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Properties of turbulence driven by random external force in the Burgers model

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Contents

Abstract. **This paper reviews the statistical properties and calculates the velocity structure functions of flows produced by a large-scale random scaling force in the Burgers model.**

Keywords: turbulence, statistics, Burgers equation, scaling

1. Introduction

Through the years, the Burgers equation has been a subject of thorough studies dealing with various physical applications. Numerous reviews and monographs have been dedicated to it (see, e.g., Refs [1–3]). In particular, the related research is reflected in a series of publications in *Physics–Uspekhi* [4–6]. In the present work, we limit ourselves to questions related to hydrodynamical turbulence, which are still insufficiently studied in our opinion. The Burgers equation is known to be the simplest variant of the Navier–Stokes equation. For many years by analyzing the stochastic properties of this equation, researchers hoped to shed light on the problem of hydrodynamical turbulence and conjure up a physical picture of a turbulent cascade. One of the 'models' of the cascade relies on the singularity. It has been proposed that the time evolution described by the Navier–Stokes equation forced on large scales results (in the limit of zero viscosity) in the appearance of singularity [7, 8]. Namely this singularity underlies the Kolmogorov scaling. It is well known that in the absence of viscosity solutions to the Burgers equation can develop a singularity $v \propto x^{1/3}$ in a finite time, but this singularity lasts an instant only and, according to numerical simulations [1], has no impact on statistically stationary properties of the solution.

A traditional approach to turbulence excitation in Burgers's equation resorts to adding a random force to its right hand side:

$$v_t + v v_x = \nu v_{xx} + g(x, t)\,. \tag{1}$$

Such an approach was discussed, for example, in Refs [4, 5].

We assume that equation (1) is normalized to characteristic problem parameters; then, ν represents the inverse Reynolds number. The random force $g(x, t)$ is assumed to be Gaussian and delta-correlated in time:

$$\big\langle g(x_1, t_1)\, g(x_2, t_2) \big\rangle = \kappa(x_1 - x_2)\, \delta(t_1 - t_2)\,.$$

The assumption that the random force $g(x, t)$ is large-scale implies that the correlation length, i.e., the decay scale of the function $\kappa(x_1 - x_2)$, exceeds substantially the viscous scale. In dimensionless units, the correlation length equals unity. This difference in scales assumes the existence of an inertial range

$$\nu^{-1} \ll l \ll 1\,,$$

analogous to the inertial range of the Navier–Stokes equation for Kolmogorovian turbulence.

In addition to the case of large-scale force confined to small wave numbers, it is of interest to analyze solutions of the Burgers equation driven by a scaling force. For such a force, the Fourier image of the correlator $\kappa(x_1 - x_2)$ is written out as a power law:

$$\kappa(k) = 2D_0 |k|^{\beta}\,.$$

The exponent β in the last formula defines the scaling properties of a driving. In the case of $\beta > 0$, the force operates on small scales; for example, $\beta = 2$ corresponds to

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the thermal noise in the Kardar–Parisi–Zhang (KPZ) model [9]. It is well known that the problem in this case can be solved (see, for example, book [10]) and its solution demonstrates a simple scaling. We are interested in the case of $\beta < 0$. The force then operates on large scales (it is assumed that the divergence is cut on this scale). In this case, the renormgroup method does not work [11] and nonlinearity plays a decisive role. For $\beta < -3$, the driving force is differentiable, and according to numerical simulations the solution becomes piecewise continuous and contains a finite number of jumps, whereas the exponents of structure functions take the form $\xi_n = \min(1, n)$.

For a nondifferentiable external force, i.e., a force with diverging correlations of derivatives as $k \to \infty$ $(-3 < \beta < 0)$, the presence of jumps and dimensional considerations lead to the following structure function exponents [12]: $\xi_n = \min(1, -n\beta/3)$. However, in neither case has a rigorous derivation for the scaling of structure functions been proposed. Notice also that dimensional considerations do not always work for the Burgers equation. Indeed, for a large-scale random forcing, the third correlation function $S_3 = \langle (\delta v(l))^3 \rangle$ obeys the Kolmogorov law $\langle (\delta v(l))^3 \rangle = \varepsilon l$, where $\varepsilon = \kappa(0)$ is the energy flux [1]. Dimensional considerations analogous to those used for turbulence governed by the Navier–Stokes equation with regard to the finite number of jumps would lead to $\xi_n = \min(1, n/3)$, but the observed dependence is notably different: $\xi_n = \min(1, n)$.

