u^2 + \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu . \tag{29}$$

We consider the sphere (hyperboloid) embedded into the Euclidean (pseudo-Euclidean) space $\mathbb{R}^{p+1,q}$ by means of the equation

$$u^2 + \eta_{\mu\nu} x^\mu x^\nu = b , \qquad b = \text{const} \neq 0 . \tag{30}$$

To simplify the calculations, we ignore the signs and domains of the definition of the radicand, which depend on the constant b and the signature of the metric $\eta_{\mu\nu}$. Both signs and the signature can be properly dealt with in each particular case.

We introduce spherical coordinates $\{x^\alpha\} \mapsto \{r, \chi^1, \ldots, \chi^{p+q-1}\}$, where r is the radial coordinate and χ denotes angular coordinates in the Euclidean (pseudo-Euclidean) space $\mathbb{R}^{p,q} \subset \mathbb{R}^{p+1,q}$. Then metric (29) and embedding equation (30) take the form

$$\mathrm{d}s^2 = \mathrm{d}u^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega , \tag{31}$$

$$u^2 + r^2 = b , \tag{32}$$

where $\mathrm{d}\Omega(\chi, \mathrm{d}\chi)$ is the angular part of the Euclidean metric (whose explicit form is not important here). Equation (32) yields the relations

$$u = \pm\sqrt{b - r^2} \quad \Rightarrow \quad \mathrm{d}u = \mp \frac{r\,\mathrm{d}r}{\sqrt{b - r^2}} .$$

Substituting $\mathrm{d}u$ in (31) yields the induced metric:

$$\mathrm{d}s^2 = \frac{b\,\mathrm{d}r^2}{b - r^2} + r^2\,\mathrm{d}\Omega . \tag{33}$$

Now, we transform the radial coordinate $r \mapsto \rho$ as

$$r := \frac{\rho}{1 + \rho^2/(4b)} \quad \Rightarrow \quad \mathrm{d}r = \frac{1 - \rho^2/(4b)}{\left[1 + \rho^2/(4b)\right]^2} .$$

Then the induced metric takes the conformally Euclidean (pseudo-Euclidean) form

$$\mathrm{d}s^2 = \frac{\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Omega}{\left[1 + \rho^2/(4b)\right]^2} .$$

Returning to the Cartesian coordinates $\{\rho, \chi^1, \ldots, \chi^{p+q-1}\} \mapsto \{x^\alpha\}$, we find metric (28), where

$$R = -\frac{2}{b} .$$

The above construction shows that the metric on a constant-curvature space is locally isometric to that of either the Euclidean (pseudo-Euclidean) space ($R = 0$), or the sphere $\mathbb{S}^{p+q}$, or the hyperboloid $\mathbb{H}^{p+q}$, depending on the metric signature and the sign of the scalar curvature.

Euclidean (pseudo-Euclidean) metric (29) and the hypersurfaces defined by Eqn (30) are invariant under the rotation group $\mathbb{O}(p+1, q)$ transformations. Hence,

$$\dim \mathbb{O}(p+1, q) = \frac{n(n+1)}{2}, \qquad n := p + q,$$

and, in accordance with Theorem 3.1, the number of independent Killing vectors is maximal and the space of constant curvature is homogeneous and isotropic.

4. Symmetric tensors on a constant-curvature space

It was shown in Section 3 that a homogeneous and isotropic n-dimensional manifold is necessarily a constant-curvature space with the maximal number $n(n+1)/2$ of linearly independent Killing vector fields. Such spaces are common in applications. Moreover, they can carry other tensor fields, for example, matter fields in general relativity. In order to have a symmetric model, it is necessary to impose a symmetry condition on both the metric and other fields. In this section, we find conditions such that the simplest tensor fields on a constant-curvature space are also homogeneous and isotropic.

