Physics – Uspekhi **59** (2) 109 – 120 (2016)

REVIEWS OF TOPICAL PROBLEMS PACS numbers: 05.10.Gg, 05.40.Jc, **44.25.+f**, 92.60.Fm, 92.60.hk

System of convective thermals as a generalized ensemble of Brownian particles

A N Vulfson, O O Borodin

DOI: 10.3367/UFNe.0186.201602a.0113

Contents

Abstract. The system of thermals that makes the fine structure of a turbulent convective layer of a fluid is considered. A simplified probabilistic-geometrical approach is outlined that uses measurements along the observation line to determine the average in-plane parameters of the system. A dynamic equation for an isolated thermal interacting with its environment is derived. A Langevin equation similar to the stochastic equation for an ensemble of 'fast' Brownian particles is constructed for a system of thermals. The nonlinear Langevin equation for such a system leads to the associated kinetic form of the Fokker–Planck equation. It is shown that the stationary solution of the kinetic Fokker–Planck equation is identical to the Maxwell distribution and approximately consistent with the distributions measured in the turbulent convective layer of the atmosphere.

Keywords: stochastic ensemble of convective thermals, nonlinear Langevin equation, Fokker–Planck equation, Maxwell velocity distribution for an ensemble of convective thermals

A N Vulfson, O O Borodin Oil and Gas Research Institute, Russian Academy of Sciences,
ul. Gubkina 3, 119333 Moscow, Russian Federation;
Higher School of Economics National Research University,
ul. Myasnitskaya 20, 101000 Moscow, Russian Federation
E-mail: vulfson@ipng.ru, borodin@ipng.ru

Received 20 August 2015, revised 30 November 2015
Uspekhi Fizicheskikh Nauk **186** (2) 113 – 124 (2016)
DOI: 10.3367/UFNr.0186.201602a.0113
Translated by S D Danilov; edited by A M Semikhatov

1. Introduction

An ensemble of Brownian particles is the best known stochastic ensemble of physical kinetics. Its studies, begun with the research in [1–3], were comprehensively completed in [4, 5].

We assume that spherical Brownian particles with fixed masses and radii move with certain velocities in a medium with uniform density, temperature, and viscosity. Let the Brownian particles be of microscopic or ultra-microscopic size such that their motion in a viscous fluid corresponds to small Reynolds numbers, $\mathrm{Re} \leqslant 1$. According to the Stokes approximation, the drag force p_{md} is proportional to the velocity of particle motion for constant radius and viscosity, i.e., $p_{\mathrm{md}} \sim \mathrm{Re}$, and the related Langevin equation is linear.

A mathematical model of an ensemble of 'fast' Brownian particles was proposed in Refs [6–9]. In this model, the drag force depends arbitrarily on the particle motion velocity, i.e., $p_{\mathrm{md}} = p_{\mathrm{md}}(\mathrm{Re})$, and the corresponding Langevin equation is nonlinear. The application of this model to relativistic Brownian motion is considered in Refs [10, 11].

We note that classical ensembles of Brownian particles are 'thermodynamical' systems because they are placed in a thermostat with a fixed temperature T. Such 'thermodynamical' systems are characterized by a constant value of the thermal velocity squared, $\overline{v^2} = 3k_{\mathrm{B}}T/(2m) = \mathrm{const}$, where k_{B} is the Boltzmann constant and m is the particle mass.

There also exist non-'thermodynamical' stochastic systems, i.e., the systems lacking connection to a thermostat, which can be regarded as analogs of Brownian particles. It is known that in the case of well-developed turbulent convection, a system of localized buoyant vortices (thermals) always

develops over a horizontally homogeneous heated surface; they uniformly populate the convective layer, forming its fine structure [12–14].

The isolines of vertical velocity fields and temperature pulsations separate a sufficiently well-defined warmer rising thermal from the ambient medium. The form of isolines suggests that the thermal be considered a body of rotation around a vertical axis and its form be characterized by the radius of the largest horizontal cross section. This observation allows interpreting the system of thermals as an ensemble of particles.

Convective thermals are spawned from a rather thin surface layer adjacent to the heated surface and rise from there under the action of the Archimedes force. Importantly, the process of their formation is related to surface layer instability and is manifestly stochastic. The interaction of convective vortices among themselves and with their neighborhood is also random, making the system of thermals a stochastic ensemble.

A substantial part of the convective layer is occupied by the so-called layer of intense mixing, where the magnitude of the second turbulent moment of vertical velocity stays practically constant, $\overline{w^2} = \text{const}$. This analogy between a thermostat and the mixed layer allows using random forces of the same structure in the Langevin equations for a system of thermals and for an ensemble of Brownian particles.

In atmospheric conditions, penetrating turbulent convection develops over a heated horizontally homogeneous surface of land or the ocean. The existence of atmospheric thermals and a fine structure in the convective layer were first noted in [15]. Systematic processing of airborne measurements of temperature pulsations in rising thermals was first carried out in Ref. [12]. Measurements with the help of airplane laboratories present the most efficient and comprehensive method to explore turbulent convection in the atmosphere (see, e.g., Refs [16, 17]).

Warmer atmospheric thermals are detected rather reliably by lidars [18] and Doppler radars [19]. Their characteristic sizes range from several dozen centimeters to several dozen meters. The characteristic amplitudes of velocities and positive temperature pulsations are of the order of 0.5 m s^{-1} and 0.3 °C.

Systems of thermals evolve in the oceanic boundary layer subjected to abrupt surface cooling. However, their chaotic motion bears a descending character in this case [20]. An idea of what an ensemble of thermals looks like can be gained from laboratory modeling at large Rayleigh numbers. The results of laboratory experiments [21], presented in Fig. 1, clearly exhibit the chaotic character of the motion of thermals.

In this review, it is assumed that convective thermals are localized warm vortices rising with random vertical velocities. It is additionally assumed that thermals are of equal size and buoyancy. Thus, a system of convective thermals is considered a generalized ensemble of Brownian particles. In the framework of the model proposed, a special Langevin equation is used to describe the system of thermals, with a nonlinear dissipative force and a random force of the structure known for the system of Brownian particles. In other words, the stochastic force in the equation of motion for thermals is the square root of the Einstein diffusion coefficient times Gaussian white noise.

This Langevin equation gives rise to the kinetic Fokker–Planck equation in the phase space of vertical velocities for the probability density of the thermals of a stochastic ensemble. The interpretation of the stochastic integral in the Langevin equation is chosen such that the probability density of thermals satisfies the K-form of the associated Fokker–Planck equation with variable coefficients (see Refs [9; 22, pp. 292–294; 23]). Later, similar equations were considered in [24, 25].

Figure 1. Ensemble of thermals in the form of dense salt fingers descending in a water layer, according to Ref. [21]. The descending motion of thermals is visualized by adding fluorescein to the salt and illuminating the fingers through a slit.

From a kinetic standpoint, the use of the K-form of the Fokker–Planck equation with variable coefficients is essential, because only for this equation do nonstationary distributions over vertical velocities converge to the Maxwell distribution at large times.

The equilibrium distribution of convective thermals over vertical velocities is constructed as a stationary solution of the Fokker–Planck equation in the K-form. It is shown that the Maxwell distribution over vertical velocities, constructed theoretically, agrees qualitatively and quantitatively with known empirical distributions for ascending motions found in field experiments.

The possibility of extending the methods of physical kinetics to turbulent flows of homogeneous fluids was discussed in monographs [26, 27]. The method proposed there turned out to be efficient for describing homogeneous isotropic turbulence. The application of the kinetic approach to the problem of turbulent convection was considered for the first time in Refs [28, 29]. The results of research presented in this review continue and substantiate this work.

2. Turbulent convective layer and its parameters

We consider a static layer of liquid or gas in the field of gravity, bounded from below by a flat surface. Under uniform heating, a turbulent convective layer is formed over the surface.

Let t be the time; x, y, z be the coordinates of a Cartesian reference system with the z axis directed opposite to the gravitational acceleration g and the axes x and y lying on the flat heated surface; $\mathbf{u}$ and w be the components of the velocity vector along the plane xy and the z axis; $\rho(x, y, z, t)$, $\Theta(x, y, z, t)$, and $p(x, y, z, t)$ be the respective local values of the density, potential temperature, and pressure, with $\bar{\rho}(z)$, $\bar{\Theta}(z)$, and $\bar{p}(z)$ being the reference values of density, potential temperature, and pressure, related through the equation of state for an ideal gas and the equation of statics; and $\Theta' = \Theta - \bar{\Theta}$ and $p' = p - \bar{p}$ be the deviations of the potential temperature and pressure from their static background values. The modified pressure and dimensionless pulsations of the potential temperature are denoted as

$$\Phi = \frac{p'}{\bar{\rho}}\,, \qquad \theta = \frac{\Theta'}{\bar{\Theta}}\,. \tag{1}$$

Then the equations of the Boussinesq convection theory in the form given in Refs [30, 31] with a fixed neutrally stratified potential temperature profile take the form

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla\Phi\,, \quad \dfrac{\mathrm{d}w}{\mathrm{d}t} = -\dfrac{\partial\Phi}{\partial z} + g\theta\,, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 0\,, \quad \nabla\mathbf{u} + \dfrac{\partial w}{\partial z} = 0\,. \end{cases} \tag{2}$$

Averaging possible products of w and θ over the area, we obtain various turbulent moments of the convective layer.

