Physics – Uspekhi **59** (5) 475 – 479 (2016)

METHODOLOGICAL NOTES PACS numbers: **01.40. – d**, **01.65. + g**, **04.20. – q**

Another route to the Lorentz transformations

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DOI: 10.3367/UFNe.2015.11.037648

Contents

Abstract. **This paper uses the Galilean relativity principle and the dependence of the rate of a clock on its velocity to derive the Lorentz transformations (LTs). Analyzing different ways of deriving the LTs provides different perspectives on them and their implications, as well as making them more accessible to a wide range of readers with an interest in relativistic physics.**

Keywords: relativity principle, Galilean transformations, Lorentz transformations, special theory of relativity

1. Introduction

In the search for fundamental laws, we conduct experiments, analyze the data obtained, and figure out which equations most precisely fit our observations. If new experimental data are not described by known equations, we devise other, more acceptable, ones. For example, such laws as the Galilean principle of relativity (1632), the Galilean transformations (1638), and Newton's three laws of motion (1687) discovered when analyzing experimental findings of their own and predecessors laid the basis for a coherent and consistent theory called Galileo–Newton classical mechanics.

The principles of classical mechanics seemed to be eternal. However, advances in the study of electromagnetic phenomena in the 19th century led to the discovery of the Maxwell equations (1861). These equations turned out to be non-invariant with respect to the Galilean transformations, changing their form when passing from one reference frame to another. It followed that, if the Maxwell equations are true, then the range of applicability of the Galilean transformations should be restricted. The foundations of science began to shake. J C Maxwell, when developing his equations, supposed that they are valid only in the preferred absolute frame of reference linked with a hypothetical aether — the postulated medium for the propagation of electromagnetic waves. A contradiction between the preferred frame of reference and the Galilean principle of relativity forced both experimental and theoretical physicists to search for the aether and new transformations of space and time. A few alternatives were suggested to identify new transformations.

(1) Both the Galilean principle of relativity and transformations are correct. The Maxwell equations are wrong.

(2) The Maxwell equations are valid in the aether reference frame; the speed of light is independent of the light source velocity and the direction of radiation propagation. The Maxwell equations change their form in the reference frames moving relative to the aether. Both the Galilean principle of relativity and the transformations are violated.

(3) The principle (postulate) of relativity formulated by H Poincaré (1904) holds true: "The principle of relativity, according to which the laws of physical phenomena must be the same for a stationary observer as for one carried along in a uniform motion of translation, so that we have no means, and can have none, of determining whether or not we are being carried along in such a motion" [1] (Poincaré statement is given here according to its translation in the collected articles [2]).[1] It was assumed that an all pervasive aether exists causing length contraction and time dilation of objects and clocks moving through it.

The problem of both the absolute frame of reference linked with aether and new transformations was to be resolved experimentally. The most significant and convincing negative results were obtained by Michelson and Morley in their famous experiments on revealing the motion of Earth relative to aether [4, 5]. The experiments were conducted from 1881 until 1929. The aether was elusive, while the Galilean principle of relativity was strengthening its position.

In order to find new transformations, various hypotheses requiring experimental check were put forwarded. In particular, there was the hypothesis of object's length contraction,

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Received 13 May 2014, revised 18 November 2015
Uspekhi Fizicheskikh Nauk **186** (5) 537 – 541 (2016)
DOI: 10.3367/UFNr.2015.11.037648
Translated by K Alkalaev; edited by A Radzig

[1] The Poincaré principle of relativity is essentially the same as the Galilean principle of relativity: "... a man below decks on a ship cannot tell whether the ship is docked or is moving smoothly through the water" [3]. The Poincaré principle of relativity considers involving not only mechanical phenomena but also those of an electromagnetic and gravitational nature. In his turn, Galileo did not rule them out either. The electromagnetic field equations were known in the integral form. The speed of light was measured with a sufficiently high accuracy. Both formulations assumed that all inertial reference frames are equivalent. Therefore, from now on we use the same term for them — the Galilean principle of relativity.