Let

$$T = \mathrm{d}x^\alpha \otimes \ldots \otimes \mathrm{d}x^\beta \, T_{\alpha \ldots \beta}$$

be an arbitrary tensor field on a constant-curvature space $\mathbb{S}$. To be specific, we consider covariant tensor fields. We assume that an isometry $\imath : x \mapsto x'$ is given. Then the requirement that a given tensor field is symmetric with respect to the isometry group has the same form as for metric (2):

$$T(x) = \imath^* T(x'),$$

where $\imath^*$ is the map of differential forms. This condition has the component form

$$T_{\alpha \ldots \beta}(x) = \frac{\partial x'^\gamma}{\partial x^\alpha} \cdots \frac{\partial x'^\delta}{\partial x^\beta} \, T_{\gamma \ldots \delta}(x'). \qquad (34)$$

Let an infinitesimal isometry be generated by a Killing tensor field $K = K^\alpha \partial_\alpha$. Then symmetry condition (34) means that the corresponding Lie derivative vanishes:

$$\mathsf{L}_K T = 0. \qquad (35)$$

The same symmetry condition must be satisfied for any tensor field with both covariant and contravariant indices.

We now consider the simplest cases common in applications.

Example 4.1. Let a differentiable scalar field (function) $\varphi(x) \in \mathcal{C}^1(\mathbb{S})$ be defined on a constant-curvature space $\mathbb{S}$. The vanishing Lie derivative condition then takes the form

$$K^\alpha(x) \partial_\alpha \varphi(x) = 0.$$

An invariant scalar field must be constant, $\varphi = \text{const}$, on the whole $\mathbb{S}$ because the Killing vector field components $K^\alpha(x)$ can take arbitrary values at any point $x \in \mathbb{S}$. Thus, a homogeneous and isotropic scalar field on a constant-curvature space $\mathbb{S}$ is constant: $\varphi(x) = \text{const}$ for all $x \in \mathbb{S}$.

Example 4.2. We consider a differentiable covector field $A = \mathrm{d}x^\alpha A_\alpha$. Then invariance condition (35) takes the form

$$K^\beta \partial_\beta A_\alpha + \partial_\alpha K^\beta A_\beta = 0.$$

We choose Killing vectors such that the equality $K^\beta(x) = 0$ is satisfied at an arbitrary but fixed point $x \in \mathbb{S}$. Moreover, Killing vectors can be chosen such that the partial derivatives $\partial_\beta K_\alpha$ are arbitrary and antisymmetric at a given point. Because $\partial_\alpha K^\beta = \nabla_\alpha K^\beta$ at a given point, the equalities

$$\partial_\alpha K^\beta A_\beta = \partial_\alpha K_\beta A^\beta = \partial_\gamma K_\beta (\delta_\alpha^\gamma A^\beta)$$

hold. It follows that

$$\delta_\alpha^\gamma A^\beta = \delta_\alpha^\beta A^\gamma,$$

because the construction works for any point of $\mathbb{S}$. Contracting the indices α and γ yields the equality

$$nA^\beta = A^\beta.$$

Thus, except for the trivial case $n = 1$, lowering the index yields $A_\alpha = 0$. Consequently, if a covector field is homogeneous and isotropic, it vanishes identically.

The same is true for vector fields $X = X^\alpha \partial_\alpha$: a homogeneous and isotropic vector field on a constant-curvature space $\mathbb{S}$ necessarily vanishes.

Example 4.3. As a third example, we consider a differentiable second-rank covariant tensor $T_{\alpha\beta}$. We assume no symmetry in the indices α and β. The Lie derivative of a second-rank tensor is given by

$$\mathsf{L}_K T_{\alpha\beta} = K^\gamma \partial_\gamma T_{\alpha\beta} + \partial_\alpha K^\gamma T_{\gamma\beta} + \partial_\beta K^\gamma T_{\alpha\gamma}.$$

As in the preceding case, we choose the Killing vector such that the relation $K^\gamma(x) = 0$ is satisfied at a point $x \in \mathbb{S}$ and the partial derivatives $\partial_\alpha K_\beta$ are antisymmetric. Then, equating the Lie derivative to zero, we find

$$\delta_\alpha^\delta T^\gamma{}_\beta + \delta_\beta^\delta T_\alpha{}^\gamma = \delta_\alpha^\gamma T^\delta{}_\beta + \delta_\beta^\gamma T_\alpha{}^\delta.$$