Let h be the convective layer height, and $gS_\theta = g\overline{\theta w}_0$ the buoyancy flux per unit area of the underlying surface, of the dimension [m^2 s^{-3}]. The availability of parameters gS_θ and h allows introducing the Deardorff parameters for the velocity and buoyancy [32] in the convective layer:

$$w_\mathrm{D} = h^{1/3}(gS_\theta)^{1/3}, \quad g\theta_\mathrm{D} = h^{-1/3}(gS_\theta)^{2/3}\,. \tag{3}$$

In what follows, we deal with a convective layer that is developed sufficiently well over it vertical extent. In such layers, the Deardorff parameters can be treated as constant: $w_\mathrm{D} = \mathrm{const}$ and $g\theta_\mathrm{D} = \mathrm{const}$.

The lower part of the turbulent convective layer, $0 < z/h < 0.5$, is referred to as a mesolayer of intense convection. This layer is permeated by an ensemble of evolving thermals. In the layer above, $0.5 < z/h < 1.0$, the convective thermals move inertially because, on crossing the level of intense convection $z/h = 0.5$, the mean buoyancy of the thermals drops substantially. The qualitative difference between the lower and upper parts of the convective turbulent layer is fully confirmed by laboratory measurements [33].

Turbulence in the layer of intense convection is described by the theory of local self-similarity (see Ref. [34]). According to Ref. [35], the second turbulent moment of the vertical velocity in the turbulent layer can be approximately expressed as

$$\frac{\overline{w^2}}{w_\mathrm{D}^2} = \lambda_{ww}\left(\frac{z}{h}\right)^{2/3}\left(1 - 0.8\,\frac{z}{h}\right)^2. \tag{4}$$

The coefficient $\lambda_{ww} = 1.8$ was obtained in the process of direct measurements of the near-surface layer in the Minnesota 1973 Atmospheric Boundary Layer Experiment and the Air-Mass Transportation Experiment 1975 (for details, see Refs [35, 36]).

Some data from field measurements of the second vertical velocity moment in the atmospheric mesolayer of intense convection [35] are presented in Fig. 2, together with the results of laboratory experiments [37, 38]. These data indicate that the dependence of $\overline{w^2}$ on the height z is relatively weak in the layer $0.1 < z/h < 0.5$.

The observed scatter of points, which is a result of the stochastic character of convection, accompanies all measurements in turbulent flows.

Using the classification in Ref. [39], we single out the following layers in the mesolayer of intense convection:

(1) the layer of free convection $0 < z/h < 0.1$;

(2) the well-mixed layer $0.1 < z/h < 0.5$.

For developed turbulence, the layer of free convection is the fluid layer adjacent to the heated surface. It is defined as the layer of constant flux, $gS_\theta = \mathrm{const} > 0$.

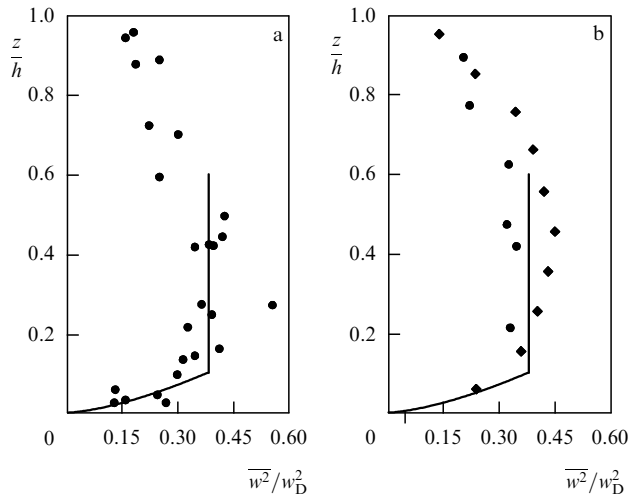

Figure 2. Values of dimensionless second moments of the vertical velocity. The vertical lines correspond to the relation $\overline{w^2}/w_\mathrm{D}^2 = 0.38$. (a) The dots represent empirical values of w^2/w_D^2 in the atmosphere based on the data of AMTEX-1975 according to Ref. [35]. (b) The diamonds and dots represent empirical values of $\overline{w^2}/w_\mathrm{D}^2$ in the fluid according to respective laboratory experiments in Refs [37] and [38].

The turbulent moments in the layer of free convection $0 < z/h < 0.1$ can be determined on the basis of the self-similarity theory (see Refs [34, 40–43] for details).

The turbulent moments in the mixed layer $0.1 < z/h < 0.5$, of interest for physical kinetics, satisfy the relations

$$\bar{w} = 0\,, \quad \frac{\mathrm{d}\bar{\theta}}{\mathrm{d}z} = 0\,, \quad \overline{w^2} = 0.38 w_\mathrm{D}^2\,. \tag{5}$$

The first equality in (5), following from the continuity equation, is valid across the entire convective layer. The second equality in (5) shows that the stratification of the potential temperature is close to the neutral one within the mixed layer, which allows this layer to be considered an analog of a homogeneous fluid layer. The third equality in (5) indicates the spatial uniformity of the second moment of vertical velocity (see Fig. 2), which allows considering the mixed layer to be an analog of a thermostat in statistical mechanics.

3. Empirical mean parameters of the system of convective thermals in the mixed layer and their approximation

It is essential that airborne measurements of the parameters characterizing systems of thermals are performed along certain horizontal lines of observation [12, 14, 16]. When a system of thermals is carried by the wind, the data obtained at meteorological towers and masts [13, 14] also correspond to measurements along some horizontal line of observations.

Let $\tilde{R}_l$ and $g\tilde{\theta}_l$ be the mean radii and buoyancies for thermals in the ensemble along the line of observations l located at a level z. The measurements in [12, 16, 45] provided field data on the mean values of $\tilde{R}_l$ and $\tilde{\theta}_l$ as functions of the dimensionless height z/h (Fig. 3).

According to Refs [16, 45], the mean radius $\tilde{R}_l$ and buoyancy $g\tilde{\theta}_l$ vary sharply in the free convective layer $0 < z/h < 0.1$ and remain practically constant in the layer of

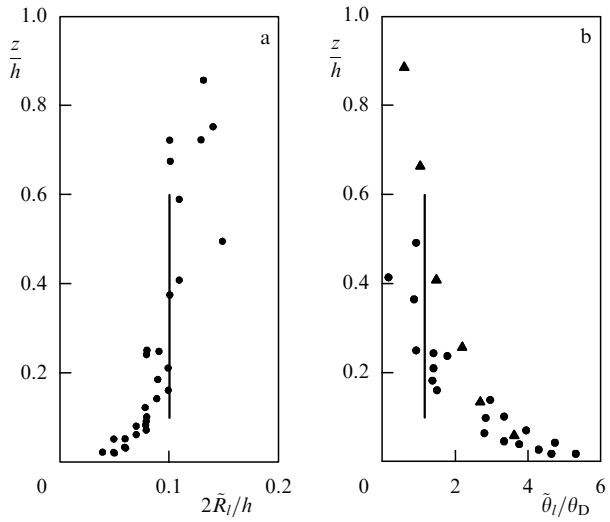

Figure 3. The dependence of mean values of (a) the diameters $2\tilde{R}_l/h$ and (b) the buoyancy $\tilde{\theta}_l/\theta_{\rm D}$ on the dimensionless height z/h for a system of warm updrafts in a convective layer based on observational data in Ref. [16] (dots) and Ref. [45] (triangles). The vertical lines correspond to the approximation (6) of data in the mixed layer.

intense mixing $0.1 < z/h < 0.5$ (see Fig. 3). Using similarity theory, we assume that the mean radius and buoyancy in the mixed layer are approximately described by

$$\frac{2\tilde{R}_l}{h} = 0.1\,, \qquad \frac{g\tilde{\theta}_l}{g\theta_{\rm D}} = 1.12\,. \tag{6}$$

The first relation in (6) shows that the mean diameter of thermals $2\tilde{R}_l$ is the thickness of the free convection layer $0.1h$. The second relation in (6) means that the mean dimensionless temperature of thermals $\tilde{\theta}_l$ is defined by the Deardorff parameter $\theta_{\rm D}$. A comparison of approximations (6) with observational data [16, 45] is presented in Fig. 3.