2. Generating function and generating functional

The goal of this study is the consistent derivation of the above-mentioned scaling laws. Earlier, for the derivation of structure functions, A Polyakov proposed using the generating n-point function [13]

$$Z_n(\lambda_j, x_j, t) = \left\langle \exp\left(\sum \lambda_j v_j(x_j)\right)\right\rangle . \tag{2}$$

The correlation functions follow from expression (2) by differentiation over λ_i at $\lambda_i = 0$:

$$\langle v_{i_1} \ldots v_{i_k} \rangle = \frac{\partial^k Z_n}{\partial \lambda_{i_1} \ldots \partial \lambda_{i_k}} \bigg|_{\lambda_i = 0} .$$

This approach was generalized for the case of generating functional [14]. The analysis of this functional was performed for large λ, which gives the possibility of finding an instanton solution [13, 14], yet does not allow computations of structure functions.

We also consider the generating functional, but to analyze it we resort to a different method — multiscale expansions [15].

The generating functional is written out as

$$Z\big[\eta(x), t\big] = \left\langle \exp\left(\int \eta(x)\, v(x,t)\, \mathrm{d}x\right)\right\rangle . \tag{3}$$

If we select the functions $\eta(x) = \sum_i \lambda_i \delta(x - x_i)$ as delta-functions centered at different points x_i and having amplitudes λ_i, the result is Eqn (2). As usual, averaging is carried out over an ensemble of random forcing realizations. However, if velocity correlation functions have a finite correlation length, the ergodic hypothesis is valid and the mean can be computed via spatial integration, namely

$$Z\big[\eta(x), t\big] = \left\langle \exp\left(\int \eta(x)\, v(x,t)\, \mathrm{d}x\right)\right\rangle$$
$$= \lim_{L\to\infty} \frac{1}{L} \int_0^L \exp\left(\int \eta(x)\, v(x - y, t)\, \mathrm{d}x\right) \mathrm{d}y .$$

Let us consider a stationary equation. In this case, the variational equation for the generating functional takes the form

$$\int \mathrm{d}x\, \eta(x) \left\{ \partial_x \frac{\delta^2 Z}{\delta \eta^2(x)} - \nu \partial_{xx} \frac{\delta Z}{\delta \eta(x)} - \frac{1}{2}\, \gamma(x) Z\big[\eta(x)\big] \right\} = 0 . \tag{4}$$

Here, we introduced the function $\gamma(x) = \int \kappa(x - y)\eta(y)\, \mathrm{d}y$. Notice that the function $\gamma(x)$ is simply the function $\eta(x)$ smoothed over the correlation scale. This equation also contains a 'fast' scale as $\nu \to 0$, which is proportional to $1/\nu$. We introduce $y = x/\nu$ and seek a solution as an expansion in the small parameter using the method of multiscale expansions [15]. With this aim, we assume that the number Z is about unity and that the functions $A(x) = \delta^2 Z/\delta\eta^2(x)$ and $B(x) = \delta Z/\delta\eta(x)$ depend on many scaled variables $A = A(y_0, y_1, \ldots)$, $B = B(y_0, y_1, \ldots)$; in view of the above, accordingly, $\gamma(x) = \gamma(y_1, y_2, \ldots)$. Following the method of multiscale expansions, we represent the derivative as $\partial_y = \partial_{y_0} + \nu \partial_{y_1} + \ldots$. Since the variables $y_0, y_1, \ldots$ are asymptotically very different, differentiation or integration with respect to them is carried out independently. Inserting the expansion into Eqn (4), we find

$$\int \mathrm{d}y_0\, \mathrm{d}y_1 \ldots \eta(y_0, y_1, \ldots) \Big[(\partial_{y_0} + \nu \partial_{y_1} + \ldots) A(y_0, y_1, \ldots)$$
$$- (\partial_{y_0 y_0} + 2\partial_{y_0}\partial_{y_1}) B(y_0, y_1, \ldots) - \nu\, \frac{1}{2}\, \gamma(y_1, y_2, \ldots) Z \Big] = 0 .$$