Contracting the indices α and δ and lowering γ yields

$$(n-1) T_{\gamma\beta} + T_{\beta\gamma} = g_{\beta\gamma} T, \qquad T := T_\alpha{}^\alpha.$$

Transposing the indices β and γ and subtracting the resulting expression, we obtain

$$(n-2)(T_{\gamma\beta} - T_{\beta\gamma}) = 0.$$

It follows that when $n \neq 2$, an invariant second-rank tensor is symmetric. Using this symmetry, we find that

$$T_{\alpha\beta} = \frac{T}{n} g_{\alpha\beta}.$$

Because the trace T is a scalar, it must be constant by symmetry arguments from the first example. Therefore, a homogeneous and isotropic second-rank tensor on a constant-curvature space is given by

$$T_{\alpha\beta} = C g_{\alpha\beta}, \qquad C = \text{const}. \qquad (36)$$

This formula holds at $n \geqslant 3$, and for the symmetric part, at $n = 2$.

In the two-dimensional case, a homogeneous and isotropic covariant tensor can have an antisymmetric part proportional to the totally antisymmetric second-rank tensor $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$:

$$T_{[\alpha\beta]} = -T_{[\beta\alpha]} = C\varepsilon_{\alpha\beta} ,$$

if the symmetry under space reflections is disregarded. The sign of the totally antisymmetric tensor changes under reflections: $\varepsilon_{\alpha\beta} \mapsto -\varepsilon_{\alpha\beta}$. Therefore, a homogeneous and isotropic tensor invariant under reflections in two dimensions has the same form (36) as in the higher-dimensional case.

Homogeneous and isotropic contravariant second-rank tensors and mixed-symmetry tensors

$$T^{\alpha\beta} = Cg^{\alpha\beta}, \qquad T^{\alpha}{}_{\beta} = C\delta^{\alpha}_{\beta}$$

can be considered along the same lines. The resulting expressions for homogeneous and isotropic tensors are used in cosmological models, where $T_{\alpha\beta}$ plays the role of the matter stress–energy tensor.

5. Manifolds with maximally symmetric submanifolds

In many physical applications, for example, in cosmology, a Riemannian (pseudo-Riemannian) manifold $\mathbb{M}$, $\dim \mathbb{M} = n$, is a topological product of two manifolds, $\mathbb{M} = \mathbb{R} \times \mathbb{S}$, where $\mathbb{R}$ is the real line identified with the time and $\mathbb{S}$ is a constant-curvature space. For any $t \in \mathbb{R}$, there is a submanifold $\mathbb{S} \subset \mathbb{M}$. Because $\mathbb{S}$ is a constant-curvature space, it is homogeneous and isotropic. The corresponding isometry group is generated by $n(n-1)/2$ Killing vectors on $\mathbb{S}$, where $n := \dim \mathbb{M}$. In this section, we find the most general form of the metric on $\mathbb{M}$ that is invariant under the transformation group generated by the isometry group of the submanifold $\mathbb{S}$.

Let x^{μ}, $\mu = 1, \ldots, n-1$ be coordinates on the constant-curvature space $\mathbb{S}$. Then the metric on $\mathbb{S}$ is $\overset{\circ}{g}_{\mu\nu}(x)$. By construction, it is invariant under the isometry group generated by the Killing vector fields $K_i = K_i^{\mu}(x)\partial_{\mu}$, $i = 1, \ldots, n(n-1)/2$.

We suppose that a sufficiently smooth metric g of Lorentzian signature is defined on the whole $\mathbb{M} = \mathbb{R} \times \mathbb{S}$, and $t \in \mathbb{R}$ is the time coordinate, i.e., $g_{00} > 0$. We also suppose that all $t = \text{const}$ sections are space-like. Moreover, we assume that the restriction of the metric g to $\mathbb{S}$ coincides with $\overset{\circ}{g}_{\mu\nu}$ for any fixed time. Obviously, such a metric has the form

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{0\nu} \\ g_{\mu 0} & h_{\mu\nu} \end{pmatrix} , \tag{37}$$

where $g_{00}(t,x)$ and $g_{0\mu}(t,x) = g_{\mu 0}(t,x)$ are arbitrary functions of t and x, and $h_{\mu\nu}(t,x)$ is a constant-curvature metric on $\mathbb{S}$, where t is a parameter. All the metric components are supposed to be sufficiently smooth in both t and x. The matrix

$$h_{\mu\nu} - \frac{g_{0\mu}g_{0\nu}}{g_{00}}$$

is negative definite, because the metric $g_{\alpha\beta}$ is of the Lorentzian signature. Moreover, the matrix $h_{\mu\nu}$ is also negative definite by construction.

