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Energy–momentum tensor of the electromagnetic field in dispersive media

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Abstract. **We study the relation between the energy–momentum tensor of the electromagnetic field and the group velocity of quasi-monochromatic waves in a nonabsorptive, isotropic, spatially and temporally dispersive dielectric. It is shown that the Abraham force acting on a dielectric is not needed for the momentum conservation law to hold if the dielectric is free of external charges and currents and if the Abraham momentum density is used. The energy–momentum tensor turns out to be symmetric, and the Maxwell stress tensor is expressed either in terms of the momentum density vector and the group velocity or in terms of the energy density and the group velocity. The stress tensor and the energy density are essentially dependent on the frequency and wave vector derivatives of the functions that describe the electromagnetic properties of the medium (i.e., the dielectric permittivity and the magnetic permeability). The obtained results are applicable to both ordinary and left-handed media. The results are compared with those of other authors. The pressure a wave exerts on the interface between two media is calculated. For both ordinary and left-handed media, either 'radiation pressure' or 'radiation attraction' can occur at the interface, depending on the material parameters of the two media. For liquid dielectrics, the striction effect is considered.**

Keywords: energy–momentum tensor, dispersive media, group velocity, light pressure, striction effect

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1. Introduction

The energy–momentum tensor is a fundamental concept characterizing the electromagnetic field in the vacuum and media. The temporal components of this tensor represented in the four-dimensional form determine the energy and momentum densities and the energy flux density. The spatial components give the momentum flux density. These quantities play an important role in all electrodynamic phenomena, and the energy–momentum tensor and related questions have been discussed in various aspects by many authors [1–15].

The energy–momentum tensor in a dispersive medium was constructed only for spectral intervals corresponding to 'transparency windows', i.e., with dissipation neglected [1]. The electromagnetic energy density in the medium in these regions is described by the Brillouin formula

$$w = \frac{1}{16\pi}\left(\frac{\partial}{\partial\omega}\big(\omega\epsilon_{\alpha\beta}(\mathbf{k},\omega)\big)E_\alpha^* E_\beta + \mathbf{B}^*\mathbf{B}\right), \tag{1}$$

containing the permittivity tensor $\epsilon_{\alpha\beta}(\mathbf{k},\omega)$ that relates the Fourier harmonics of the macroscopic electric field $\mathbf{E}$ and the generalized electric induction vector $\tilde{\mathbf{D}}$. The Brillouin formula pertains to a wave packet with harmonics whose frequencies and wave vectors are located in narrow intervals $\Delta\omega \ll \omega$ and $\Delta k \ll k$ in vicinities of the specified values of $\mathbf{k}$ and ω. In (1), averaging over the main period $T = 2\pi/\omega$ is performed. In the absence of spatial dispersion and using the permittivity $\varepsilon(\omega)$ and the magnetic permeability $\mu(\omega)$, the Brillouin formula takes the usual form

$$w = \frac{1}{16\pi}\left[\frac{\partial}{\partial\omega}\big(\omega\epsilon(\omega)\big)\mathbf{E}^*\mathbf{E} + \frac{\partial}{\partial\omega}\big(\omega\mu(\omega)\big)\mathbf{H}^*\mathbf{H}\right], \tag{2}$$

where $\mathbf{H} = \mathbf{B}/\mu(\omega)$.

We note that both the setting of the problem of the energy density in a dispersive medium and its solution are approximate. In expressions (1) and (2) for the energy density, small terms of the order of $\Delta\omega/\omega \ll 1$, $\Delta k/k \ll 1$ are omitted in Maxwell's equations (for completeness, we reproduce the corresponding calculations in Section 4). As regards the setting of the problem, the limitations are even more significant. This is explained by the fact that the 'field in matter' physical system consists of two constantly interacting subsystems. The main limitations are the smallness of the external field compared to internal fields in matter (linear responses to the field are considered), the closeness of the medium to statistical equilibrium (the Gibbs distribution for the unperturbed medium), and the absence of strong dissipation, providing the free penetration of the field into the medium. Weak dissipation can be considered using the perturbation theory. For strong dissipation, as pointed out in [1], the energy–momentum tensor probably cannot be expressed only in terms of the permittivity tensor. It can contain, for example, a macroscopic parameter of the medium such as the heat capacity.

For the momentum density $\mathbf{g}$ of the electromagnetic field, as evidenced by the history of the formation of this important concept, difficulties emerged due to different definitions of this quantity proposed by Minkowski and Abraham [8–10] (see also the discussion of this question in [12]). Although the Abraham representation

$$\mathbf{g} = \frac{1}{16\pi c}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^* - \frac{\omega}{c}\frac{\partial \epsilon_{\alpha\beta}}{\partial \mathbf{k}} E_\alpha^* E_\beta\right) \tag{3}$$

is already firmly adopted at present, leading to a symmetric four-dimensional energy–momentum tensor and to the conservation of the angular momentum in an isotropic medium, we present some additional arguments in Section 6 in favor of this expression for temporally and spatially dispersive media. In the absence of spatial dispersion, the momentum density takes the form

$$\mathbf{g} = \frac{1}{16\pi c\mu(\omega)}\left(\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*\right). \tag{4}$$

The Minkowski representation for the momentum density and its generalization to dispersive media are discussed in Section 5.

Most controversy surrounds the concept of the momentum flux density in a dispersive medium. Pitaevskii concluded in his classic paper [16] that the momentum flux density in an alternating electromagnetic field in a dispersive medium is expressed the same as for static fields with the static permittivity ε and magnetic permeability μ replaced by the corresponding frequency-dependent quantities $\varepsilon(\omega)$ and $\mu(\omega)$, and the products $E_\alpha E_\beta$ replaced by those averaged over the field period,

$$E_\alpha E_\beta \to \overline{E_\alpha E_\beta}\,, \qquad H_\alpha H_\beta \to \overline{H_\alpha H_\beta}\,. \tag{5}$$

This gives the stress tensor for a liquid dielectric

$$\sigma^{\mathrm{P}}_{\alpha\beta} = \frac{1}{4\pi}\left(\varepsilon(\omega)\overline{E_\alpha E_\beta} + \mu(\omega)\overline{H_\alpha H_\beta}\right) - \frac{1}{8\pi}\left(\varepsilon - \rho\frac{\partial\varepsilon}{\partial\rho}\right)\overline{E^2}\delta_{\alpha\beta} - \frac{1}{8\pi}\left(\mu - \rho\frac{\partial\mu}{\partial\rho}\right)\overline{H^2}\,\delta_{\alpha\beta}\,, \tag{6}$$

where ρ is the dielectric mass density. The same method of passing to an alternating field was recommended in a recent review by Makarov and Rukhadze [17].

The tempting simplicity of this result casts some doubts, however. It is known that wave packets in a dispersive medium are transferred (together with their energy and momentum) at the group velocity $\mathbf{u} = \mathrm{d}\omega/\mathrm{d}\mathbf{k}$, which can be written in an isotropic medium without spatial dispersion in terms of the frequency derivatives of the permittivity and magnetic permeability:

$$\mathbf{u} = \frac{c}{\mathrm{d}\left(\omega\sqrt{\varepsilon(\omega)\mu(\omega)}\right)/\mathrm{d}\omega}\,\frac{\mathbf{k}}{k}\,. \tag{7}$$

Therefore, the absence of the group velocity in the expression for the momentum flux density brings up questions. We note that the role of the group velocity was discussed in the well-known review [18] for the relativistic generalization of this velocity and construction of the energy–momentum tensor in the four-dimensional formalism. The results in [18] contain the group velocity.

We also note that unlike the phase velocity having the exact geometrical (and physical) meaning, the group velocity is an approximate concept. This velocity appears as a linear term in the expansion of a quasimonochromatic wave frequency $\omega(k)$ in the wave vector [see below (21)]. However, we recall the approximations mentioned above that were used in calculations of the main component of the energy–momentum tensor, i.e., the energy density. The use of the group velocity is within the limits of the original assumptions about the smallness of the parameters $\Delta\omega/\omega \ll 1$ and $\Delta k/k \ll 1$ and is appropriate in solving the formulated problem. The group velocity is quite illustrative and plays an important role in various problems concerning the propagation of waves.

The absence of the group velocity in Pitaevskii's formulas can be related to the use of the quasistationary model with a capacitor with an alternating voltage applied to its plates. It is clear that in the quasistationary approximation, waves do not propagate in the capacitor and the group velocity cannot be manifested. But it becomes significant beyond this approximation. The reference to the book by Landau and Lifshitz [1, p. 384], where Pitaevskii's work is also considered, does not remove doubts: "To satisfy the quasistationarity conditions, the circuit size should be small compared to the wavelength c/ω. This restriction, however, is not essential in nature and does not minimize the generality of the conclusion made."

Moreover, direct verification shows that stress tensor (6) does not ensure the electromagnetic field momentum conservation law, which must hold in a homogeneous medium without dissipation. This disadvantage of expression (6) manifests itself not only in the presence of dispersion but also in nondispersive media, i.e., when the permittivity $\varepsilon > 0$ and the magnetic permittivity $\mu > 0$ are independent of the frequency and wave vector (see expressions (71)–(74) and the corresponding text in Section 7). To satisfy the momentum conservation law by specifying the momentum density in Abraham's representation (3), (4) and specifying the stress tensor by Pitaevskii's expression (6), it is necessary to introduce the additional Abraham force applied to matter into the balance equation. In the case of a narrow packet of transverse eigenmodes of a dielectric, which is the most important case for dispersive media, there are no reasonable physical grounds for such a complication of the theory and

the introduction of phantom quantities into it. It is more natural to modify the stress tensor by using the group velocity that has a clear physical meaning.

Therefore, it is reasonable to refine the form of the energy–momentum tensor in a dispersive medium. This is also stimulated by the increasing use of metamaterials in which the temporal and spatial dispersions play a key role (see, e.g., review [19]). We note at once that we consider this class of artificial materials, which is diverse and complicated to describe, using the simplest isotropic model. The model assumes that (i) these materials can be described using the negative frequency-dependent permittivity $\varepsilon(\omega) < 0$ and magnetic permeability $\mu(\omega) < 0$, and (ii) the group velocity in these materials, unlike that in ordinary isotropic dielectrics, is directed oppositely to the phase velocity and wave vector. The last condition can be satisfied only in dispersive media. Otherwise, the phase and group velocities have the same magnitude and direction.