We expand the functions A and B in a power series in ν and collect terms with the same power in ν:

$$\int \mathrm{d}y_0\, \eta(y_0, y_1, \ldots)(\partial_{y_0} A_0 - \partial_{y_0 y_0} B_0) = 0 , \tag{5}$$

$$\int \mathrm{d}y_0\, \eta(y_0, y_1, \ldots) \Big[\partial_{y_1} A_0 + \partial_{y_0} A_1 - \partial_{y_0 y_0} B_1 - 2\partial_{y_0}\partial_{y_1} B_0$$
$$- \frac{1}{2}\, \gamma(y_1, \ldots) Z \Big] = 0 . \tag{6}$$

Equations (5) and (6) should be satisfied for any function $\eta(y_0, y_1, \ldots)$. Setting in Eqn (5) $\eta(y_0, y_1, \ldots) = \delta(y_0 - y)$, where y is an arbitrary point, we get

$$\partial_y\, \frac{\delta^2 Z}{\delta \eta^2(y)} - \partial_{yy}\, \frac{\delta Z}{\delta \eta(y)} = 0 .$$

It should be clear that if this equation is obeyed, formula (5) is valid for any function η. A solution to the last equation can easily be found:

$$Z = \left\langle \exp\left[\int \eta(x) \sum U_i(x - x_i)\, \mathrm{d}x \right] \right\rangle ,$$

where

$$\frac{\mathrm{d}U_i^2}{\mathrm{d}y} - \frac{\mathrm{d}^2 U_i}{\mathrm{d}y^2} = 0 .$$

The last equation is valid in the vicinity of some point set x_i. We consider the case $x_i - x_{i-1} \approx 1 \gg \nu$. In this way, we disregard the interaction between the points i and $i-1$ in the exact solution. As we shall see, the interaction is exponentially weak, $\propto \exp(-1/\nu)$, and, consequently, can be ignored in the asymptotic expansion.

The solution to the above equation is well known:

$$U_i = V_0 \, \frac{1 - \exp(V_0 y)}{1 + \exp(V_0 y)} \, .$$

Here, $V_0(x) = V_0(i)$ is a function slowly varying on the scale $1/\nu$; moreover, x_i, in the vicinity of which this solution is valid, is also a slowly varying function.

Thus, one obtains

$$U(x) = \sum_i U_i(x) \, , \quad U_i(x) = V_0(i) \, \frac{1 - \exp\left[V_0(i)(x - x_i)/\nu\right]}{1 + \exp\left[V_0(i)(x - x_i)/\nu\right]} \, .$$

The positions of jumps x_i and their amplitudes $V_0(i)$ are not defined in the approximation considered. Because of the homogeneity of the random force, we assume that x_i are distributed with uniform probability along the x-axis and the mean should be understood as a spatial mean. To find the relationships for the parameters of jumps (stepwise functions or steps), we take advantage of an equation that follows exactly from the averaged Burgers's equation. Multiplying the equation by $V^2(x)$ and $V^2(x')$, after standard manipulations we get

$$\partial_l S_3 - \nu \partial_{ll} S_2 = \kappa \, . \tag{7}$$

Here, as usual, the third and second correlation functions are

$$S_3 = \frac{1}{L} \int_0^L \big(U(x+l) - U(x)\big)^3 \, \mathrm{d}x \, ,$$

$$S_2 = \frac{1}{L} \int_0^L \big(U(x+l) - U(x)\big)^2 \, \mathrm{d}x \, .$$

We fix ν and let $l \to 0$; since the functions U_i are smooth, it is obvious that $S_n \propto l^n$, and $S_3 \to 0$ as $l \to 0$. For the function S_2 we obtain

$$-\nu \sum_i \frac{1}{L} \int_0^L \left(\frac{\partial U_i}{\partial x}\right)^2 \mathrm{d}x = \kappa(0) \, .$$

As $\nu \to 0$, the contributions to the integral are from steps only, so we split the integral over intervals L_i and, performing integration for the function given above, find

$$\frac{1}{N} \sum_i^N \frac{V_0^3(i)}{3L_0} = -\kappa(0) \, , \tag{8}$$

where $V_0(i)$ is the amplitude of the step function, N is the number of steps, and $L_0 = L/N$ is the mean distance between them.