which greatly influenced the search for the Lorentz transformations [G Fitzgerald (1889), H Lorentz (1982)]. Also, the hypothesis of time dilation of clocks was gaining acceptance [W Voight (1887), J Larmor (1898)]. Both hypotheses were motivated by the corollaries for the solutions of electrodynamic problems described by the Maxwell equations along with nonrelativistic electron dynamics (a 'contraction' of electromagnetic fields induced by charged particles uniformly moving along the longitudinal direction [Heaviside (1888)], a 'wrong' dependence of the energy of fields on velocities of uniformly moving particles, a dependence of the orbital period of the electron on the molecular velocity). The Galilean transformations were to be replaced with new and more complicated transformations characterized by the change in space and time coordinate scales:

$$\begin{aligned} &x = f_x(x', y', z', t', v)\,, \quad y = f_y(x', y', z', t', v)\,, \\ &z = f_z(x', y', z', t', v)\,, \quad t = f_t(x', y', z', t', v)\,, \\ &x' = f_x(x, y, z, t, -v)\,, \quad y' = f_y(x, y, z, t, -v)\,, \\ &z' = f_z(x, y, z, t, -v)\,, \quad t' = f_t(x, y, z, t, -v)\,, \end{aligned} \qquad (1)$$

where v is the velocity of the reference frame K' moving relative to the other reference frame K, so that their appropriate coordinate axes are parallel to each other.

Taking into account the sign of the relative velocity of frames, we can show that the last four equations in Eqn (1) follow from the first four equations, along with the Galilean principle of relativity.

It turned out that the above system of equations can be solved for f_α, but not necessarily uniquely. As far back as 1887, Voight became the first to find such coordinate transformations which involved time $t'|_{v \neq 0} \neq t$ and left invariant the wave equation for a free electromagnetic field [6]. However, the Voight transformations turned out to be inconsistent with the thought experiment: scales along transverse axes were not invariant (which contradicts the Galilean principle of relativity), while the time scale depended on the Lorentz factor squared.

The transformations consistent with experimental data were found by J Larmor in 1897 [7],[2] and H Lorentz in 1899 and 1904 [8,9]. H Poincaré suggested referring to them as the Lorentz transformations. Their derivation was based on the assumption that the Maxwell equations (the wave equation in electrodynamics) are invariant under the Lorentz transformations.[3] However, Lorentz only partly understood the essence of his transformations. It was H Poincaré who realized their deep significance [10]. Prior to that, Poincaré has already noted the essential problem of moving clock synchronizing in the relativity theory. He also stressed how important it is to keep the Galilean principle of relativity for the inertial frames of reference, in particular, he proposed a few more detailed formulations of the principle. The Lorentz transformations were interpreted by Poincaré as rotations in four-dimensional spacetime. In particular, he noted that they possess the group properties, as a consequence of the Galilean principle of relativity. Later on, the Lorentz transformations were derived by A Einstein in 1905 by combining the Galilean principle of relativity and the principle of the constant speed of light (assuming also as Poincaré that the Galilean principle of relativity is valid for all electrodynamic phenomena). Einstein's derivation was given in a concise form and therefore attracted the attention of many researchers. Moreover, Einstein invented and popularized for a wide circle of readers the clock paradox (the twin paradox).

In 1910, Wladimir Sergeevich Ignatowsky (1875–1942), a Russian and Soviet physicist and mathematician, deduced the Lorentz transformations without the utilization of electrodynamics. He only used the Galilean principle of relativity, the linear dependence between space and time coordinates of moving and rest frames, and the group theory (the axiomatic approach to developing the theory and the use of three inertial frames of reference) [12, 13]. A year later Philipe Frank and Herman Rothe published in *Annalen der Physik* the paper [14] where they further developed the results of Ignatowsky. In this study, they noted the occurrence of a more general linear-fractional transformation between two inertial frames, so that both the Galilean and the Lorentz transformations are just particular cases.