2. Initial equations

We consider an isotropic, nonabsorbing, statistically homogeneous, temporally and spatially dispersive medium. To advance to the high-frequency region, the field-induced macroscopic current $\mathbf{j}_{\text{int}}$ in a dielectric should be described as a unified phenomenon, without separation into polarization and magnetization currents. Therefore, it is convenient to use the system of Maxwell's equations with three field vectors $\mathbf{E}$, $\mathbf{B}$, and $\tilde{\mathbf{D}}$ [1, 20, 21], where $\tilde{\mathbf{D}}$ is the generalized electric induction vector, expressed in terms of the total current produced by particles in the medium:

$$\tilde{\mathbf{D}}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) + 4\pi \int_{-\infty}^{t} \mathbf{j}_{\text{int}}(\mathbf{r}, t')\, \mathrm{d}t' , \tag{8}$$

or

$$\tilde{\mathbf{D}}(\mathbf{r}, t) = \int_{-\infty}^{t} \mathrm{d}t' \int \mathrm{d}^3 r'\, \epsilon_{\alpha\beta}(\mathbf{r} - \mathbf{r}', t - t')\, E_\beta(\mathbf{r}', t') . \tag{9}$$

Here, $\epsilon_{\alpha\beta}(\mathbf{r}, t)$ is the linear response function for a homogeneous and stationary medium, having the form

$$\begin{aligned} \epsilon_{\alpha\beta}(\mathbf{k}, \omega) &= \varepsilon_{\text{l}}(k, \omega)\, \frac{k_\alpha k_\beta}{k^2} + \varepsilon_{\text{t}}(k, \omega) \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \right) , \\ \tilde{D}_\alpha(\mathbf{k}, \omega) &= \epsilon_{\alpha\beta}(\mathbf{k}, \omega) E_\beta(\mathbf{k}, \omega) \end{aligned} \tag{10}$$

in the Fourier representation, where ε_{l} and ε_{t} are the longitudinal and transverse permittivities. The tensor $\epsilon_{\alpha\beta}$ is symmetric in nongyrotropic media. The dependence of ε_{t} and ε_{l} on the modulus $k = |\mathbf{k}|$ is due to the assumed isotropy of the dielectric.

In the case of an ideally transparent (nonabsorbing) medium, the tensor $\epsilon_{\alpha\beta}(\mathbf{k}, \omega)$ is real. The dissipation of the electromagnetic energy is described by the imaginary parts ε_{l}'' and ε_{t}''. Because the real and imaginary parts of the linear response functions are connected by the Kramers–Kronig dispersion relations, a medium in which dissipation would be absent at any field frequencies does not exist. However, media that are weakly dissipative in certain spectral regions (transparency windows) can exist. We consider just such regions, assuming that the medium is nondissipative.

If the field in the medium is described by four vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$, and $\mathbf{B}$ without spatial dispersion, then the total permittivity tensor contains both the permittivity $\varepsilon(\omega)$ and the magnetic permeability $\mu(\omega)$, which are independent of $\mathbf{k}$, but the tensor itself retains the dependence on the wave vector $\mathbf{k}$,

$$\epsilon_{\alpha\beta}(\mathbf{k}, \omega) = \varepsilon(\omega)\delta_{\alpha\beta} + \left(\frac{ck}{\omega}\right)^2 \left(1 - \frac{1}{\mu(\omega)}\right) \left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right). \tag{11}$$

The dependence on $\mathbf{k}$ appears because the tensor $\epsilon_{\alpha\beta}$ is related to the total current of the medium and describes both electrical and magnetic properties of the medium. By comparing (10) and (11), we find a relation between permittivities ε_{t}, ε_{l} and ε, μ:

$$\varepsilon_{\text{l}}(\omega) = \varepsilon(\omega) , \qquad \varepsilon_{\text{t}}(k, \omega) = \varepsilon(\omega) + \left(\frac{ck}{\omega}\right)^2 \left(1 - \frac{1}{\mu(\omega)}\right) . \tag{12}$$

Maxwell's equations with three field vectors in the presence of external charges ρ_{ext} and currents $\mathbf{j}_{\text{ext}}$ have the form

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} , \tag{13}$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \tilde{\mathbf{D}}}{\partial t} + \frac{4\pi}{c}\, \mathbf{j}_{\text{ext}}(\mathbf{r}, t) , \tag{14}$$

$$\nabla \tilde{\mathbf{D}} = 4\pi \rho_{\text{ext}}(\mathbf{r}, t) , \qquad \nabla \mathbf{B} = 0 . \tag{15}$$

By using this system of equations, we write the balance equation for the energy transferred to the field from an external source,

$$-\mathbf{j}_{\text{ext}}\, \mathbf{E} = \frac{1}{4\pi} \left(\mathbf{B}\, \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\, \frac{\partial \tilde{\mathbf{D}}}{\partial t} \right) + \frac{c}{4\pi}\, \nabla\, (\mathbf{E} \times \mathbf{B}) . \tag{16}$$

The force density with which the field acts on external charges and currents is

$$\begin{aligned} \mathbf{f}_{\text{ext}} = \rho_{\text{ext}} \mathbf{E} + \frac{1}{c}\, \mathbf{j}_{\text{ext}} \times \mathbf{B} &= -\frac{1}{4\pi c} \left(\frac{\partial \tilde{\mathbf{D}}}{\partial t} \times \mathbf{B} \right) \\ &+ \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{4\pi}\, \mathbf{E}(\nabla \tilde{\mathbf{D}}) \\ &- \frac{1}{4\pi c} \left(\tilde{\mathbf{D}} \times \frac{\partial \mathbf{B}}{\partial t} \right) - \frac{1}{4\pi}\, \tilde{\mathbf{D}} \times (\nabla \times \mathbf{E}) . \end{aligned} \tag{17}$$

The last two terms, whose sum vanishes, are included for symmetry.

3. Wave packet in a dispersive medium

In a dispersive medium, it is necessary to consider an electromagnetic field in the form of a narrow wave packet, as in the derivation of Brillouin formula (1). Otherwise, it is difficult to express the quantities of interest in terms of the permittivities $\varepsilon_{\text{t}}(k, \omega)$ and $\varepsilon_{\text{l}}(k, \omega)$ depending on the frequency and wave number, which are calculated in microscopic theory. Other methods for introducing permittivities are considered, e.g., in [22, 23]. The vectors of the electromagnetic field for a narrow wave packet in the Fourier space can be written in the form of Fourier integrals over small frequency and wave-vector intervals $\alpha \leqslant \Delta\omega \ll \omega$ and $q \leqslant \Delta k \ll k$:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= \int \boldsymbol{\mathcal{E}}(\mathbf{k} + \mathbf{q}, \omega + \alpha) \exp\left[\mathrm{i}(\mathbf{k} + \mathbf{q})\mathbf{r} - \mathrm{i}(\omega + \alpha)t\right] \frac{\mathrm{d}^3 q\, \mathrm{d}\alpha}{(2\pi)^4} \\ &= \boldsymbol{\mathcal{E}}(\mathbf{r}, t) \exp(\mathrm{i}\varphi) , \quad \varphi(\mathbf{r}, t) = \mathbf{k}\mathbf{r} - \omega t . \end{aligned} \tag{18}$$

The amplitude $\boldsymbol{\mathcal{E}}(\mathbf{r},t)$ is determined by the sources ρ_{ext} and $\mathbf{j}_{\text{ext}}$. It varies in space and time slowly compared to the fast phase factor $\exp(\mathrm{i}\varphi)$. In particular, the coordinate and time derivatives of $\boldsymbol{\mathcal{E}}(\mathbf{r},t)$ are of the order of $\Delta k/k$ and $\Delta\omega/\omega$, unlike the derivatives of the phase factor.

But in regions where $\rho_{\text{ext}} = j_{\text{ext}} = 0$, the coordinate and time dependences of the amplitude can be specified. In these regions, the Fourier harmonics can only be the eigenmodes of the dielectric. Therefore, the quantities $\mathbf{k}$ and ω are related by the dispersion equation that follows from the system of Maxwell's equations (13)–(15) and has the form

$$\omega^2(k)\varepsilon_{\text{t}}(k,\omega) = c^2k^2 \tag{19}$$

for transverse waves. The Fourier integral in relation (18) becomes three-dimensional,

$$\mathbf{E}(\mathbf{r},t) = \int \boldsymbol{\mathcal{E}}(\mathbf{k}+\mathbf{q})\exp\left[\mathrm{i}(\mathbf{k}+\mathbf{q})\mathbf{r} - \mathrm{i}\omega(\mathbf{k}+\mathbf{q})t\right]\frac{\mathrm{d}^3q}{(2\pi)^3}\,. \tag{20}$$

With the group velocity $\mathbf{u}$ introduced standardly as

$$\omega(\mathbf{k}+\mathbf{q}) \approx \omega(k) + \mathbf{u}\mathbf{q}\,, \qquad \mathbf{u} = \frac{\mathrm{d}\omega}{\mathrm{d}\mathbf{k}}\,, \tag{21}$$

we obtain a wave packet with a slowly varying amplitude, which propagates with the group velocity $\mathbf{u}$:

$$\begin{aligned} \boldsymbol{\mathcal{E}}(\mathbf{r}-\mathbf{u}t) &= \int \boldsymbol{\mathcal{E}}(\mathbf{k}+\mathbf{q})\exp\left[\mathrm{i}\mathbf{q}(\mathbf{r}-\mathbf{u}t)\right]\frac{\mathrm{d}^3q}{(2\pi)^3}\,, \\ \mathbf{E}(\mathbf{r},t) &= \boldsymbol{\mathcal{E}}(\mathbf{r}-\mathbf{u}t)\exp(\mathrm{i}\varphi)\,. \end{aligned} \tag{22}$$

In the approximation used here, the wave packet does not change its shape.

The generalized electric induction vector $\tilde{\mathbf{D}}$ can be expressed in terms of the Fourier harmonics of the linear response and electric field strength:

$$\begin{aligned} \tilde{D}_\alpha &= \exp(\mathrm{i}\varphi)\int \epsilon_{\alpha\beta}(\mathbf{k}+\mathbf{q},\omega+\alpha)\mathcal{E}_\beta(\mathbf{k}+\mathbf{q},\omega+\alpha) \\ &\quad\times \exp(\mathrm{i}\mathbf{q}\mathbf{r} - \mathrm{i}\alpha t)\,\frac{\mathrm{d}^3q\,\mathrm{d}\alpha}{(2\pi)^4}\,. \end{aligned} \tag{23}$$

We calculate the integral to find the electric induction in the first order in the parameters $\Delta k/k$, $\Delta\omega/\omega$:

$$\begin{aligned} \tilde{D}_\alpha(\mathbf{r},t) &= \epsilon_{\alpha\beta}(\mathbf{k},\omega)\mathcal{E}_\beta(\mathbf{r},t)\exp(\mathrm{i}\varphi) - \mathrm{i}\exp(\mathrm{i}\varphi) \\ &\quad\times\left(\frac{\partial\epsilon_{\alpha\beta}}{\partial\mathbf{k}}\,\boldsymbol{\nabla} - \frac{\partial\epsilon_{\alpha\beta}}{\partial\omega}\,\frac{\partial}{\partial t}\right)\mathcal{E}_\beta(\mathbf{r},t)\,. \end{aligned} \tag{24}$$

If the wave packet consists of eigenmodes of the dielectric, then the slow amplitude of the electric field has the argument $\mathbf{r}-\mathbf{u}t$, i.e., $\mathcal{E}_\beta(\mathbf{r},t) = \mathcal{E}_\beta(\mathbf{r}-\mathbf{u}t)$.

