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Stochastic structure formation in random media

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Contents

Abstract. **Stochastic structure formation in random media is considered using examples of elementary dynamical systems related to the two-dimensional geophysical fluid dynamics (Gaussian random fields) and to stochastically excited dynamical systems described by partial differential equations (lognormal random fields). In the latter case, spatial structures (clusters) may form with a probability of one in almost every system realization due to rare events happening with vanishing probability. Problems involving stochastic parametric excitation occur in fluid dynamics, magnetohydrodynamics, plasma physics, astrophysics, and radiophysics. A more complicated stochastic problem dealing with anomalous structures on the sea surface (rogue waves) is also considered, where the random Gaussian generation of sea surface roughness is accompanied by parametric excitation.**

Keywords: stochastic equations, intermittency, Lyapunov characteristic parameter, typical realization curve, dynamical localization, statistical topography, clustering

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"The book of Nature is written
in the language of mathematics."
Galileo Galilei

1. Introduction

Humanity exists in a certain realization of stochastic spatio–temporal chaos. Yet physicists believe that the basic dynamical equations of mechanics, fluid dynamics, magnetohydrodynamics, electrodynamics, acoustics, optics, radiophysics, etc. describe the actual evolution of the world in space and time.

In 2014, the scientific community celebrated a jubilee — the 450th anniversary of the birth of the great scientist Galileo [1], who contended that Nature formulates its laws in the language of mathematics. And the equations of dynamics in space and time indubitably embody one of the main manifestations of mathematics in physics! If Galileo was right, then a question arises as to how the laws of Nature can 'be rectified' from these equations without analyzing possible physical mechanisms of concrete phenomena. This question can only be answered with a rigorous statistical analysis. Namely such an approach is demonstrated in this work.

We begin by formulating the *main task* of a statistical analysis of stochastic dynamical systems in the way we understand it: *to find out, based on a relevant statistical analysis, such general properties of stochastic dynamical systems that are manifested with a unit probability, i.e., for almost all realizations of the systems being considered.* This is related to the fact that we commonly do not possess an ensemble for averaging, and specialists in numerical modeling and experimentalists alike deal only with separate

realizations of random processes and fields. Traditional statistical averaging gives, as a rule, 'the mean over a hospital ward'. There are, of course, exceptions (see Section 1.1).

In stochastic dynamical systems described by equations in partial derivatives, a stochastic structure formation in space and time may take place in some events with a unit probability for individual realizations of the fields involved. Such processes and phenomena, occurring with a unit probability, we will call *coherent*. This kind of 'statistical coherence' can be considered some organization of a complex dynamical system, while singling out its *statistically stable characteristics* is analogous to the introduction of the concept of *coherence*, understood as *self-organization* in multicomponent systems arising from chaotic interactions among their elements (see, for example, book [2]).

This work deals with three types of the simplest dynamical systems: systems related to Gaussian random fields, to stochastic parametric excitation, and to stochastic parametric excitation fed by Gaussian pumping (the combined event). They are all described by equations in partial derivatives.

Note that even in Gaussian random fields one may encounter nontrivial situations, atypical for ordinary Gaussian noise. Such phenomena occur, for example, in two-dimensional problems of geophysical fluid dynamics in the rotating fluid with random bottom topography (see, for example, papers [3, 4]) and in the problem of anomalous structures on the sea surface (see Section 6.2).

1.1 Two-dimensional geophysical fluid dynamics

A simple manifestation of coherent phenomena corresponds to a two-dimensional incompressible flow of an ideal fluid on the plane $\mathbf{R} = (x, y)$, described by a streamfunction $\psi(\mathbf{R}, t)$ satisfying the equation

$$\frac{\partial}{\partial t}\,\Delta\psi(\mathbf{R}, t) = J\big(\Delta\psi(\mathbf{R}, t); \psi(\mathbf{R}, t)\big)\,, \quad \psi(\mathbf{R}, 0) = \psi_0(\mathbf{R})\,, \qquad (1)$$

where Δ is the Laplacian, and $J\big(\psi(\mathbf{R}, t); \varphi(\mathbf{R}, t)\big)$ is the Jacobian of two functions [5]:

$$J\big(\psi(\mathbf{R}, t); \varphi(\mathbf{R}, t)\big) = \frac{\partial\psi(\mathbf{R}, t)}{\partial x}\,\frac{\partial\varphi(\mathbf{R}, t)}{\partial y} - \frac{\partial\varphi(\mathbf{R}, t)}{\partial x}\,\frac{\partial\psi(\mathbf{R}, t)}{\partial y}\,.$$

Nonlinear interactions would drive hydrodynamic system (1) to statistical equilibrium. Realizing that the process whereby this equilibrium is established involves numerous interactions between perturbations on different scales, one may suppose that in the simplest case of a statistically homogeneous and isotropic initial random field $\psi_0(\mathbf{R})$ such an equilibrium distribution will be a Gaussian one, and the task reduces to defining its parameters. The random streamfunction $\psi(\mathbf{R}, t)$ stays spatially homogeneous and isotropic in the process of evolution. Since $\psi(\mathbf{R}, t)$ is defined up to the additive constant, its statistical characteristics can be described by the one-time structure function

$$D_\psi(\mathbf{R} - \mathbf{R}', t) = \Big\langle \big[\psi(\mathbf{R}, t) - \psi(\mathbf{R}', t)\big]^2 \Big\rangle = 2\big(B_\psi(0, t) - B_\psi(\mathbf{R} - \mathbf{R}', t)\big)\,,$$

where $B_\psi(\mathbf{R} - \mathbf{R}', t) = \langle\psi(\mathbf{R}, t)\psi(\mathbf{R}', t)\rangle$ is the spatial correlation function of field $\psi(\mathbf{R}, t)$.

If we assume that the field $\psi(\mathbf{R}, t)$ has the distribution of a Gaussian statistically homogeneous and isotropic field described by the steady-state structural function

$$D_\psi(R) = \lim_{t\to\infty} D_\psi(\mathbf{R}, t)\,,$$

then the following equation can be obtained for this function [2]:

$$(\Delta_q + \lambda)\Delta_q D_\psi(q) = 0\,, \qquad (2)$$

where the separation constant λ has the dimension of an inverse length squared, $q = |\mathbf{R} - \mathbf{R}'|$, and Δ_q is the radial part of the Laplace operator.

There are two possible solutions of equation (2) that correspond to positive and negative values of the constant λ: $\lambda = k_0^2 > 0$ and $\lambda = -k_0^2 < 0$.

If $\lambda = k_0^2 > 0$, equation (2) is reduced to

$$\Delta_q D_\psi(q) = C J_0(k_0 q)\,,$$

where C is a dimensional constant, and $J_0(z)$ is the Bessel function of the first kind. In this case, the function $D_\psi(q)$ is determined by solving the Poisson equation, and the result for the spectral density of the streamfunction is as follows:

$$E(k) = E\delta(k - k_0)\,,$$

which corresponds to random structures with a *certain fixed spatial scale*. In the case considered, these structures are *vortices*, i.e., we are dealing with structure formation through *vortex genesis*.

For $\lambda = -k_0^2 < 0$, equation (2) reduces to a similar equation

$$\Delta_q D_\psi(q) = C K_0(k_0 q)\,,$$

but with a modified Bessel function of the second kind (the Macdonald function) $K_0(z)$ on the right-hand side with dimensional parameters k_0 and C. The related spectral density of the streamfunction now takes the form

$$E(k) = \frac{E_0}{k^2 + k_0^2}\,,$$

which corresponds to the Gibbs distribution with two integrals of motion — those of energy and vorticity squared (enstrophy) of a velocity field (see, for example, Refs [6, 7]).

Acting in a similar way, it is also possible to consider equilibrium states for quasigeostrophic flows (in the presence of rotation) with a random topography of underlying surface for a one- and two-layer fluid flows (Fig. 1), where we encounter a similar situation.

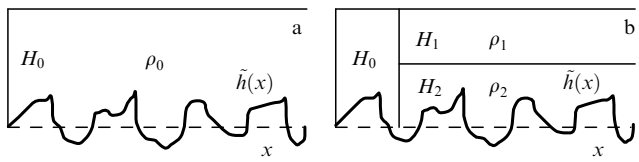

Figure 1. Schematics of one- (a) and two-layer (b) models of hydrodynamical flows. H_0, H_1, and H_2 are the thicknesses of the fluid layers with densities ρ_0, ρ_1, and ρ_2, respectively; $\tilde{h}(x)$ is the function describing the topography of the bottom.

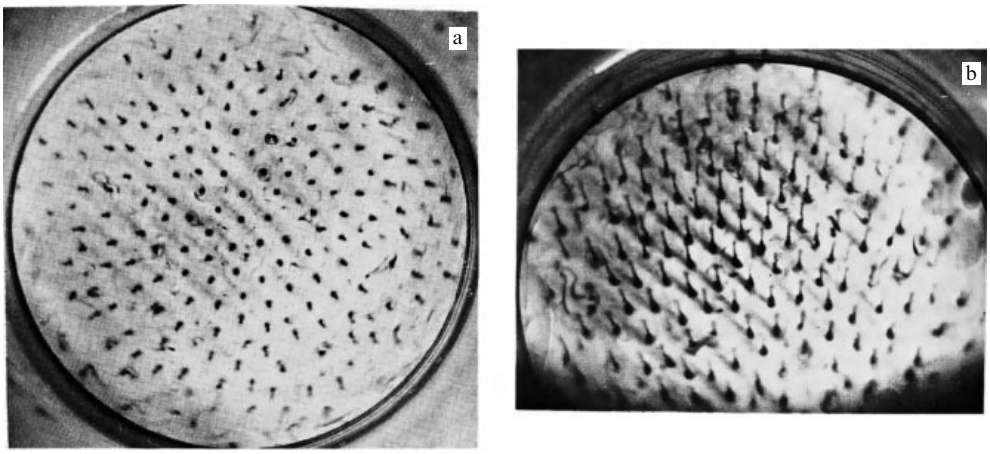

Figure 2. System of regular convecting vortices on a rotating platform: (a) top view, and (b) side view. (Taken from monograph [12].)

For a one-layer model, the fluid motion in the two-dimensional plane $\mathbf{R} = (x, y)$ is described by the *streamfunction* that satisfies the equation

$$\frac{\partial}{\partial t}\,\Delta\psi(\mathbf{R}, t) + \beta_0\,\frac{\partial}{\partial x}\,\psi(\mathbf{R}, t) = J\big(\Delta\psi(\mathbf{R}, t) + h(\mathbf{R}); \psi(\mathbf{R}, t)\big)\,,$$

where the parameter β_0 is the meridional derivative of the local Coriolis parameter f_0, and $h(\mathbf{R}) = f_0\tilde{h}(\mathbf{R})/H_0$. Here, $\tilde{h}(x)$ is the function describing the bottom topography $\tilde{h}(\mathbf{R})$, and H_0 is the mean thickness of a fluid layer. This equation describes the *barotropic* motion of a fluid. In a more general case, to explore *baroclinic* motions, a two-layer model of quasigeostrophic flows is used, which is governed by the system of equations

$$\frac{\partial}{\partial t}\big[\Delta\psi_1 - \alpha_1 F(\psi_1 - \psi_2)\big] + \beta_0\,\frac{\partial\psi_1}{\partial x} = J\big(\Delta\psi_1 - \alpha_1 F(\psi_1 - \psi_2); \psi_1\big)\,,$$

$$\frac{\partial}{\partial t}\big[\Delta\psi_2 - \alpha_2 F(\psi_2 - \psi_1)\big] + \beta_0\,\frac{\partial\psi_2}{\partial x} = J\big(\Delta\psi_2 - \alpha_2 F(\psi_2 - \psi_1) + f_0\alpha_2 h; \psi_2\big)\,,$$

where the additional parameters $\alpha_1 = 1/H_1$ and $\alpha_2 = 1/H_2$ are the inverse layer thicknesses, and $F = f_0^2\rho/g(\Delta\rho)$, $\Delta\rho/\rho = (\rho_2 - \rho_1)/\rho_0 > 0$.

Thus, there are already *two fixed scales* in a two-layer fluid.

It seems plausible to assume that such structures have been observed in experimental studies in rotating fluids (see, for example, Refs [8–10], review [11], and monographs [12, 13]), and also in numerical simulations (see, for example, Ref. [14]). As illustrations, we present Figs 2–5 which, in our opinion, correspond to the situation described. Figure 6 shows an example of structure formation in the field of surface flows in the Baltic Sea [15, 16].

1.2 Parametrically excited dynamical systems

We turn now to the statistical analysis of stochastic dynamical systems related to random parametric excitation in space and time. Such systems, appearing in many branches of physics, can be described by ordinary differential equations, as well as by partial differential equations. Stochastic structure formation for such systems in random media in the form of *clustering* is related to the parametric excitation of various

Figure 3. System of irregular convective vortices on a rotating platform for a rotational velocity larger than in Fig. 2. (Taken from monograph [12].)

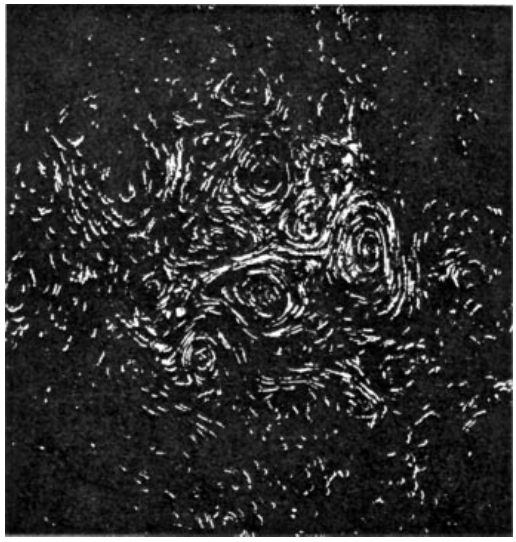

Figure 4. Streak pattern of velocities in a baroclinically unstable two-layer stratified fluid on a rotating platform. (Taken from Ref. [11].)

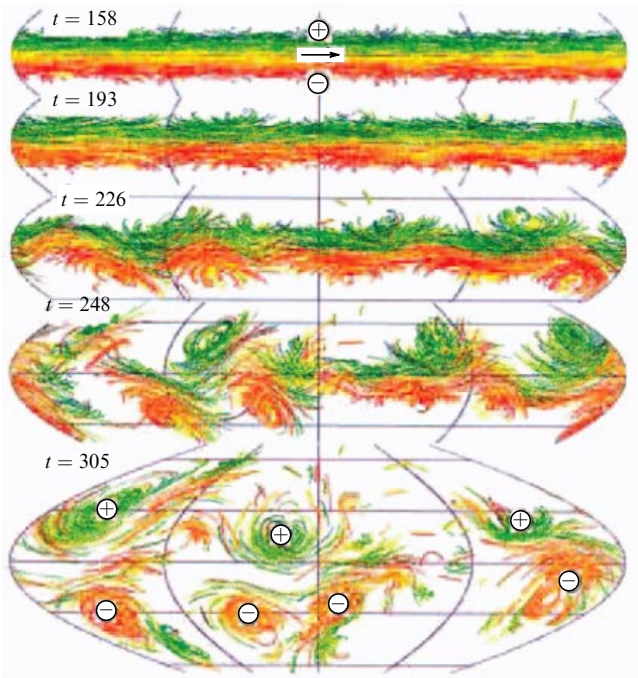

Figure 5. (Color online.) Formation of eddies on a sphere driven by an unstable shear flow. (Taken from Ref. [14].)

Figure 6. Domain of the submesoscale surface vorticity field in the Baltic Sea.

physical fields in these media. *Clustering of a particular field implies the appearance of compact regions with large field values against the background of surrounding areas with relatively low field values*. Statistical averaging, expectedly, destroys all the information on clusters. Such challenges occur in fluid dynamics (*a passive scalar tracer in a turbulent flow*), in magnetohydrodynamics (*a passive vector tracer — magnetic field in a turbulent flow*), and in the propagation of waves of various origins (acoustic and radio waves, light and laser radiation) in random media. All these issues are commonly considered in the *kinematic approximation* and share the following two most important traits.

(1) At fixed points in space, the field realizations in time are random processes which possess a specific character: they have the shape of peaks that appear at random instants of time. The intervals between them are characterized by low intensity and long duration. Such a realization of a random process in time for any location in space stems from the lognormal one-time distribution of probabilities, which has a slightly sloping 'tail'. The large but rare outliers (fluctuations) come from these tails. The main statistical characteristics of the processes being considered are the one-time probability density, one-time moment functions, typical realization curve characterizing the key features in the behavior of realizations of random processes, and Lyapunov exponent. In one-dimensional tasks described by ordinary differential equations with initial or boundary conditions, only such physical phenomena as *dynamical localization* can be observed in a number of cases (see Section 2).