Let $\widehat{R}_a$ and $\widehat{\theta}_a$ be mean values of the radius and buoyancy for an ensemble of thermals on the horizontal plane a located at a level z. Obviously, the cross section of a system of spherical thermals by the horizontal plane $z = \text{const}$ forms a system of randomly placed circles in plane a. An aircraft conducting measurements crosses only a part of these circles along random chords. Because the length of a chord never exceeds the diameter, we can argue that $\widehat{R}_a \geqslant \tilde{R}_l$. The correspondence between the mean parameters on a plane and along a line was thoroughly discussed in Refs [12–14]. In what follows, we assume that

$$\widehat{R}_a = \frac{4}{\pi}\,\tilde{R}_l\,, \qquad \widehat{\theta}_a = \tilde{\theta}_l\,. \tag{7}$$

A justification of (7) based on the method of geometric probabilities is presented in Appendices A and B.

It follows from Eqns (5)–(7) that the mean parameters $\widehat{R}_a$, $g\widehat{\theta}_a$, and $\overline{w^2}$ satisfy the relation

$$\frac{2}{3}\,g\widehat{\theta}_a\widehat{R}_a = 0.04753 w_{\rm D}^2 = 0.125\,\overline{w^2}\,. \tag{8}$$

The existence of an equality similar to (8) also follows from laboratory studies with isolated bubbles rising in a homogeneous fluid [46, 47]. These experiments led to the empirical relation

$$\frac{2}{3}\,g\widehat{\theta}R_{\rm s} = \alpha_{\rm S}\widehat{w}^{\,2}\,, \tag{9}$$

where $\widehat{w}$ and $\widehat{\theta}$ are the volume mean values of the vertical velocity and potential temperature pulsations, $R_{\rm s}$ is the maximum bubble cross section radius, and $\alpha_{\rm S}$ is a constant dimensionless coefficient: $\alpha_{\rm S} \simeq 0.2$ according to Ref. [46] and $\alpha_{\rm S} \simeq 0.26$ according to Ref. [47].

In analogy with the model of an ideal gas, we consider an idealized ensemble of 'isolated' thermals, each moving as if the others were absent. We use the averaged invariant (9) to describe the ensemble of thermals with differing values of buoyancy and radii; then $(2/3)\langle g\widehat{\theta}R_{\rm s}\rangle = \alpha_{\rm S}\langle\widehat{w}^2\rangle$. Because thermals ascend in the ambient fluid that is practically at rest, we can set $\langle\widehat{w}^2\rangle = \overline{w^2}$. As a consequence, the averaged invariant for the ensemble of isolated thermals can be written as

$$\frac{2}{3}\,g\widehat{\theta}_a\widehat{R}_a = 0.26\overline{w^2}\,. \tag{10}$$

Obviously, the thermals of a real convective ensemble interact with each other, because they are situated sufficiently close (see Fig. 1) Thus, the mean distance between convective thermals is approximately $3\widehat{R}_a$ according to Ref. [12]. The thermals of the ensemble therefore expand less than their free isolated counterparts. This explains the reduced value of the numerical factor in the right-hand side of (8) with respect to the numerical factor in the right-hand side of Eqn (10).

4. Equation of motion for an isolated thermal in the mixed layer

An isolated thermal is a basis element of turbulent convection (see Refs [48–50]). We consider dynamical equations for an isolated thermal in more detail.

Let $\psi(x, y, z, t) = 0$ be an arbitrary smooth closed surface with the area $S_\psi = S_\psi(t)$, and let $V_\psi = V_\psi(t)$ be the volume bounded by this surface; we also let $\mathbf{n}$ be the normal, $D_{\rm n}$ be the normal velocity of the surface, and $b = b(x, y, z, t)$ be an arbitrary substance filling the volume V_ψ. Following Ref. [51], we use the theorem on differentiation of a moving volume, whence

$$\frac{\rm d}{{\rm d}t}\int_{V_\psi} b\,{\rm d}v = \int_{V_\psi}\frac{\partial b}{\partial t}\,{\rm d}v + \int_{S_\psi} bD_{\rm n}\,{\rm d}s\,. \tag{11}$$

With Eqn (11), integrating the equations of motion and the entropy conservation equation in Boussinesq form (2) over an arbitrary volume $V_\psi = V_\psi(t)$, we find

$$\begin{cases} \dfrac{\rm d}{{\rm d}t}\displaystyle\int_{V_\psi} w\,{\rm d}v + \int_{S_\psi} w(v_{\rm n} - D_{\rm n})\,{\rm d}s = g\int_{V_\psi}\theta\,{\rm d}v + \frac{p_{\rm md}}{\rho}\,, \\ \dfrac{\rm d}{{\rm d}t}\displaystyle\int_{V_\psi}\theta\,{\rm d}v + \int_{S_\psi}\theta(v_{\rm n} - D_{\rm n})\,{\rm d}s = 0\,. \end{cases} \tag{12}$$

Here, $v_{\rm n}$ is the normal velocity on the moving surface, $p_{\rm md}$ is the pressure-related surface force exerted on the thermal by the ambient medium, which is different from the Archimedes force, and $\rho = \text{const}$ is the ambient density.

We introduce the parameters averaged over the volume V_ψ: the mean velocity $\widehat{w}$ and the perturbation of the dimensionless potential temperature $\widehat{\theta}$,

$$\widehat{w} = \frac{1}{V_\psi}\int_{V_\psi} w\,{\rm d}v\,, \qquad \widehat{\theta} = \frac{1}{V_\psi}\int_{V_\psi}\theta\,{\rm d}v\,. \tag{13}$$

Substituting (13) in Eqns (12), we obtain

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\,\widehat{w}\,V_\psi + \displaystyle\int_{S_\psi} w(v_\mathrm{n} - D_\mathrm{n})\,\mathrm{d}s = g\widehat{\theta}\,V_\psi + \dfrac{p_\mathrm{md}}{\rho}\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\,\widehat{\theta}\,V_\psi + \displaystyle\int_{S_\psi} \theta(v_\mathrm{n} - D_\mathrm{n})\,\mathrm{d}s = 0\,. \end{cases} \tag{14}$$

A convective thermal can be identified either as the closed surface $w = 0$ or as the closed surface $\theta = 0$. Both definitions lead to some simplification in Eqns (14).

Following Ref. [12], we assume that a rising thermal is the relatively warmer spatial domain $\theta \geqslant 0$ bounded by the moving surface $\theta = 0$. Then, for $\psi = \theta$, we obtain

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\,\widehat{w}\,V_\theta + \displaystyle\int_{S_\theta} w(v_\mathrm{n} - D_\mathrm{n})\,\mathrm{d}s = g\widehat{\theta}\,V_\theta + \dfrac{p_\mathrm{md}}{\rho}\,, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\,\widehat{\theta}\,V_\theta = 0\,. \end{cases} \tag{15}$$

We assume that at any instant of time, the thermal is a sphere with a variable radius R and the center of mass $\widehat{z}$. It is also assumed that the thermal maintains its spherical form in the process of vertical motion. Then

$$\begin{aligned} &\frac{\mathrm{d}\widehat{z}}{\mathrm{d}t} = \widehat{w}\,, \qquad \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\mathrm{d}R}{\mathrm{d}\widehat{z}}\,\frac{\mathrm{d}\widehat{z}}{\mathrm{d}t} = \alpha_R(\widehat{z})\widehat{w}\,, \\ &\frac{1}{V_\theta}\,\frac{\mathrm{d}V_\theta}{\mathrm{d}t} = \frac{3}{R}\,\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{3}{R}\,\alpha_R(\widehat{z})\widehat{w}\,, \end{aligned} \tag{16}$$

where $\alpha_R(\widehat{z})$ is the entrainment coefficient for the thermal.

We next set $\alpha_R(\widehat{z}) = \alpha_R = \mathrm{const}$. According to experimental observations, the entrainment coefficient is $\alpha_R = 0.2-0.26$ above individual buoyant thermals (for details, see Refs [46, 47]).

In the approximation where the ambient medium does not move, $w = 0$ and $\Phi = 0$, and therefore the surface integral as well as the surface interaction force between the thermal and the ambient medium are absent, $p_\mathrm{md} = 0$. From Eqns (15) and (16), it then follows that

$$\begin{cases} \dfrac{\mathrm{d}\widehat{z}}{\mathrm{d}t} = \widehat{w}\,, \quad \dfrac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = g\widehat{\theta} - \dfrac{3}{R}\,\alpha_R\widehat{w}^{\,2}\,, \\ \dfrac{\mathrm{d}\widehat{\theta}}{\mathrm{d}t} = -\dfrac{3}{R}\,\alpha_R\widehat{w}\widehat{\theta}\,, \quad \dfrac{\mathrm{d}R}{\mathrm{d}t} = \alpha_R\widehat{w}\,. \end{cases} \tag{17}$$

The model of thermal in (17), based on the approximation of an ambient medium at rest, was considered in Refs [48–50].