The magnetic induction $\mathbf{B}$ is expressed in terms of $\mathbf{E}$ with the help of Maxwell's equation (13). In the zeroth order, the relation between the vectors is the same as in the case of monochromatic waves,

$$\boldsymbol{\mathcal{B}} = \frac{c}{\omega}\,\mathbf{k}\times\boldsymbol{\mathcal{E}}\,, \tag{25}$$

where $\boldsymbol{\mathcal{B}}$ is the slowly varying amplitude of the magnetic induction. Taking the first-order correction into account, we obtain

$$\begin{aligned} \boldsymbol{\mathcal{B}} &= \frac{c}{\omega}\,\mathbf{k}\times\boldsymbol{\mathcal{E}} - \frac{\mathrm{i}c}{\omega}\left(\boldsymbol{\nabla}\times\boldsymbol{\mathcal{E}} + \frac{\mathbf{k}}{\omega}\times\frac{\partial\boldsymbol{\mathcal{E}}}{\partial t}\right), \\ \mathbf{B}(\mathbf{r},t) &= \boldsymbol{\mathcal{B}}(\mathbf{r},t)\exp(\mathrm{i}\varphi)\,. \end{aligned} \tag{26}$$

In regions without external charges and currents, $\boldsymbol{\mathcal{B}}(\mathbf{r},t) = \boldsymbol{\mathcal{B}}(\mathbf{r}-\mathbf{u}t)$.

4. Averaging the energy transfer equation and the group velocity

We calculate the current density induced by a wave packet with electric field (18) in a homogeneous medium:

$$\begin{aligned} \mathbf{j}_\alpha^{\text{int}}(\mathbf{r},t) &= \int \kappa_{\alpha\beta}(\mathbf{r}-\mathbf{r}',t-t')E_\beta(\mathbf{r}',t')\,\mathrm{d}^3r'\,\mathrm{d}t' \\ &= \exp\left[\mathrm{i}(\mathbf{k}\mathbf{r}-\omega t)\right]\int \kappa_{\alpha\beta}(\mathbf{r}-\mathbf{r}',t-t') \\ &\quad\times\exp\left[-\mathrm{i}\mathbf{k}(\mathbf{r}-\mathbf{r}') + \mathrm{i}\omega(t-t')\right]\mathcal{E}_\beta(\mathbf{r}',t')\,\mathrm{d}^3r'\,\mathrm{d}t'\,. \end{aligned} \tag{27}$$

Here, the tensor function of the linear response $\kappa_{\alpha\beta}$ has the meaning of a generalized electric conduction. In the Fourier representation, these tensors are related by the well-known expression

$$\kappa_{\alpha\beta}(\mathbf{k},\omega) = -\frac{\mathrm{i}\omega}{4\pi}\left(\epsilon_{\alpha\beta}(\mathbf{k},\omega) - \delta_{\alpha\beta}\right). \tag{28}$$

The slowly varying amplitude can be expanded in a power series in the vicinity of a point $(\mathbf{r},t)$, through the first-order terms,

$$\boldsymbol{\mathcal{E}}(\mathbf{r}',t') \approx \boldsymbol{\mathcal{E}}(\mathbf{r},t) + \left[(\mathbf{r}'-\mathbf{r})\boldsymbol{\nabla}\right]\boldsymbol{\mathcal{E}}(\mathbf{r},t) + (t'-t)\frac{\partial\boldsymbol{\mathcal{E}}(\mathbf{r},t)}{\partial t}\,.$$

Substituting this expression in (27) and integrating gives the induced current in terms of the slowly varying amplitude and also the Fourier transform of the linear response function and its frequency and wave vector derivatives:

$$\begin{aligned} \mathbf{j}_\alpha^{\text{int}}(\mathbf{r},t) &= \kappa_{\alpha\beta}(\mathbf{k},\omega)E_\beta(\mathbf{r},t) + \exp\left[\mathrm{i}(\mathbf{k}\mathbf{r}-\omega t)\right] \\ &\quad\times\left[\frac{\partial}{\partial\omega}\,\frac{\omega(\epsilon_{\alpha\beta}-\delta_{\alpha\beta})}{4\pi}\,\frac{\partial}{\partial t} - \frac{\omega}{4\pi}\,\frac{\partial\epsilon_{\alpha\beta}}{\partial k_\gamma}\,\nabla_\gamma\right]\mathcal{E}_\beta(\mathbf{r},t)\,. \end{aligned} \tag{29}$$

Averaging the product $\mathbf{j}_{\text{int}}\mathbf{E}$ over the main period gives the result

$$\begin{aligned} \overline{\mathbf{j}_{\text{int}}\mathbf{E}} &= \frac{1}{16\pi}\left(\frac{\partial}{\partial\omega}(\omega\epsilon_{\alpha\beta}) - \delta_{\alpha\beta}\right)\frac{\partial}{\partial t}\,E_\alpha^*E_\beta \\ &\quad - \frac{\omega}{16\pi}\,\boldsymbol{\nabla}\left(\frac{\partial\epsilon_{\alpha\beta}}{\partial\mathbf{k}}\,E_\alpha^*E_\beta\right). \end{aligned} \tag{30}$$

The zeroth-order term in (29) makes no contribution if the electromagnetic energy dissipation is neglected. By using the previous result in averaging the initial equation (16), we obtain the energy balance in a dispersive medium:

$$\begin{aligned} -\overline{\mathbf{j}_{\text{ext}}\mathbf{E}} &= \frac{\partial}{\partial t}\left[\frac{1}{16\pi}\left(\frac{\partial}{\partial\omega}(\omega\epsilon_{\alpha\beta})E_\alpha^*E_\beta + \mathbf{B}^*\mathbf{B}\right)\right] \\ &\quad + \operatorname{div}\left[\frac{c}{16\pi}\left(\mathbf{E}\times\mathbf{B}^* + \mathbf{E}^*\times\mathbf{B} - \frac{\omega}{c}\,\frac{\partial\epsilon_{\alpha\beta}}{\partial\mathbf{k}}\,E_\alpha^*E_\beta\right)\right]. \end{aligned} \tag{31}$$

We see that the energy density w (the quantity under the time derivative) and the energy flux density $\boldsymbol{\gamma}$ (under the divergence) obtained here from Maxwell's equations are described by relations (1) and (3) (the latter expression should be multiplied by c^2). They depend on slowly varying amplitudes and do not contain rapidly changing phase factors.

It is convenient to represent w and $\boldsymbol{\gamma}$ in a simpler form in terms of the electric field vector $\mathbf{E}$ by eliminating the vector $\mathbf{B}$ with the help of relations (25) and (26) for the wave packet obtained from Maxwell's equations. In this case, we do not go beyond the above-mentioned approximations, because relations (1) and (3) themselves are approximate, which is obvious from the foregoing. Using (25) and (26), we find

$$w = \frac{1}{16\pi}\,\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}\left(\varepsilon_{\mathrm{t}} + \frac{\partial}{\partial\omega}\,\omega\varepsilon_{\mathrm{t}}\right), \tag{32}$$

$$\boldsymbol{\gamma} = \frac{c}{8\pi}\,\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}\sqrt{\varepsilon_{\mathrm{t}}}\left(1 - \frac{k}{2\varepsilon_{\mathrm{t}}}\frac{\partial\varepsilon_{\mathrm{t}}}{\partial k}\right)\mathbf{n}, \qquad \mathbf{n} = \frac{\mathbf{k}}{k}. \tag{33}$$

Expressions (32) and (33) are applicable in regions where external field sources are absent, $\mathbf{j}_{\mathrm{ext}} = 0$, $\rho_{\mathrm{ext}} = 0$.

The ratio $\boldsymbol{\gamma}/w$ has the form

$$\frac{\boldsymbol{\gamma}}{w} = \mathbf{n}v_{\mathrm{ph}}\,\frac{\varepsilon_{\mathrm{t}} - (k/2)\,\partial\varepsilon_{\mathrm{t}}/\partial k}{\varepsilon_{\mathrm{t}} + (\omega/2)\,\partial\varepsilon_{\mathrm{t}}/\partial\omega}, \qquad v_{\mathrm{ph}} = \frac{c}{\sqrt{\varepsilon_{\mathrm{t}}}}, \tag{34}$$

where v_{ph} is the phase velocity. Expressions (34), like $\boldsymbol{\gamma}$ and w separately, do not contain the longitudinal permittivity ε_{l}, which is quite natural because the magnetic vector $\mathbf{B}$ is absent in longitudinal oscillations and they cannot freely propagate. As regards the fraction in the right-hand side, according to the meaning of this ratio, it is the energy transfer rate in a dispersive medium, i.e., the group velocity

$$\mathbf{u} = \frac{\mathrm{d}\omega}{\mathrm{d}\mathbf{k}} = \frac{\mathbf{k}c}{k\sqrt{\varepsilon_{\mathrm{t}}}}\,\frac{\varepsilon_{\mathrm{t}} - (k/2)\,\partial\varepsilon_{\mathrm{t}}/\partial k}{\varepsilon_{\mathrm{t}} + (\omega/2)\,\partial\varepsilon_{\mathrm{t}}/\partial\omega}. \tag{35}$$

This can be easily verified by using dispersion equation (19) for transverse waves and the formula $\mathbf{u} = \mathrm{d}\omega/\mathrm{d}\mathbf{k}$:

$$\omega(\mathbf{k}) = \frac{ck}{\sqrt{\varepsilon_{\mathrm{t}}(k,\omega)}}, \qquad \frac{\mathrm{d}\omega}{\mathrm{d}\mathbf{k}} = \mathbf{n}\,\frac{c}{\sqrt{\varepsilon_{\mathrm{t}}}} - \frac{ck}{2\varepsilon_{\mathrm{t}}^{3/2}}\left(\frac{\partial\varepsilon_{\mathrm{t}}}{\partial\mathbf{k}} - \frac{\partial\varepsilon_{\mathrm{t}}}{\partial\omega}\,\frac{\mathrm{d}\omega}{\mathrm{d}\mathbf{k}}\right), \tag{36}$$

which gives (35). The group velocity in isotropic dispersive media, as follows from its explicit form, can be directed both along the wave vector and in the opposite direction.