(2) The structure formation itself of a stochastic field takes place in physical space and is described through a related statistical analysis based on the ideas of statistical topography of a stochastic field. In the simplest problem formulation, under statistical homogeneity in space, all one-point statistical characteristics of a random field are independent of spatial locations. Accordingly, the equation for the one-point probability density of a random field coincides in form with the equation for the probability density of a random process at each point in space, although the sense of these equations is substantially different. Relatedly, the statistical analysis of these equations should also be completely different.

A detailed discussion of these questions can be found in monographs [17–20] and articles [21–24].

First of all, a question arises as to whether or not such physical phenomena as localization and clustering occur in individual realizations of the processes and fields being considered, and if yes, then over which characteristic time (or on which spatial scales).

The phenomenon of structure formation in stochastic, parametrically excited dynamical systems on its own is well known in physics. For example, solutions of one-dimensional problems on parametric excitation, described by ordinary differential equations, are random processes.

The simplest dynamical system of that kind defines a lognormal random process $y(t;\alpha)$ described by a first-order ordinary stochastic differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, y(t;\alpha) = \left\{-\alpha + z(t)\right\} y(t;\alpha), \quad y(0;\alpha) = 1\,, \tag{3}$$

where $z(t)$ is a Gaussian random process of *white noise* with the parameters

$$\left\langle z(t)\right\rangle = 0\,, \quad B_z(t-t') = \left\langle z(t)z(t')\right\rangle = 2D\delta(t-t')\,.$$

The solution to equation (3) takes the form

$$y(t;\alpha) = \exp\left(-\alpha t + \int_0^t \mathrm{d}\tau\, z(\tau)\right). \tag{4}$$

It should be noted that the change in the sign of parameter α in Eqn (4) is statistically equivalent to passing to the process $1/y(t)$ [25].

Figure 7 presents realizations of the lognormal random process $y(t;\alpha)$ given by formula (4) for positive and negative values of the parameter α and $|\alpha|/D = 1$ (the dashed lines correspond to the functions $\exp(-Dt)$ for $\alpha > 0$, and $\exp(Dt)$ for $\alpha < 0$). The presence of rare, but strong, spikes (fluctuations) with respect to the dashed curves in the directions of both large and small values can be seen in Fig. 7. This

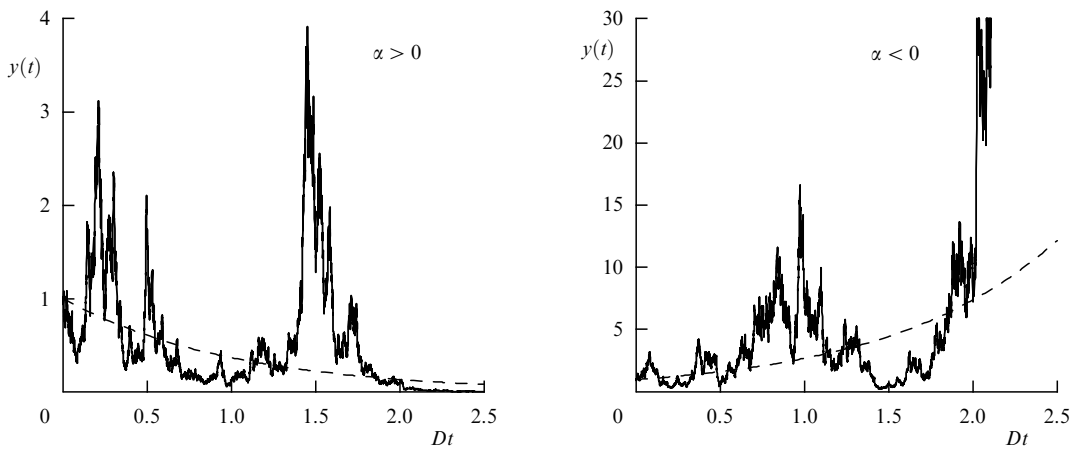

Figure 7. Realizations of the lognormal process $y(t)$ for $\alpha > 0$, $\alpha < 0$, and $|\alpha|/D = 1$.

property of random processes is called *intermittency*; it was intensively studied in the 1980s (see, for example, Refs [26, 27]). A detailed discussion of this question is given, for example, in monograph [19] and paper [28]. The curve with respect to which we identify the outliers (fluctuations) will be referred to as the *typical realization curve*.

The authors of practically every one of the numerous articles exploring the properties of intermittency cite Zel'dovich et al. [27] when turning to the notion of 'intermittency'. The term *intermittency* on its own emerged in studies of the velocity field and temperature spots in turbulent media [29, 30] (as, for instance, stated in Ref. [27]). However, even at that time it was already well known that one-point distributions of velocity fields and temperature fluctuations are close to Gaussian ones (see, for example, Ref. [31]). The term *intermittency* is certainly a telling one, and it characterizes the time variability of a random field at a fixed location in space, i.e., the variability of a random process with respect to its mean value.

At that time, certainly, it was also known that stochastic instability (parametric excitation) could occur in dynamical systems as a consequence of fluctuations in the internal parameters of the system. However, for a long time, up to the 1980s, nobody took interest in these questions. The achievement of the authors of Ref. [27] is that they, in all probability, were the first to draw attention to the possibility of stochastic structure formation as a consequence of such parametric excitation, which had been known at that time from various kinds of observations.

The abstract to paper [27] states, "The processes of instability in random media are characterized by formation of specific structures in which a growing quantity reaches *record*-high values. Despite the rareness of such concentrations namely they confine the main part of integral characteristics of a growing quantity (the mean value, the mean squared value, etc.). The appearance of such structures is referred to as the phenomenon of intermittency." Thus, in Ref. [27] strong rare outliers (fluctuations) are termed *specific structures*, while the process proper whereby these structures (outliers or fluctuations) form is called the phenomenon of *intermittency*.

In our understanding, *intermittency* is a general property of all random processes, independent of the amplitude of possible fluctuations, while *structure formation constitutes a certain type of evolution of stochastic dynamical systems in space and time*.

Reference [27] treated these notions as identical. At present, for example, some scientists call large rare outliers (fluctuations) characteristic of both the stochastic linear and nonlinear Leontovich equations (see below) rogue (freak) waves (see, for example, the lectures by V E Zakharov [32]). A rogue wave is undoubtedly a phenomenon of spatio–temporal watermass clustering, and it should be considered on the basis of appropriate statistical analysis of the evolution of random fields.

It should be noted that the statistical theory of stationary extremal statistical processes is an independent branch of probability theory (see, for example, review [33]). However, in our opinion, this area has nothing to do with stochastic structure formation in space and time.

A fundamental feature of stochastic dynamical systems described by partial differential equations is that their solutions comprise random fields in space and time. The difficulty in explaining processes of structure formation in these systems is related to two factors. First, at any fixed point in space the random field constitutes a random process in time. Second, for any fixed instant of time, the random field represents a random process over its spatial coordinates. Intermittency (i.e., variability) occurs namely for random processes (with respect to time or spatial coordinate); it is a general property of any random process irrespective of the *nature of its origin*.

In this study, the intermittency of a random process is understood as a more or less uninterrupted alteration of outliers (fluctuations) of this process toward larger as well as lower values with respect to the deterministic curve — *the curve of typical realization*, which is the median of the integral probability distribution function (see Section 2.1). In this case, a lognormal, parametrically excited random process can exponentially decay with time in individual realizations (certainly, with some fluctuations), which corresponds to the phenomenon of *dynamical localization*. An exponential growth of a random process with time is also possible, which corresponds to the absence of dynamical localization. A peculiarity of a lognormal random process is the presence of rare anomalously high spikes (fluctuations) on the curves of the process (see Fig. 7), related to the long sloped 'tail' of the probability density (see Section 2). All traditional statistical characteristics, such as moment and correlation functions of arbitrary order, result from these fluctuations.

By introducing the notion of the typical realization curve for a random process, we return to the historical sense of the

concept of intermittency, which is general for all random processes and has a strict probabilistic definition and a transparent physical sense.

Examples of stochastic dynamical systems. One of the simplest physical problems related to parametric excitation is that of diffusion of a particle or an ensemble of particles in a random velocity field $\mathbf{u}(\mathbf{r}, t)$ with given statistical properties, in the kinematic approximation (see, for example, monographs [17–20], which provide an extensive bibliography of problems considered), diffusion described by the system of ordinary differential equations

$$\frac{\mathrm{d}\mathbf{r}(t)}{\mathrm{d}t} = \mathbf{u}\big(\mathbf{r}(t), t\big)\,, \qquad \mathbf{r}(0) = \mathbf{r}_0\,. \tag{5}$$

Numerical simulations of this problem indicate that the dynamics of the ensemble of particles can essentially differ, depending on whether the random velocity field is divergent or not. For a concrete realization of a nondivergent stationary velocity field $\mathbf{u}(\mathbf{r})$, particles (the two-dimensional case) uniformly spread in a circle only mix in the domain bounded by a deformed contour. The contour becomes only strongly filamented with time, acquiring a fractal character. For a potential velocity field $\mathbf{u}(\mathbf{r})$, particles uniformly spread over a square at the initial instant of time form cluster regions as the system evolves with time. We stress that cluster formation in this case is a purely kinematic effect. Obviously, on averaging over an ensemble of realizations of a random velocity field this feature of particle's dynamics will disappear.

The *Anderson dynamical localization* is also known for eigenfunctions of the one-dimensional stationary Schrödinger equation with a random potential [34, 35]. Accordingly, the *dynamic localization* of wave intensity takes place in a boundary-value problem related to waves in random layered media in a spatial interval $[L_0, L]$ [36]. This case is described by a stochastic Helmholtz equation

$$\frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} + k^2\big(1 + \varepsilon(x)\big)u(x) = 0 \tag{6}$$

with the boundary conditions that the field and its derivative be continuous at the layer boundary:

$$u(L) + \frac{\mathrm{i}}{k}\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\bigg|_{x=L} = 2\,, \qquad u(L_0) - \frac{\mathrm{i}}{k}\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}\bigg|_{x=L_0} = 0\,.$$

As for random fields, we may introduce a generalization of lognormal random process (4), extending it to a lognormal random field according to the formula

$$f(\mathbf{r}, t; \alpha) = f_0(\mathbf{r}) \exp\left(-\alpha t + \int_0^t \mathrm{d}\tau\, z(\mathbf{r}, \tau)\right), \tag{7}$$

where $z(\mathbf{r}, t)$ is the Gaussian random field delta-correlated in time with a zero mean and the correlation function

$$B_z(\mathbf{r} - \mathbf{r}', t - t') = \big\langle z(\mathbf{r}, t) z(\mathbf{r}', t')\big\rangle = 2D(\mathbf{r} - \mathbf{r}')\delta(t - t')\,. \tag{8}$$

This field satisfies the first-order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}, t; \alpha) = \big(-\alpha + z(\mathbf{r}, t)\big) f(\mathbf{r}, t; \alpha), \quad f(\mathbf{r}, 0; \alpha) = f_0(\mathbf{r})\,, \tag{9}$$

which parametrically depends on the position of point $\mathbf{r}$ in space.

Notice that the studies [26, 27] on intermittency mentioned above considered the equation

$$\frac{\mathrm{d}f(\mathbf{r}, t)}{\mathrm{d}t} = z(\mathbf{r}, t) f(\mathbf{r}, t) + \mu_f \Delta f(\mathbf{r}, t) \tag{10}$$

as their model problem, where μ_f is the dynamic diffusion coefficient for the field $f(\mathbf{r}, t)$. This equation with random breeding and diffusion is typical of problems in biology and the kinetics of chemical and nuclear reactions (see, for example, Ref. [37]).

If $f_0(\mathbf{r}) = 1$, all one-point statistical characteristics of this filed are independent of $\mathbf{r}$.

At the initial stage of diffusion, the solution to problem (10) is given by function (7) with $\alpha = 0$ and $f_0(\mathbf{r}) = 1$:

$$f(\mathbf{r}, t) = \exp\left(\int_0^t \mathrm{d}\tau\, z(\mathbf{r}, \tau)\right), \tag{11}$$

which is statistically equivalent to the random process $y(t; 0)$ (4) for one-point statistical characteristics. As will be shown further, there is no structure formation in this case. But the general feature of intermittency remains.

It should be noted that adding the 'destruction' effect to equation (10) with the help of the term $-\alpha f(\mathbf{r}, t; \alpha)$ for $\alpha > 0$ leads to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{r}, t; \alpha) = \big(-\alpha + z(\mathbf{r}, t)\big) f(\mathbf{r}, t; \alpha) + \mu_f \Delta f(\mathbf{r}, t; \alpha)\,, \tag{12}$$

the solution of which at the initial stage is already described by formula (7). In this case, as shown further, stochastic structure formation in the form of clustering becomes possible.

The inclusion of randomness into the medium parameters spawns stochasticity in the physical fields themselves. Individual realizations, for example, of lognormal scalar two-dimensional field $f(\mathbf{R}, t)$ (11), where $\mathbf{R} = \{x, y\}$, resemble a complex mountain landscape with randomly distributed peaks, troughs, ridges, and passes. Figure 8 illustrates examples of two numerically simulated realizations of random fields with different statistical structures.

Clustering in random physical fields arises first of all in problems of *turbulent transport* in a random velocity field $\mathbf{u}(\mathbf{r}, t)$. In particular, *clustering* may occur for both *a passive scalar tracer* (the field of density) [17–20, 38] and a *vector tracer* (magnetic field energy) in the framework of *kinematic approximation* [17–20, 39]. The input stochastic equations in these cases are the continuity equation for the tracer density field $\rho(\mathbf{r}, t)$, viz.

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\,\mathbf{u}(\mathbf{r}, t)\right)\rho(\mathbf{r}, t) = \mu_\rho \Delta\rho(\mathbf{r}, t)\,, \quad \rho(\mathbf{r}, 0) = \rho_0(\mathbf{r})\,, \tag{13}$$

and the induction equation for the solenoidal magnetic field $\mathbf{H}(\mathbf{r}, t)$ [40]:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\,\mathbf{u}(\mathbf{r}, t)\right)\mathbf{H}(\mathbf{r}, t) = \left(\mathbf{H}(\mathbf{r}, t)\,\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{u}(\mathbf{r}, t) + \mu_H \Delta\mathbf{H}(\mathbf{r}, t)\,,$$
$$\mathbf{H}(\mathbf{r}, 0) = \mathbf{H}_0(\mathbf{r})\,, \tag{14}$$

where μ_ρ and μ_H are the dynamic diffusion coefficients for the density field and the magnetic field, respectively. Here, $\mathbf{u}(\mathbf{r}, t)$

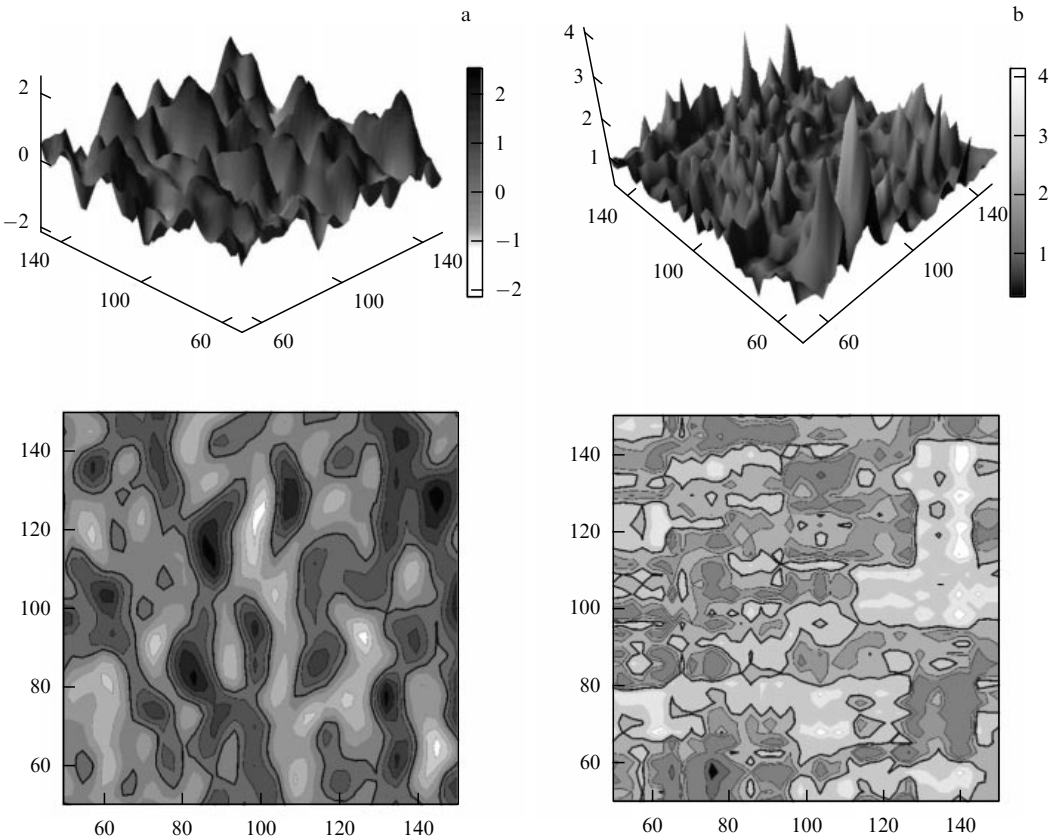

Figure 8. Realizations of (a) a Gaussian field $\ln f(\mathbf{r}, t)$, and (b) a lognormal field $f(\mathbf{r}, t)$ (11) and their topographic level lines. The thick contours correspond to zero isolines in panel (a), and to the unit field in panel (b).

is the field of turbulent velocities with given statistical properties, which is assumed to be homogeneous and isotropic in space, and stationary in time.