We note that for $\theta = 0$, Boussinesq equations (2) become the equations of a homogeneous fluid. Therefore, in constructing a model for the motion of a thermal with its interaction with a neutrally stratified ambient medium taken into account, it is reasonable to rely on the equations of motion for a rigid body in a homogeneous medium.

It is known that the equation describing the uniform motion of a rigid body in a homogeneous fluid [52] includes the vortex drag force

$$p_\mathrm{md} = -\frac{1}{2}\,c_\mathrm{d}^0 S\rho\widehat{w}^{\,2}\,, \tag{18}$$

where $\rho = \mathrm{const}$ is the fluid density, S is the area of the largest cross section, and c_d^0 is the effective coefficient of vortex drag exerted on the rigid body. In particular, for a uniformly moving sphere of constant radius R,

$$p_\mathrm{md} = -\frac{1}{2}\,c_\mathrm{d}^0\pi R^2\rho\widehat{w}^{\,2} = -\frac{3}{8}\,c_\mathrm{d}^0 V\rho\,\frac{\widehat{w}^{\,2}}{R}\,, \qquad V = \frac{4}{3}\,\pi R^3\,. \tag{19}$$

With (19), the equation of motion for a sphere in a homogeneous ideal fluid under the action of the buoyancy force can be written as

$$\rho_\mathrm{S} V\,\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = -(\rho_\mathrm{S} - \rho)Vg - \frac{3}{8}\,c_\mathrm{d}^0\rho V\,\frac{\widehat{w}^{\,2}}{R}\,, \tag{20}$$

where $\rho_\mathrm{S} = \mathrm{const}$ is the sphere density.

In [53, 54], the motion of a thermal is described by using the equation of motion for a sphere with constant radius (20) with the vortex drag coefficient $c_\mathrm{d} \neq c_\mathrm{d}^0$. In the framework of vortex model [53], a theoretical estimate $c_\mathrm{d} \leqslant 9/8$ is available for the dimensionless vortex drag coefficient.

We mention that Eqn (20) includes vortex drag force (18) but not the force exerted on the rigid sphere by the ambient fluid due to the added mass. This force arises for accelerated motion and is manifested as an opposing force equal to the added mass times the acceleration. If the rigid body is a sphere, the more precise expression for the opposing force p_md becomes

$$\begin{cases} p_\mathrm{md} = -\mu\rho V\,\dfrac{\mathrm{d}\widehat{w}}{\mathrm{d}t} - \dfrac{1}{2}\,c_\mathrm{d}^0 S\rho\widehat{w}^{\,2}\,, \\ \mu = \dfrac{1}{2}\,, \quad V = \dfrac{4}{3}\,\pi R^3\,, \quad S = \pi R^2\,. \end{cases} \tag{21}$$

Modifying (20) by taking opposing force (21) into account (including the added mass), we obtain

$$\left(\rho_\mathrm{S} + \frac{1}{2}\,\rho\right) V\,\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = -(\rho_\mathrm{S} - \rho)Vg - \frac{3}{8}\,c_\mathrm{d}^0\rho V\,\frac{\widehat{w}^{\,2}}{R}\,. \tag{22}$$

For a hard sphere with the density close to that of the fluid, $\rho_\mathrm{S} \approx \rho$, Eqn (22) takes the limit form

$$\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = -\frac{2}{3}\,g\,\frac{\rho_\mathrm{S} - \rho}{\rho} - \frac{1}{4}\,c_\mathrm{d}^0\,\frac{\widehat{w}^{\,2}}{R}\,. \tag{23}$$

Notably, relation (23) has a hydrodynamic origin, as shown in monograph [55].

We consider the case where a sphere with a rigid shell can change its radius. In this situation, in order to construct a heuristic equation of motion, it is reasonable to use an analog of the Meshcherskii equation instead of Eqn (22),

$$\rho_\mathrm{S}\,\frac{\mathrm{d}}{\mathrm{d}t}(\widehat{w}V) = -(\rho_\mathrm{S} - \rho)Vg - \frac{3}{8}\,c_\mathrm{d}^0\rho V\,\frac{\widehat{w}^{\,2}}{R} - \frac{1}{2}\,\rho V\,\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t}\,. \tag{24}$$

With Eqns (16), Eqn (24) is modified to

$$\left(\rho_\mathrm{S} + \frac{1}{2}\,\rho\right)\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = -(\rho_\mathrm{S} - \rho)\,g - \frac{3}{8}\,c_\mathrm{d}^0\rho\,\frac{\widehat{w}^{\,2}}{R} - 3\rho_\mathrm{S}\,\frac{\widehat{w}}{R}\,\frac{\mathrm{d}R}{\mathrm{d}t}\,. \tag{25}$$

The question of the motion of a sphere with a rigid shell of variable radius in an ideal fluid is considered in [55], where it is shown that with the use of the hydrodynamical method and a set of simplifying assumptions, the equation of motion becomes similar to Eqn (25). An approximate equation proposed in Ref. [55] can be obtained by replacing the factor 3 in the last term in Eqn (25) by 3/2.

We construct a heuristic equation for a sphere with a rigid shell of variable radius assuming that the sphere density is close to that of the fluid, $\rho_S \approx \rho$. Relying on Eqn (25) and taking entrainment coefficient (16) into account, we obtain

$$\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = -\frac{2}{3}\, g\, \frac{\rho_S - \rho}{\rho} - \left(\frac{1}{4}\, c_d^0 + 2\alpha_R\right) \frac{\widehat{w}^2}{R}\,. \tag{26}$$

Based on the foregoing, we turn to the description of the motion of thermals in a moving adiabatic medium, $\mathrm{d}\bar{\theta}/\mathrm{d}z = 0$, $\theta = 0$.

Using Eqn (19), we approximate the surface integral in Eqn (15) as

$$\int_{S_\theta} w(v_n - D_n)\,\mathrm{d}s = \frac{3}{8}\, c_d^1 V \frac{\widehat{w}^2}{R}\,. \tag{27}$$

Inserting (27) in Eqn (15) and taking the effect of added mass into account, we obtain

$$\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = \frac{2}{3}\, g\theta - \left(\frac{1}{4}\, c_d + 2\alpha_R\right) \frac{\widehat{w}^2}{R}\,. \tag{28}$$

We note that the vortex drag coefficient $c_d = c_d^0 + c_d^1$ can differ to some degree from that for a hard sphere, c_d^0.

In the approximation of convection theory, relation (28) is fully analogous to the equation of motion for a sphere with a rigid shell of variable radius (26).

We conclude that the system of equations for an individual thermal moving in a mixed layer and interacting with a neutrally stratified medium can be written as

$$\begin{cases} \dfrac{\mathrm{d}\widehat{z}}{\mathrm{d}t} = \widehat{w}\,, \quad \dfrac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = \dfrac{2}{3}\, g\widehat{\theta} - \left(\dfrac{1}{4}\, c_d + 2\alpha_R\right) \dfrac{\widehat{w}^2}{R}\,, \\ \dfrac{\mathrm{d}\widehat{\theta}}{\mathrm{d}t} = -\dfrac{3}{R}\, \alpha_R \widehat{w}\,\widehat{\theta}\,, \quad \dfrac{\mathrm{d}R}{\mathrm{d}t} = \alpha_R \widehat{w}\,, \quad c_d \leqslant \dfrac{9}{8}\,. \end{cases} \tag{29}$$

The model of an isolated thermal in (29) is similar to those of thermals in Refs [53, 54, 56], which use slightly different forms of the equation of motion.

5. Equations of motion for thermals of an ensemble in the mixed layer

We now consider an ensemble of thermals in a mixed layer. Obviously, the thermals of the ensemble move in a random medium and interact with each other, and therefore they differ somewhat from isolated mixed-layer thermals.

We suppose that the action of surrounding thermals on thermal i can be described by including a random rapidly oscillating force $q_i(\widehat{w}_i, t)$ in equation of motion (29),

$$\begin{cases} \dfrac{\mathrm{d}\widehat{\theta}_i}{\mathrm{d}t} = -\dfrac{3}{R_i}\, \alpha_R \widehat{w}_i\, \widehat{\theta}_i\,, \quad \dfrac{\mathrm{d}R_i}{\mathrm{d}t} = \alpha_R \widehat{w}_i\,, \\ \dfrac{\mathrm{d}\widehat{w}_i}{\mathrm{d}t} = \dfrac{2}{3}\, g\widehat{\theta}_i - \gamma(\widehat{w}_i, R_i)\widehat{w}_i + q_i(\widehat{w}_i, t)\,, \\ \gamma(\widehat{w}_i, R_i) = \left(\dfrac{c_d}{4} + 2\alpha_R\right) \dfrac{\widehat{w}_i}{R_i}\,, \quad c_d \leqslant \dfrac{9}{8}\,. \end{cases} \tag{30}$$

Here, $\gamma(\widehat{w}_i, R_i)$ is the mobility coefficient, which linearly depends on the velocity and has the dimension [s^{-1}].