Our calculations show that the group velocity naturally appears in the study of energy characteristics of transverse waves and completely corresponds to the initial approximation concerning a 'narrow' wave packet in the Fourier space. As mentioned above, in the absence of spatial dispersion, the group velocity can be calculated from (7).

5. Averaging the momentum transfer equation and the Minkowski representation

Equation (17) is bilinear in the field vectors, which must be real-valued. We take the complex description of fields in (18)–(26) into account and represent Eqn (17) in terms of the real parts of the field vectors,

$$\begin{aligned}\mathbf{f}_{\mathrm{ext}} = &-\frac{1}{16\pi c}\,\frac{\partial}{\partial t}\left(\tilde{\mathbf{D}}^*\times\mathbf{B} + \tilde{\mathbf{D}}\times\mathbf{B}^*\right)\\ &-\frac{1}{16\pi}\left[\tilde{\mathbf{D}}^*\times(\boldsymbol{\nabla}\times\mathbf{E}) + \tilde{\mathbf{D}}\times(\boldsymbol{\nabla}\times\mathbf{E}^*)\right.\\ &\left.+\,\mathbf{B}^*\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B}^*) - \mathbf{E}^*(\boldsymbol{\nabla}\tilde{\mathbf{D}}) - \mathbf{E}(\boldsymbol{\nabla}\tilde{\mathbf{D}}^*)\right].\end{aligned} \tag{37}$$

Here, the terms containing rapidly oscillating factors $\exp(\pm 2\mathrm{i}\varphi)$ are omitted. Averaging the terms with time derivatives over the main period $T = 2\pi/\omega$ gives

$$\begin{aligned}&\frac{1}{T}\int_t^{t+T}\frac{\partial}{\partial t'}\left[\tilde{\mathbf{D}}^*\times\mathbf{B} + \tilde{\mathbf{D}}\times\mathbf{B}^*\right]\mathrm{d}t'\\ &\quad = \frac{1}{T}\left[\boldsymbol{\mathcal{D}}^*\times\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{D}}\times\boldsymbol{\mathcal{B}}^*\right]_t^{t+T} \approx \frac{\partial}{\partial t}\left[\boldsymbol{\mathcal{D}}^*\times\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{D}}\times\boldsymbol{\mathcal{B}}^*\right],\end{aligned} \tag{38}$$

where $\boldsymbol{\mathcal{D}}$ without a tilde denotes the slowly varying electric induction amplitude. The terms containing $\exp(\pm 2\mathrm{i}\varphi)$ give second-order terms after averaging, which should be omitted. In averaging terms involving coordinate derivatives, we should take into account that the differentiation of the exponential $\boldsymbol{\nabla}\exp(\pm\mathrm{i}\varphi) = \pm\mathrm{i}\mathbf{k}\exp(\pm\mathrm{i}\varphi)$ gives not only zeroth-order terms but also terms of the next orders in $\Delta k/k \ll 1$. The order-of-magnitude estimate of individual terms can be done using the identities

$$\mathrm{i}\mathbf{k}\boldsymbol{\mathcal{E}} + \boldsymbol{\nabla}\boldsymbol{\mathcal{E}} = 0, \qquad \mathrm{i}\mathbf{k}\boldsymbol{\mathcal{B}} + \boldsymbol{\nabla}\boldsymbol{\mathcal{B}} = 0, \tag{39}$$

which follow from the equalities $\boldsymbol{\nabla}\tilde{\mathbf{D}} = 0$ and $\boldsymbol{\nabla}\mathbf{B} = 0$. The first of these is applicable in regions where external charges are absent.

The zeroth-order terms mutually cancel. The second-order terms should be omitted. As a result, using relations (10) and (24), we obtain the groups of first-order terms averaged over the period:

$$\begin{aligned}&\frac{1}{T}\int_t^{t+T}\left[\tilde{\mathbf{D}}^*\times(\boldsymbol{\nabla}\times\mathbf{E}) + \tilde{\mathbf{D}}\times(\boldsymbol{\nabla}\times\mathbf{E}^*) - \mathbf{E}^*(\boldsymbol{\nabla}\tilde{\mathbf{D}}) - \mathbf{E}(\boldsymbol{\nabla}\tilde{\mathbf{D}}^*)\right]\mathrm{d}t'\\ &= -\varepsilon_{\mathrm{t}}(k,\omega)\left[\boldsymbol{\nabla}(\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^*) - (\boldsymbol{\mathcal{E}}^*\boldsymbol{\nabla})\boldsymbol{\mathcal{E}} - (\boldsymbol{\mathcal{E}}\boldsymbol{\nabla})\boldsymbol{\mathcal{E}}^* - \boldsymbol{\mathcal{E}}(\boldsymbol{\nabla}\boldsymbol{\mathcal{E}}^*) - \boldsymbol{\mathcal{E}}^*(\boldsymbol{\nabla}\boldsymbol{\mathcal{E}})\right]\\ &- \mathrm{i}\varepsilon_{\mathrm{t}}\left[\boldsymbol{\mathcal{E}}^*(\mathbf{k}\boldsymbol{\mathcal{E}}) - \boldsymbol{\mathcal{E}}(\mathbf{k}\boldsymbol{\mathcal{E}}^*) - \boldsymbol{\mathcal{E}}^*(\boldsymbol{\nabla}\boldsymbol{\mathcal{D}}) - \boldsymbol{\mathcal{E}}(\boldsymbol{\nabla}\boldsymbol{\mathcal{D}}^*) -\right.\\ &\left.- \mathbf{k}\left\{\frac{\partial\varepsilon_{\mathrm{t}}}{\partial k}\,\frac{1}{k}\left[\boldsymbol{\mathcal{E}}(\mathbf{k}\boldsymbol{\nabla})\boldsymbol{\mathcal{E}}^* + \boldsymbol{\mathcal{E}}^*(\mathbf{k}\boldsymbol{\nabla})\boldsymbol{\mathcal{E}}\right] - \frac{\partial\varepsilon_{\mathrm{t}}}{\partial\omega}\left(\boldsymbol{\mathcal{E}}\,\frac{\partial\boldsymbol{\mathcal{E}}^*}{\partial t} + \boldsymbol{\mathcal{E}}^*\,\frac{\partial\boldsymbol{\mathcal{E}}}{\partial t}\right)\right\}.\end{aligned} \tag{40}$$

The terms containing the magnetic induction $\mathbf{B}$ have a simpler structure:

$$\begin{aligned}&\frac{1}{T}\int_t^{t+T}\left[\mathbf{B}^*\times(\boldsymbol{\nabla}\times\mathbf{B}) + \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B}^*)\right]_\alpha\mathrm{d}t'\\ &\quad = \frac{\partial}{\partial x_\beta}\left[(\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^*)\delta_{\alpha\beta} - \mathcal{B}_\alpha\mathcal{B}_\beta^* - \mathcal{B}_\alpha^*\mathcal{B}_\beta\right].\end{aligned} \tag{41}$$

Using results (38)–(41), we can average momentum balance equation (37). Terms under the time derivative take the form

$$\mathbf{g}^{\mathrm{M}} = \frac{1}{16\pi c}\left(\boldsymbol{\mathcal{D}}^*\times\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{D}}\times\boldsymbol{\mathcal{B}}^* + \mathbf{n}ck\,\frac{\partial\varepsilon_{\mathrm{t}}}{\partial\omega}\,\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}\right) \tag{42}$$

after averaging. Expression (42) is a generalization of the Minkowski momentum density [8, 9] to a dispersive medium.

To verify this, we take into account that in the absence of dispersion ($\partial\varepsilon/\partial\omega = 0$, $\partial\mu/\partial\omega = 0$) for transverse waves, according to equalities (10)–(12) and (19),

$$\tilde{\boldsymbol{\mathcal{D}}} = \varepsilon\mu\mathbf{E} = \mu\mathbf{D}\,, \tag{43}$$

where $\mathbf{D} = \varepsilon\mathbf{E}$ is the ordinary electric induction used in Maxwell's equations with four field vectors. The vectors $\mathbf{E}$ and $\mathbf{B}$ are the same in both variants of the equation representation. Using (12), the last term in the right-hand side of (42) can be written in the form

$$\mathbf{n}ck\,\frac{\partial\varepsilon_{\rm t}}{\partial\omega}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}) = -(\mu-1)(\mathbf{D}^*\times\boldsymbol{\mathcal{B}} + \mathbf{D}\times\boldsymbol{\mathcal{B}}^*)\,. \tag{44}$$

Substituting (43) and (44) in (42), we obtain the Minkowski momentum density averaged over the main period. In the modern notation (see [12, 21]), we have the expression

$$\mathbf{g}^{\rm M} = \frac{1}{16\pi c}(\mathbf{D}^*\times\mathbf{B} + \mathbf{D}\times\mathbf{B}^*) \tag{45}$$

containing the nongeneralized (ordinary) electric and magnetic inductions.

Returning to expression (42) in the general case, we apply it to a wave packet composed of the eigenmodes of a dielectric (i.e., assume that the electromagnetic field source ($\rho_{\rm ext}$, $\mathbf{j}_{\rm ext}$) is located outside the region under study). By eliminating the vectors $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{B}}$ with the help of relations (10), (19), (22), and (25), we simplify the equation to the form

$$\mathbf{g}^{\rm M} = \frac{\mathbf{n}}{8\pi c}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}})\varepsilon_{\rm t}^{3/2}\left(1 + \frac{\omega}{2\varepsilon_{\rm t}}\frac{\partial\varepsilon_{\rm t}}{\partial\omega}\right), \qquad \mathbf{n} = \frac{\mathbf{k}}{k}\,. \tag{46}$$

Passing to the permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$, we obtain

$$\mathbf{g}^{\rm M} = \frac{\mathbf{n}}{8\pi c}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}})\varepsilon(\omega)\sqrt{\varepsilon\mu}\left(1 + \frac{\omega}{2\varepsilon}\frac{\partial\varepsilon}{\partial\omega} + \frac{\omega}{2\mu}\frac{\partial\mu}{\partial\omega}\right). \tag{47}$$

In the vacuum, all the representations of the vector $\mathbf{g}^{\rm M}$ presented above describe the momentum flux density of the wave packet,

$$\mathbf{g}^{\rm vac} = \frac{1}{16\pi c}(\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{H}}^* + \boldsymbol{\mathcal{E}}^*\times\boldsymbol{\mathcal{H}})\,, \tag{48}$$

averaged over the main period.