We stress that in the analysis of these and similar equations of mathematical physics, considered further in this article, we will not be interested in their direct solutions or physical mechanisms giving birth to one physical phenomenon or another. Our goal is to learn whether the input equations on their own contain information on the possibility (or impossibility) of stochastic structure formation in random media with a unit probability, i.e., for almost all realizations of their solutions.

It should be noted that a scalar density field always experiences clustering in a compressible fluid flow. Figure 9 displays the pattern of cluster structure of the Universe, taken from the Internet, which in all probability is directly related to the clustering of cosmic matter in random velocity fields. This question is discussed in Section 4.1.

The formation of stochastic structures may also take the form of a *caustic structure* of the wave field intensity in problems involving waves propagating in randomly inhomogeneous media in the framework of the Leontovich complex-valued stochastic parabolic equation (see, for example, Refs [17, 18, 41]):

$$\frac{\partial}{\partial x}\, u(x, \mathbf{R}) = \frac{\mathrm{i}}{2k}\, \Delta_{\mathbf{R}} u(x, \mathbf{R}) + \frac{\mathrm{i}k}{2}\, \varepsilon(x, \mathbf{R}) u(x, \mathbf{R})\,, \qquad u(0, \mathbf{R}) = u_0(\mathbf{R})\,, \tag{15}$$

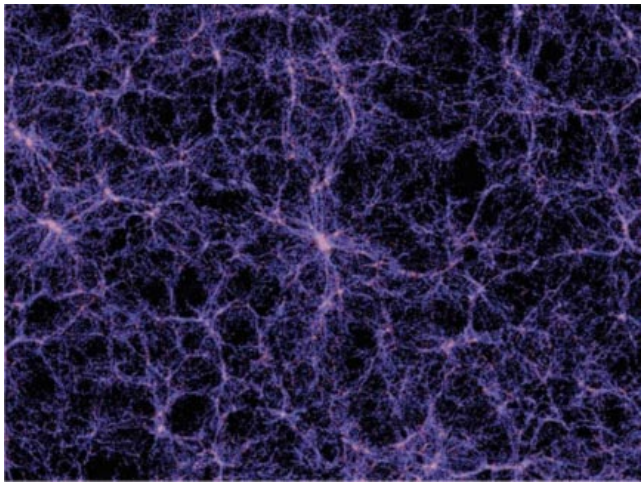

Figure 9. Cluster structure of the Universe.

where x is the coordinate in the direction of wave propagation, $\mathbf{R}$ is the coordinate in the transverse plane, and $\varepsilon(x, \mathbf{R})$ is the deviation of dielectric constant from unity.

We mention that this same equation becomes a nonstationary Schrödinger equation with the random potential $\varepsilon(x, \mathbf{R})$ on the replacement of x with time t.

If we introduce the amplitude and phase of a wave field according to the formula

$$u(x, \mathbf{R}) = A(x, \mathbf{R}) \exp\big(\mathrm{i}S(x, \mathbf{R})\big)\,,$$

Figure 10. Caustics in a swimming pool (a), and in shallow water (b).

the equation for the wave field intensity $I(x,\mathbf{R}) = |u(x,\mathbf{R})|^2$ takes the following form:

$$\frac{\partial}{\partial x}\, I(x,\mathbf{R}) + \frac{1}{k}\nabla_{\mathbf{R}}\{\nabla_{\mathbf{R}} S(x,\mathbf{R}) I(x,\mathbf{R})\} = 0\,, \quad I(0,\mathbf{R}) = I_0(\mathbf{R}). \tag{16}$$

Equation (16) coincides in form with the continuity equation (13) for the tracer density field in a random potential flow in the absence of dynamic diffusivity and, accordingly, the wave field intensity should undergo clustering. This problem is discussed in more detail in Section 3.2.1.

It should also be emphasized that a nonlinear generalization of equation (15) which corresponds to a monochromatic nonlinear *problem on wave self-action* in randomly inhomogeneous media, is described by the Leontovich complex-valued nonlinear parabolic equation (the *nonlinear Schrödinger equation*)

$$\begin{aligned} &\frac{\partial}{\partial x}\, u(x,\mathbf{R}) = \frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}} u(x,\mathbf{R}) + \frac{\mathrm{i}k}{2}\,\varepsilon\big(x,\mathbf{R}; I(x,\mathbf{R})\big) u(x,\mathbf{R})\,, \\ &u(0,\mathbf{R}) = u_0(\mathbf{R})\,. \end{aligned} \tag{17}$$

For equation (17), the wave field intensity $I(x,\mathbf{R})$ is also described by equation (16) [but, clearly, with another phase function $S(x,\mathbf{R})$], so that the intensity should experience clustering, too.

In particular, since equation (16) does not depend on the form of function $\varepsilon(x,\mathbf{R})$, then even if $\varepsilon(x,\mathbf{R}) = 0$ for the initial condition $u(0,\mathbf{R}) = u_0(\mathbf{R})$ the caustic structure formation, as is well known, takes place, which is regularly observed in swimming pools or in shallow water. In this case, equations (15) and (17) take the form

$$\frac{\partial}{\partial x}\, u(x,\mathbf{R}) = \frac{\mathrm{i}}{2k}\,\Delta_{\mathbf{R}} u(x,\mathbf{R})\,, \quad u(0,\mathbf{R}) = u_0(\mathbf{R})\,.$$

The solution to the last equation is the function

$$\begin{aligned} u(x,\mathbf{R}) &= \exp\left(\frac{\mathrm{i}x}{2k}\,\Delta_{\mathbf{R}}\right) u_0(\mathbf{R}) \\ &= \frac{k}{2\pi \mathrm{i}x}\int \mathrm{d}\mathbf{R}'\, \exp\left[\frac{\mathrm{i}k}{2x}(\mathbf{R}-\mathbf{R}')^2\right] u_0(\mathbf{R}')\,, \end{aligned} \tag{18}$$

and for a plane incident wave the initial condition is $|u_0(\mathbf{R})| = 1$, i.e., the condition $u_0(\mathbf{R}) = \exp(\mathrm{i}S_0(\mathbf{R}))$, where $S_0(\mathbf{R})$ is the field of the random initial phase. In this case, spatial fluctuations in the initial distribution of the wave phase transform into a caustic structure in the wave field intensity. The case is known as the *random phase screen*. Examples of such clustering are given in Fig. 10.

Dynamical systems (13)–(17) are conservative and preserve integral characteristics such as the total tracer mass $M = \int \mathrm{d}\mathbf{r}\, \rho(\mathbf{r},t)$, the magnetic field flux $\int \mathrm{d}\mathbf{r}\, \mathbf{H}(\mathbf{r},t)$, and wave field power $I = \int \mathrm{d}\mathbf{R}\, I(x,\mathbf{R})$.

For homogeneous initial conditions $\rho_0(\mathbf{r}) = \rho_0$, $\mathbf{H}_0(\mathbf{r}) = \mathbf{H}_0$, and $u_0(\mathbf{R}) = u_0$, and for random parameters that are statistically homogeneous in space, the corollary of the conservative character of dynamical systems (13)–(17) is the equalities

$$\big\langle \rho(\mathbf{r},t)\big\rangle = \rho_0\,, \quad \big\langle \mathbf{H}(\mathbf{r},t)\big\rangle = \mathbf{H}_0\,, \quad \big\langle I(x,\mathbf{R})\big\rangle = I_0 = |u_0|^2\,.$$

A peculiarity of equations (13) and (14) is the parametric excitation of both the density field $\rho(\mathbf{r},t)$ (in a compressible fluid flow) and the magnetic field energy $E(\mathbf{r},t) = \mathbf{H}^2(\mathbf{r},t)$ (for a turbulent flow of fluid) with time in *each realization*, which has come to be known as the *stochastic dynamo* (see, for example, Ref. [40]).

At the initial stages of dynamical system evolution such parametric excitation is accompanied by an increase with time of all traditional statistical characteristics of problem solutions, such as the moment function of the density field $\langle \rho^n(\mathbf{r},t)\rangle$ and magnetic field energy $\langle E^n(\mathbf{r},t)\rangle$, as well as their correlation functions of arbitrary order. As the distance is increased, the moments of radiation power $\langle I^n(x,\mathbf{R})\rangle$ grow in random media as well.

The dynamic diffusion effects for the density and magnetic field are insignificant through the early phases of their evolution and, neglecting them, we arrive at the first-order partial differential equations

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\,\mathbf{u}(\mathbf{r},t)\right)\rho(\mathbf{r},t) = 0\,, \quad \rho(\mathbf{r},0) = \rho_0(\mathbf{r})\,, \tag{19}$$

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathbf{r}}\,\mathbf{u}(\mathbf{r},t)\right)\mathbf{H}(\mathbf{r},t) = \left(\mathbf{H}(\mathbf{r},t)\,\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{u}(\mathbf{r},t)\,, \\ &\mathbf{H}(\mathbf{r},0) = \mathbf{H}_0(\mathbf{r})\,. \end{aligned} \tag{20}$$

However, namely at small times *spatial structures* may develop in individual realizations of respective fields!

As an illustration of structure formation in a magnetic field, we present Fig. 11 and an excerpt found on the web [42]:

"*What does puzzle astrophysicists so strongly?*

Contrary to hypotheses formed for fifty years, at the boundary of a planetary system observers encountered a

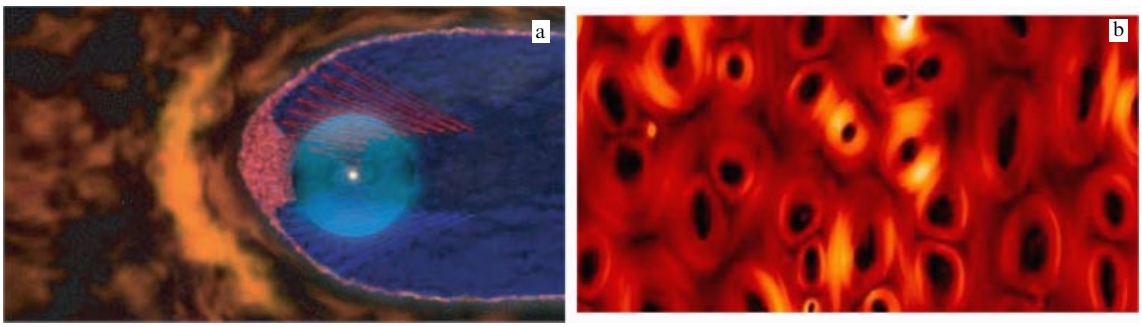

Figure 11. (Color online.) Magnetic field configuration on the boundary of a heliosphere as it looks, in all probability, in reality. A conditional interpretation (a), and a reconstruction of a magnetic bubble system (b).

boiling foam of locally magnetized areas each of hundreds of millions kilometers in extent, which form a nonstationary cellular structure in which magnetic field lines are permanently breaking and recombining to form new areas — *magnetic bubbles*."

Questions touching stochastic structure formation for the magnetic field energy are considered in Section 4.2.2.

1.3 Statistical characteristics of a random velocity field $\mathbf{u}(\mathbf{r}, t)$

The random velocity field $\mathbf{u}(\mathbf{r}, t)$ will be considered Gaussian, statistically homogeneous and isotropic in space, and stationary in time, with the respective correlation and spectral functions

$$B_{ij}(\mathbf{r}-\mathbf{r}', t-t') = \langle u_i(\mathbf{r}, t) u_j(\mathbf{r}', t') \rangle = \int \mathrm{d}\mathbf{k}\, E_{ij}(\mathbf{k}, t-t') \exp\left[\mathrm{i}\mathbf{k}(\mathbf{r}-\mathbf{r}')\right], \qquad E_{ij}(\mathbf{k}, t) = \frac{1}{(2\pi)^3} \int \mathrm{d}\mathbf{r}\, B_{ij}(\mathbf{r}, t) \exp(-\mathrm{i}\mathbf{k}\mathbf{r}). \tag{21}$$

In the general case of an arbitrary random velocity field $\mathbf{u}(\mathbf{r}, t)$, the spectral function $E_{ij}(\mathbf{k}, t)$ has the form

$$E_{ij}(\mathbf{k}, t) = E^{\mathrm{s}}(k, t)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) + \frac{k_i k_j}{k^2} E^{\mathrm{p}}(k, t), \tag{22}$$

where $E^{\mathrm{s}}(k, t)$ and $E^{\mathrm{p}}(k, t)$ are, respectively, the solenoidal and potential components of the velocity field spectral function.

The variance of the velocity field in this case takes the form

$$\sigma_{\mathbf{u}}^2 = B_{ii}(\mathbf{0}, 0) = \int \mathrm{d}\mathbf{k}\, E_{ii}(\mathbf{k}, 0) = \int \mathrm{d}\mathbf{k}\, \left[(d-1)E^{\mathrm{s}}(k, 0) + E^{\mathrm{p}}(k, 0)\right], \tag{23}$$

where d is the space dimension, with implied summation taken over twice repeating indices.

Let us introduce the function

$$B_{ij}(\mathbf{r}) = \int_0^\infty \mathrm{d}\tau\, B_{ij}(\mathbf{r}, \tau) \tag{24}$$

important for the further statistical analysis, which defines all the statistical characteristics of the problem solution in the diffusion approximation (see, for example, monographs [17–20]). Then, the following relations are valid:

$$B_{ii}(\mathbf{0}) = D_0 = \int_0^\infty \mathrm{d}\tau\, B_{ii}(\mathbf{0}, \tau) = \int_0^\infty \mathrm{d}\tau \int \mathrm{d}\mathbf{k}\, E_{ii}(\mathbf{k}, \tau) = \sigma_{\mathbf{u}}^2 \tau_0 = \sigma_{\mathrm{s}}^2 \tau_{\mathrm{s}} + \sigma_{\mathrm{p}}^2 \tau_{\mathrm{p}}, \tag{25}$$

where $\sigma_{\mathbf{u}}^2$ and τ_0 are the variance and the time correlation radius of a random velocity field, σ_{s}^2 and σ_{p}^2 are, respectively, the variances of the solenoidal and potential components of the velocity field, and τ_{s} and τ_{p} are their time correlation radii.