We use the approximation $\alpha_R = 0$ assuming that the interaction of any thermal with the neighboring elements of the ensemble limits the variation in its radius as the height is changed. In this situation, the individual radii of thermals in the ensemble stay fixed, which fully agrees with the approximation of experimental data (Fig. 3a).

In the approximation $\alpha_R = 0$, the dynamical equations for the ensemble of thermals in the mixed layer take the form

$$\begin{cases} \dfrac{\mathrm{d}\widehat{\theta}_i}{\mathrm{d}t} = 0\,, \quad \dfrac{\mathrm{d}R_i}{\mathrm{d}t} = 0\,, \\ \dfrac{\mathrm{d}\widehat{w}_i}{\mathrm{d}t} = \dfrac{2}{3}\, g\widehat{\theta}_i - \gamma(\widehat{w}_i, R_i)\widehat{w}_i + q_i(\widehat{w}_i, t)\,, \\ \gamma(\widehat{w}_i, R_i) = \dfrac{c_d}{4}\, \dfrac{\widehat{w}_i}{R_i}\,, \quad c_d \leqslant \dfrac{9}{8}\,. \end{cases} \tag{31}$$

Let all the thermals in the ensemble share identical values of temperature pulsations $\widehat{\theta}_a$ and radius $\widehat{R}_a$. These assumptions allow integrating the first two equations of system (31) with mean parameters (6) and (7). Omitting the index i of the individual thermal, we obtain

$$\begin{cases} \dfrac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = \dfrac{2}{3}\, g\widehat{\theta}_a - \gamma(\widehat{w})\widehat{w} + q(\widehat{w}, t)\,, \\ \gamma(\widehat{w}) = \dfrac{1}{4}\, c_d\, \dfrac{\widehat{w}}{\widehat{R}_a}\,, \quad \dfrac{2}{3}\, g\widehat{\theta}_a \widehat{R}_a = \dfrac{1}{8}\, \overline{w^2}\,, \quad c_d \leqslant \dfrac{9}{8}\,. \end{cases} \tag{32}$$

In the framework of the proposed ensemble model, the mixed-layer thermals have equal buoyancy, radii, and random vertical velocities. Based on measurements in [12, 16, 45], the number of convective thermals in the mixed layer can also be considered constant.

We specify the magnitude of a rapidly oscillating force $q(\widehat{w}, t)$, in analogy with that for an ensemble of Brownian particles, by using the product of the generalized Einstein diffusion coefficient $D_E(\widehat{w}) = \gamma(\widehat{w})(\overline{w^2})$ and a time-dependent function $\xi(t)$, where $\xi(t)$ is Gaussian white noise. Thus, following [8, 57, 58], we obtain

$$\begin{cases} q(\widehat{w}, t) = \sqrt{D_E(\widehat{w})}\, \xi(t)\,, \quad D_E(\widehat{w}) = \gamma(\widehat{w})(\overline{w^2})\,, \\ \langle \xi(t) \rangle = 0\,, \quad \langle \xi(t)\xi(t') \rangle = 2\delta(t - t')\,. \end{cases} \tag{33}$$

Here, the Einstein diffusion coefficient $D_E(\widehat{w})$ is expressed in units [$\mathrm{m}^2\ \mathrm{s}^{-3}$], $\xi(t)$ is a random function of time with dimension [$\mathrm{s}^{-1/2}$], $\delta(t - t')$ is the Dirac delta function, t and t' are arbitrary time instants, and angular brackets denote ensemble averaging. The relation $\langle \xi(t) \rangle = 0$ in Eqns (33) naturally follows from the validity of the equation of motion (29) for an isolated thermal being applied to the motion of thermals in the ensemble. The last relation in Eqns (33) expresses the property of being delta-correlated.

The coefficients of system (33) allow some transformations of the buoyancy parameter $g\widehat{\theta}_a$ in the equations of motion. Indeed, let β be the numerical factor satisfying $\beta c_d = 1$. It then follows from equality (8) that

$$\begin{aligned} \frac{2}{3}\, g\widehat{\theta}_a &= \frac{1}{8}\, \frac{\overline{w^2}}{\widehat{R}_a} = \frac{1}{2}\, \beta\, \frac{c_d}{4}\, \frac{\overline{w^2}}{\widehat{R}_a} = \frac{1}{2}\, \beta\, \frac{\partial}{\partial \widehat{w}} \left(\frac{1}{4}\, \frac{c_d}{\widehat{R}_a}\, \widehat{w}\, \overline{w^2}\right) \\ &= \frac{1}{2}\, \beta\, \frac{\partial}{\partial \widehat{w}}\, \gamma(\widehat{w})\overline{w^2} = \frac{1}{2}\, \beta\, \frac{\partial}{\partial \widehat{w}}\, D_E(\widehat{w})\,. \end{aligned} \tag{34}$$

Manipulations with Eqns (32) using Eqns (33) and (34) lead to a nonlinear stochastic equation for the ensemble of convec-

tive thermals in the form [24]

$$\frac{\mathrm{d}\widehat{w}}{\mathrm{d}t} = \frac{1}{2}\,\beta\,\frac{\partial}{\partial\widehat{w}}\,D_{\mathrm{E}}(\widehat{w}) - \gamma(\widehat{w})\widehat{w} + \sqrt{D_{\mathrm{E}}(\widehat{w})}\,\xi(t)\,,$$
$$\gamma(\widehat{w}) = \frac{c_{\mathrm{d}}}{4}\,\frac{\widehat{w}}{\widehat{R}_a}\,,\quad D_{\mathrm{E}}(\widehat{w}) = \gamma(\widehat{w})(\overline{w^2})\,,\quad \beta c_{\mathrm{d}} = 1\,,\quad c_{\mathrm{d}} \leqslant \frac{9}{8}\,,$$
$$\langle\xi(t)\rangle = 0\,,\quad \langle\xi(t)\xi(t')\rangle = 2\delta(t-t')\,. \tag{35}$$

Setting $c_{\mathrm{d}} = 1$, we find $\beta = 1$. In this case, the special form of (35) is similar to the nonlinear Langevin equation in a thermostat in the interpretation of Stratonovich, proposed in Refs [7–9, 23].

It is essential that nonlinear Langevin equation (35) is valid in the layer of intense mixing, $0.1h \leqslant z \leqslant 0.5h$.

6. Vertical homogeneity of the velocity probability density of convective thermals

The presence of a random force in Eqn (35) necessitates a statistical description of the ensemble of convective thermals of the elementary layer.

Let a be a horizontal plane located at an arbitrary level z, $0.1h \leqslant z \leqslant 0.5h$. We can define the probability density $f_{aw}(z,\widehat{w},t)$ in the plane a.

We assume that $n_a^w(z,\widehat{w},t)\,\mathrm{d}\widehat{w}$ is the number of thermals per unit area of plane a with velocities in the interval from $\widehat{w}$ to $\widehat{w}+\mathrm{d}\widehat{w}$. Let $n_a(z,t)$ be the total number of thermals per unit area of plane a, or the concentration of thermals in the plane. Using a statistical definition, we suppose that the probability of a thermal appearing in an elementary phase volume $\mathrm{d}\widehat{w}$ on the plane a is given by

$$f_{aw}(z,\widehat{w},t)\,\mathrm{d}\widehat{w} = \frac{n_a^w(z,\widehat{w},t)\,\mathrm{d}\widehat{w}}{n_a(z,t)}\,,$$
$$\int_0^\infty n_a^w(z,\widehat{w},t)\,\mathrm{d}\widehat{w} = n_a(z,t)\,. \tag{36}$$

We assume that the approximation based on separation of variables is valid for the functions $n_a^w(z,\widehat{w},t)$,

$$n_a^w(z,\widehat{w},t) = M_a(z)\,m_a^w(\widehat{w},t)\,, \tag{37}$$

where $M_a(z)$ and $m_a^w(\widehat{w},t)$ are the multipliers into which it is factored. Relation (37) is referred to as the condition of vertical homogeneity in what follows.

Substituting (37) in Eqn (36), we arrive at the approximation for the concentration of thermals in the plane:

$$n_a(z,t) = M_a(z)m_a(t)\,,\quad m_a(t) = \int_0^\infty m_a^w(\widehat{w},t)\,\mathrm{d}\widehat{w}\,. \tag{38}$$

Here, the function $M_a(z)$ characterizes the change in the concentration of thermals with height at a fixed time instant t. The function $m_a(t)$ characterizes the change in the concentration of thermals with time at a fixed level z.

The existence of approximation (38) for the concentration of atmospheric thermals in the layer of intense mixing $0.1h \leqslant z \leqslant 0.5h$ is confirmed by measurements [12] (Fig. 4).