First of all, we continue the action of the isometry group from $\mathbb{S}$ to the whole $\mathbb{M}$ as follows. We suppose that the Killing vector field components $K_i^{\mu}(t,x)$ parametrically depend on t. We define the action of infinitesimal isometries on $\mathbb{M}$ by the relations

$$\begin{aligned} &t \mapsto t' = t , \\ &x^{\mu} \mapsto x'^{\mu} = x^{\mu} + \epsilon K^{\mu}, \qquad \epsilon \ll 1 , \end{aligned} \tag{38}$$

where K is an arbitrary Killing vector from the Lie algebra generated by the vectors K_i. In other words, the isometry transformations do not shift points on the real axis $t \in \mathbb{R} \subset \mathbb{M}$. This means that Killing vectors are continued to the whole of $\mathbb{M}$ such that the extra component is absent: $K^0\partial_0 = 0$. The continuation is nontrivial if the Killing vector fields become parametrically dependent on t. The resulting Lie algebra of Killing vector fields continued to $\mathbb{M}$ is the same.

Example 5.1. In four dimensions, Killing vector fields continued to the whole $\mathbb{M} = \mathbb{R} \times \mathbb{S}$ generate the isometry group $(\mathbb{M}, \mathbb{G})$, where

$$\mathbb{G} = \begin{cases} \mathbb{SO}(4) , & \mathbb{S} = \mathbb{S}^3 \text{ (sphere)}, \\ \mathbb{ISO}(3) , & \mathbb{S} = \mathbb{R}^3 \text{ (Euclidean space)}, \\ \mathbb{SO}(3,1) , & \mathbb{S} = \mathbb{H}^3 \text{ (two-sheeted hyperboloid)}. \end{cases}$$

This example is important in cosmology.

We can now define a homogeneous and isotropic space–time.

Definition. *A space–time* $(\mathbb{M}, g)$ *is called homogeneous and isotropic if*

(1) *the manifold is the topological product* $\mathbb{M} = \mathbb{R} \times \mathbb{S}$, *where* $\mathbb{R}$ *is the time axis and* $\mathbb{S}$ *is a three-dimensional constant-curvature space endowed with a negative-definite metric*;

(2) *the metric* g *is invariant under transformations* (38) *generated by the isometry group of* $\mathbb{S}$.

We find the most general form of a homogeneous and isotropic metric of the universe.

Theorem 5.1. *Let metric* (37) *on* $\mathbb{M} = \mathbb{R} \times \mathbb{S}$ *be sufficiently smooth and invariant under transformations* (38). *Then, in some neighborhood of any point, a coordinate system exists such that the metric is block-diagonal*:

$$ds^2 = dt^2 + h_{\mu\nu}\, dx^{\mu}\, dx^{\nu} , \tag{39}$$

where $h_{\mu\nu}(t,x)$ *is a constant-curvature metric on* $\mathbb{S}$ *for all* $t \in \mathbb{R}$. *Moreover, the Killing vector field components are independent of time.*

Proof. Let x^{μ} be coordinates on $\mathbb{S}$. We fix one of the hypersurfaces $t = \text{const}$. The corresponding tangent vector has spatial components only: $X = X^{\mu}\partial_{\mu}$. The corresponding orthogonal vector $n^{\alpha}\partial_{\alpha}$ must satisfy the relation

$$n^0 X^{\nu} g_{0\nu} + n^{\mu} X^{\nu} g_{\mu\nu} = 0 .$$

This equality must be satisfied for all tangent vectors X, thereby giving rise to spatial components of normal vectors

$$n^{\mu} = -n^0 g_{0\nu} \hat{g}^{\mu\nu} ,$$

where $\hat{g}^{\mu\nu}$ is the inverse spatial metric, $\hat{g}^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$. It is easy to show that normal vectors are time-like.