Later on, in a statistical analysis of the problem, we will need the second spatial derivatives of the correlation function of a random velocity field $\mathbf{u}(\mathbf{r}, t)$ at the zero argument. By virtue of the assumptions about the characteristics of random field $\mathbf{u}(\mathbf{r}, t)$, the following tensor equality holds for these derivatives (see, for example, monographs [17–20]):

$$-\frac{\partial^2 B_{ij}(0)}{\partial r_k \partial r_l} = \frac{D^{\mathrm{s}}}{d(d+2)}\left[(d+1)\delta_{kl}\delta_{ij} - \delta_{ki}\delta_{lj} - \delta_{kj}\delta_{li}\right] + \frac{D^{\mathrm{p}}}{d(d+2)}\left(\delta_{kl}\delta_{ij} + \delta_{ki}\delta_{lj} + \delta_{kj}\delta_{li}\right), \tag{26}$$

where in the three-dimensional case the parameters D^{s} and D^{p} have the form

$$D^{\mathrm{s}} = \int \mathrm{d}\mathbf{k}\, k^2 E^{\mathrm{s}}(k) = 4\pi \int_0^\infty \mathrm{d}k\, k^4 E^{\mathrm{s}}(k) = \frac{1}{2}\int_0^\infty \mathrm{d}\tau\, \langle \boldsymbol{\omega}(\mathbf{r}, t+\tau)\boldsymbol{\omega}(\mathbf{r}, t)\rangle,$$
$$D^{\mathrm{p}} = \int \mathrm{d}\mathbf{k}\, k^2 E^{\mathrm{p}}(k) = 4\pi \int_0^\infty \mathrm{d}k\, k^4 E^{\mathrm{p}}(k) = \int_0^\infty \mathrm{d}\tau \left\langle \frac{\partial \mathbf{u}(\mathbf{r}, t+\tau)}{\partial \mathbf{r}} \frac{\partial \mathbf{u}(\mathbf{r}, t)}{\partial \mathbf{r}} \right\rangle, \tag{27}$$

where $\boldsymbol{\omega}(\mathbf{r}, t) = \operatorname{rot} \mathbf{u}(\mathbf{r}, t)$ is the vorticity, $\partial \mathbf{u}(\mathbf{r}, t)/\partial \mathbf{r}$ is the divergence of velocity field, and

$$E^{\mathrm{s}}(k) = \int_0^\infty \mathrm{d}\tau\, E^{\mathrm{s}}(k, \tau), \qquad E^{\mathrm{p}}(k) = \int_0^\infty \mathrm{d}\tau\, E^{\mathrm{p}}(k, \tau).$$

The coefficients D^{s} and D^{p}, defined by relationships (27), can be written out through the statistical characteristics of the

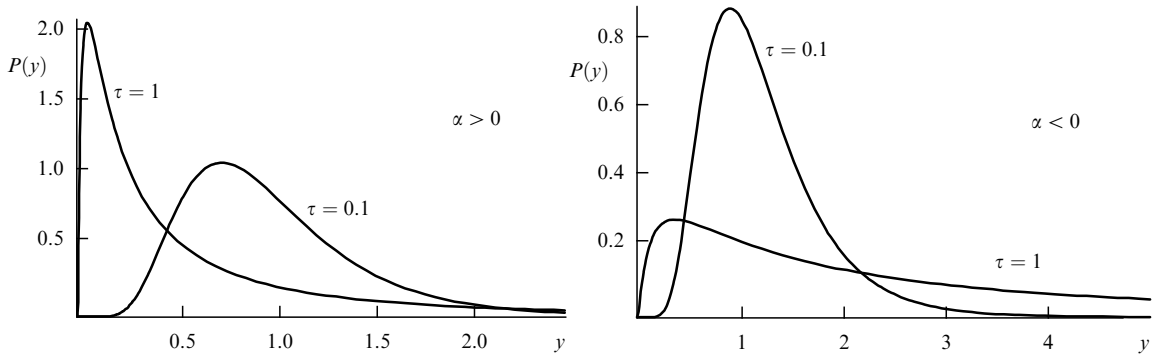

Figure 12. Lognormal probability distributions (34) for the parameter $|\alpha|/D = 1$ and dimensionless time $\tau = Dt = 0.1$ and 1.

velocity field derivatives as

$$D^{\mathrm{s}} = \frac{1}{2}\,\sigma_{\boldsymbol{\omega}}^2 \tau_{\boldsymbol{\omega}}\,, \qquad D^{\mathrm{p}} = \sigma_{\operatorname{div}\mathbf{u}}^2 \tau_{\operatorname{div}\mathbf{u}}\,. \tag{28}$$

We are interested in two examples of the random velocity field:

(1) incompressible hydrodynamic turbulence;

(2) potential hydrodynamic fields.

A particular case of the potential random field is exemplified by wave turbulence, where the correlation function of the velocity field is given by the following expression

$$B_{ij}(\mathbf{r}, t) = \int \mathrm{d}\mathbf{k}\; \frac{k_i k_j}{k^2}\, E^{\mathrm{p}}(k) \exp\left(-\lambda k^2 t\right) \cos\left(\mathbf{kr} - \omega(\mathbf{k}) t\right), \tag{29}$$

where $\omega = \omega(\mathbf{k}) > 0$ defines the dispersion curve for wave motions, and the parameter λ describes wave damping.

The variance of the velocity field in this case takes the form

$$\sigma_{\mathbf{u}}^2 = \left\langle \mathbf{u}^2(\mathbf{r}, t) \right\rangle = \int \mathrm{d}\mathbf{k}\, E^{\mathrm{p}}(k)\,, \tag{30}$$

and the quantity analogous to that in formula (24) is defined as

$$\begin{aligned} B_{ij}(\mathbf{r}) &= \int_0^\infty \mathrm{d}t\, B_{ij}(\mathbf{r}, t) \\ &= \int \mathrm{d}\mathbf{k}\; \frac{k_i k_j}{k^2} \left[E_1^{\mathrm{p}}(k) \cos(\mathbf{kr}) + E_2^{\mathrm{p}}(k) \sin(\mathbf{kr})\right], \end{aligned} \tag{31}$$

where

$$\begin{aligned} E_1^{\mathrm{p}}(k) &= E^{\mathrm{p}}(k)\, \frac{\lambda k^2}{\lambda^2 k^4 + \omega^2(k)}\,, \\ E_2^{\mathrm{p}}(k) &= E^{\mathrm{p}}(k)\, \frac{\omega(k)}{\lambda^2 k^4 + \omega^2(k)}\,. \end{aligned} \tag{32}$$

The detection and description of the phenomenon of spatial structure formation (clustering) in individual realizations of random fields prove to be possible only by analyzing one-time and one-point probability densities of solutions to equations given above if one resorts to the ideas of statistical topography. We consider first the statistical description of lognormal random processes.

2. Lognormal processes, intermittency, and dynamical localization

The one-time probability density $P(y, t; \alpha) = \langle \delta(y(t; \alpha) - y) \rangle$ of lognormal process (4) obeys the Fokker–Planck equation

$$\begin{aligned} &\frac{\partial}{\partial t}\, P(y, t; \alpha) = \left(\alpha\, \frac{\partial}{\partial y}\, y + D\, \frac{\partial}{\partial y}\, y\, \frac{\partial}{\partial y}\, y\right) P(y, t; \alpha)\,, \\ &P(y, 0; \alpha) = \delta(y - 1)\,, \end{aligned} \tag{33}$$

the solution of which, naturally, depends on the parameter α:

$$P(y, t; \alpha) = \frac{1}{2y\sqrt{\pi D t}} \exp\left\{-\frac{\ln^2\left[y \exp(\alpha t)\right]}{4Dt}\right\}. \tag{34}$$

Probability distribution (34) implies a substantially different behavior for $\alpha > 0$ and $\alpha < 0$. The common feature in both cases is only the appearance of long, moderately sloping 'tails' at large t, indicating an increasing role of large excursions of processes $y(t; \alpha)$ in the formation of one-time statistics. The plots of logarithmically normal probability densities (34) for $\alpha > 0$ and $\alpha < 0$ for the parameter $|\alpha|/D = 1$ and dimensionless time $\tau = Dt = 0.1$ and 1 are given in Fig. 12.

Accordingly, the integral probability distribution function is given by the expression

$$\begin{aligned} F(y, t; \alpha) &= \int_{-\infty}^{y} \mathrm{d}y'\, P(t; y') = \mathcal{P}\big(y(t; \alpha) < y\big) \\ &= \Pr\left\{\frac{1}{\sqrt{2Dt}} \ln\left[y \exp(\alpha t)\right]\right\}, \end{aligned} \tag{35}$$

where the function $\Pr(z)$ is the *probability integral* defined as

$$\Pr(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{d}x\, \exp\left(-\frac{x^2}{2}\right). \tag{36}$$

Obviously $\Pr(\infty) = 1$ and $\Pr(0) = 1/2$. The asymptotic form of the probability integral for $z \to \pm\infty$ can easily be found from expression (36), namely

$$\begin{aligned} \Pr(z)_{z\to\infty} &\approx 1 - \frac{1}{z\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right), \\ \Pr(z)_{z\to-\infty} &\approx \frac{1}{|z|\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right). \end{aligned} \tag{37}$$

From equation (33), we can also easily derive the equality

$$\alpha = -\lim_{t\to\infty} \frac{\partial\langle \ln y(t;\alpha)\rangle}{\partial t} . \tag{38}$$

It follows then that the parameter α in equation (33) coincides with the Lyapunov characteristic exponent (38) for the lognormal random process $y(t;\alpha)$ (4) (see, for example, reviews [43, 44]).

2.1 Typical realization curve of a random process

The statistical characteristics of the process $z(t)$ at a fixed time instant t are described by the probability density $P(z,t)$ and the integral probability distribution function $F(Z,t) = \int_{-\infty}^{Z} \mathrm{d}z'\, P(z',t)$.

The typical realization curve for the random process $z(t)$ is referred to as the deterministic curve $z^*(t)$, which is the *median of the integral probability distribution function* and is defined through the solution of the algebraic equation

$$F\big(z^*(t),t\big) = \frac{1}{2} . \tag{39}$$

This implies, on the one hand, that for any time instant t the probability $\mathcal{P}\{z(t) > z^*(t)\} = \mathcal{P}\{z(t) < z^*(t)\} = 1/2$.

On the other hand, the median has a specific property that for any interval (t_1, t_2) the random process $z(t)$ 'winds round' the curve $z^*(t)$ so that the mean time during which $z(t) > z^*(t)$ coincides with that when the reverse inequality $z(t) < z^*(t)$ is true (Fig. 13), i.e. one has

$$\langle T_{z(t)>z^*(t)}\rangle = \langle T_{z(t)<z^*(t)}\rangle = \frac{1}{2}(t_2 - t_1) .$$

The curve $z^*(t)$, needless to say, may differ essentially from any individual realization of the process $z(t)$ and does not describe the amplitude of possible excursions. Thus, the typical realization curve $z^*(t)$ of a random process $z(t)$, obtained with the help of the one-time probability density, is defined nevertheless over the entire time interval $t \in (0,\infty)$, and is namely that deterministic curve relative to which the intermittency is enfolding.

The typical realization curve (39) for a Gaussian random process $z(t)$ coincides with the expectation of the process $z(t)$, i.e., $z^*(t) = \langle z(t)\rangle$, while the typical realization curve for a lognormal process $f(t) = \exp(z(t))$ is defined by the equality

$$f^*(t) = \exp(\langle z(t)\rangle) = \exp(\langle \ln f(t)\rangle) .$$

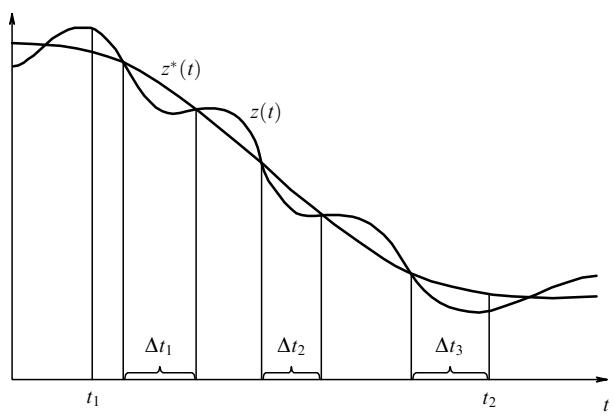

Figure 13. Regarding the definition of a typical realization curve for a random process.

As a consequence, the typical realization curve of lognormal process (4) is described by the formula

$$f^*(t) = \exp\big(\langle \ln f(t)\rangle\big) = \exp(-\alpha t) ,$$

which coincides with the Lyapunov exponential function.

For $\alpha > 0$, the typical realization curve decays exponentially with time and, in the opposite case, $\alpha < 0$, grows exponentially; namely these functions are plotted in Fig. 7 with dashed lines. At $\alpha = 0$, the intermittency takes place with respect to the line $f^*(t) = 1$.

2.2 Dynamical localization

We note that *for one-dimensional problems the positivity of the Lyapunov characteristic index α corresponds well to the physical phenomenon of dynamical localization* (*clustering*).

For the problem involving diffusion of an ensemble of particles (5) in a Gaussian random velocity field $\mathbf{u}(\mathbf{r},t)$, the typical realization curve for the distance between two particles is given by the exponential function of time:

$$l^*(t) = l_0 \exp\left\{\frac{1}{d(d+2)}\left[D^{\mathrm{s}} d(d-1) - D^{\mathrm{p}}(4-d)\right] t\right\} , \tag{40}$$

where d is the space dimension, and the coefficients D^{s} and D^{p} are described by formulas (27) (see, for example, Refs [17–20]).

From the last formula, it follows that in the two-dimensional case ($d = 2$) the expression

$$l^*(t) = l_0 \exp\left[\frac{1}{4}(D^{\mathrm{s}} - D^{\mathrm{p}})t\right]$$

substantially depends on the sign of the difference $D^{\mathrm{s}} - D^{\mathrm{p}}$. In particular, for a nondivergent velocity field ($D^{\mathrm{p}} = 0$), we have an exponentially growing typical realization curve, which corresponds to particles running away exponentially fast at small distances between them. In the other limiting case — for a potential velocity field ($D^{\mathrm{s}} = 0$) — the typical realization curve will be an exponentially decaying one; hence, the obvious tendency of particles to 'coalesce'. Thus, the condition for clustering in the two-dimensional case reduces to holding the inequality $D^{\mathrm{s}} < D^{\mathrm{p}}$.

In the three-dimensional case ($d = 3$), from Eqn (40) it follows that

$$l^*(t) = l_0 \exp\left[\frac{1}{15}(6D^{\mathrm{s}} - D^{\mathrm{p}})t\right] ,$$

and the typical realization curve will decay with time if the condition $D^{\mathrm{p}} > 6D^{\mathrm{s}}$, which is more demanding than in the two-dimensional case, is satisfied.

In the one-dimensional case, one finds $l^*(t) = l_0 \exp(-D^{\mathrm{p}} t)$, and the typical realization curve always decays with time, for in this case the velocity field is always potential.

For a boundary-value problem (6) dealing with a plane wave incident on a half-space of a random layered medium, the wave field intensity $I(x) = |u(x)|^2$ is a lognormal random process with the typical realization curve $I^*(x) = 2\exp[-D(L-x)]$, where the parameter $D = k^2\sigma_\varepsilon^2 l_0/2$ (σ_ε^2 is the $\varepsilon(x)$ process variance, and l_0 is its correlation radius) for the model of Gaussian random process $\varepsilon(x)$ with the correlation function $\langle \varepsilon(x)\varepsilon(x')\rangle = 2D\delta(x-x')$. The coefficient of wave transmission through a sufficiently thick layer of a random medium decays exponentially for the problem

considered, and the half-space of a randomly inhomogeneous medium ($L_0 \to -\infty$) completely reflects the wave incident on it. In this case, the wave field intensity $I(x)$ is statistically equivalent to the random process $2y(t;\alpha)$ at $\alpha = D$, and its realization resembles the mirror reflection of Fig. 7 ($\alpha > 0$). And yet, certainly, the moments of wave field intensity exponentially grow with distance from the wave source into the medium. We mention that in monograph [35] this effect was established through the analysis of the Lyapunov exponent for the problem at hand, which matches the typical realization curve for the lognormal process.

We also mention that the quantity inverse to the diffusion coefficient D of this problem, defining a natural length scale related to medium random inhomogeneities, is commonly termed the *localization length*, $l_{\rm loc} = 1/D$.

3. Lognormal fields, statistical topography, and clustering

3.1 Lognormal random fields

Let us consider now a positive lognormal random field $f(\mathbf{r}, t)$, whose one-point probability density

$$P(\mathbf{r}, t; f) = \left\langle \delta\big(f(\mathbf{r}, t) - f\big)\right\rangle$$

is governed by the equation

$$\frac{\partial}{\partial t} P(\mathbf{r}, t; f) = \left(D_0 \frac{\partial^2}{\partial \mathbf{r}^2} + \alpha \frac{\partial}{\partial f} f + D_f \frac{\partial}{\partial f} f \frac{\partial}{\partial f} f\right) P(\mathbf{r}, t; f) \tag{41}$$

with the initial condition $P(\mathbf{r}, 0; f) = \delta(f - f_0(\mathbf{r}))$, where D_0 is the diffusion coefficient in the $\mathbf{r}$-space, and the coefficients α and D_f characterize diffusion in the f-space. The parameter α can both differ from zero and be equal to it (the critical case). The change in the sign of α for one-point characteristics implies the transition from the field $f(\mathbf{r}, t)$ to the field $\tilde{f}(\mathbf{r}, t) = 1/f(\mathbf{r}, t)$.

The solution of equation (41) is written out as follows:

$$P(\mathbf{r}, t; f) = \frac{1}{2f\sqrt{\pi D_f t}} \exp\left(D_0 t \frac{\partial^2}{\partial \mathbf{r}^2}\right) \times \exp\left\{-\frac{\ln^2\left[f \exp(\alpha t)/f_0(\mathbf{r})\right]}{4D_f t}\right\}. \tag{42}$$

Note that for a positive conservative random field $f(\mathbf{r}, t)$ satisfying the condition $\int \mathrm{d}\mathbf{r}\, f(\mathbf{r}, t) = \int \mathrm{d}\mathbf{r}\, f_0(\mathbf{r})$, the parameter $\alpha = D_f$ and equation (41) can be recast as

$$\frac{\partial}{\partial t} P(\mathbf{r}, t; f) = \left(D_0 \frac{\partial^2}{\partial \mathbf{r}^2} + \alpha \frac{\partial^2}{\partial f^2} f^2\right) P(\mathbf{r}, t; f). \tag{43}$$

Needless to say, the property of intermittency is always exhibited by any random field $f(\mathbf{r}, t)$, as well. For any fixed point in space $\mathbf{r}$, the temporal evolution of $f(\mathbf{r}, t)$ is a random process for which all the above holds.