To derive the Lorentz transformations, the authors of Refs [12–14] started only from the Galilean principle of relativity, which carried no quantitative data. The resulting Lorentz transformations involved an arbitrary constant n, so that the quantity $\tilde{C} = 1/\sqrt{n}$ had the dimensionality of velocity, entering the equations as $v/\tilde{C}$. The $\tilde{C}$ could be defined from the Maxwell equations experimentally verified by Hertz in 1988, provided they are invariant under the Lorentz transformations when the constants in the Maxwell equations and the Lorentz transformations are equal to each other ($\tilde{C} = c$, c is the speed of light). Otherwise, one could fix the constant experimentally by measuring the dependences of the particle or molecular excited state lifetimes on velocities.[4]

To find the unknown constant, Ignatowsky considered the electrodynamic equations for a uniformly moving point charge. Nor did he use the Galilean principle of relativity. He showed that "for a fixed observer the corresponding equipotential surface of the charge field is the Heaviside ellipsoid with the ratio of longitudinal and transverse axes equal to $1/\gamma = [1 - (v/c)^2]^{1/2}$, while for the charge-comoving observer the surface is a sphere" [12]. Then, based on the length contraction of moving objects, he found that $n^{-2} = \tilde{C} = c$.

[2] J Larmor was not only one of the first to find the spacetime coordinate transformations, but also one of the few who already in 1900 consciously perceived the Lorentz transformations. For example, he calculated in book [11] the orbital period of an electron in the second order in the molecular velocity and noted that in this case the orbit goes from circle to ellipse. According to H Lorentz, these results were considered real and not just a trick.

[3] A bias current was conjectured by Maxwell in those days when all currents were considered to be closed. Later on, when nonclosed currents were also taken into consideration, the introduction of the bias current was shown to be necessary, being the consequence of the charge conservation law. Experiments on the generation of electromagnetic waves [H R Hertz (1887)] confirmed the need for the bias current. Therefore, the Maxwell equations turned out to be relativistically invariant even before special relativity was invented. Moreover, the Maxwell equations aided the development of the special theory of relativity.

[4] W S Ignatowsky writes: "... a numerical value and sign of n can be found experimentally. ... as we did not focus on a particular physical phenomenon, we can determine n from any phenomenon, thereby obtaining the same value, because n is the universal constant." This constant was derived from the most general principle. Ignatowsky also noted that within his derivation "the optics lose their unique position with respect to the principle of relativity. It follows that the principle of relativity is more general, as it depends on the universal constant alone and not on a particular physical phenomenon" [12].

Hence, the Lorentz transformations were completely described. They formed the basis of the new, logically closed, special theory of relativity, which was developed by many outstanding mathematicians and experimental and theoretical physicists.

In the present note, we describe a physical approach to deriving the Lorentz transformations in a concise and simple way, making them more accessible for general readers. The approach is based on the Galilean principle of relativity and uses the fact that the rate of a clock depends on its velocity. We also analyze other ways of deriving the Lorentz transformations and discuss their implications.

2. Derivation of the Lorentz transformations based on time dilation of a moving clock

Let us introduce two inertial frames of reference: the lab fixed frame K, and the frame K' moving along the x-axis with velocity v (see Fig. 1). The x-, y-, z-axes of the frame K and the corresponding axes of the frame K' are co-directional, their coordinate origins coinciding at the instants of time $t = t' = 0$. The clocks in both coordinate systems are identical and synchronized.