We now let $\nu = 0$, S_3 transforms into steps, and the correlator on each of them is $\propto V_0^3(i)/L_0 l$ (the correlator of the function $\operatorname{sign} x$ with itself). We get the same relationship (8). Thus, a relationship is found between the amplitudes of steps, energy dissipation, and mean distance between the steps. This relationship resembles the Kolmogorov law (see the discussion below).

We turn now to Eqn (6). Take $\eta(y_0, y_1, \ldots) = \eta(y_1, \ldots)$ to be a slowly varying function. We integrate over y_0 and y_1. Equation (6) will then gain diverging (secular) terms proportional to y_0 as $y_0 \to \infty$. According to the general principle of the construction of asymptotic expansions, they should be set to zero, which gives an equation for slowly varying functions:

$$\int \eta(y_1) \, \mathrm{d}y_1 \, \partial_{y_1} \frac{\delta^2 Z}{\delta \eta^2(y_1)} = \frac{1}{2} \int \mathrm{d}y_1 \, \mathrm{d}y \, \kappa(y_1 - y) \eta(y) \eta(y_1) \, Z \, . \tag{9}$$

3. N-point distribution functions

We found the form of the generating functional in the leading approximation with respect to the parameter $1/\nu$ and derived equation (9) in the next approximation. Further, we will limit ourselves to the construction of the generating function. With this goal, as mentioned above, we set $\eta(y_1) = \sum \lambda_i \delta(y_1 - x_i)$ in equation (9) to obtain [13]

$$\sum_j \lambda_j \frac{\partial}{\partial \lambda_j} \left(\frac{1}{\lambda_j} \frac{\partial Z}{\partial x_j}\right) = \sum_{i,j} \kappa(x_i - x_j) \lambda_i \lambda_j Z \, . \tag{10}$$

Following Polyakov [13], let us introduce the function F, so that

$$Z = \lambda_1 \ldots \lambda_N F(\lambda_1 x_1, \ldots, \lambda_N x_n) \, .$$

We insert this expression into equation (10) to get

$$\sum \frac{\partial^2}{\partial x_i \partial \lambda_i} F = \sum \kappa(x_i - x_j) \lambda_i \lambda_j F \, . \tag{11}$$

In what follows, it is more convenient to consider other functions F_N which are the Laplace transform of F:

$$F_N = \big\langle \theta\big(u_1 - v(x_1)\big) \theta\big(u_2 - v(x_2)\big) \ldots \theta\big(u_n - v(x_n)\big) \big\rangle \, .$$

The N-point distribution functions are then given by the relationships

$$P_n = \frac{\partial^n F_N}{\partial u_1 \ldots \partial u_n} \, . \tag{12}$$

On applying the Laplace transform to equation (11), we obtain

$$\sum u_k \partial_{x_k} F_N - \sum \kappa(x_i - x_j) \partial^2_{u_i u_j} F_N = 0 \, . \tag{13}$$

Equation (13) is linear and can easily be solved. We note, however, that here we are dealing with an asymptotic expansion which is obtained under the assumption that the terms A_0 and B_0 are of the same order. This is violated in equation (13) for small velocities, $u_i \approx 0$, unless the velocities tend to zero consistently with the gradients of functions. As we shall see, namely this consistency sets the scaling.

Let us consider the two-point function F_2, introducing variables $l = x_1 - x_2$ and $v = u_1 - u_2$. Assuming that the pair function F_2 depends only on these variables, we get

$$v \partial_l F_2 = \big(2\kappa(0) - \kappa(l) - \kappa(-l)\big) \partial^2_{vv} F_2 \, . \tag{14}$$

According to Eqn (12), the equation for the probability density P_2 will take the form

$$\partial_l P_2 = \big(2\kappa(0) - \kappa(l) - \kappa(-l)\big) \partial^2_{vv} \left(\frac{P_2}{v}\right) .$$

Consider the case when the random force is localized on large scales, then $2\kappa(0) - \kappa(l) - \kappa(-l) = \kappa_1 l^2$. Introducing

$S_n = \int v^n P_2 \, \mathrm{d}v$, where n is an arbitrary number, we find

$$\frac{\mathrm{d}S_n}{\mathrm{d}l} = \kappa_1 l^2 n(n-1) S_{n-3} \,.$$

It is easy to verify that the chain of structure functions in this relationship allows the scaling solution $S_n \propto l^n$. It should be recalled that this scaling is governed by the slowly varying part of the distribution function. The fast part, as we saw earlier, always gives $S_n \propto l$. We therefore get a bifractal scaling: $\xi_n = \min(1, n)$.