For a spatially, statistically homogeneous problem which corresponds to the initial field distribution $f_0(\mathbf{r}) = f_0$, no one-point statistical characteristics of the field $f(\mathbf{r}, t)$ depend on the point $\mathbf{r}$, and the positivity of the index

$$\alpha = -\lim_{t\to\infty} \frac{\partial\left\langle \ln f(\mathbf{r}, t)\right\rangle}{\partial t}$$

for the lognormal field $f(\mathbf{r}, t)$ implies that, at any location in space, the realizations of this field decay with time despite large rare excursions, which occur for a lognormal process. In this case, the characteristic time of field decay is $t \sim 1/\alpha$. *But if the field decays almost everywhere, it must be concentrated somewhere, i.e., clustering should take place*. For a negative parameter α, the field grows at every fixed point in space.

In the last case, probability density (42) does not depend on $\mathbf{r}$ and is described by the equation

$$\frac{\partial}{\partial t} P(t; f) = \left(\alpha \frac{\partial}{\partial f} f + D_f \frac{\partial}{\partial f} f \frac{\partial}{\partial f} f\right) P(t; f), \qquad P(0; f) = \delta(f - f_0), \tag{44}$$

with the solution

$$P(t; f) = \frac{1}{2f\sqrt{\pi D_f t}} \exp\left\{-\frac{\ln^2\left[f \exp(\alpha t)/f_0\right]}{4D_f t}\right\}. \tag{45}$$

Thus, for a spatially homogeneous problem statement, the one-point statistical characteristics of a random field $f(\mathbf{r}, t)$ are statistically equivalent to those of the lognormal process $f(t;\alpha)$ with the probability density (45). A specific feature of this distribution is the appearance of a long gently sloping 'tail' for $D_f t \gg 1$, which indicates the increased role of large fluctuations of the process $f(t;\alpha)$ in forming one-time statistics (see Fig. 12). For this distribution, all moment functions exponentially grow with time and, in particular, at $n = 1$ and for $D_f > \alpha$ the expectation is given by

$$\left\langle f(\mathbf{r}, t)\right\rangle = f_0 \exp\left[(D_f - \alpha)t\right],$$

whereas the quantity α is the Lyapunov characteristic index.

Figure 14 plots schematically random realizations of the field $f(\mathbf{r}, t)$ for the parameter α with different signs.

One can describe spatial clustering in almost any realization of the random field $f(\mathbf{r}, t)$ by resorting to the ideas of statistical topography.

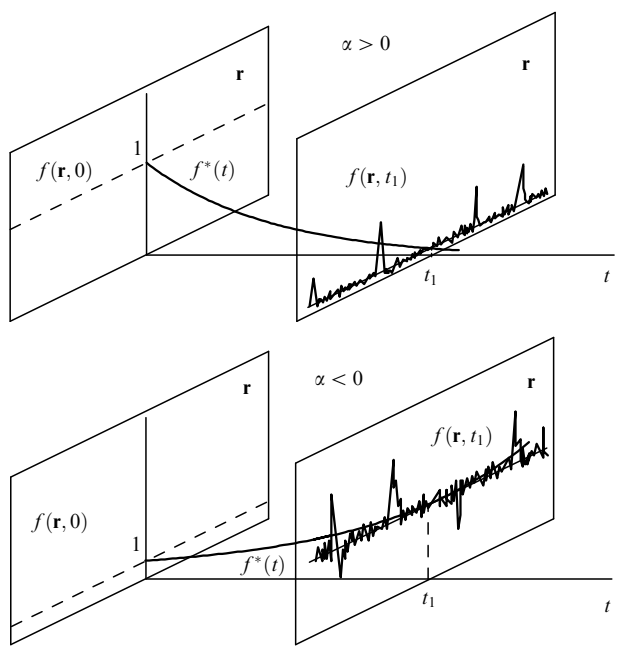

Figure 14. Schematics of the behavior of random realizations of the field $f(\mathbf{r}, t)$ for $\alpha > 0$ and $\alpha < 0$.

3.2 Statistical topography of random fields

The main subject of study in the statistical topography of random fields, just as in the traditional topography of massifs, is a system of contours — isolines (in the two-dimensional case) or isosurfaces (in three dimensions) defined by the equality $f(\mathbf{r}, t) = f = \text{const}$.

To analyze the system of contours (we limit ourselves to the two-dimensional case $\mathbf{r} = \mathbf{R}$ for simplicity), it is convenient to introduce the Dirac delta-function constrained to these contours:

$$\varphi(\mathbf{R}, t; f) = \delta\big(f(\mathbf{R}, t) - f\big)\,, \tag{46}$$

called the *indicator function*.

Such quantities as the total area bounded by the level lines of the areas where the random field $f(\mathbf{R}, t)$ exceeds the given level f, i.e., $f(\mathbf{R}, t) > f$:

$$S(t; f) = \int \mathrm{d}\mathbf{R}\, \theta\big(f(\mathbf{R}, t) - f\big) = \int \mathrm{d}\mathbf{R} \int_f^\infty \mathrm{d}f'\, \varphi(\mathbf{R}, t; f')\,,$$

and the total 'mass' of the field comprised in these regions, namely

$$\begin{aligned} M(t; f) &= \int \mathrm{d}\mathbf{R}\, f(\mathbf{R}, t)\theta\big(f(\mathbf{R}, t) - f\big) \\ &= \int \mathrm{d}\mathbf{R} \int_f^\infty \mathrm{d}f'\, f' \varphi(\mathbf{R}, t; f')\,, \end{aligned}$$

where $\theta(f(\mathbf{R}, t) - f)$ is the Heaviside theta-function, may be expressed in terms of function (46).

The mean value of the indicator function (46) over an ensemble of realizations of the random field $f(\mathbf{R}, t)$ defines the one-time and one-point in space probability density [17–20]

$$P(\mathbf{R}, t; f) = \big\langle \delta\big(f(\mathbf{R}, t) - f\big)\big\rangle\,,$$

which directly sets the ensemble mean values of quantities $S(t; f)$ and $M(t; f)$:

$$\big\langle S(t; f)\big\rangle = \int \mathrm{d}\mathbf{R} \int_f^\infty \mathrm{d}f'\, P(\mathbf{R}, t; f')\,,$$

$$\big\langle M(t; f)\big\rangle = \int \mathrm{d}\mathbf{R} \int_f^\infty \mathrm{d}f'\, f' P(\mathbf{R}, t; f')\,.$$

Information on the detailed structure of field $f(\mathbf{R}, t)$ can be obtained by additionally considering its spatial gradient $\mathbf{p}(\mathbf{R}, t) = \nabla f(\mathbf{R}, t)$. So, for example, the quantity

$$l(t; f) = \oint \mathrm{d}l = \int \mathrm{d}\mathbf{R}\, \big|\mathbf{p}(\mathbf{R}, t)\big| \delta\big(f(\mathbf{R}, t) - f\big) \tag{47}$$

describes the total length of contours. The integrand in Eqn (47) is described by the extended indicator function

$$\varphi(\mathbf{R}, t; f, \mathbf{p}) = \delta\big(f(\mathbf{R}, t) - f\big)\delta\big(\mathbf{p}(\mathbf{R}, t) - \mathbf{p}\big)\,, \tag{48}$$

and the mean value of $l(t; f)$ [see formula (47)] is related to the joint one-time probability density of field $f(\mathbf{R}, t)$ and its gradient $\mathbf{p}(\mathbf{R}, t)$, which is obtained by averaging the indicator function (48) over an ensemble of realizations, which gives the function

$$P\big(\mathbf{R}, t; f, \mathbf{p}\big) = \big\langle \delta\big(f(\mathbf{R}, t) - f\big)\, \delta\big(\mathbf{p}(\mathbf{R}, t) - \mathbf{p}\big)\big\rangle\,.$$

By additionally considering spatial derivatives of the second order, the total number of contours $f(\mathbf{R}, t) = f = \text{const}$ can be estimated with the help of approximate formula (up to not-closing lines)

$$\begin{aligned} N(t; f) &= N_{\text{in}}(t; f) - N_{\text{out}}(t; f) \\ &= \frac{1}{2\pi} \int \mathrm{d}\mathbf{R}\, \kappa(t, \mathbf{R}; f) \big|\mathbf{p}(\mathbf{R}, t)\big| \delta\big(f(\mathbf{R}, t) - f\big)\,, \end{aligned} \tag{49}$$

where $N_{\text{in}}(t; f)$ ($N_{\text{out}}(t; f)$) is the number of contours for which the vector $\mathbf{p}$ is directed along the inner (outer) normal, and $\kappa(t, \mathbf{R}; f)$ is the curvature of the level line.

3.2.1 Conditions of cluster structure formation. We now discuss the conditions of stochastic structure formation in parametrically excited random fields. It is clear that for a *positive field* $f(\mathbf{R}, t)$ *in general* the condition of cluster formation with a unit probability, i.e., for almost all realizations, is the simultaneous tendency of being fulfilled, as $t \to \infty$, for asymptotic equalities

$$\big\langle S(t; f)\big\rangle \to 0\,, \quad \big\langle M(t; f)\big\rangle \to \int \mathrm{d}\mathbf{R}\, \big\langle f(\mathbf{R}, t)\big\rangle\,.$$

The lack of structure formation corresponds to the simultaneous tendency towards satisfying the asymptotic equalities as $t \to \infty$:

$$\big\langle S(t; f)\big\rangle \to \infty\,, \quad \big\langle M(t; f)\big\rangle \to \int \mathrm{d}\mathbf{R}\, \big\langle f(\mathbf{R}, t)\big\rangle\,.$$

For a spatially homogeneous field $f(\mathbf{R}, t)$, the one-point probability density $P(\mathbf{R}, t; f)$ does not depend on $\mathbf{R}$; in this case, statistical means of all the expressions (without integration over $\mathbf{R}$) will describe specific (per unit area) values of the respective quantities.

Thus, the specific mean area $\langle s_{\text{hom}}(t; f)\rangle$, where the random field $f(\mathbf{R}, t)$ exceeds a given level f, coincides with the event probability at any point in space, $f(\mathbf{R}, t) > f$:

$$\big\langle s_{\text{hom}}(t; f)\big\rangle = \big\langle \theta\big(f(\mathbf{R}, t) - f\big)\big\rangle = \mathcal{P}\big\{f(\mathbf{R}, t) > f\big\}\,,$$

so that the mean specific area offers a geometric interpretation of the probability of the event $f(\mathbf{R}, t) > f$, which is, clearly, independent of point $\mathbf{R}$. As a consequence, the conditions of clustering for the *homogeneous* case reduce to the tendency of being valid, as $t \to \infty$, for asymptotic equalities

$$\big\langle s_{\text{hom}}(t; f)\big\rangle = \mathcal{P}\big\{f(\mathbf{r}, t) > f\big\} \to 0\,, \quad \big\langle m_{\text{hom}}(t; f)\big\rangle \to \big\langle f(t)\big\rangle\,,$$

whereas the absence of clustering is linked to the tendency of being valid, as $t \to \infty$, for the asymptotic equalities

$$\big\langle s_{\text{hom}}(t; f)\big\rangle = \mathcal{P}\big\{f(\mathbf{r}, t) > f\big\} \to 1\,, \quad \big\langle m_{\text{hom}}(t; f)\big\rangle \to \big\langle f(t)\big\rangle\,.$$

Thus, *clustering in a spatially homogeneous problem is a phenomenon* (*occurring with a unit probability, i.e., for almost all realizations of a random positive field*) *spawned by a rare event occurring with vanishing probability*.

In this case, the mere presence of rare events serves as a *trigger* which initiates the structure formation process, *while the structure formation on its own is the intrinsic property of a random medium*, i.e., is in essence the *law of Nature* [21–24].

The characteristic time of cluster structure formation in space is determined by the character of the asymptotic expressions given above at large times. Now, this time is

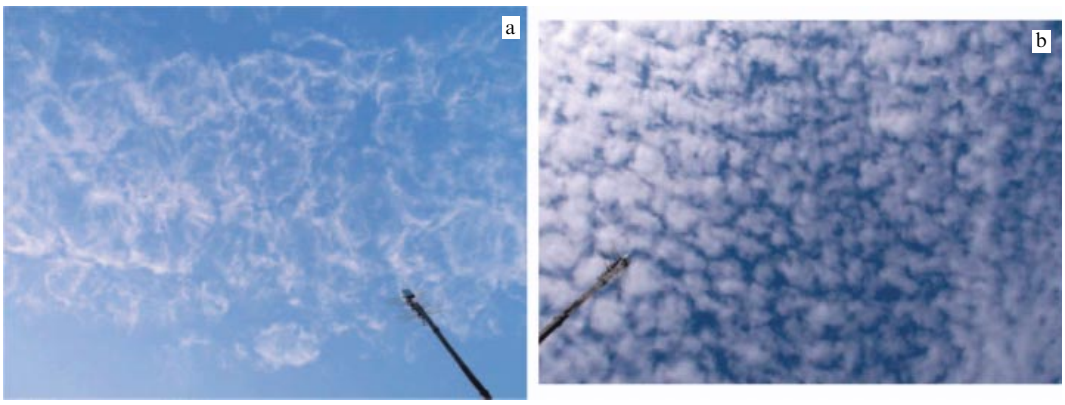

Figure 15. Photos of the cluster structure in a cloudy sky: (a) cluster structure close to an 'ideal' one, and (b) 'deformed' clusters. The black stripes in the bottom corners are parts of a ground-based antenna. Photos are taken on 15 June and 2 August 2013 at 21:00 on the coast of the Sea of Azov. (Courtesy of V A Dovzhenko.)

defined not only by the Lyapunov statistical characteristic index α, but also by the diffusion coefficient D_f in phase space of a positive field $f(\mathbf{r}, t)$. It is certainly larger than the characteristic time of realization decay at any fixed point in space.

For concrete physical dynamical systems, the description of clustering in physical fields reduces, therefore, to computation of the stochastic Lyapunov index α and diffusion coefficient D_f, which is, generally speaking, a rather tedious task for concrete partial differential equations.

In the presence of clustering, the field is simply absent over a large portion of space! It should be clear that the conditions above on the presence or absence of clustering in the field $f(\mathbf{R}, t)$ have nothing to do with the parametric growth of statistical characteristics like moment or correlation functions of arbitrary order, as time progresses.

The above criterion of ideal clustering (in analogy with ideal fluid dynamics) corresponds to the dynamics of cluster formation in dynamical systems governed, generally speaking, by partial differential equations of the first order (the Eulerian representation). This ideal structure emerges as a very narrow band (in the two-dimensional case) or very narrow tubes (in the three-dimensional case).

Notice that the first-order partial differential equations can be solved in general by the methods of characteristics. This corresponds to the Lagrangian description of dynamical systems. In this case, the characteristic curves described by ordinary differential equations can, naturally, have different peculiarities and even singularities. The conditions for the appearance of such peculiarities in the Lagrangian description do not have direct connections to the phenomenon of clustering in space and time, i.e., in the Eulerian representation.

However, in real physical systems, various additional factors related to the generation of spatial derivatives of a random field may become visible later; they *distort* but *do not eliminate* this picture of clustering. In particular, a situation is possible where the respective probability density approaches steady-state regime $P(\mathbf{R}; f)$ as $t \to \infty$. In this case, the functionals of the form

$$\langle S(f) \rangle = \int \mathrm{d}\mathbf{R} \int_f^\infty \mathrm{d}f' \, P(\mathbf{R}; f') \,,$$

$$\langle M(f) \rangle = \int \mathrm{d}\mathbf{R} \int_f^\infty \mathrm{d}f' f' P(\mathbf{R}; f')$$

cease to describe further distortions of the cluster picture. We need to study the temporal evolution of the functionals related to the spatial derivatives of the field $f(\mathbf{R}, t)$, such as the total contour length and the number of contours.

As an example of 'ideal' and 'deformed' clustering in Nature, we present here photos of the cluster structure of a cloudy sky[1] (Fig. 15) and a lava lake in volcano craters (Fig. 16). We note that a statistical theory describing volcanic lava dynamics, as far as we know, does not exist at all. A similar pattern is demonstrated by structures (Figs 17, 18) related to the parabolic Leontovich (Schrödinger) equation (15) (see, for example, monographs [17–20]).