Suppose that the Galilean principle of relativity holds true. From this it follows that the space is homogeneous and isotropic, while the reference frames are linearly related. Moreover, from the Galilean principle of relativity it is inferred that the Lorentz transformations posses the group properties. This is because the form of the Lorentz transformations is not changed when passing from one frame to another directly or through yet another frame.[5]

For the observers in each frame of reference, the scales in transverse directions y, z and y', z' in the frames K and K' can be compared by simultaneously placing one scale on the other. Then, according to the Galilean principle of relativity, their lengths do not depend on velocity v, i.e., the scales are relativistically invariant. From this it follows that

$$y' = y\,, \quad z' = z\,. \tag{2}$$

We rely upon the experimental fact that a clock at rest with respect to the moving frame K' is g times slower than the identical clock in the fixed frame K. Here, the time dilatation factor $g = g(v/\tilde{c})$ is the unknown even function of velocity v of the moving clock, and $\tilde{c}$ is the unknown constant with the dimensionality of velocity. Also, one has $g(0) = 1$, $g|_{v/\tilde{c} \ll 1} = 1 + k(v/\tilde{c})^2 + \ldots,\ k > 0$. Then, using the clocks located along the x-axis, the observers in K will find that in the time period t a clock moving past them and located in the origin of K' will show time[6]

$$t' = \frac{t}{g}\,. \tag{3}$$

Thus, the clock in the coordinate origin of the moving frame K' during the time period t according to the clock of K frame covers the distance $l = x = vt$ and shows time $t' = t/g$. Then, the observers in the moving frame K' will find that, according to their clock, during time t' the resting clock in the origin of K frame moving along the negative direction of x'-axis reaches the point $x' = -vt'$, and, therefore, covers the distance $l' = |x'| = vt'$, which by formula (3) is equal to $l' = l/g$. From this it follows that for the observers in K' the length scale along the longitudinal axis in K frame looks g times shorter than that of K'. Of course, we suppose that the length scales, as well as the clocks in both frames K and K', are completely identical.

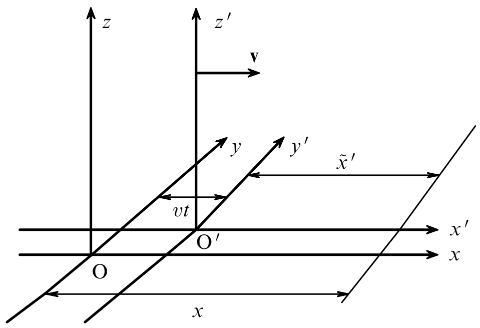

Figure 1. Rest frame of reference $K(x, y, z)$ and a moving frame of reference $K'(x', y', z')$.

On the other hand, from the Galilean principle of relativity it follows that objects in the frame K' observed from the frame K also look contracted to the same number of times as time dilatation factor g. Then, combining the effect of time dilation in moving clocks and the Galilean principle of relativity, we find that moving objects look contracted to g times in the longitudinal direction.[7]

The lengths of segments $x = \mathrm{O}x$ and $\tilde{x}' = \mathrm{O}'x'$ measured in the fixed frame K at the instant of time t are related as $\tilde{x}' = x - vt$ (see Fig. 1). The length of the segment x' measured in its own frame of reference K' is g times longer than its length measured in the frame K moving past it, i.e., $x' = g\tilde{x}'$. It follows that coordinates are related as

$$x' = g(x - vt)\,. \tag{4}$$

Using formula (4) and the Galilean principle of relativity applied to measurements in the moving frame K', we derive the analogous relation among coordinates x, x', t':

$$x = g(x' + vt')\,. \tag{5}$$

In expression (5), an account was taken of the fact that the velocity of K relative to K' has the same value but opposite sign.

The system of four equations (2), (4), and (5) relates three coordinates and time of events occurring in both fixed and moving frames. Therefore, we have found relativistic transformations of space and time coordinates from the fixed frame K to the moving frame K' in a concise and simple way. The unknown time dilatation factor $g = g(v/\tilde{c})$ can be found experimentally.

Equations (4), (5) depend on time implicitly. An explicit dependence of time in K' on the longitudinal coordinate and time in K can be found by substituting coordinate x' given by

[5] Notice that the Lorentz transformations form a group only in the case of parallel velocities.

[6] Time dilation occurs for both a uniformly moving clock and an accelerating clock, e.g., in a circular accelerator.