For the convective layer height $h \approx 2500$ m, the layer of intense mixing $0.1 \leqslant z/h \leqslant 0.5$ is bounded by the heights 250 m $\leqslant z \leqslant$ 1250 m. Thus, the data plotted in Fig. 4 should be considered an empirical justification of approximation (38) and the condition of vertical homogeneity (37).

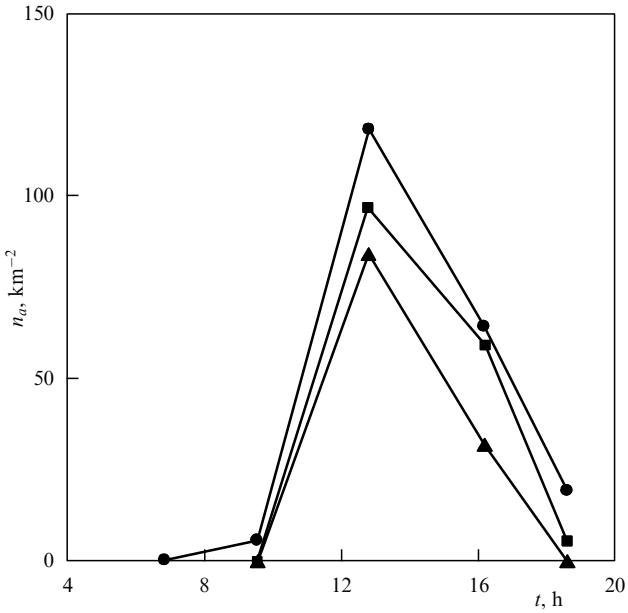

Figure 4. Diurnal evolution of the concentration of thermals in the mixed layer. The dots, squares, and triangles represent the respective results of measurements of the concentration of thermals at the height of 300, 500, and 1000 m.

Substituting relations (37) and (38) in definition (36), we obtain

$$f_{aw}(z,\widehat{w},t)\,\mathrm{d}\widehat{w} = \frac{n_a^w(z,\widehat{w},t)\,\mathrm{d}\widehat{w}}{n_a(z,t)} = \frac{m_a^w(\widehat{w},t)\,\mathrm{d}\widehat{w}}{m_a(t)} = f_w(\widehat{w},t)\,\mathrm{d}\widehat{w}\,. \tag{39}$$

Hence, for any level z, the equality $f_{aw}(z,\widehat{w},t) \equiv f_w(\widehat{w},t)$ holds; it expresses the property of vertical homogeneity of the probability density in the layer of intense mixing $0.1h \leqslant z \leqslant 0.5h$.

7. Probability density of the distribution of convective thermals over velocities. Nonlinear Langevin equation and the kinetic Fokker–Planck equation

The metric space $\Omega_w^+ = \{\widehat{w} : 0 \leqslant \widehat{w} < \infty\}$ is referred to as the Maxwell space in what follows. An ensemble of convective thermals in the Maxwell space is depicted by a 'cloud' of particles. As the ensemble of convective thermals moves, the cloud of points flows in the phase space as a continuous medium with the density $f_w(\widehat{w},t)$.

We introduce the phase fluid whose velocity in the metric Maxwell space is defined by the Langevin equation like Eqn (35).

Monographs [8, 57–59] show that for ensembles of particles satisfying the Langevin equations, the probability densities $f_w(\widehat{w},t)$ satisfy the associated Fokker–Planck equations. We note that in the presence of Gaussian white noise in the stochastic Langevin equation, the correlator of Eqn (33) includes the Dirac generalized function. The process $\xi(t)$ is therefore a generalized stochastic process. A mathematical description of generalized stochastic processes is possible only on the basis of an integral representation. The stochastic integral is defined as the limit of the integral sum. A rectangle in the integral sum can be computed in

several ways, depending on the distribution of height in the elementary subdivision interval (for details, see Ref. [60]). Hence, the uniqueness of the computation of the integral sum is controlled by the internal parameter $0 \leqslant \alpha \leqslant 1$, which is related to the interpretation of the stochastic integral. In particular, the case $\alpha = 0$ corresponds to the choice of the height of the rectangle on the left boundary of the elementary subdivision interval and leads to the Itô interpretation. The case $\alpha = 1/2$ corresponds to the height selection in the middle of the elementary subdivision interval and leads to the Stratonovich interpretation. The case $\alpha = 1$ corresponds to the height selection on the right boundary of the elementary subdivision interval and leads to the Hänggi–Klimontovich interpretation (for details, see Refs [10, 60]).

According to Ref. [24], for a fixed parameter $0 \leqslant \alpha \leqslant 1$, the Fokker–Planck equation associated with Eqn (35) takes the form

$$\frac{\partial f_w}{\partial t} = \frac{\partial}{\partial \widehat{w}} \left(\gamma(\widehat{w}) \widehat{w} f_w + D_{\mathrm{E}}(\widehat{w}) \frac{\partial f_w}{\partial \widehat{w}} \right) - (\beta + 2\alpha - 2) \frac{1}{2} \frac{\partial f_w}{\partial \widehat{w}} \frac{\partial D_{\mathrm{E}}(\widehat{w})}{\partial \widehat{w}} . \tag{40}$$

If we choose the parameter of interpretation α from the condition

$$\beta + 2\alpha - 2 = 0 , \tag{41}$$

then the related Fokker–Planck equation takes the form of the kinetic equation

$$\frac{\partial f_w}{\partial t} = \frac{\partial}{\partial \widehat{w}} \left(\gamma(\widehat{w}) \widehat{w} f_w + D_{\mathrm{E}}(\widehat{w}) \frac{\partial f_w}{\partial \widehat{w}} \right) . \tag{42}$$

Equation (42) was considered in Refs [7–9] (see also Refs [61, 62]).

In the framework of the vortex model [53], the vortex drag coefficient c_{d} remains undefined, but is bounded by the inequality $c_{\mathrm{d}} \leqslant 9/8$. We assume the vortex drag coefficient $c_{\mathrm{d}} = 0.5$; then $\beta = 1/c_{\mathrm{d}} = 2$ and hence $\alpha = 0$. In this case, nonlinear Langevin equation (35) corresponds to the Itô interpretation.

If we assume the vortex drag coefficient $c_{\mathrm{d}} = 1$, then $\beta = 1/c_{\mathrm{d}} = 1$ and hence $\alpha = 1/2$. In this case, Langevin equation (35) corresponds to the Stratonovich interpretation.

We consider the properties of the K-form of the Fokker–Planck equation (42) in more detail. It is obvious that any solution of Eqn (42), by construction, satisfies the normalization condition

$$\int_0^\infty f_w(\widehat{w}, t) \, \mathrm{d}\widehat{w} = 1 . \tag{43}$$

Additionally, solving kinetic Fokker–Planck equation (42) requires that initial and boundary conditions be set. As the initial condition, we can use an arbitrary positive function $f_w^0 = f_w^0(\widehat{w})$ such that

$$f_w(\widehat{w}, 0) = f_w^0(\widehat{w}) > 0 , \quad \int_0^\infty f_w^0(\widehat{w}) \, \mathrm{d}\widehat{w} = 1 . \tag{44}$$

We take the following relations as the boundary conditions:

$$\lim_{\widehat{w} \to +\infty} \left[\gamma(\widehat{w}) \widehat{w} f_w + D_{\mathrm{E}}(\widehat{w}) \frac{\partial f_w}{\partial \widehat{w}} \right] = 0 , \quad \lim_{\widehat{w} \to 0} \left(D_{\mathrm{E}}(\widehat{w}) \frac{\partial f_w}{\partial \widehat{w}} \right) = 0 . \tag{45}$$

The boundary conditions in form (45) can be regarded as the condition that the full diffusive flux vanishes in the phase space. The first limit in Eqns (45) implies that the function $f_w(\widehat{w}, t)$ decays sufficiently fast as $\widehat{w} \to +\infty$.

Integrating Fokker–Planck equation (42) over the velocity space Ω_w^+ with boundary conditions (45) leads to the equality

$$\frac{\partial}{\partial t} \int_0^\infty f_w \, \mathrm{d}\widehat{w} = -\gamma(\widehat{w}) \widehat{w} f_w \Big|_0 = 0 . \tag{46}$$

This confirms that boundary conditions (45) are consistent with normalization condition (43).

8. Generalized Maxwell distribution for an ensemble of convective thermals. Comparison with experimental data

The kinetic form of Fokker–Planck equation (42) admits a unique stationary solution satisfying boundary conditions (45) in the form of a generalized one-dimensional Maxwell distribution,

$$f_w^\infty(\widehat{w}) \, \mathrm{d}\widehat{w} = \sqrt{\frac{2}{\pi}} (\overline{w^2})^{-1/2} \exp\left(-\frac{1}{2} \frac{\widehat{w}^2}{\overline{w^2}} \right) \mathrm{d}\widehat{w} , \quad 0 \leqslant \widehat{w} < \infty . \tag{47}$$

The classical one-dimensional Maxwell velocity distribution for ideal gas molecules and Brownian particles follows from distribution (47) if we assume that the mean parameter $\overline{w^2}$ is proportional to the temperature of the medium.