To begin with, we note that if a plane wave propagates in a random medium, then in the approximation that the random field $\varepsilon(x, \mathbf{R})$ is delta-correlated in the longitudinal direction and its correlation function $B_\varepsilon(x - x', \mathbf{R} - \mathbf{R}') = \langle \varepsilon(x, \mathbf{R}) \varepsilon(x', \mathbf{R}') \rangle$ has the form

$$B_\varepsilon(x, \mathbf{R}) = A(\mathbf{R}) \delta(x), \quad A(\mathbf{R}) = \int_{-\infty}^{\infty} \mathrm{d}x \, B_\varepsilon(x, \mathbf{R}) \,,$$

equation (15) for the mean field $\langle u(x, \mathbf{R}) \rangle$ and the second-order coherence function

$$\Gamma_2(x, \boldsymbol{\rho}) = \left\langle u\left(x, \mathbf{R} + \frac{1}{2}\boldsymbol{\rho}\right) u^*\left(x, \mathbf{R} - \frac{1}{2}\boldsymbol{\rho}\right) \right\rangle$$

lead to the expressions

$$\langle u(x, \mathbf{R}) \rangle = u_0 \exp\left(-\frac{\gamma x}{2}\right),$$

$$\Gamma_2(x, \boldsymbol{\rho}) = |u_0|^2 \exp\left(-\frac{k^2 x D(\rho)}{4}\right),$$

which do not depend on the wave field diffraction; here, $\gamma = (k^2/4) A(0)$ is the *extinction coefficient*, and the function $D(\rho) = A(0) - A(\rho)$ is linked to the structure function of the random field $\varepsilon(x, \mathbf{R})$. Simultaneously, a statistical scale ρ_{cog} called the *coherence radius of field* $u(x, \mathbf{R})$ and defined by the condition $(1/4) k^2 x D(\rho_{\mathrm{cog}}) = 1$ appears in the plane perpendicular to the direction of wave propagation. The coherence radius depends on the wavelength, the distance travelled by the wave, and the medium statistical parameters.

[1] Yet there is no particle clustering in the Lagrangian description!

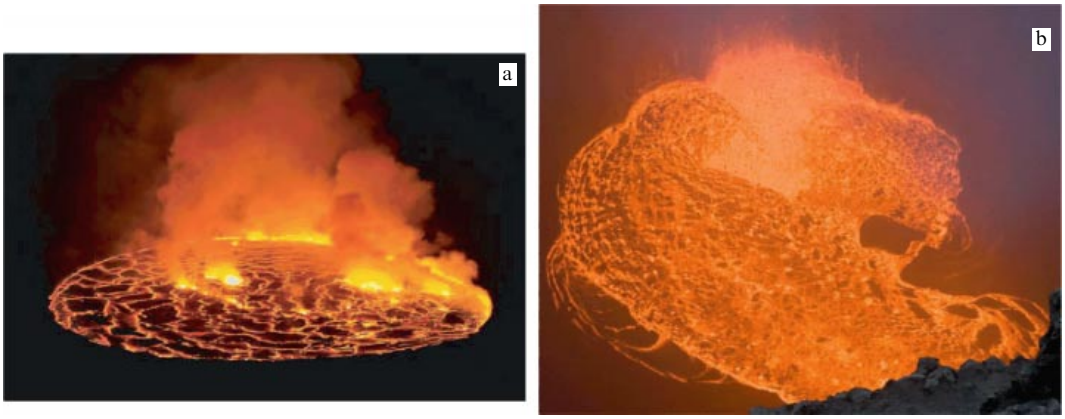

Figure 16. (a) A lake of boiling lava in the Nyiragongo Crater in the Great Lakes region of Africa (http://www.boston.com/bigpicture/2011/02/nyiragongo_crater_journey_to_t.html). (b) Lava lake in Halema'uma'u Crater on Kilauea (Photo courtesy of Hawaiian Volcano Observatory, USGS). These images can be found on the sites http://bigpicture.ru/?p-128340, http://pacificislandparks.com/2010/01/20/more-amazing-lava-lake-photos/.

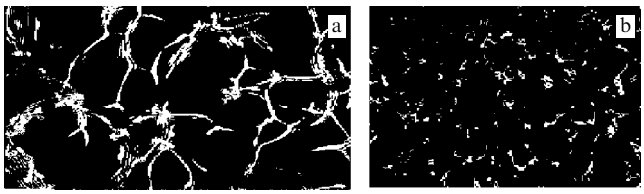

Figure 17. Transverse section of a laser beam passing through a turbulent medium (under laboratory conditions) in the region of strong focusing (a), and in the region of strong (saturated) fluctuations (b).

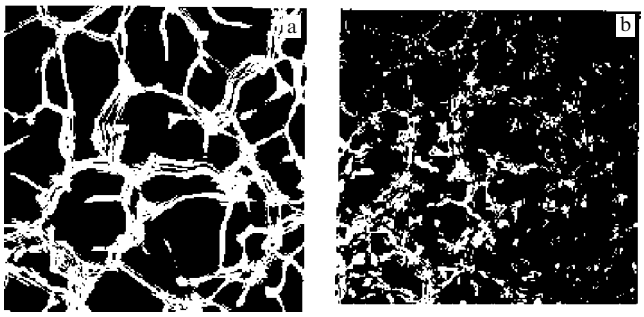

Figure 18. Results of numerical modeling with the help of a system of phase screens (18): (a) transverse section of a laser beam propagating through a turbulent medium in the region of strong focusing (a), and in the region of strong (saturated) fluctuations (b).

Notice that a complete solution to the problem of computation of the statistical characteristics of solutions to equation (15) for $x \to \infty$ was found already in 1977 in Ref. [45] (see also Refs [17–20, 46]) by resorting to the continual integral representation of solutions to equation (15).

Obviously, the probability distribution for the wave field intensity has a lognormal character if the distance travelled by the incident plane wave remains small, and in this case the stochastic structure formation (clustering) ensues.

As the distance increases, the nonlinear character of the equation for the complex-valued phase needs to be taken into account. This region of fluctuations, referred to as *the region of strong focusing*, is extremely difficult for analytical analyses. For even larger distances of wave propagation, the statistical characteristics of wave intensity reach a saturated regime; the respective spatial domain is referred to as the *region of strong intensity fluctuations*.

In this region, the statistical characteristics of wave field intensity cease to depend on the distance and take the form ($u_0 = 1$)

$$\langle I^n(x, \mathbf{R}) \rangle = n!\,, \quad P(x, I) = \exp(-I)\,.$$

Reference [45] computed the spatial correlation function of wave field intensity $I(x, \mathbf{R}) = |u(x, \mathbf{R})|^2$ for $x \to \infty$ ($\boldsymbol{\rho} = \mathbf{R}' - \mathbf{R}''$):

$$B_I(x, \boldsymbol{\rho}) = \langle I(x, \mathbf{R}')\, I(x, \mathbf{R}'') \rangle - 1 = |\Gamma_2(x, \boldsymbol{\rho})|^2 = \exp\left(-\frac{k^2 x}{2} D(\rho)\right), \tag{50}$$

which is also independent of the wave field diffraction. Now, in this problem, in addition to the spatial scale ρ_{cog}, a second characteristic spatial scale appears, $r_0 = x/(k\rho_{\text{cog}})$. However, numerous attempts by experimenterlists, continuing into the present, to associate these scales with the patterns displayed in Figs 17 and 18 have not led to success. And it is clear why! Clustering of the wave field intensity is certainly affected by diffraction, which is, however, in no way reflected in the form of its correlation function (50).

From the standpoint of statistical topography, the mean specific area of regions inside which $I(x, \mathbf{R}) > I$ and the mean specific power confined there are constant and in the limit of $x \to \infty$ do not describe the behavior of wave field intensity in individual realizations. Besides, no information is gained in this case by transition to the statistically equivalent random process. For this case, an explanation of the wave field structure in individual realizations was proposed only 20 years later (in 1997) in Ref. [47] (see also Refs [17–20]), by resorting to the analysis of such quantities as the specific mean length of contours and specific mean number of contours of wave field intensity, which are described by functionals like (47) and (49) and are connected with spatial derivatives of wave field intensity. These functions continue to increase with distance also in the region of strong intensity fluctuations, and, consequently, contour fragmentation takes place, as observed in laboratory experiments and through numerical modeling as well.

3.2.2 Statistical topography of lognormal random fields. In the analysis of one-point statistical characteristics in the spatially

homogeneous case, it is generally rewarding to take into account that the random field $f(\mathbf{R}, t)$ is statistically equivalent to some random process $f(t)$ with the same statistical characteristics.

If a one-point probability density of the random field $f(\mathbf{r}, t)$ (42) is known, one can also obtain general information on the spatial structure of random field $f(\mathbf{r}, t)$. In particular, such functionals of the random field $f(\mathbf{r}, t)$ as the common mean volume (in three dimensions) or area (in two dimensions) of the region, where $f(\mathbf{r}, t) > f$, and the common mean 'mass' of the field comprised there, are described as

$$\langle V(t, f)\rangle = \int \mathrm{d}\mathbf{r} \int_f^\infty \mathrm{d}f'\, P(\mathbf{r}, t; f')\,,$$

$$\langle M(t, f)\rangle = \int \mathrm{d}\mathbf{r} \int_f^\infty \mathrm{d}f'\, f' P(\mathbf{r}, t; f')\,.$$

The values of these functionals do not depend on diffusion in the $\mathbf{r}$-space (the coefficient D_0), and for probability distribution (42) we find the expressions

$$\begin{aligned} \langle V(t, f)\rangle &= \int \mathrm{d}\mathbf{r}\ \Pr\left\{\frac{1}{\sqrt{2Dt}}\ln\left[\frac{f_0(\mathbf{r})}{f}\exp(-\alpha t)\right]\right\}, \\ \langle M(t, f)\rangle &= \exp\left[(D-\alpha)t\right] \\ &\quad\times \int \mathrm{d}\mathbf{r}\, f_0(\mathbf{r})\Pr\left\{\frac{1}{\sqrt{2Dt}}\ln\left(\frac{f_0(\mathbf{r})}{f}\exp\left[(2D-\alpha)t\right]\right)\right\}, \end{aligned} \tag{51}$$

where the probability integral $\Pr(z)$ is defined by equality (36).

Taking now into account the asymptotics of function $\Pr(z)$ (37), one can analyze how functionals (51) evolve with time. Namely, for $t \to \infty$, the asymptotics of the mean volume decays with time, for $\alpha > 0$, according to the law

$$\langle V(t, f)\rangle \approx \frac{1}{\alpha}\sqrt{\frac{D}{\pi f^{\alpha/D} t}}\exp\left(-\frac{\alpha^2 t}{4D}\right)\int \mathrm{d}\mathbf{r}\sqrt{f_0^{\alpha/D}(\mathbf{r})}\,.$$

For $\alpha < 0$, the mean volume occupies the entire space as $t \to \infty$.

For the common mean 'mass', in the limit $t \to \infty$ we get the asymptotics (in the most interesting case — if $\alpha < 2D$)

$$\begin{aligned} \langle M(t, f)\rangle &\approx \exp\left[(D-\alpha)t\right]\int \mathrm{d}\mathbf{r}\, f_0(\mathbf{r}) \\ &\times\left\{1 - \frac{1}{2D-\alpha}\sqrt{\frac{D}{\pi t}\left(\frac{f}{f_0(\mathbf{r})}\right)^{(2D-\alpha)/D}}\exp\left[-\frac{(2D-\alpha)^2 t}{4}\right]\right\}. \end{aligned}$$

As a consequence, for $\alpha > 0$, clusters contain the overall mean 'mass' in the limit $t \to \infty$.

For homogeneous initial conditions, the respective expressions taken without integration over $\mathbf{r}$ describe the specific values of the volume comprising large excursions and their common 'mass' per unit volume, i.e.

$$\begin{aligned} \langle v_{\mathrm{hom}}(t, f)\rangle &= \langle \theta(f(\mathbf{r}, t) - f)\rangle = \mathcal{P}\{f(\mathbf{r}, t) > f\} \\ &= \Pr\left\{\frac{1}{\sqrt{2Dt}}\ln\left[\frac{f_0}{f}\exp(-\alpha t)\right]\right\}, \\ \langle m_{\mathrm{hom}}(t, f)\rangle &= f_0\exp\left[(D-\alpha)t\right] \\ &\quad\times \Pr\left\{\frac{1}{\sqrt{2Dt}}\ln\left(\frac{f_0}{f}\exp\left[(2D-\alpha)t\right]\right)\right\}. \end{aligned} \tag{52}$$

If we select a section level $f > f_0$, then at the initial instant of time $\langle v_{\mathrm{hom}}(0, f)\rangle = 0$ and $\langle m_{\mathrm{hom}}(0, f)\rangle = 0$. Spatial perturbations of the random field $f(\mathbf{r}, t)$ evolve later, and in the limit $t \to \infty$ we arrive at the asymptotic expressions ($2D > \alpha$)

$$\begin{aligned} \langle v_{\mathrm{hom}}(t, f)\rangle &= \mathcal{P}\{f(\mathbf{r}, t) > f\} \\ &\approx \begin{cases} \dfrac{1}{\alpha}\sqrt{\dfrac{D}{\pi t}\left(\dfrac{f_0}{f}\right)^{\alpha/D}}\exp\left(-\dfrac{\alpha^2 t}{4D}\right), & \alpha > 0\,, \\ 1 - \dfrac{1}{|\alpha|}\sqrt{\dfrac{D}{\pi t}\left(\dfrac{f}{f_0}\right)^{|\alpha|/D}}\exp\left(-\dfrac{\alpha^2 t}{4D}\right), & \alpha < 0\,, \end{cases} \end{aligned} \tag{53}$$

$$\begin{aligned} \langle m_{\mathrm{hom}}(t, f)\rangle &\approx f_0\exp\left[(D-\alpha)t\right] \\ &\times\left\{1 - \frac{1}{2D-\alpha}\sqrt{\frac{D}{\pi t}\left(\frac{f}{f_0}\right)^{(2D-\alpha)/D}}\exp\left[-\frac{(2D-\alpha)^2 t}{4D}\right]\right\}. \end{aligned} \tag{54}$$

Thus, for $\alpha > 0$, the specific common volume tends to zero, while the specific common 'mass' confined within it tends to the mean 'mass' in the entire space. This corresponds to the criterion of structure formation with a unit probability through 'ideal clustering' of the field $f(\mathbf{r}, t)$ being considered. *In this case, the random field $f(\mathbf{r}, t)$ is practically absent in the dominant part of the space*. The field characteristic decay time at any fixed point in space can be estimated as $\alpha t \sim 1$, and the characteristic time of field cluster structure formation as $\alpha t \sim \max\{4\xi, 4\xi/(2\xi - 1)^2\}$, where $\xi = D/\alpha$.

If $\alpha < 0$, clustering is lacking and only general amplification of the random field $f(\mathbf{r}, t)$ takes place everywhere in space. Thus, *chaos remains chaos in that case*! Only zeros of field $f(\mathbf{r}, t)$ cluster.

Let us mention that the following theorem holds.

A conservative positive parametrically excited lognormal field in a statistically homogeneous case always produces clusters with a unit probability, i.e., for almost all its realizations.

Indeed, in this case, $f(\mathbf{r}, t) = \exp(\ln f(\mathbf{r}, t))$; hence, one has

$$\langle f(\mathbf{r}, t)\rangle = \langle \exp(\ln f(\mathbf{r}, t))\rangle = \exp\left(\langle \ln f(\mathbf{r}, t)\rangle + \frac{1}{2}\sigma^2_{\ln f(\mathbf{r},t)}\right),$$

where $\sigma^2_{\ln f(\mathbf{r},t)}$ is the variance of the random field $\ln f(\mathbf{r}, t)$. Now taking into account the conservative character of this field leads to

$$\langle \ln f(\mathbf{r}, t)\rangle + \frac{1}{2}\sigma^2_{\ln f(\mathbf{r},t)} = \ln f_0\,,$$

and we find for the typical realization curve

$$f^*(\mathbf{r}, t) = \exp\left(\langle \ln f(\mathbf{r}, t)\rangle\right) = f_0\exp(-\alpha t)\,,$$

where the Lyapunov characteristic parameter

$$\alpha = \lim_{t\to\infty}\frac{1}{2t}\sigma^2_{\ln f(\mathbf{r},t)} > 0\,,$$

and the problem is that of computing it from the respective dynamic equation. Since, as pointed out earlier, for the conservative field $f(\mathbf{r}, t)$ the parameter $\alpha = D$ [see Eqn (43)], the characteristic time of cluster structure formation is $\alpha t \sim 4$, which is four times larger than the characteristic field decay time at almost any point in space.

For instance, for cluster formation in a density field, we have dynamic equation (19).

With complex-valued parabolic equation (15), we have the equation for the wave field intensity — the continuity equation (16) and, consequently, introducing the amplitude level as $\chi(\mathbf{R}, x) = \ln A(\mathbf{R}, x)$, we find for a plane wave the Lyapunov exponent in the form

$$I^*(\mathbf{R}, x) = I_0 \exp(-2\alpha x),$$

where the parameter

$$\alpha = \lim_{x\to\infty} \frac{1}{x}\, \sigma^2_{\chi(\mathbf{R},x)},$$

and $\sigma^2_{\chi(\mathbf{R},x)}$ is the variance of the amplitude level, computed in the framework of the first approximation of the smooth perturbation method proposed by S M Rytov (see, for example, books [17–20]).

As an application of the theory elaborated here, we consider concrete physical stochastic transport phenomena in random media spawned through the parametric action of the medium on the respective dynamical system.