[7] The inverse proposition is also valid: time dilation of moving clocks is a consequence of both the length contraction of moving objects and the Galilean principle of relativity. In this case, the resting observer in the frame K finds an object of length d contracted to g times is passing by him during time $\Delta t = d/gv = \Delta t'/g$, where $\Delta t' = d/v$ is the time, according to the clock of the moving frame K', it takes for a length scale adopted to the frame K' to pass by a K-frame observer.

expression (4) into formula (5):

$$t' = g\left(t - \frac{xv}{C^2}\right), \tag{6}$$

where $C^2 = v^2g^2/(g^2 - 1)$. Similarly, substituting coordinate x given by equation (5) into formula (4), we will find the explicit dependence of time in frame K on the longitudinal coordinate and time in frame K':

$$t = g\left(t' + \frac{x'v}{C^2}\right). \tag{7}$$

The relation $C^2 = C^2(v, g)$ can be transformed into the following:

$$g = \frac{1}{\sqrt{1 - (v/C)^2}}. \tag{8}$$

The set of equations (2), (4)–(7) is the analogue of direct and inverse Lorentz transformations. The set can be treated in the same way as the ordinary Lorentz transformations. In formulas (2), (4)–(7), the dependence of function g on velocity can be found from experimental data presented in tables and figures. This dependence can be approximated by appropriate analytical functions, e.g., $g = \gamma(\beta)$, where $\gamma(\beta) = (1 - \beta^2)^{-1/2}$, $\beta = v/c$.[8] Two of four equations (4)–(7) are independent. Equations for x, t and x', t' are direct and inverse analogues of the Lorentz transformations.

In what follows, we show that $C(v, g)$ in expression (8) is constant. To this end, we take advantage of the synchronization of clocks by translating them along the x'-axis from one point in the reference frame K' to the other one, with the time dilatation factor being $g_\delta = g + \delta g = g(v) + \partial g \Delta v/\partial v$, where $\Delta v = v_\delta - v \ll v$, and v_δ being the velocity of the clocks moving apart in the reference frame K (see, monograph [16, p. 24]).[9] In this case, the rate of a clock moving with factor g_δ will be different from that of a clock moving with factor g. Thus, according to formula (3), a moving clock is delayed by $\Delta t' = -\Delta t \Delta g/g^2 = -\Delta t \Delta v(\partial g/\partial v)/g^2 = -(\tilde{x}'/g^2)(\partial g/\partial v)$, where $\tilde{x}' = \Delta t \Delta v = x'/g$, with respect to a clock in the coordinate origin of K'. Therefore, time readings in the moving frame K' will be given by the clock (3) positioned in the origin and time dilation $\Delta t'$: $t' = t/g - (\tilde{x}'/g^3)(\partial g/\partial v)$. From this it follows that the time readings on the clock in the reference frame K are given by

$$t = g\left(t' + \frac{x'}{g^3}\frac{\partial g}{\partial v}\right). \tag{9}$$

Observing the clock synchronization in the fixed frame from the moving frame of reference, we find that the similar dependence appears according to the Galilean principle of relativity:

$$t' = g\left(t - \frac{x}{g^3}\frac{\partial g}{\partial v}\right). \tag{10}$$

[8] It should be noted that, according to nonelectrodynamic experimental data on the muon lifetime (see, e.g., paper [15]), the factor $g(v/\tilde{c})$ with a great accuracy ($\sim 10^{-3}$) coincides with the body relativistic factor $\gamma(v)$, while the quantity $C(v, g)$ is approximately constant and equal to the speed of light c.

[9] Translating clocks can also be done with velocity v_δ essentially different from v provided time dilation is taken into account.

Equations (9) and (10) may be employed with the same success as equations (6) and (7). In spite of their different analytical forms, they are identical. As the second terms in formulas (6) and (10) are equal to each other, we produce the equation $(g^2 - 1)/(vg^2) = (1/g^3)(\partial g/\partial v)$ [or $\mathrm{d}g/\mathrm{d}v = g(g^2 - 1)/v$], which is solved as

$$g(v) = \frac{1}{\sqrt{1 - (v/\tilde{C})^2}}, \tag{11}$$

where $\tilde{C}$ is a constant of integration. It follows that the quantity $C(v, g)$ introduced in Eqn (8) does not depend on velocity: $C(v, g) = \tilde{C} = \text{const}$.