It is important for further consideration that all the coordinate derivatives in the right-hand side of (37) are expressed in the form of the divergence $\partial\sigma^{\rm M}_{\alpha\beta}/\partial x_\beta$ of the symmetric second-rank tensor

$$\sigma^{\rm M}_{\alpha\beta} = \frac{1}{16\pi}\left[\mathcal{E}^*_\alpha\mathcal{E}_\beta + \mathcal{E}_\alpha\mathcal{E}^*_\beta - \left(\varepsilon_{\rm t}\delta_{\alpha\beta} - n_\alpha n_\beta k\,\frac{\partial\varepsilon_{\rm t}}{\partial k}\right)\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}} + \mathcal{B}_\alpha\mathcal{B}^*_\beta + \mathcal{B}^*_\alpha\mathcal{B}_\beta - \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^*\delta_{\alpha\beta}\right]. \tag{49}$$

On passing from a dielectric to the vacuum $\varepsilon_{\rm t} = 1$, the vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ transform into the macroscopic field strengths $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$, the vector $\mathbf{g}$ transforms into the electromagnetic field momentum density, and $\sigma_{\alpha\beta}$ transforms into the Maxwell stress tensor for the vacuum averaged over the main period (see, e.g., [24]), which differs in sign from the momentum flux density. Finally, for $f_{\rm ext} = 0$, the momentum balance in differential form (37) takes the form of the continuity equation

$$\frac{\partial g^{\rm M}_\alpha}{\partial t} - \frac{\partial\sigma^{\rm M}_{\alpha\beta}}{\partial x_\beta} = 0\,. \tag{50}$$

In regions where $\mathbf{f}_{\rm ext} \neq 0$, Eqn (50) acquires a source:

$$\frac{\partial g^{\rm M}_\alpha}{\partial t} - \frac{\partial\sigma^{\rm M}_{\alpha\beta}}{\partial x_\beta} = -\bar{\mathbf{f}}_{\rm ext}\,. \tag{51}$$

In this case, the form of $\mathbf{g}^{\rm M}$ and $\sigma^{\rm M}_{\alpha\beta}$ becomes complicated, and they can be calculated only when the field sources $\rho_{\rm ext}$ and $\mathbf{j}_{\rm ext}$ are specified.

Equation (50) expresses the conservation law for the volume integral of the vector $\mathbf{g}^{\rm M}$ following from macroscopic Maxwell's equation, and it can be written in the integral form

$$\frac{\rm d}{{\rm d}t}\int_{(V)} g^{\rm M}_\alpha\,{\rm d}V = \oint_{(S)}\sigma^{\rm M}_{\alpha\beta}\,{\rm d}S_\beta\,. \tag{52}$$

Because we started from the expression for the force density applied to external charges and currents, the left-hand side of (52) can be treated as an increase in the momentum related to the electromagnetic field in the volume V. The right-hand side of (52) gives the momentum flux flowing into this volume through a closed surface that bounds the volume.

The tensor $\sigma^{\rm M}_{\alpha\beta}$ can be considerably simplified using relation (25), which allows expressing all the summands in terms of the electric field:

$$\mathcal{B}^*_\alpha\mathcal{B}_\beta + \mathcal{B}_\alpha\mathcal{B}^*_\beta = \varepsilon_{\rm t}\left[2(\delta_{\alpha\beta} - n_\alpha n_\beta)\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^* - \mathcal{E}^*_\alpha\mathcal{E}_\beta - \mathcal{E}_\alpha\mathcal{E}^*_\beta\right]. \tag{53}$$

This gives the compact expression

$$\sigma^{\rm M}_{\alpha\beta} = -\frac{n_\alpha n_\beta}{8\pi}\,\varepsilon_{\rm t}\left(1 - \frac{k}{2\varepsilon_{\rm t}}\frac{\partial\varepsilon_{\rm t}}{\partial k}\right)\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^*\,. \tag{54}$$

In all the equalities starting from (38), averaging was performed over the field period. The dielectric was assumed isotropic, incompressible, and having a constant density and temperature, but the temporal and spatial dispersion was taken into account.

We recall that the tensor under the divergence sign is defined ambiguously. An arbitrary second-rank tensor $\tilde{\sigma}_{\alpha\beta}$ having zero divergence ($\partial\tilde{\sigma}_{\alpha\beta}/\partial x_\beta = 0$) can be added to it. The most general form of such a tensor is the divergence $\tilde{\sigma}_{\alpha\beta} = \partial\lambda_{\alpha\beta\gamma}/\partial x_\gamma$ of some third-rank tensor, $\lambda_{\alpha\beta\gamma}(x,y,z) = -\lambda_{\alpha\gamma\beta}$, antisymmetric in the indices β and γ.

However, as Fock has shown in [25], a tensor like $\lambda_{\alpha\beta\gamma}$, bilinear in field vectors, can be constructed only if derivatives of these vectors are included in it. Because of this, Fock postulates a principle to be satisfied by the energy–momentum tensor of the electromagnetic field: *it must be a function of the state of the system under study*. In our case, this means that the tensor must depend on field vectors that, together with other parameters (the mass density and temperature or the mass density and specific entropy), completely describe the state of the dielectric. The derivatives of the field vectors are not among the parameters determining the state. This principle eliminates the ambiguity introduced by terms like $\partial\lambda_{\alpha\beta\gamma}/\partial x_\gamma$. The principle formulated by Fock is also important for the choice of the energy flux density $\boldsymbol{\gamma}$, because the Poynting vector is initially under the divergence sign, and the ambiguity mentioned above also concerns it.

If expression (35) for the group velocity is used, the Minkowski momentum flux density of the electromagnetic field takes the form of the product $g_\alpha^{\mathrm{M}} u_\beta$ of the corresponding component of the momentum density and the propagation velocity of the wave packet and allows representing this tensor in the simplest and clearest symmetric form:

$$\sigma_{\alpha\beta}^{\mathrm{M}} = -g_\alpha^{\mathrm{M}} u_\beta = -g_\beta^{\mathrm{M}} u_\alpha \,. \tag{55}$$

Again, as in Section 4, the group velocity emerged from Maxwell's equations for the wave packet. In the given case, it is the momentum transfer velocity. There is no mystery in the constant appearance of the group velocity in transfer equations for the packets of quasimonochromatic waves. After averaging over the period, physical quantities bilinear in the field under study contain only slowly varying field amplitudes depending not on $\mathbf{r}$ and t separately but on the single argument $\mathbf{r} - \mathbf{u}t$ including the group velocity. This circumstance was pointed out in Section 3. Therefore, any vector $\mathbf{G}$ depending on the slow amplitudes $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$ satisfies the continuity equation

$$\frac{\partial G_\alpha}{\partial t} - \frac{\partial \Sigma_{\alpha\beta}}{\partial x_\beta} = 0 \,, \tag{56}$$

where $\Sigma_{\alpha\beta} = -G_\alpha u_\beta$ can be treated as the flux density of the vector $\mathbf{G}$. The tensor $\Sigma_{\alpha\beta}$ is symmetric if the vector $\mathbf{G}(\mathbf{r} - \mathbf{u}t) = G(\mathbf{r} - \mathbf{u}t)\mathbf{n}$ is directed along the wave vector $\mathbf{k} = k\mathbf{n}$. Such an ambiguity in the choice of the momentum density of the electromagnetic field and its flux in a dispersive medium shows that the use of the continuity equation only, which is obtained from Maxwell's equations, is insufficient for determining the quantities mentioned above. It is necessary to additionally apply some general physical principles.

The tensor $\sigma_{\alpha\beta}^{\mathrm{M}}$ taken with the opposite sign is the spatial part of the four-dimensional energy–momentum tensor of the electromagnetic field in a transparent medium. The stress tensor in review [18] is written in a similar form, including the group velocity. But in our opinion, the momentum density is written in [18] incorrectly and the explicit expressions for these quantities in terms of the permittivity are not presented.

6. Symmetry of the energy–momentum 4-tensor and the momentum density

Although the quantities $\mathbf{g}^{\mathrm{M}}$ and $\sigma_{\alpha\beta}^{\mathrm{M}}$ are related by continuity equation (50), they cannot be treated as the momentum density and the momentum flux density in a dispersive medium. Besides the ambiguity discussed at the end of Section 5, this is explained by the fact that the four-dimensional energy–momentum tensor containing these quantities is not symmetric:

$$T_{ik}^{\mathrm{M}} = \begin{pmatrix} w & -c\mathbf{g}^{\mathrm{M}} \\ -\dfrac{\boldsymbol{\gamma}}{c} & -\sigma_{\alpha\beta}^{\mathrm{M}} \end{pmatrix} . \tag{57}$$

Here, $i, k = 0, 1, 2, 3$, $\alpha, \beta = 1, 2, 3$, and w is the energy density (Brillouin formula (1)) which, with the use of tensor (10), takes the form

$$w = \frac{1}{16\pi}\left(\frac{\partial}{\partial\omega}\,\omega\varepsilon_{\mathrm{t}}(k,\omega)\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{B}}^*\boldsymbol{\mathcal{B}}\right) , \tag{58}$$

or, if the vector $\boldsymbol{\mathcal{B}}$ is eliminated with the help of Maxwell's equations,

$$\begin{aligned} w &= \frac{1}{16\pi}\,\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}\left(\varepsilon_{\mathrm{t}} + \frac{\partial}{\partial\omega}\,\omega\varepsilon_{\mathrm{t}}\right) , \\ w &= \frac{\varepsilon(\omega)}{8\pi}\left[1 + \frac{\omega}{2}\left(\frac{1}{\varepsilon}\,\frac{\partial\varepsilon}{\partial\omega} + \frac{1}{\mu}\,\frac{\partial\mu}{\partial\omega}\right)\right]\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}} . \end{aligned} \tag{59}$$

The last expression ignores the spatial dispersion and uses the permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$ depending only on frequency. We note that although frequency derivatives entering the above formulas can have different signs, the electromagnetic energy density $w \geqslant 0$ cannot be negative; otherwise, the field could be generated without any energy being spent by the source.