We now draw a geodesic through each point of the space-like hypersurface $x \in \mathbb{S}$ along the normal direction. We choose the geodesic length s as the time coordinate. Without loss of generality, we can assume that the initial space-like hypersurface corresponds to $s = 0$. Thus, we have built a coordinate system $\{x^\alpha\} = \{x^0 := s, x^\mu\}$ in some neighborhood of the hypersurface $\mathbb{S}$.

By construction, the lines $x^\alpha(\tau)$ of the form $\{x^0 = s, x^\mu = \text{const}\}$, where $\tau := s$, are geodesics with the velocity vector $\dot{x}^\alpha = \delta_0^\alpha$. From the geodesic equation

$$\ddot{x}^\alpha = -\Gamma_{\beta\gamma}{}^\alpha \dot{x}^\beta \dot{x}^\gamma ,$$

it follows that $\Gamma_{00}{}^\alpha = 0$ in the coordinate system under consideration. Lowering the index α, we find an equation for the metric components:

$$\partial_0 g_{0\alpha} - \frac{1}{2}\,\partial_\alpha g_{00} = 0 . \tag{40}$$

By construction, the time-like tangent vector ∂_0 has unit length. It follows that $g_{00} = 1$. Then Eqn (40) takes the form $\partial_0 g_{0\mu} = 0$. This differential equation can be solved with the initial condition $g_{0\mu}(s = 0) = 0$, because the vector n is perpendicular to the initial hypersurface. For differentiable functions $g_{0\mu}$, the equation has a unique solution $g_{0\mu} = 0$. Thus, the metric is of block-diagonal form (39) in the resulting coordinate system.

The hypersurfaces $t = \text{const}$ given above are called *geodesically parallel*.

So far, we have ignored the properties of constant-curvature surfaces. The proof is general and implies that locally there exists a 'temporal gauge' for the metric (or the synchronous coordinate system).

By construction, the zeroth component of the Killing vector vanishes, $K^0(0, x) = 0$, on the hypersurface $s = 0$. The $(0, 0)$-component of the Killing equations, which can be more conveniently written in form (9), yields the equation $\partial_s K^0(s, x) = 0$. For sufficiently smooth functions, this equation with the initial condition $K^0(0, x) = 0$ has a unique solution, $K^0(s, x) = 0$, for all s admissible in the coordinate system. As a result, all hypersurfaces $s = \text{const}$ in some neighborhood of the initial hypersurface have constant curvature.

If metric (39) is block-diagonal, then the $(0, \mu)$-components of Killing equations (9) take the form $\partial_s K^\mu = 0$. It follows that the Killing vector field is independent of time.

The spatial (μ, ν)-components of the Killing equations are satisfied because K is a Killing vector field on $\mathbb{S}$.

Returning to the original notation $s \mapsto t$, we obtain metric (39).

Hilbert introduced coordinates in which the metric takes block-diagonal form (39) (see Eqn (22) in his paper [12]). The resulting coordinate system was called Gaussian. However, the corresponding spatial sections were not constant-curvature spaces, and Killing vector fields were not considered.

If the metric is block-diagonal, Eqn (39), and $K = K^\mu \partial_\mu$, then Killing equations (9) split into temporal and spatial components:

$$(\alpha, \beta) = (0, 0) : \qquad 0 = 0 , \tag{41}$$

$$(\alpha, \beta) = (0, \mu) : \qquad h_{\mu\nu}\partial_0 K^\nu = 0 , \tag{42}$$

$$(\alpha, \beta) = (\mu, \nu) : \quad h_{\mu\rho}\partial_\nu K^\rho + h_{\nu\rho}\partial_\mu K^\rho + K^\rho \partial_\rho h_{\mu\nu} = 0 . \tag{43}$$

Theorem 5.2. *Under the assumptions of Theorem* 5.1, *metric* (39) *has the form*

$$ds^2 = dt^2 + a^2 \overset{\circ}{g}_{\mu\nu}\, dx^\mu\, dx^\nu , \tag{44}$$

where $a(t) > 0$ is an arbitrary sufficiently smooth function (*the scale factor*) *and $\overset{\circ}{g}_{\mu\nu}(x)$ is a constant-curvature metric depending only on spatial coordinates $x \in \mathbb{S}$.*