4. Stochastic transport phenomena in a random velocity field

4.1 Clustering of the density field in a random velocity field

Stochastic structure formation in a spatially homogeneous statistical problem on diffusion of the density field $\rho(\mathbf{r}, t)$ in a random velocity field is described by equation (19). In this case, the one-point probability density of field $\rho(\mathbf{r}, t)$ is independent of spatial coordinate $\mathbf{r}$ and is described by the following equation (see, for example, Refs [17–20])

$$\frac{\partial}{\partial t} P(t;\rho) = D_\rho \frac{\partial^2}{\partial \rho^2}\, \rho^2 P(t;\rho), \quad P(0;\rho) = \delta(\rho - \rho_0), \tag{55}$$

where the diffusion coefficient in the ρ-space, $D_\rho = D^{\mathrm{p}}$, is given by equality (27). Equation (55) coincides with equation (44) at the fulfillment of equalities

$$\alpha = D = D_\rho = D^{\mathrm{p}},$$

where the quantity D^{p} is determined by the potential spectral component of the velocity field. Consequently, the one-point probability density of the density field is lognormal, with the probability density and respective integral distribution function:

$$\begin{aligned} P(t;\rho) &= \frac{1}{2\rho\sqrt{\pi\tau}} \exp\left\{-\frac{\ln^2\left[\rho\exp(\tau)/\rho_0\right]}{4\tau}\right\}, \\ F(t;\rho) &= \Pr\left\{\frac{\ln\left[\rho\exp(\tau)/\rho_0\right]}{2\sqrt{\tau}}\right\}, \end{aligned} \tag{56}$$

where the parameter $\tau = D_\rho t$.

For one-point characteristics of the density field $\rho(\mathbf{r}, t)$, as already pointed out, the problem is statistically equivalent to the analysis of a random process, and in this case all the moment functions at any fixed point in space grow exponentially with time for $n > 0$, as well as for $n < 0$:

$$\langle \rho(\mathbf{r}, t)\rangle = \rho_0, \quad \langle \rho^n(\mathbf{r}, t)\rangle = \rho_0^n \exp\left[n(n-1)\tau\right], \tag{57}$$

and the typical realization curve for the density field, which coincides with the Lyapunov exponent, exponentially decays with time at any fixed point in space:

$$\rho^*(t) = \exp\left(\langle \ln\rho(\mathbf{r}, t)\rangle\right) = \rho_0 \exp(-\tau),$$

which indicates that the density field decays with a unit probability (i.e., in almost all realizations of the density field) in arbitrary divergent flows at any fixed point in space. Then, the characteristic decay time for the density field $\tau \sim 1$. Note that equation (55) for the probability density corresponds to the Eulerian description of a density field. It should be recalled that in the Lagrangian description of the density field the system of characteristic curves (particles) (5) does not necessarily undergo clustering (see Section 2.2).

The formation of density field statistics at any fixed point in space (moment and correlation functions) occurs through density field fluctuations around the typical realization curve. Thus, in the case of a compressible flow (in a divergent velocity field) the density field always undergoes clustering with a unit probability. The specific mean area (volume) of the region where $\rho(\mathbf{r}, t) > \rho$ is expressed as

$$\begin{aligned} \langle s_{\mathrm{hom}}(t,\rho)\rangle &= \int_\rho^\infty \mathrm{d}\rho'\, P(t;\rho') = \mathcal{P}\{\rho(\mathbf{r}, t) > \rho\} \\ &= \Pr\left\{\frac{\ln\left[\rho_0\exp(-\tau)/\rho\right]}{\sqrt{2\tau}}\right\}, \end{aligned} \tag{58}$$

and the specific mean 'mass' of the tracer confined in this region is described as

$$\begin{aligned} \frac{\langle m_{\mathrm{hom}}(t,\rho)\rangle_{\mathrm{hom}}}{\rho_0} &= \frac{1}{\rho_0}\int_\rho^\infty \rho'\,\mathrm{d}\rho'\, P(t;\rho') \\ &= \Pr\left\{\frac{\ln\left[\rho_0\exp(\tau)/\rho\right]}{\sqrt{2\tau}}\right\}. \end{aligned} \tag{59}$$

From Eqns (58) and (59) it follows that for $\tau \gg 1$ the mean specific area (volume) decays according to the law

$$\langle s_{\mathrm{hom}}(t,\rho)\rangle = \mathcal{P}\{\rho(\mathbf{r}, t) > \rho\} \approx \sqrt{\frac{\rho_0}{\pi\rho\tau}}\, \exp\left(-\frac{\tau}{4}\right), \tag{60}$$

whereas almost all tracer 'mass' is accumulated within it:

$$\frac{\langle m_{\mathrm{hom}}(t,\rho)\rangle}{\rho_0} \approx 1 - \sqrt{\frac{\rho}{\pi\rho_0\tau}}\, \exp\left(-\frac{\tau}{4}\right), \tag{61}$$

which corresponds to the physical phenomenon of density field clustering in a random velocity field. As follows from formulas (60) and (61), the characteristic time of cluster structure formation in the tracer field is four times that of the characteristic decay time for the density field at any fixed point in space ($\tau \sim 4$).

Note that even in an incompressible fluid, the density field will experience clustering in hydrodynamic flows if the tracer is 'buoyant', if the finite inertia of the density field is taken into account, and for multiphase fluid flows, i.e., always when a potential spectral component arises in the tracer velocity field, which is different from the velocity field of the fluid proper. This case corresponds, for example, to a cloudy sky (see Fig. 15), where the nature of stochastic character of air masses is completely irrelevant — be it developed convection

or atmospheric turbulence. Judging by the time the photo was taken (see Fig. 15), we have the second case. But there is no particle clustering in the *Lagrangian description*!

Thus, for the continuity equation

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \mathbf{r}}\,\mathbf{V}(\mathbf{r},t)\right)\rho(\mathbf{r},t)=0\,,\qquad \rho(\mathbf{r},0)=\rho_0(\mathbf{r})\,,\tag{62}$$

describing the density field of a passive scalar tracer $\rho(\mathbf{r},t)$ moving in a random hydrodynamic flow with the velocity $\mathbf{V}(\mathbf{r},t)$ and having a potential component, tracer clustering always occurs with a unit probability independently of the dynamic equation governing the velocity field $\mathbf{V}(\mathbf{r},t)$.

For example, in the case of tracer $\rho(\mathbf{r},t)$ with low inertia, the velocity field $\mathbf{V}(\mathbf{r},t)$ can be described by the phenomenological equation (see Ref. [48])

$$\left(\frac{\partial}{\partial t}+\mathbf{V}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V}(\mathbf{r},t)=-\lambda\big[\mathbf{V}(\mathbf{r},t)-\mathbf{u}(\mathbf{r},t)\big]\,,\tag{63}$$

where $\mathbf{u}(\mathbf{r},t)$ is the velocity field of the hydrodynamic flow itself, and the parameter $\tau=1/\lambda$ is the known Stokes time depending on the size of tracer particles and molecular viscosity. Equation (63) is that of a *simple wave* with linear friction and a random force related to the hydrodynamic flow. A peculiarity of this equation consists in the fact that it is valid only in the asymptotic limit $\lambda\to\infty$. This implies that the parameter $\lambda\tau_0\gg 1$, where τ_0 is the temporal correlation radius for the hydrodynamic velocity field $\mathbf{u}(\mathbf{r},t)$ and, consequently, the approximation of temporally delta-correlated field $\mathbf{u}(\mathbf{r},t)$ is inapplicable for the statistical problem description, and the finiteness of the temporal correlation radius τ_0 for the field $\mathbf{u}(\mathbf{r},t)$ needs to be taken into account.

Under the assumption that the variance of the random velocity field $\sigma_{\mathbf{u}}^2=\langle\mathbf{u}^2(\mathbf{r},t)\rangle$ is sufficiently small for large values of parameter λ (low inertia of particles), equation (63) can be linearized with respect to the function $\mathbf{V}(\mathbf{r},t)\approx\mathbf{u}(\mathbf{r},t)$, and we then arrive at a simpler vector equation

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\mathbf{u}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V}(\mathbf{r},t)\\&\quad=-\left(\mathbf{V}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{u}(\mathbf{r},t)-\lambda\big[\mathbf{V}(\mathbf{r},t)-\mathbf{u}(\mathbf{r},t)\big]\,.\end{aligned}$$

In this approximation, the probability density of the density field is described by an equation like Eqn (43):

$$\begin{aligned}&\frac{\partial}{\partial t}\,P(\mathbf{r},t;\rho)=\left(D_0\,\frac{\partial^2}{\partial \mathbf{r}^2}+D_\rho\,\frac{\partial^2}{\partial \rho^2}\,\rho^2\right)P(\mathbf{r},t;\rho)\,,\\&P(\mathbf{r},0;\rho)=\delta\big(\rho_0(\mathbf{r})-\rho\big)\,,\end{aligned}\tag{64}$$

where D_0 and D_ρ are the diffusion coefficients:

$$\begin{aligned}&D_0=\frac{1}{d}\int_0^\infty \mathrm{d}\tau\,\big\langle\mathbf{V}(\mathbf{r},t+\tau)\mathbf{V}(\mathbf{r},t)\big\rangle=\frac{1}{d}\,\tau_{\mathbf{V}}\big\langle\mathbf{V}^2(\mathbf{r},t)\big\rangle\,,\\&D_\rho=\int_0^\infty \mathrm{d}\tau\left\langle\frac{\partial\mathbf{V}(\mathbf{r},t+\tau)}{\partial\mathbf{r}}\,\frac{\partial\mathbf{V}(\mathbf{r},t)}{\partial\mathbf{r}}\right\rangle=\tau_{\operatorname{div}\mathbf{V}}\left\langle\left(\frac{\partial\mathbf{V}(\mathbf{r},t)}{\partial\mathbf{r}}\right)^2\right\rangle,\end{aligned}\tag{65}$$

and $\tau_{\mathbf{V}}$ and $\tau_{\operatorname{div}\mathbf{V}}$ are the temporal correlation radii for the random fields $\mathbf{V}(\mathbf{r},t)$ and $\partial\mathbf{V}(\mathbf{r},t)/\partial\mathbf{r}$.

For an incompressible fluid flow in the diffusion approximation, diffusion coefficients (65) are described by the expressions [48]

$$\begin{aligned}&D_0=\frac{1}{d}\,\tau_{\mathbf{V}}\big\langle\mathbf{V}^2(\mathbf{r},t)\big\rangle=\frac{1}{d}\,\tau_0 B_{ii}(0,0)=\frac{d-1}{d}\,\tau_0\int\mathrm{d}\mathbf{k}\,E(k,0)\,,\\&D_\rho=\tau_{\operatorname{div}\mathbf{V}}\left\langle\left(\frac{\partial\mathbf{V}(\mathbf{r},t)}{\partial\mathbf{r}}\right)^2\right\rangle=\frac{4}{\lambda}\,\frac{d^2-1}{d(d+2)}\,D_1D_2(\lambda)\,,\end{aligned}\tag{66}$$

where the coefficient

$$D_1=-\frac{\tau_0}{d-1}\big\langle\mathbf{u}(\mathbf{r},t)\Delta\mathbf{u}(\mathbf{r},t)\big\rangle$$

does not depend on the parameter λ. The coefficient $D_2(\lambda)$, if $\lambda\tau_0\gg 1$, is defined by the expression

$$D_2(\lambda)=-\frac{1}{\lambda(d-1)}\,\big\langle\mathbf{u}(\mathbf{r},t)\,\Delta\mathbf{u}(\mathbf{r},t)\big\rangle\,.$$

Thus, we see that the coefficient $D_\rho\sim\sigma_{\mathbf{u}}^4$ in equation (64). And the vortex component of field $\mathbf{u}(\mathbf{r},t)$ first generates the vortex component of the field $\mathbf{V}(\mathbf{r},t)$ by a direct linear mechanism without a contribution from advection, and only then do the vortex component of the field $\mathbf{V}(\mathbf{r},t)$ generate a divergent component of the field $\mathbf{V}(\mathbf{r},t)$ through the advection mechanism.

Note that for particles with low inertia in the presence of buoyancy and gravity forces the tracer velocity field $\mathbf{V}(\mathbf{r},t)$ in the hydrodynamic flow $\mathbf{u}(\mathbf{r},t)$ is described by the equations (see, for example, Ref. [49])

$$\left(\frac{\partial}{\partial t}+\mathbf{V}(\mathbf{r},t)\frac{\partial}{\partial \mathbf{r}}\right)\mathbf{V}(\mathbf{r},t)=-\lambda\big[\mathbf{V}(\mathbf{r},t)-\mathbf{u}(\mathbf{r},t)\big]+\mathbf{g}\left(1-\frac{\rho_0}{\rho_{\mathrm{p}}}\right),\tag{67}$$

where g is the gravitational acceleration, and ρ_{p} and ρ_0 are the densities of tracer particles and the medium, respectively.

The velocity $\mathbf{v}$ of tracer sedimentation or floating up, directed, as a rule, vertically, is determined by the balance between the buoyancy and viscous friction forces for a moving tracer and is described by the formula

$$\frac{\mathbf{g}}{\lambda}\left(1-\frac{\rho_0}{\rho_{\mathrm{p}}}\right)=\mathbf{v}\,.$$

Writing now $\mathbf{V}(\mathbf{r},t)=\mathbf{v}+\mathbf{v}(\mathbf{r},t)$, where $\mathbf{v}(\mathbf{r},t)$ are the fluctuations of a tracer velocity field with respect to $\mathbf{v}$, for the system of equations (62) and (67) the one-point probability density $P(\mathbf{r},t;\rho)$ is described by equation (64), where the diffusion coefficient $D_\rho(\mathbf{v})$, depending on the sedimentation velocity $\mathbf{v}$, is now given by the expression [50]

$$D_\rho(\mathbf{v})=\frac{4(d+1)}{d(d+2)(d-1)\lambda^2}\,\frac{\partial^2 B_{\alpha\alpha}^{(\mathbf{u})}(\mathbf{0},0)}{\partial\mathbf{r}^2}\int_0^\infty\mathrm{d}\tau\,\frac{\partial^2 B_{\beta\beta}^{(\mathbf{u})}(\mathbf{v}\tau,\tau)}{\partial\mathbf{r}^2}\,.\tag{68}$$

Consequently, the presence of tracer sedimentation leads to a reduction in the diffusion coefficient $D_\rho(\mathbf{v})$, i.e., to a larger clustering time. In reality, the clustering of sedimenting weakly inertial particles explains numerous phenomena in Nature, for example, the spot structure of radioactive precipitation after the Chernobyl catastrophe (see, for example, Ref. [51]). The spotty structure of sand precipitation over the Indian Ocean after sand storms in African deserts is also well known.

We note that in review [52], dealing with the large-scale structure of the Universe, the vector equation of a simple

wave (63), with no right-hand side but under the assumption that the velocity field $\mathbf{V}(\mathbf{r},t)$ is potential, is 'glued' to the continuity equation (62). The randomness in this case is attributed to fluctuations in the initial conditions. Clearly, in this case the tracer field undergoes clustering with a probability of unity in almost every its realization. However, the compliance of this fact with the observed distribution of galaxies can in no way be treated as confirmation of the relevance of the velocity field model used, based on the simple wave equation.

We stress that the clustering of a tracer field with a unit probability is realized for any model of a velocity field (linear or nonlinear) in the presence of a potential component in the velocity field.

4.2 Probabilistic description of a magnetic field and its energy in a random velocity field

4.2.1 Probabilistic description of a magnetic field. Let us consider now a probabilistic description of a magnetic field based on dynamic equation (20) in a statistically homogeneous case. Just as for the density field, we will assume that the random component of a velocity field $\mathbf{u}(\mathbf{r},t)$ is a divergent ($\operatorname{div}\mathbf{u}(\mathbf{r},t)\neq 0$) random Gaussian field, homogeneous and isotropic in space and stationary delta-correlated in time.

In this case, the one-point probability density of vector magnetic field $P(t;\mathbf{H})$, which does not depend on the spatial variable $\mathbf{r}$, is described by the equation [17–20, 25, 39]

$$\frac{\partial}{\partial t}\,P(t;\mathbf{H})=\left\{D_1\,\frac{\partial^2}{\partial H_k\,\partial H_l}\,H_lH_k+D_2\,\frac{\partial^2}{\partial H_l\,\partial H_l}\,H_k^2\right\}P(t;\mathbf{H})\,, \tag{69}$$

where D_1 and D_2 are the diffusion coefficients in the $\{\mathbf{H}\}$-space:

$$D_1=\frac{(d^2-2)D^{\mathrm{p}}-2D^{\mathrm{s}}}{d(d+2)}\,,\qquad D_2=\frac{(d+1)D^{\mathrm{s}}+D^{\mathrm{p}}}{d(d+2)}\,.$$

Equation (69) governs the dynamics of mean energy $\langle E(\mathbf{r},t)\rangle=\langle\mathbf{H}^2(\mathbf{r},t)\rangle$ with time and yields the expression for the covariance of the magnetic field components $\langle W_{ij}(\mathbf{r},t)\rangle=$ $=\langle H_i(\mathbf{r},t)H_j(\mathbf{r},t)\rangle$:

$$\left\langle E(\mathbf{r},t)\right\rangle=E_0\exp\left[2\,\frac{d-1}{d}(D^{\mathrm{s}}+D^{\mathrm{p}})t\right],$$

$$\frac{\left\langle W_{ij}(\mathbf{r},t)\right\rangle}{\left\langle E(\mathbf{r},t)\right\rangle}=\frac{1}{d}\,\delta_{ij}+\left(\frac{W_{ij}(0)}{E_0}-\frac{1}{d}\delta_{ij}\right)\times\exp\left[-2\,\frac{(d+1)D^{\mathrm{s}}+D^{\mathrm{p}}}{d+2}\,t\right].$$

Thus, the mean magnetic field energy grows exponentially with time, and the isotropization of the magnetic field also develops exponentially. Note that the spectral components of the velocity field enter these exponents in an additive way. Obviously, this feature is preserved for all other correlations of the magnetic field and its energy.