Consider a general case when the correlation function can be written in a power-law form. The correlator in this case takes the form $\kappa(l) = \kappa_0 - \kappa_1 |l|^{-1-\beta}$ for $\beta < -1$, and $\kappa = \kappa_1 l^{-1-\beta}$ for $-1 < \beta < 0$.

If $-1 < \beta < 0$, stationary equation (14) assumes in the limit $l \to 0$ the form

$$v\,\partial_l F_2 = \kappa_1 l^{-\beta-1} \partial^2_{vv} F_2 \,.$$

Just as in the previous case, it is easy to see that the structure functions defined by the slow part of the distribution function take the scaling form

$$\langle v^n \rangle \propto l^{-\beta n/3} \,. \tag{15}$$

Thus, for forcing in the scaling form the power-law exponents of the structure function also show the bifractal scaling: $\xi_n = \min(1, \beta n/3)$.

4. Conclusions

We saw that the method of multiscale expansions allows the distribution functions to be decomposed into fast and slow constituents. The fast function is responsible for the correlation of jumps in Burgers's equation, whereas the slow one is related to smooth fluctuations in the case of a large-scale force or the scaling $\beta n/3$ for the power-law spectrum of a driving force.

The standard similarity considerations for the Navier–Stokes turbulence rely on relationship (7) [16]. We see that the latter is also valid for the Burgers equation, but it does not give the $n/3$ scaling for structure functions. Notice, however, that the Kolmogorovian scaling is valid not only for S_3, but also for the correlation of velocity jumps at discontinuities, residing in relationship (8). Indeed, the energy flux $\kappa(0)$ does not depend on the scale of the correlation function, which is on the order of L in this consideration. The correlation length can therefore be varied substantially, with an energy flux keeping fixed; in this case, the mean cube of velocity jumps is proportional to $\kappa(0)L$, according to relation (8).

A great deal of attention in the literature was devoted to the case of $\beta = 1$ [1], for it formally corresponds to the Kolmogorov scaling (up to a logarithmic correction). Furthermore, the scaling exponents of structure functions $\langle |v|^n \rangle$ are described by a smooth curve with high accuracy (see Fig. 1) in this case. It is important that at $n = 3$ the respective structure function exponent $\xi_3 = 0.85$ and stays practically constant if the grid is varied from 2^{-16} to 2^{-18} (the respective values are displayed in blue and red circles). The observed contradiction with the theoretical consideration was eliminated in Ref. [17]. Its authors noticed that the bifractal structure is retrieved if the structure functions are considered under the condition $vl > 0$. Notice that this property is also present in our solution — a stationary solution to equation (14) derived here exists if $vl > 0$.

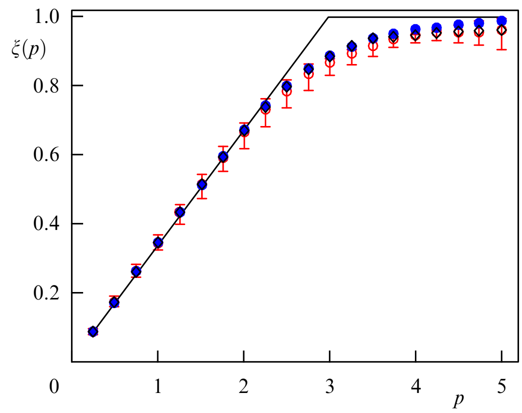

Figure 1. (Color online.) The power-law exponents of structure functions $S_{\mathrm{mod}}(p) = \langle |v|^p \rangle$ as a function of p at $\beta = 1$. It can be seen that the Kolmogorov law is not obeyed for the correlation functions $S_{\mathrm{mod}}(p)$, because $\xi_3 = 0.85$. Reference [17] reports that the curve $\xi(p)$ in the plot is described as $\xi(p) = \min(p/3, 1)$, if the structure functions are computed under the condition of $vl > 0$.

It should also be noted that the Kolmogorov law for the third correlation function (7) is derived not for $\langle |v|^3 \rangle$ but for $\langle v^3 \rangle$. As mentioned above, the scaling $n/3$ of the Burgers equation at $\beta = 1$ discussed here is not related to energy dissipation and hence has little in common with the turbulence in the framework of the Navier–Stokes equation.

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