Proof. Killing equations (43) are satisfied because $h_{\mu\nu}(t, x)$ is a constant-curvature metric on $\mathbb{S}$ for all $t \in \mathbb{R}$. Theorem 5.1 asserts that Killing vector fields are independent of time. Thus, differentiating equations (43) in time, we find the relation

$$\dot{h}_{\mu\rho}\partial_\nu K^\rho + \dot{h}_{\nu\rho}\partial_\mu K^\rho + K^\rho \partial_\rho \dot{h}_{\mu\nu} = 0 .$$

This implies that the time derivative of the metric $\dot{h}_{\mu\nu}$ is a homogeneous and isotropic second-rank tensor. Example 4.3 says that the time derivative must be proportional to the metric itself:

$$\dot{h}_{\mu\nu} = f h_{\mu\nu} , \tag{45}$$

where $f(t)$ is a sufficiently smooth function of time.

If $f = 0$, the proof is trivial, and the metric is already of form (44) for $a = \text{const}$.

Letting $f \neq 0$, we introduce a new temporal coordinate $t \mapsto t'$ defined by the differential equation

$$dt' = f(t)\, dt .$$

Equation (45) then takes the form

$$\frac{dh_{\mu\nu}}{dt'} = h_{\mu\nu} .$$

The general solution is given by

$$h_{\mu\nu}(t', x) = C \exp(t') \overset{\circ}{g}_{\mu\nu}(x), \qquad C = \text{const} \neq 0 ,$$

where $\overset{\circ}{g}_{\mu\nu}(x)$ is a constant-curvature metric on $\mathbb{S}$, which is independent of time. Hence, representation (44) follows.

Theorem 1.1 follows from Theorems 5.1 and 5.2.

6. Example

The explicit form of the Friedmann metric for a homogeneous and isotropic universe, Eqn (44), depends on coordinates on the constant-curvature space. The Friedmann metric in the stereographic coordinates is diagonal:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{a^2 \eta_{\mu\nu}}{(1 + b_0 x^2)^2} \end{pmatrix} , \tag{46}$$

where $b_0 = -1, 0, 1$, $\eta_{\mu\nu} := \operatorname{diag}(- - -)$ is the negative-definite Euclidean metric and $x^2 := \eta_{\mu\nu} x^\mu x^\nu \leqslant 0$. Because the metric on spatial cross sections is negative definite, the values $b_0 = -1, 0, 1$ correspond to the respective spaces of negative, zero, and positive curvature. In the cases of positive and negative curvature, the stereographic coordinates are defined on the whole Euclidean space $x \in \mathbb{R}^3$. In the positive-curvature case, the stereographic coordinates are defined in the interior of the ball $|x^2| < 1/b_0$.

We transform the coordinates as $x^\mu \mapsto x^\mu/a$. The resulting metric takes a nondiagonal form, while the conformal factor disappears:

$$g = \begin{pmatrix} 1 + \dfrac{\dot{b}^2 x^2}{4b^2(1+bx^2)^2} & \dfrac{\dot{b}x_\nu}{2b(1+bx^2)^2} \\ \dfrac{\dot{b}x_\mu}{2b(1+bx^2)^2} & \dfrac{\eta_{\mu\nu}}{(1+bx^2)^2} \end{pmatrix}, \qquad (47)$$

where

$$b(t) := \frac{b_0}{a^2(t)} \qquad (48)$$

and the dot denotes the time derivative.

We see that the metric of a homogeneous and isotropic universe can be nondiagonal without the conformal factor. Moreover, the scalar curvature of spatial cross sections, which is proportional to $b(t)$, explicitly depends on time.