From equations (9)–(11), we find direct and inverse equations for the time transformations from one frame of reference to another:

$$t' = g\left(t - \frac{vx}{\tilde{C}^2}\right), \tag{12}$$

$$t = g\left(t' + \frac{vx'}{\tilde{C}^2}\right). \tag{13}$$

Combining expressions (5) and (7), we find how the longitudinal particle's velocity v'_x is transformed when passing from the moving frame K' to the rest frame K:

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{v'_x + v}{1 + v'_x v/C^2}. \tag{14}$$

If we pass from the reference frame K' to the reference frame K'' which moves relative to K' at the velocity v'', and apply the Galilean principle of relativity to require that the dependence of v_x on v''_x keeps the form (14), then it is easy to see that we arrive at the same relation: $C(v) = \tilde{C} = \text{const}$. There are also other ways to show that this quantity remains constant.

Having identified the constant $\tilde{C}$, we obtained the generalized Lorentz transformations, which are just Ignatowsky transformations developed without the involvement of electrodynamics. According to relationship (14), the physical meaning of the constant $\tilde{C}$ is the limiting velocity of signal transmission in the special theory of relativity.

3. Conclusion

In this paper, based on the Galilean principle of relativity, we have shown how space and time coordinates transform when going over from one inertial frame of reference to another [see formulas (2), (4)–(7)]. To this end, we have assumed that the rate of a clock is identified with some unknown time dilatation factor $g(v)$, which depends on the velocity of clock motion. This is analogous to the Lorentz transformations in which the factor $g(v)$ substitutes for the relativistic factor $\gamma(v)$. The factor $g(v)$ is to be determined either from experimental data organized in tables or from analytical expressions approximating the numerical data.

Then, using the clock synchronization, we have explicitly deduced the same rule of time reading transformations from the moving frame K' to the rest frame but in a different representation. Leaning upon the condition that these clock time readings are the same with those found earlier, we have defined the time dilatation factor $g(v)$. As expected, the form of $g(v)$ coincides with that of the relativistic factor $\gamma(v)$. In this case, the analogue of the Lorentz transformations goes into the Lorentz transformations found by Ignatowsky with an unknown fundamental constant having the dimensionality of velocity. The constant can be obtained in experiments not related to measuring the speed of light or other electrodynamic measurements phenomena. From this it follows that the Galilean principle (postulate) of relativity is fundamental in the sense that any electrodynamic, gravitational, or other existing or future laws must be invariant under the generalized Lorentz transformations — the Ignatowsky transformations.

We believe that describing classical physics in this manner is more general, natural, and clear for those who just started to study the special theory of relativity. For a wide circle of general readers, the axiomatic derivation of the Lorentz transformations may seem a bit unnatural, while a somewhat lengthy derivation based on the Lorentz invariance of the Maxwell equations may obscure the nature of the Lorentz transformations. In our opinion, various issues and paradoxes of special relativity are easier to discuss at this stage, while some of them naturally disappear.[10]

It is interesting to note that particular complicated expressions for the Lorentz transformations and the corresponding relativistic factor entering them follow from the Galilean principle of relativity, which is independent of any experimentally measured quantities. In fact, those expressions follow just from Galileo's terrestrial observations and experiments with an observer looking through a window of the uniformly moving boat. On the other hand, some additional experimental data (for a single point) are required only to fix the unknown fundamental constant of the Lorentz transformations. It is surprising how much information the Galilean principle of relativity contains and what an important role it plays in Nature. Some of the issues considered in the present paper can be found in papers [17–19] (and references cited therein), where they are interpreted along the same or similar lines.

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[10] For example, the most difficult twin paradox is easily overcome just because we started from the experimentally verified time dilatation effect for moving objects.