It can be shown that stationary Maxwell solution (47) is the limit of the general nonstationary solution of Fokker–Planck equation (42) as $t \to +\infty$.

The empirical vertical velocity probability densities in the atmospheric mixed layer, as obtained by field measurements, are given in Refs [63–65]. However, comparison with Maxwell distribution (47) is only possible for those data whose reprocessing allows representing the empirical probability density in the form $\varphi(\upsilon)$, where $\upsilon = \widehat{w}/(\overline{w^2})^{1/2}$.

The distribution over vertical velocities in an ensemble of convective thermals on the lower boundary of the atmospheric mixed layer ($z = 100$ m $\approx 0.1h$) was constructed in Ref. [63]. Generalized Maxwell distribution (47) is juxtaposed with the results of airborne measurements [63] in Fig. 5a.

The distribution over vertical velocities in an ensemble of thermals in the mixed layer at the levels $z/h = 0.42$ and $z/h = 0.55$ was constructed in Ref. [64]. A comparison of generalized Maxwell distribution (47) with reprocessed results of balloon measurements [64] is plotted in Fig. 5b.

In the convective turbulent layer, the region of ascending motion is populated solely with buoyant thermals. The results presented in Fig. 5 show that in the region of ascending motion $\Omega_w^+ = \{\widehat{w} \colon 0 \leqslant \widehat{w} < \infty\}$, normal curve (47) can be used to approximate the experimental data. Thus, generalized one-dimensional Maxwell distribution (47) is indeed realized in an ensemble of thermals in the convective mixed layer.

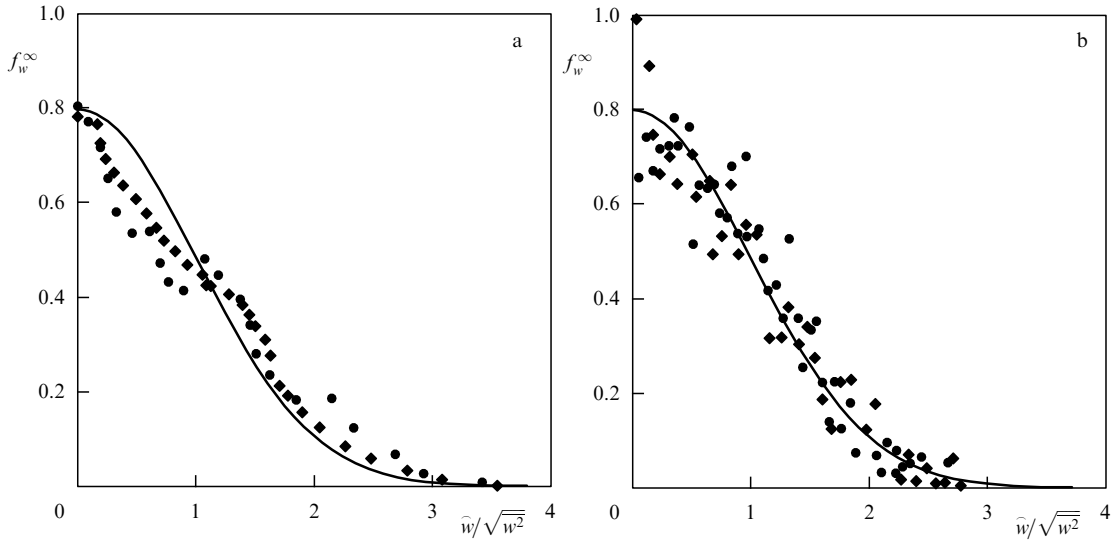

Figure 5. (a) Probability density of the vertical velocity distribution according to experimental data [63]. The dots and diamonds respectively correspond to airborne measurements along horizontal directions at a height of 100 m in directions parallel and perpendicular to the wind direction. Generalized Maxwell distribution (47) is shown by the solid line. (b) The probability density of the vertical velocity distribution according to experimental data [64]. The dots and diamonds respectively correspond to balloon measurements recomputed to the along-wind direction at the levels of $z/h = 0.42$ and $z/h = 0.55$. Generalized Maxwell distribution (47) is shown by the solid line.

9. Conclusions

The experimentally observed Maxwell velocity distribution in a system of updrafts allows treating the system of convective thermals as an ensemble of 'fast' Brownian particles. The rigorous development of such a kinetic approach opens up new possibilities for the description of moments of the turbulent convective layer.

The theoretical derivation of the one-dimensional Maxwell distribution for a system of convective updrafts convincingly demonstrates the possibility of constructively using physical kinetics methods to describe non-'thermodynamical' stochastic systems, i.e., systems that are not connected to a thermostat. This result is without a doubt of general physical significance.

The authors are indebted to the anonymous referee for a number of constructive remarks. The work was supported by the grant 15-05006849-a from the Russian Foundation for Basic Research.

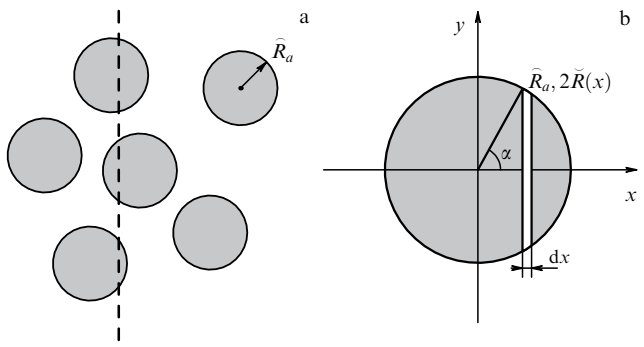

Figure 6. Mutual location of cross sections of horizontally identical thermals and the line of observations in the plane $z = \text{const}$. (a) The crossing of the system of horizontally identical thermals with the line of observation. (b) The crossing of an individual thermal by a random chord.

Appendix A. Mean radii for a system of thermals on a plane and along a line

We relate the mean length of a semichord $\tilde{R}_l$ to the model mean radius $\widehat{R}_a$ of a system of thermals on the plane. To simplify geometrical consideration, we replace each spherical thermal by a vertical cylinder with the circular cross-section radius $\widehat{R}_a$ and height $4\widehat{R}_a/3$, i.e., of equal volume.

Obviously, the cross section of the system of model cylindrical thermals by the plane $z = \text{const}$ forms a system of identical randomly placed circles in this plane. An airplane passing along a horizontal line parallel to the y axis crosses only some of these circles (Fig. 6a). It is assumed that a model circular isolated thermal is crossed along a random chord $x = \text{const}$ of the length $2\breve{R}(x)$ (Fig. 6b).

Let $\varphi(x)\,\mathrm{d}x$ be the probability of the chord passing through the interval $[x, x + \mathrm{d}x]$ perpendicular to the line of observation (Fig. 6b). From geometrical considerations, it then follows that

$$\varphi(x)\,\mathrm{d}x = \frac{\mathrm{d}x}{2\widehat{R}_a}\,, \qquad \int_{-\widehat{R}_a}^{\widehat{R}_a} \varphi(x)\,\mathrm{d}x = 1\,. \tag{A.1}$$

Let $\tilde{R}_l$ be the mean length of a semichord along the line of observations, given by the expectation value

$$\tilde{R}_l = \int_{-\widehat{R}_a}^{\widehat{R}_a} \breve{R}(x)\,\varphi(x)\,\mathrm{d}x = \frac{1}{\widehat{R}_a}\int_0^{\widehat{R}_a} \breve{R}(x)\,\mathrm{d}x\,, \tag{A.2}$$

where $\varphi(x)\,\mathrm{d}x$ is the probability of the chord passing through the interval $\mathrm{d}x$.

We transform (A.2) into polar coordinates $\widehat{R}_a, \alpha$ (Fig. 6b). Then

$$x = \widehat{R}_a \cos\alpha\,, \quad \mathrm{d}x = -\widehat{R}_a \sin\alpha\,\mathrm{d}\alpha\,, \quad \breve{R}(x) = \widehat{R}_a \sin\alpha\,. \tag{A.3}$$

Table 1. Height dependence of mean radii for a system of atmospheric thermals.

Flight height, m	10	30	50	100	300	500	1000	2000	3000
Number of measurements	761	2480	7611	8728	4748	4007	2656	1409	523
Mean radii of thermals, m	21.0	24.5	27.5	30.5	34.0	35.0	36.0	37.0	40.5

Substituting (A.3) in (A.2), we obtain

$$\tilde{R}_l = -\widehat{R}_a \int_{\pi/2}^{0} \sin^2\alpha\,\mathrm{d}\alpha = \widehat{R}_a \int_0^{\pi/2} \sin^2\alpha\,\mathrm{d}\alpha$$
$$= \widehat{R}_a \int_0^{\pi/2} \frac{1}{2}(1-\cos 2\alpha)\,\mathrm{d}\alpha\,. \qquad \text{(A.4)}$$

Integration of (A.4) leads to the relation

$$\tilde{R}_l = \frac{\pi}{4}\,\widehat{R}_a\,. \qquad \text{(A.5)}$$

Equality (A.5) links the observed mean length $\tilde{R}_l$ of a semichord with the model mean radius $\widehat{R}_a$ of the system of thermals and is a justification of relation (7).