The generalized Poynting vector (the energy flux density) can be written as (see [1, 20, 21] and the results in Section 4)

$$\boldsymbol{\gamma} = \frac{c}{16\pi}\left(\boldsymbol{\mathcal{E}}^* \times \boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{B}}^* - \frac{\omega}{c}\,\frac{\partial\epsilon_{\alpha\beta}}{\partial\mathbf{k}}\,\mathcal{E}_\alpha^*\mathcal{E}_\beta\right) \tag{60}$$

and takes the form

$$\boldsymbol{\gamma} = \frac{c}{8\pi}\,\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}\sqrt{\varepsilon_{\mathrm{t}}}\left(1 - \frac{k}{2\varepsilon_{\mathrm{t}}}\,\frac{\partial\varepsilon_{\mathrm{t}}}{\partial k}\right)\mathbf{n} \tag{61}$$

for a packet of transverse waves (61). The ratio of (61) and (59) is equal to group velocity (35). If the permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$ are independent of the wave number, expression (61) can be written as

$$\boldsymbol{\gamma} = \frac{c}{16\pi}\left(\boldsymbol{\mathcal{E}}^* \times \boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}^*\right) , \tag{62}$$

where $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{B}}/\mu(\omega)$, or as

$$\boldsymbol{\gamma} = \frac{c}{8\pi\mu}\left(\frac{ck}{\omega}\right)(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}})\mathbf{n} = \frac{c\sqrt{\varepsilon\mu}}{8\pi\mu}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}})\mathbf{n} . \tag{63}$$

We note that $\sqrt{\varepsilon\mu}/\mu \neq \sqrt{\varepsilon/\mu}$ if $\varepsilon < 0$ and $\mu < 0$ (metamaterials) and the positive value of the square root is considered, as usual. The direction of the vector $\boldsymbol{\gamma}$, like the momentum flux density $\mathbf{g}$, can coincide with the direction $\mathbf{n}$ of the wave vector or can be opposite.

Returning to energy–momentum tensor (57) and comparing the components $T_{0\alpha}^{\mathrm{M}}$ and $T_{\alpha 0}^{\mathrm{M}}$ using expressions (46) and (61), we conclude that this tensor is not symmetric, $T_{0\alpha}^{\mathrm{M}} \neq T_{\alpha 0}^{\mathrm{M}}$, but the stress tensor is symmetric, $\sigma_{\alpha\beta}^{\mathrm{M}} = \sigma_{\beta\alpha}^{\mathrm{M}}$. The nonsymmetricity of T_{ik}^{M} is not surprising, because it is well known that the Minkowski energy–momentum tensor is also nonsymmetric in the absence of dispersion. Many authors, starting with Abraham [9], consider the nonsymmetricity of the tensor an essential disadvantage of the Minkowski treatment and use the Abraham formula (see [1])

$$\mathbf{g}^{\mathrm{A}} = \frac{1}{16\pi c}\left(\boldsymbol{\mathcal{E}}^* \times \boldsymbol{\mathcal{H}} + \boldsymbol{\mathcal{E}} \times \boldsymbol{\mathcal{H}}^*\right) \tag{64}$$

for the momentum density (its value is averaged over the period). In this case, an additional term — the Abraham force — appears in the momentum balance equation.

The symmetricity of the energy–momentum 4-tensor is required for the conservation of the angular momentum of a closed matter + field system and for representing it in the traditional illustrative form in terms of the momentum itself. This property is obvious for a system consisting of an

electromagnetic field and charged particles in a vacuum. In the case of a dielectric at rest, the momentum density contains contributions from the field itself and the electron subsystem of the dielectric interacting with its nuclear core, and the requirement that the 4-tensor be symmetric for such a nonclosed system is not so obvious. Notably, Polevoi and Rytov [18] do not believe that this requirement is obligatory: "In such a formulation of the problem, there is no physical foundation for requiring the symmetricity of T_{ik}." The author of [26] also resolutely expresses his opinion against the Abraham tensor, even denying its tensor status.

However, the main properties of the symmetry of a dielectric medium required for the angular momentum conservation, homogeneity and isotropy, are preserved in the case under study. These properties are inherent in a space filled with a homogeneous and isotropic nonabsorbing dielectric treated as a macroscopic 'continuous medium'. Therefore, there are no grounds to give up the symmetricity of the energy–momentum 4-tensor. In addition, in our case of a wave packet in a transparent dielectric, an additional argument appears in favor of the symmetricity of the energy–momentum tensor related to the ambiguity of determining the momentum density from the continuity equation. This ambiguity is eliminated by subjecting the energy–momentum tensor to the condition of symmetry that it, of course, has in the vacuum model and, according to the arguments presented above, should have in the case of a homogeneous and isotropic dielectric medium under study. The symmetry condition leads to the relation $T_{0\alpha} = T_{\alpha 0}$ or, in the three-dimensional case, to the relation $c\mathbf{g} = \boldsymbol{\gamma}/c$. Using (61) and (63), we find

$$\mathbf{g} = \frac{\mathbf{n}}{8\pi c}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}})\sqrt{\varepsilon_{\mathrm{t}}}\left(1 - \frac{k}{2\varepsilon_{\mathrm{t}}}\frac{\partial\varepsilon_{\mathrm{t}}}{\partial k}\right) \tag{65}$$

in the presence of spatial dispersion, and

$$\mathbf{g} = \frac{\mathbf{n}}{8\pi c\mu}\sqrt{\varepsilon\mu}\,(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}) \tag{66}$$

if the permittivity $\varepsilon(\omega)$ and permeability $\mu(\omega)$ depending only on the frequency are used. For such a choice of the momentum density, its vector is directed along the phase velocity $\mathbf{v}_{\mathrm{ph}} = c(\mathbf{k}/k)\sqrt{\varepsilon\mu}$ in an ordinary medium and in the opposite direction in left-handed media ($\varepsilon < 0$, $\mu < 0$). But in both cases, the direction of $\mathbf{g}$ coincides with the group velocity direction, whose projection on $\mathbf{k}$ can have both signs. The energy–momentum tensor contains frequency derivatives of $\varepsilon(\omega)$, $\mu(\omega)$, and $\varepsilon_{\mathrm{t}}(k,\omega)$ not only in the component $T_{00} = w$ (the energy density), but also in the stress tensor

$$\sigma_{\alpha\beta} = -T_{\alpha\beta} = -g_\alpha u_\beta = -g_\beta u_\alpha \tag{67}$$

because of the presence of the group velocity in it. We represent this tensor in two forms according to two expressions (65) and (66) for the momentum density:

$$\begin{aligned} \sigma_{\alpha\beta} &= -\frac{\mathbf{un}}{8\pi c}\,(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}})\sqrt{\varepsilon_{\mathrm{t}}}\left(1 - \frac{k}{2\varepsilon_{\mathrm{t}}}\frac{\partial\varepsilon_{\mathrm{t}}}{\partial k}\right)n_\alpha n_\beta\,, \\ \sigma_{\alpha\beta} &= -\frac{\mathbf{un}}{8\pi c\mu}\sqrt{\varepsilon\mu}\,(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}})\,n_\alpha n_\beta\,. \end{aligned} \tag{68}$$

Expressions (68) contain only the electric field. It is also easy to write a symmetric expression containing both electric and magnetic field strengths. Because $g_\alpha = \gamma_\alpha/c^2 = wu_\alpha/c^2$, from (68) we obtain the totally symmetric expression

$$\sigma_{\alpha\beta} = -\frac{u_\alpha u_\beta w}{c^2}\,, \tag{69}$$

where the energy density w is described by Brillouin formula (1) or (2).

7. Comparison with the stress tensor obtained by other authors

We return to Pitaevskii stress tensor (6) and write it for a wave packet in the notation used here:

$$\begin{aligned} \sigma^{\mathrm{P}}_{\alpha\beta} &= \frac{\varepsilon(\omega)}{8\pi}\left(\mathcal{E}_\alpha\mathcal{E}^*_\beta + \mathcal{E}^*_\alpha\mathcal{E}_\beta - \frac{1}{2}\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^*\delta_{\alpha\beta}\right) \\ &\quad + \frac{1}{8\pi\mu(\omega)}\left(\mathcal{B}_\alpha\mathcal{B}^*_\beta + \mathcal{B}^*_\alpha\mathcal{B}_\beta - \frac{1}{2}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{B}}^*\delta_{\alpha\beta}\right). \end{aligned} \tag{70}$$

Here, striction terms, which are insignificant for the problem under study, are omitted, i.e., we assume that the medium is incompressible, $\rho = \mathrm{const}$. Using expression (25), we simplify the expression for the stress tensor to

$$\sigma^{\mathrm{P}}_{\alpha\beta} = -n_\alpha n_\beta\,\frac{\varepsilon(\omega)}{8\pi}\,\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^*\,. \tag{71}$$

We have shown that the continuity equation

$$\frac{\partial g_\alpha}{\partial t} = \frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta} \tag{72}$$

is satisfied, where g_α is the Abraham momentum density given by expression (4) or expression (66) for a wave packet, and $\sigma_{\alpha\beta}$ is determined by the second expression in (68). This means that the total momentum of the packet $\mathbf{G} = \int\mathbf{g}\,\mathrm{d}V$ is conserved. Pitaevskii stress tensor (71) differs from (68) both in the absence of dispersion and in its presence:

$$\frac{\sigma^{\mathrm{P}}_{\alpha\beta}}{\sigma_{\alpha\beta}} = \frac{c}{u}\sqrt{\varepsilon\mu} \neq 1\,. \tag{73}$$

The stress tensor in a dispersive medium, like energy density (1), contains the frequency and wave-vector derivatives introduced into the theory via the group velocity. The continuity equation with the Pitaevskii tensor is violated:

$$\frac{\partial g_\alpha}{\partial t} \neq \frac{\partial\sigma^{\mathrm{P}}_{\alpha\beta}}{\partial x_\beta}\,. \tag{74}$$

The equality can be recovered either by using our tensor (68) or by adding the density of some force (the Abraham force) to the right-hand side of (74). In this case, the volume integral $\int\mathbf{g}\,\mathrm{d}V$ is not conserved, and it cannot be considered the total momentum of the system but can be treated only as some part of the momentum. The momentum turns out to be distributed between the electromagnetic field and matter. The principle of this distribution (the construction of the tensor $\sigma^{\mathrm{P}}_{\alpha\beta}$) is based on the quasistationary model, inapplicable to wave phenomena, and therefore has an arbitrary character. In fact, the momentum density and flux are produced by combined oscillations of the electromagnetic field and charged particles of matter. A consistent separation of the momentum into two parts pertaining to only the field or matter seems impossible, because these interacting subsystems freely exchange the physical characteristics, including momentum.

We also compare our result with the results concerning the electromagnetic field obtained by Polevoi and Rytov [18]. Their formulas (51) and (53) in our notation take the form

$$w = T_{00}\,, \tag{75a}$$

$$\gamma_\alpha = wu_\alpha = -cT_{\alpha 0}\,, \tag{75b}$$

$$\sigma_{\alpha\beta} = -g_\alpha u_\beta = -T_{\alpha\beta}\,, \tag{75c}$$

$$g_\alpha = \frac{w}{\omega}\,k_\alpha = -T_{0\alpha}\,. \tag{75d}$$

According to equality (75c), the relation of the stress tensor to the momentum density and group velocity corresponds to our result. But the momentum density $\mathbf{g}$ is defined such that the energy–momentum tensor is nonsymmetric, which is especially pointed out by the authors themselves. As mentioned above, it is difficult to agree with this.