Now, we simply discard off-diagonal elements, choose $g_{00} = 1$, and add the scale factor. Then the metric takes the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{a^2\eta_{\mu\nu}}{(1+bx^2)^2} \end{pmatrix}. \qquad (49)$$

This metric contains two arbitrary independent functions of time: $a(t) > 0$ and $b(t)$. It is nondegenerate for any b, including zero. All $t = \text{const}$ sections of the corresponding space–time are obviously spaces of constant curvature and are therefore homogeneous and isotropic. The metric is interesting because, in general, it can be used to analyze solutions passing through the zero $b = 0$. If such solutions exist, the spatial cross sections change the curvature from positive to negative values and conversely during the time evolution.

We cannot eliminate an arbitrary function $b(t)$ by means of a coordinate transformation without producing off-diagonal terms.

There is a curious situation. On one hand, all spatial sections of metric (49) are homogeneous and isotropic. On the other hand, any homogeneous and isotropic metric must have form (1). The key is that metric (49) is in general not homogeneous and isotropic. Indeed, each $t = \text{const}$ section of the space–time $\mathbb{M}$ is a constant-curvature space, and spatial (μ, ν)-components of Killing equations (32) are satisfied, but the mixed $(0, \mu)$-components are not. In the stereographic coordinates, six independent Killing vectors of the spatial section are expressed as

$$\begin{aligned} \hat{K}_{0\mu} &= (1 + bx^2)\partial_\mu - \frac{2}{b}\, x_\mu x^\nu \partial_\nu\,, \\ \hat{K}_{\mu\nu} &= x_\mu \partial_\nu - x_\nu \partial_\mu\,, \end{aligned} \qquad (50)$$

where the indices $\mu, \nu = 1, 2, 3$ label Killing vector fields. The first three Killing vectors generate translations at the coordinate origin $x^2 = 0$, while the last three generate rotations. We see that the first three Killing vector fields explicitly depend on time through the function $b(t)$, while Eqns (42) are not satisfied.

There is another method to see that metric (49) is not homogeneous and isotropic. Direct calculation yields the scalar curvature:

$$R = -\frac{24b}{a^2} + 6\left[\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} - \frac{1}{1+bx^2}\left(4\,\frac{\dot{a}\dot{b}x^2}{a} + \ddot{b}x^2\right) + 3\,\frac{\dot{b}^2x^4}{(1+bx^2)^2}\right],$$

which explicitly depends on x, and the metric is therefore not homogeneous and isotropic.

This example shows that the homogeneity and isotropy of spatial sections are not sufficient for the complete four-dimensional metric to be homogeneous and isotropic. An equivalent definition is as follows.

Definition. *A space–time is called homogeneous and isotropic if*

(1) *all constant-time* $t = \text{const}$ *cross sections are constant-curvature spaces* $\mathbb{S}$;

(2) *the extrinsic curvature of hypersurfaces* $\mathbb{S} \hookrightarrow \mathbb{M}$ *is homogeneous and isotropic*.

The definition of the extrinsic curvature of embedded surfaces can be found, e.g., in [19, 23]. In our notation, the extrinsic curvature $K_{\mu\nu}$ for block-diagonal metric (39) is proportional to the time derivative of the spatial part of the metric:

$$K_{\mu\nu} = -\frac{1}{2}\,\dot{h}_{\mu\nu}\,.$$

The last definition of a homogeneous and isotropic space–time is equivalent to the definition given in Section 5. Indeed, the first condition implies that the space–time is a topological product $\mathbb{M} = \mathbb{R} \times \mathbb{S}$. It follows that the metric can be mapped into block-diagonal form (39). Then the second condition in the definition yields Eqn (45), and we can therefore follow the proof of Theorem 5.2.

We note that the second condition in the definition of a homogeneous and isotropic universe is necessary because metric (49) provides a counterexample.

7. Conclusion

In this paper, we have given two equivalent definitions of a homogeneous and isotropic space–time. We also explicitly proved Theorem 1.1, which describes the most general form of a homogeneous and isotropic metric up to a coordinate transformation. This is the Friedmann metric. Although the theorem is known, its proof and corresponding definitions are difficult to find in the literature. It seems that the proof of Theorem 5.2 and the second definition of a homogeneous and isotropic space–time were not known before. The proof of Theorem 5.2 is simple, but not simpler than the one in book [16]. However, it is well adapted to proving the equivalence of the definitions.

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