The reasoning above, valid for an ensemble of identical thermals cylindrical in shape, can also be repeated for an ensemble of spherical thermals. We do not present a description of this more complicated variant in order not to encumber this appendix with too much detail.

The dependence of the mean radius $\widehat{R}_a$ on the height z, computed from the data in Ref. [12], is presented in Table 1.

For the height of convective layer $h \approx 2500$ m, the layer of intense mixing $0.1 \leqslant z/h \leqslant 0.5$ is bounded by the heights 250 m $\leqslant z \leqslant$ 1250 m. It is essential that the mean radii of the system of ascending thermals are considered constant, $\widehat{R}_a = \text{const}$, in the convective layer of intense mixing.

Appendix B. Mean buoyancy of a system of thermals on a plane and along a line

We consider a system of horizontally identical thermals with dimensionless potential temperatures that satisfy the approximation

$$\theta(r,z) = \theta_a^0(z)\sqrt{1-\left(\frac{r}{\widehat{R}_a}\right)^2}\,. \qquad \text{(B.1)}$$

Let $\zeta = r/\widehat{R}_a$ be the dimensionless polar coordinate in a circular section of a thermal. Then, with the profile shape in (B.1), the cross-sectional mean buoyancy $g\widehat{\theta}_a$ of a thermal is given by

$$g\widehat{\theta}_a(z) = \frac{1}{\pi\widehat{R}_a^2}\int_0^{2\pi}\int_0^{\widehat{R}_a} g\theta(r,z)r\,\mathrm{d}r\,\mathrm{d}\alpha = \frac{2}{\widehat{R}_a^2}\int_0^{\widehat{R}_a} g\theta(r,z)r\,\mathrm{d}r$$
$$= 2g\theta_a^0\int_0^1 (1-\zeta^2)^{1/2}\zeta\,\mathrm{d}\zeta = \frac{2}{3}\,g\theta_a^0(z)\,. \qquad \text{(B.2)}$$

Equality (B.2) connects the mean dimensionless potential temperature of the system of horizontally identical thermals $\widehat{\theta}_a$ on the level z with their common amplitude θ_a^0.

Let an aircraft cross a thermal of a circular cross section along a chord $x = \text{const}$ with the length $2\breve{R}(x)$. The profile of temperature pulsations in the thermal is presented in Fig. 7. Then the temperature of the thermal along the chord $x = \text{const}$ is given by

$$\theta(x,y,z) = \theta_l(x,z)\sqrt{1-\left(\frac{y}{\breve{R}}\right)^2}\,,$$
$$\theta_l(x,z) = \theta_a^0(z)\sqrt{1-\left(\frac{x}{\breve{R}}\right)^2}\,, \qquad \text{(B.3)}$$

where $\theta_l(x,z)$ is the dimensionless random temperature of the thermal at the center of the random chord, and y is the coordinate along the chord. In this case, $\breve{\theta}(x,z)$ — the mean dimensionless temperature along an arbitrary chord $x = \text{const}$ through the circular cross section of the thermal — takes the form

$$\breve{\theta}(x,z) = \frac{1}{2\breve{R}}\int_{-\breve{R}}^{\breve{R}} g\theta(x,y,z)\,\mathrm{d}y$$
$$= \frac{\theta_l(x,z)}{\breve{R}}\int_0^{\breve{R}}\sqrt{1-\left(\frac{y}{\breve{R}}\right)^2}\,\mathrm{d}y = \frac{\pi}{4}\,\theta_l(x,z)\,. \qquad \text{(B.4)}$$

Accordingly, the mean dimensionless potential temperature $\tilde{\theta}_l(x,z)$ along the observation line l is given by the expectation value

$$\tilde{\theta}_l(z) = \int_{-\widehat{R}_a}^{\widehat{R}_a} \breve{\theta}(x,z)\,\varphi(x)\,\mathrm{d}x = \frac{1}{2\widehat{R}_a}\int_{-\widehat{R}_a}^{\widehat{R}_a}\breve{\theta}(x,z)\,\mathrm{d}x$$
$$= \frac{\pi}{4}\,\frac{1}{\widehat{R}_a}\int_0^{\widehat{R}_a}\theta_l(x,z)\,\mathrm{d}x\,. \qquad \text{(B.5)}$$

We transform (B.5) taking (B.3) into account and using polar coordinates $\widehat{R}_a, \alpha$ [see (A.3)]:

$$x = \widehat{R}_a\cos\alpha\,, \quad \mathrm{d}x = -\widehat{R}_a\sin\alpha\,\mathrm{d}\alpha\,,$$
$$\theta_l = \theta_a^0\sqrt{1-\cos^2\alpha} = \theta_a^0\sin\alpha\,. \qquad \text{(B.6)}$$

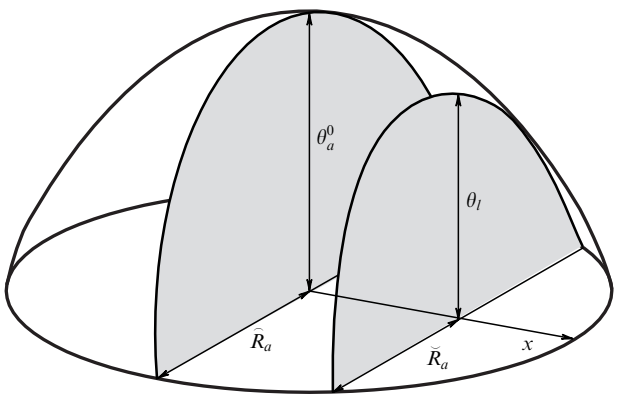

Figure 7. Surface of the dimensionless potential temperature and its cross section by a random vertical plane $x = \text{const}$; θ_a^0 and θ_l are the respective dimensionless potential temperatures at the centers of a thermal and a random chord, $\widehat{R}_a$ is the radius of the thermal, and $\breve{R}_a$ is the random semichord length.

Table 2. Height dependence of mean temperature pulsations for a system of atmospheric thermals.

Flight height, m	10	30	50	100	300	500	1000	1500	2000	2500	3000
Temperature perturbations, °C	0.70	0.32	0.28	0.25	0.17	0.13	0.12	0.11	0.10	0.18	0.19

Substituting (B.6) in (B.5) leads to the equality

$$\tilde{\theta}_l = \frac{\pi}{4}\,\theta_a^0 \int_0^{\pi/2} \sin^2\alpha\,\mathrm{d}\alpha = \frac{\pi^2}{16}\,\theta_a^0\,. \qquad \text{(B.7)}$$

With (B.2), it then follows that

$$g\tilde{\theta}_l = \frac{3\pi^2}{32}\,g\widehat{\theta}_a \simeq 0.92 g\widehat{\theta}_a\,. \qquad \text{(B.8)}$$

Equality (B.8) connects the mean buoyancy along the observation line $g\tilde{\theta}_l$ with the model buoyancy of the system of thermals in the plane, $g\widehat{\theta}_a$.

The derivation of relation (B.8) essentially relies on the assumption that the boundary of the thermal coincides with the isoline $\theta = 0$ [see equality (B.1)]. Laboratory and numerical modeling of an isolated thermal indicates that the region bounded by the surface $w = 0$ is located inside the region bounded by the surface $\theta = 0$ (for details, see, e.g., Ref. [48]). Hence, if the boundary of the thermal were defined as $w = 0$, relation (B.8) would take the form

$$g\tilde{\theta}_l = kg\widehat{\theta}_a\,, \quad k > 0.92\,. \qquad \text{(B.9)}$$

As a consequence, it is correct to use the approximation $g\tilde{\theta}_l = g\widehat{\theta}_a$. The equality $g\tilde{\theta}_l = g\widehat{\theta}_a$ links the observed mean buoyancy $g\tilde{\theta}_l$ along the observation line with the model mean buoyancy $g\widehat{\theta}_a$ of the system of thermals in a plane and offers a justification for relation (7).

The dependences of the mean temperature pulsation $\Theta_0\widehat{\theta}_a$ on the height z, computed from the data in Ref. [12], are presented in Table 2.

For the height of convective layer $h \approx 2500$ m, the layer of intense mixing $0.1 \leqslant z/h \leqslant 0.5$ is bounded by the heights 250 m $\leqslant z \leqslant$ 1250 m. It is essential that mean pulsations of the dimensionless potential temperature in a system of ascending thermals are considered constant, $\widehat{\theta}_a = \text{const}$, in the convective layer of intense mixing.

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