In concluding this section, we compare our result (69) for the stress tensor with the one for the electromagnetic field in a vacuum (see, e.g., [2, 13, 24]). For a quasimonochromatic field in a vacuum, we have the tensor averaged over the period

$$\sigma^{\mathrm{v}}_{\alpha\mu} = \frac{1}{16\pi}(\mathcal{E}^*_\alpha\mathcal{E}_\mu + \mathcal{E}_\alpha\mathcal{E}^*_\mu + \mathcal{H}^*_\alpha\mathcal{H}_\mu + \mathcal{H}_\alpha\mathcal{H}^*_\mu) - \frac{1}{16\pi}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}})\,\delta_{\alpha\mu}\,, \tag{76}$$

the relations between field vectors

$$\mathbf{n}\times\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{H}}\,,\quad |\boldsymbol{\mathcal{E}}| = |\boldsymbol{\mathcal{H}}|\,,\quad \mathbf{n} = \frac{\mathbf{k}}{k}\,,\quad u_\alpha = cn_\alpha\,, \tag{77}$$

and the field energy density

$$w = \frac{1}{16\pi}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}})\,. \tag{78}$$

Using these, we find

$$\sigma^{\mathrm{v}}_{\alpha\mu} = -wn_\alpha n_\mu\,. \tag{79}$$

Stress tensor (69) in a dielectric without losses differs from expression (79) for the vacuum by the replacement of energy density (78) with Brillouin expression (1) and unit vectors with the ratios u_α/c. Maxwell stress tensor (69), like energy density (1), contains frequency and wave-vector derivatives of the permittivity $\varepsilon_{\mathrm{t}}(k,\omega)$.

8. Light pressure at the interface of two media

Different relations characterizing the momentum of the electromagnetic field can be most directly verified by measuring the force. It seems that the most convenient object for this purpose is light pressure, which was first measured by Lebedev in 1899–1907 [27]. In this connection, we calculate the force applied to a planar interface between two media, one of which (or both) is a transparent dielectric. This problem is of interest in connection with the recent discussion [17, 26, 28, 29] about so-called light attraction, which is related by the participants of the discussion to metamaterials, i.e., artificial materials having negative permittivity and magnetic permeability, $\varepsilon(\omega) < 0$ and $\mu(\omega) < 0$, in some frequency interval (see, e.g., [28, 30] and [17, 26, 29, 31]).

We describe the interaction of a wave with the interface between media in the geometrical optics approximation, as in the derivation of Fresnel formulas [32], and exclude spatial dispersion from the consideration. This last condition is very important because spatial dispersion leads to a nonlocal interdependence of the field vectors and does not allow the boundary conditions for them to be written in the ordinary form. We also recall that we consider metamaterials in the simplest model (see the end of Section 1) and do not intend to give any detailed description of this quite diverse class of artificial materials with complicated properties.

We start from the simplest case of normal incidence of a plane quasimonochromatic wave (a wave packet) on a flat interface between two transparent dielectrics with positive permittivites and permeabilities ε_1, μ_1 and ε_2, μ_2, in the absence of temporal dispersion. The reflection coefficient R, which is calculated from Fresnel formulas, has the form

$$R = \left(\frac{\sqrt{\varepsilon_1\mu_2} - \sqrt{\varepsilon_2\mu_1}}{\sqrt{\varepsilon_1\mu_2} + \sqrt{\varepsilon_2\mu_1}}\right)^2. \tag{80}$$

For $\varepsilon_1/\varepsilon_2 = \mu_1/\mu_2$, reflection is absent, $R = 0$. At the interface, the vectors $\mathbf{E}$ and $\mathbf{H}$ are continuous, and therefore the momentum density $g = g_1 = g_2$ is also continuous, but the group velocity experiences a jump, $u_1/u_2 = \varepsilon_2/\varepsilon_1 = \mu_2/\mu_1$. Therefore, the momentum flux density also experiences a jump. The pressure P_{rad} at the interface between media is produced by the difference between momentum fluxes on the two sides of the interface,

$$P_{\mathrm{rad}} = g(u_1 - u_2) = gu_1\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\right), \tag{81}$$

where $gu_1 = \varepsilon_1\boldsymbol{\mathcal{E}}\boldsymbol{\mathcal{E}}^*/(8\pi) > 0$ is the momentum flux density incident on the wave boundary. As follows from (81), the wave exerts pressure on the interface (pushes it in the propagation direction) if $\varepsilon_2 > \varepsilon_1$. Otherwise, the interface boundary is attracted to the incident wave. Thus, 'light attraction' is also possible on the boundaries of ordinary transparent dielectrics with positive ε and μ.

We note that expression (81) describes only the pressure produced by the electromagnetic wave. This pressure is applied in a layer a few wavelengths in thickness where the field configuration changes on both sides of the geometrical interface. This layer can also be subjected to the action of nonelectromagnetic forces, for example, gravitation and hydrostatic pressure in a liquid dielectric. Expression (81) is not valid for metamaterials, because it neglects dispersion.

We next consider the case where the first medium has $\varepsilon_1 < 0$ and $\mu_1 < 0$. A quasimonochromatic wave is incident from this medium on the interface with another medium. This means that the Poynting vector, the momentum density, and the group velocity are directed to the interface. The wave vector and phase velocity are directed away from the interface. All the considerations about the directions of fluxes and velocities, in particular, for waves with oppositely directed phase and group velocities, were clearly explained in detail in the well-known lectures by Mandelshtam [33, pp. 431–437]. Thus, the momentum flux $g_1u_1 > 0$ directed from one medium to another is incident per unit time on the interface.

If the second medium absorbs all the energy incident on the interface (an absolutely black body), then for the normal incidence we have

$$P_{\mathrm{rad}} = g_1u_1 > 0\,, \tag{82}$$

i.e., the interface is subjected to light pressure applied in a layer of the order of the absorption length in thickness. In the case of a perfectly reflecting second medium, the pressure is doubled. Our results differ from statements in [17, 26, 29] that light pressure is replaced in left-handed media by 'light attraction' because of the inequality $\varepsilon < 0$. It is clear from the data presented above that this statement contradicts the momentum conservation law. This error is probably explained by the fact that Pitaevskii stress tensor (7) and our tensor (68) behave differently on passing from an ordinary medium to a left-handed medium. Pitaevskii tensor (71) changes its sign after the substitution $\varepsilon \to -\varepsilon$, $\mu \to -\mu$, whereas the sign of our tensor (68) does not change (the signs of $\mathbf{n} = \mathbf{k}/k$, ε, and μ change). The incorrect conclusion about light attraction made in [17] is explained by the use of incorrect expression (6) for the stress tensor in a dispersive medium and the neglect of the Abraham force. This force is taken into account in our calculations, being included in the stress tensor.

In using expression (66) for a left-handed medium, it is necessary to change the sign in it twice: first, to change the signs of ε and μ, and then to make the replacement $\mathbf{n} \to -\mathbf{n}$, because the direction of the wave vector in left-handed media is opposite to the direction of the energy and momentum fluxes of the wave. As regards the group velocity, in left-handed media, as in ordinary media, it is oriented in the direction of the energy and momentum fluxes and is opposite to the wave vector. This is possible only in dispersive media and, according to (7), requires the fulfillment of the condition

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\,\omega\sqrt{\varepsilon(\omega)\mu(\omega)} < 0 \quad \text{or} \quad 1 + \Pi(\omega) < 0\,,$$
$$\Pi(\omega) = \frac{\omega}{2}\left(\frac{1}{\varepsilon}\frac{\mathrm{d}\varepsilon}{\mathrm{d}\omega} + \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\omega}\right). \tag{83}$$

We note that this condition is compatible with the propagation of electromagnetic waves only in media where simultaneously $\varepsilon < 0$ and $\mu < 0$; otherwise, the electromagnetic energy density (59) becomes negative:

$$w = \frac{\varepsilon(\omega)}{8\pi}\,\mathbf{E}\,\mathbf{E}^*\big[1 + \Pi(\omega)\big] = \frac{\mu(\omega)}{8\pi}\,\mathbf{H}\,\mathbf{H}^*\big[1 + \Pi(\omega)\big]\,. \tag{84}$$

This would indicate the instability of the medium with respect to the spontaneous increase in the electromagnetic field, and such a medium cannot be realized in nature.

With the normal incidence of a quasimonochromatic wave on the interface between two transparent media, one of which is left-handed and the other ordinary, the light pressure on the interface can be calculated as the difference between the momentum density fluxes at this interface, taking the incident and reflected waves and the waves transmitted into the second medium into account:

$$P_{\mathrm{rad}} = g_1 u_1 (1 + R) - g_2 u_2 (1 - R)\,. \tag{85}$$

Here, the quantities g and u are positive and are the projections of the corresponding vectors on the normal to the interface directed from the first medium to the second. The reflection coefficient

$$R = \left|\frac{\sqrt{|\varepsilon_1|\mu_2} - \sqrt{\varepsilon_2|\mu_1|}}{\sqrt{|\varepsilon_1|\mu_2} + \sqrt{\varepsilon_2|\mu_1|}}\right|^2 \tag{86}$$

can be easily calculated using the corresponding Fresnel formulas.

For a wave incident on the boundary at an arbitrary angle θ_0, different polarizations of the incident wave should be considered. For each polarization, the vectors $\mathbf{g}$ and $\mathbf{u}$ should be projected on the normal. As a result, we obtain the expressions

$$P'_{\mathrm{rad}} = \mathbf{g}'_1\mathbf{u}_1\big(1 + R_\perp(\theta_0)\big)\cos^2\theta_0 - \mathbf{g}'_2\mathbf{u}_2\big(1 - R_\perp(\theta_0)\big)\cos^2\theta_{\mathrm{r}}\,, \tag{87}$$

$$P''_{\mathrm{rad}} = \mathbf{g}''_1\mathbf{u}_1\big(1 + R_\perp(\theta_0)\big)\cos^2\theta_0 - \mathbf{g}''_2\mathbf{u}_2\big(1 - R_\perp(\theta_0)\big)\cos^2\theta_{\mathrm{r}}\,, \tag{88}$$

where θ_{r} is the angle of refraction. The quantity $\mathbf{g}'$ in equality (87) is the momentum density produced by the field components $\mathbf{E}_\perp$ and $\mathbf{H}_\parallel$, while the vector $\mathbf{g}''$ in (88) is produced by the components $\mathbf{E}_\parallel$ and $\mathbf{H}_\perp$ of the incident wave.

The value of P_{rad} in expressions (85)–(88) can be both positive and negative, depending on the parameters of the media, and therefore both light pressure and 'light attraction' are possible.

9. Striction effect in a dispersive medium

We have assumed so far that a medium is incompressible, i.e., the mass density $\rho = \text{const}$ was assumed fixed and invariable. However, an electromagnetic field can produce internal stresses in matter, which cause variations in the density in liquids and gasses and can also produce shear deformations in solids. These are the so-called striction effects. Although, as Tamm pointed out [3], striction forces make no contribution to the total force acting on a dielectric body placed in a vacuum or surrounded by other bodies in mechanical equilibrium, their influence on internal deformations cannot be assumed small.

The striction effect was first considered by Helmholtz [34] back in the 19th century. Striction contributions to the Pitaevskii tensor (6) correspond to terms with the derivatives $\partial\varepsilon/\partial\rho$ and $\partial\mu/\partial\rho$. It is obvious that the corresponding terms also appear in our consideration if we drop the assumption about the incompressibility of the medium.

For this purpose, we rank the temporal and spatial scales of our macroscopic problem. The smallest scale $\lambda = 2\pi/k$ corresponds to the wavelength of a quasimonochromatic wave. The spatial size of the wave packet under study has a considerably larger scale, $l = 2\pi/\Delta k$. To take the striction effect in a liquid dielectric (i.e., in the absence of possible shear stresses) into account, we consider a weakly inhomogeneous medium with the density $\rho(\mathbf{r})$ depending on coordinates. We assume that the inhomogeneity scale L ($\partial\rho/\partial r \approx \rho/L$) is large compared to the wave packet size,

$$\lambda \ll l \ll L\,. \tag{89}$$

Otherwise, the preceding analysis becomes invalid, because the group velocity depends on the coordinates and the propagation of the wave packet is considerably complicated.

In the calculations presented above, we considered the limit case $L \to \infty$. Now, we assume that the scale L is finite. In this case, the quantities $\varepsilon(\omega, \rho)$ and $\mu(\omega, \rho)$ and the group velocity $u(\omega, \rho)$ become slow functions of coordinates due to the dependence $\rho(\mathbf{r})$. The state of an isotropic medium is characterized, along with the density, by another parameter, either the temperature T or the specific entropy s. We assume

that this parameter is constant, i.e., we adopt the isothermal or adiabatic conditions (depending on experimental conditions).

We calculate the time derivative of the momentum density written in the symmetric form in terms of the vectors $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$:

$$\mathbf{g} = -\frac{\mathbf{n}}{16\pi c}\left(\frac{\sqrt{\varepsilon\mu}}{\mu}\,\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}} + \frac{\sqrt{\varepsilon\mu}}{\varepsilon}\,\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}}\right). \tag{90}$$

Time enters the arguments of the fields $\boldsymbol{\mathcal{E}}(\mathbf{r}-\mathbf{u}t)$ and $\boldsymbol{\mathcal{H}}(\mathbf{r}-\mathbf{u}t)$. The differentiation gives

$$\frac{\partial\boldsymbol{\mathcal{E}}}{\partial t} \approx -u_\beta\,\frac{\partial\boldsymbol{\mathcal{E}}}{\partial x_\beta}\,,$$

where terms of the order of l/L are omitted, as are analogous expression for the magnetic field. This allows us to write the derivative of the momentum density as

$$\frac{\partial\mathbf{g}_\alpha}{\partial t} = -\frac{n_\alpha n_\beta}{16\pi}\left[\eta_{\mathrm{e}}(\omega,\rho)\frac{\partial}{\partial x_\beta}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}) + \eta_{\mathrm{m}}(\omega,\rho)\frac{\partial}{\partial x_\beta}(\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}})\right], \tag{91}$$

where the quantities

$$\eta_{\mathrm{e}}(\omega,\rho) = \frac{\sqrt{\varepsilon\mu}\,u}{\mu c}\,, \qquad \eta_{\mathrm{m}}(\omega,\rho) = \frac{\sqrt{\varepsilon\mu}\,u}{\varepsilon c} \tag{92}$$

include the permittivity and permeability, as well as the group velocity, and u is its projection on the wave-vector direction $\mathbf{k}$.

We now transform the right-hand side to the divergence of the stress tensor. Performing the identical transformation

$$\frac{\partial\mathbf{g}_\alpha}{\partial t} = -\frac{n_\alpha n_\beta}{16\pi}\frac{\partial}{\partial x_\beta}(\eta_{\mathrm{e}}\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}} + \eta_{\mathrm{m}}\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}}) + \frac{n_\alpha n_\beta}{16\pi}\left[\frac{\partial\eta_{\mathrm{e}}}{\partial\rho}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}) + \frac{\partial\eta_{\mathrm{m}}}{\partial\rho}(\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}})\right]\frac{\partial\rho}{\partial x_\beta}\,, \tag{93}$$

we integrate both parts of the resultant equality over a spatial region that completely includes the wave packet under study. Due to the conservation of the total momentum of the electromagnetic field in a medium without losses, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int g_\alpha\,\mathrm{d}V = 0\,, \tag{94}$$

which gives

$$\int\left[\frac{\partial\eta_{\mathrm{e}}}{\partial\rho}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}) + \frac{\partial\eta_{\mathrm{m}}}{\partial\rho}(\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}})\right]\frac{\partial\rho}{\partial x_\beta}\,\mathrm{d}V = -\int\rho\,\frac{\partial}{\partial x_\beta}\left[\frac{\partial\eta_{\mathrm{e}}}{\partial\rho}(\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}}) + \frac{\partial\eta_{\mathrm{m}}}{\partial\rho}(\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}})\right]. \tag{95}$$

The factor ρ in the last integral can be introduced with the accuracy used under the derivative sign, which gives an error of the order of l/L. With the integrand in (95), we can write Eqn (93) in terms of the divergence of the stress tensor

$$\sigma_{\alpha\beta} = -\frac{n_\alpha n_\beta}{16\pi}\left[\left(\eta_{\mathrm{e}} + \rho\,\frac{\partial\eta_{\mathrm{e}}}{\partial\rho}\right)\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}} + \left(\eta_{\mathrm{m}} + \rho\,\frac{\partial\eta_{\mathrm{m}}}{\partial\rho}\right)\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}}\right] \tag{96}$$

in the usual form

$$\frac{\partial\mathbf{g}_\alpha}{\partial t} = \frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}\,. \tag{97}$$

We note that, as follows from the derivation presented above, the expression $\partial\sigma_{\alpha\beta}/\partial x_\beta$ consistently takes terms of the order of $\mathcal{E}^2/l$ into account, but terms of the next order of smallness $\mathcal{E}^2/L$ are calculated inaccurately. Therefore, by neglecting the last small terms upon differentiation of (96) over the coordinates, the terms containing ρ, η_{e}, and η_{m} can be assumed constant.

If the dispersion is negligible and $\varepsilon > 0$, $\mu > 0$, and $u = c/\sqrt{\varepsilon\mu} = v_{\mathrm{ph}}$, then the stress tensor takes the form

$$\sigma_{\alpha\beta} = -\frac{n_\alpha n_\beta}{16\pi\varepsilon\mu}\left[\left(\varepsilon - \rho\,\frac{\mathrm{d}\varepsilon}{\mathrm{d}\rho}\right)\boldsymbol{\mathcal{E}}^*\boldsymbol{\mathcal{E}} + \left(\mu - \rho\,\frac{\mathrm{d}\mu}{\mathrm{d}\rho}\right)\boldsymbol{\mathcal{H}}^*\boldsymbol{\mathcal{H}}\right]. \tag{98}$$

It becomes similar to the stress tensor for static electric and magnetic fields [1], but the complete equivalence is impossible because, we are dealing with a field of transverse electromagnetic waves propagating in the specified direction $\mathbf{n}$, and the tensor is averaged over the main period of the field.

10. Conclusions

Our study shows that a packet of the eigenmodes of a nonabsorbing isotropic dielectric has simple and obvious properties. In particular, the tensor of the momentum flux density of the electromagnetic field is expressed as the momentum flux density of classical particles, i.e., as the product of the corresponding components of the momentum density and the group velocity of the packet. The latter is an analog of the velocity of particles. This makes it possible to represent the Maxwell stress tensor in the obvious form $\sigma_{\alpha\beta} = -wu_\alpha u_\beta/c^2$, which is the same for the field in a vacuum and in a dispersive medium. An analogous relation is also fulfilled for the momentum density itself in a vacuum and a medium: $g_\alpha = wu_\alpha/c^2$. In the vacuum, the ratio u_α/c is replaced by the ratio k_α/k.

In this case, of course, we should bear in my mind that the group velocity, despite its clearness and important role in the theory of wave propagation (see, in particular, the historical note by Levin [35]), is in fact an approximate concept and appears as a linear term in the frequency expansion in the wave-vector powers. Consideration of the next terms of the expansion leads to a spread of the wave packet and to corrections to most expressions containing the group velocity. But the formulation of the problem itself on the energy–momentum tensor in a dispersive medium is approximate, because it involves an expansion in the small parameters $\Delta k/k$ and $\Delta\omega/\omega$ specifying the width of a spectral interval. Therefore, the use of the group velocity is not beyond the scope of our initial assumptions. This statement was substantiated in detail in the derivation of the energy and momentum transfer equations from Maxwell's equations in Sections 4 and 5.

Correct equations for the momentum flux density allow easy calculations of the electromagnetic force applied to the interface between two media with different electromagnetic properties, in particular, with a negative permittivity and a negative magnetic permeability, $\varepsilon < 0$ and $\mu < 0$. This force (the pressure of quasimonochromatic waves) can have different directions with respect to the normal, depending on the parameters of both media.

Thus, based on the results obtained, we can make the following conclusions:

(i) The total Abraham momentum $\mathbf{G}$ of a packet of quasimonochromatic transverse waves in a nonabsorbing isotropic dielectric is conserved without introducing the additional Abraham force: $\mathbf{G} = \int\mathbf{g}\,\mathrm{d}V = \mathrm{const}$.

(ii) The tensor of the momentum flux density of the electromagnetic field is symmetric and is given by the product of the density components g_α of the momentum itself and the group velocity u_β of the packet: $T_{\alpha\beta} = g_\alpha u_\beta = g_\beta u_\alpha$.

(iii) The tensor of the momentum flux density, like the energy density, strongly depends on the frequency derivatives of functions describing the electromagnetic properties of a medium (permittivity and magnetic permeability). This dependence appears due to the group velocity.

(iv) Our results are applicable to both ordinary and left-handed media in which $\varepsilon < 0$ and $\mu < 0$ and the group and phase velocities are directed oppositely.

(v) At the interface between two media, depending on their parameters, either light pressure or light attraction is possible. It results from the pressure on the interface between two media with different electromagnetic properties, which we calculated.

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