4.2.2 Probabilistic description of magnetic field energy. The probability density of magnetic field energy $P(t;E)$ in a spatially homogeneous case is defined by the equality

$$P(t;E)=\left\langle\delta\big(E(\mathbf{r},t)-E\big)\right\rangle_{\mathbf{u}}=\left\langle\delta\big(\mathbf{H}^2(\mathbf{r},t)-E\big)\right\rangle_{\mathbf{H}}.$$

As a result, we get an equation which coincides with equation (44) [17–20, 25, 39], taken with the parameters

$$\alpha=2\,\frac{d-1}{d+2}(D^{\mathrm{p}}-D^{\mathrm{s}})\,,\quad D=4(d-1)\,\frac{(d+1)D^{\mathrm{p}}+D^{\mathrm{s}}}{d(d+2)}\,.$$

The parameter α can differ from zero (being positive or negative) or be equal to it (the critical case).

The probability density is described by formula (34) for an initial spatially homogeneous energy distribution.

Thus, in this case, the one-point statistical characteristics of energy $E(\mathbf{r},t)/E_0$ are statistically equivalent to the characteristics of a random process $E(t)$ with probability density (34).

The characteristic feature of distribution (34) lies in the appearance of a long sloping tail for $Dt\gg 1$, implying an increased role of large outliers of the process $E(t)$ in forming its one-time statistics. For this distribution, all the moments of magnetic field energy, namely

$$\left\langle E^n(t)\right\rangle=E_0^n\exp\left[-2n\,\frac{d-1}{d+2}(D^{\mathrm{p}}-D^{\mathrm{s}})t+4n^2(d-1)\,\frac{(d+1)D^{\mathrm{p}}+D^{\mathrm{s}}}{d(d+2)}\,t\right],$$

grow exponentially with time for $n>0$, as well as for $n<0$; in particular, at $n=1$, the mean specific energy is written as

$$\left\langle E(t)\right\rangle=E_0\exp\left[\frac{2(d-1)}{d}(D^{\mathrm{p}}+D^{\mathrm{s}})t\right], \tag{70}$$

or in a more suitable representation as

$$\left\langle\ln\frac{E(t)}{E_0}\right\rangle=-\alpha t=-2\,\frac{d-1}{d+2}(D^{\mathrm{p}}-D^{\mathrm{s}})t\,.$$

The parameter α is, hence, the *Lyapunov characteristic index*. In this regard, the *typical realization curve* for the random process $E(t)$, determining the behavior of magnetic field energy in concrete realizations, is exponential at any fixed spatial location:

$$E^*(t)=E_0\exp(-\alpha t)=E_0\exp\left[-2\,\frac{d-1}{d+2}(D^{\mathrm{p}}-D^{\mathrm{s}})t\right],$$

and either grows or decays with time. So, for $\alpha>0$ ($D^{\mathrm{p}}>D^{\mathrm{s}}$) the typical realization curve decays exponentially at all spatial locations, which points to the cluster structure of a magnetic field in its individual realizations of a magnetic field energy; the growth of moments in this case is brought about by rare, yet strong, ejections of energy with respect to the typical realization curve.

For $\alpha<0$ ($D^{\mathrm{p}}<D^{\mathrm{s}}$), the typical realization curve grows exponentially with time, which points to a general increase in the magnetic field energy at each point in space. Clustering in the magnetic field energy does not happen in this case. Note that namely this case is realized for an incompressible magnetohydrodynamic flow ($D^{\mathrm{p}}=0$); hence, *there is no structure formation in the incompressible case*.

The respective asymptotic expressions for the specific values of the volume of large ejections and their total energy, for homogeneous initial conditions, have the form of expressions (53) and (54) with the change $f\to E$.

This implies that for $\alpha > 0$ ($D^{\mathrm{p}} > D^{\mathrm{s}}$) the specific total volume tends to zero, while the specific total energy inside it coincides with the mean energy in the entire space. The latter points to clustering of magnetic field energy with a probability one, i.e., in almost all magnetic field realizations. Consequently, the magnetic field is simply absent over a larger portion of the space.

When $\alpha < 0$ ($D^{\mathrm{p}} < D^{\mathrm{s}}$), clustering does not happen, and there is only a general increase in the magnetic field energy everywhere in space. Note that, in this case, the inverse quantity, $1/E$, undergoes clustering, i.e., clusters of compact regions appear from where the magnetic field is expelled (magnetic zeros).

We mentioned earlier that the parameters D^{p} and D^{s} characterizing statistics of the random velocity field enter all statistical moment and correlation functions of the magnetic field energy in an additive way. This is, certainly, the consequence of equations (14) and (20) being linear. However, this fact implies that all the main (functional) relationships in such a statistical description do not *distinguish between the influence of solenoidal and potential components* of the random velocity field. This means that all the relationships derived for the mentioned statistical quantities have the same structure for both incompressible ($D^{\mathrm{p}} = 0$) and purely potential ($D^{\mathrm{s}} = 0$) flows. Nevertheless, since clustering is absent for an incompressible flow and, conversely, is present for a potential one, it is absolutely clear that the statistical characteristics mentioned above *do not contain any information on the stochastic structure formation in individual realizations of magnetic field energy, namely, on clustering*.

Furthermore, the input induction equation (14) is valid within the applicability limits of the kinematic approximation. In the presence of clustering, when the magnetic field is absent over a larger portion of space, its back reaction on the velocity field is not essential. In contrast, in the absence of clustering, when the magnetic field is generated everywhere in space, the kinematic approximation can be valid only over a small time interval, and discussing the role of the dynamic diffusion coefficient in forming the statistics of magnetic field energy in this interval is, in our opinion, simply not serious.

4.2.3 The critical case of $\alpha = 0$ ($D^{\mathrm{p}} = D^{\mathrm{s}}$). This case can be considered as *pseudoequilibrium*, in analogy with the case of equilibrium thermal noise [53, 54]. At $\alpha = 0$, the one-point probability density takes the form

$$P(t; E) = \frac{1}{2E\sqrt{\pi D t}} \exp\left[-\frac{\ln^2(E/E_0)}{4Dt}\right].$$

The random processes $E(t)$ and $1/E(t)$ are statistically equivalent. The specific mean volume tends to half of the total volume as $t \to \infty$, and the specific mean energy tends to the total mean energy.

Thus, clustering does not happen at $\alpha = 0$ ($D^{\mathrm{p}} = D^{\mathrm{s}}$) in the framework of equation (44). It is worth mentioning that this result does not seem satisfactory, for equation (44) is itself approximate, obtained under the assumption that the random velocity field is delta-correlated in time.

Accounting for the finiteness of the time correlation radius allows unequivocally judging the presence or lack of the physical phenomenon of *clustering*. The results of dedicated computations have shown that accounting for the finite time correlation radius of a velocity field clustering occurs with a probability one [54].

An analogous situation also takes place for random acoustic waves if they are not damped.

Thus, a random Gaussian acoustic field $\mathbf{u}(\mathbf{r}, t)$, statistically homogeneous and isotropic in space and stationary in time, is described by the correlation and spectral tensors ($\tau = t - t'$):

$$\left\langle u_i(\mathbf{r}, t) u_j(\mathbf{r}', t') \right\rangle = \sigma_{\mathbf{u}}^2 B_{ij}(\mathbf{r} - \mathbf{r}', \tau) = \sigma_{\mathbf{u}}^2 \int \mathrm{d}\mathbf{k}\, E_{ij}(\mathbf{k})\, f(\mathbf{k}, \mathbf{r}, \tau),$$

where $\sigma_{\mathbf{u}}^2 = \langle \mathbf{u}^2(\mathbf{r}, t) \rangle$ is the variance of the velocity field, and the function

$$f(\mathbf{k}, \mathbf{r}, \tau) = \exp\left(-\lambda(k)\tau\right) \cos\left(\mathbf{k}\mathbf{r} - \omega(k)\tau\right),$$

where $\omega(k) = ck$ is the dispersion law for acoustic waves, and c is the speed of sound.

The exponentially decaying term is related to dissipative factors in equations of fluid dynamics and magnetohydrodynamics, and $\lambda(k) = \lambda_{\mathrm{p}} k^2$.

Here, the velocity field spectral tensor contains only the potential component $E_{ij}(\mathbf{k}) = E(k) k_i k_j / k^2$. And, since the time integral for $\lambda_{\mathrm{p}} \ll c l_0$ (where l_0 is the spatial correlation radius of the velocity field) has the asymptotics

$$\int_0^\infty \mathrm{d}t\, f(k, t) = \frac{\lambda_{\mathrm{p}}}{c^2},$$

clustering of the magnetic field energy occurs with a probability one (i.e., in almost all its realizations) in the presence of small absorption.

In the absence of dissipation, the diffusion coefficient in equation (44) becomes zero, and we do not have any information on the presence or absence of *clustering*. In general, however, the following equality is valid in the absence of dissipation:

$$\int_0^\infty \mathrm{d}t \cos\left(\omega(k) t\right) = \pi \delta\left(\omega(k)\right),$$

owing to which resonances emerge between various harmonics of the acoustic wave field in higher approximations. This allows establishing in the second order of the functional perturbation method (after cumbersome manipulations) that clustering of magnetic field energy occurs with a unit probability (i.e., in almost all its realizations), and computing the characteristic time it takes for clustering to set up, $t \sim 1/\alpha_2$, where the Lyapunov characteristic parameter [54] is given by

$$\alpha_2 = \frac{\sigma_{\mathbf{u}}^2}{c^2} \int \mathrm{d}\mathbf{k}\, k^2 E(k) \left(\frac{4}{5} \lambda_{\mathrm{p}} + 76\pi^2 \frac{\sigma_{\mathbf{u}}^2}{c} k^2 E(k) \right).$$

Notice that there is no clustering of magnetic field energy in an equilibrium thermal velocity field.

We also mention that analogous computations for the density field of the passive tracer in random wave fields have indicated the presence of clustering with a probability one [55].

5. Model of a stochastic velocity field allowing analytical solutions to transport problems

Let us now consider a simple model velocity field of the form

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{v}(t)\, f(\mathbf{k}\mathbf{r}),$$

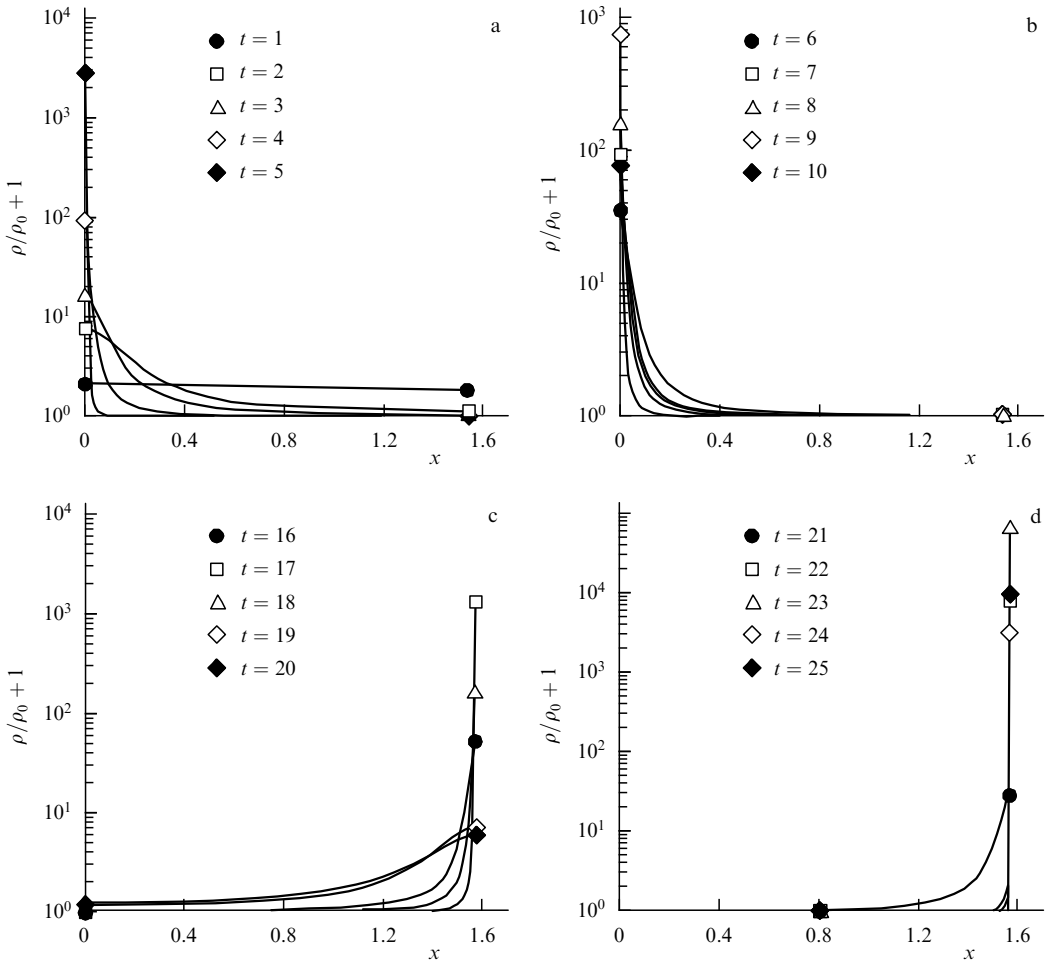

Figure 19. Spatio–temporal evolution of the Eulerian density field, described by formula (74).

where $\mathbf{v}(t)$ is a Gaussian vector random *white noise* process, and $f(\mathbf{kr})$ is a periodic function. Selecting the x-axis in the direction of vector $\mathbf{k}$, we see that in the framework of this model the velocity field depends on a single spatial coordinate, i.e., $f(\mathbf{kr}) = f(kx)$. Notice that the velocity field model of the form

$$\mathbf{u}(\mathbf{r}, t) = \mathbf{v}(t) \sin(2\mathbf{kr}), \tag{71}$$

proposed for the fist time in review [56], made possible the derivation of an analytical solution for continuity equation (19) for the scalar density field $\rho(\mathbf{r}, t)$, as well as a solution of equation (20) for the vector magnetic field and, consequently, gave the possibility of tracing the onset and evolution of cluster formation in these fields in individual realizations of the random velocity field. We note that this form of function $f(\mathbf{kr})$, corresponding to the first term of its expansion in harmonics, is routinely used in numerical simulations related to the problem.

In numerical modeling of various phenomena, a model of Gaussian vector random process $\mathbf{v}(t)$, delta-correlated in time, has been used, with the parameters

$$\langle \mathbf{v}(t) \rangle = 0, \quad \langle v_i(t) v_j(t') \rangle = 2\sigma^2 \delta_{ij} \tau_0 \delta(t - t'), \tag{72}$$

where σ^2 is the variance for each velocity component, and τ_0 its temporal correlation radius. We will adopt dimensionless variables

$$t \to k^2 \sigma^2 \tau_0 t, \quad x \to kx, \quad \langle v_i(t) v_j(t') \rangle \to 2\delta_{ij} \delta(t - t'). \tag{73}$$

5.1 Model of passive tracer diffusion

The solution of equation (19) for the density field in the case considered has the form [56]

$$\frac{\rho(x, t)}{\rho_0} = \frac{1}{\exp\big(T(t)\big) \cos^2(kx) + \exp\big(-T(t)\big) \sin^2(kx)}, \tag{74}$$

where $T(t) = 2k \int_0^t \mathrm{d}\tau \, v_x(\tau)$ is the Wiener random process.

From expression (74), it can be seen that the density field is small everywhere except for the vicinities of points $kx = n\pi/2$, where $\rho(x, t)/\rho_0 = \exp\big(\pm T(t)\big)$, i.e., the field is rather strong close to these points, granted the sign of the random factor $T(t)$.

Thus, in the problem considered, the cluster structure of the density field in the Euler description forms in the vicinity of points

$$kx = n\,\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots .$$

The results of simulations of space–time evolution experienced by a realization of the Euler density field $1 + \rho(x, t)/\rho_0$ (unity is added to eliminate difficulties in logarithmic representation for densities approaching zero) are presented in Fig. 19 in dimensionless variables (73). This

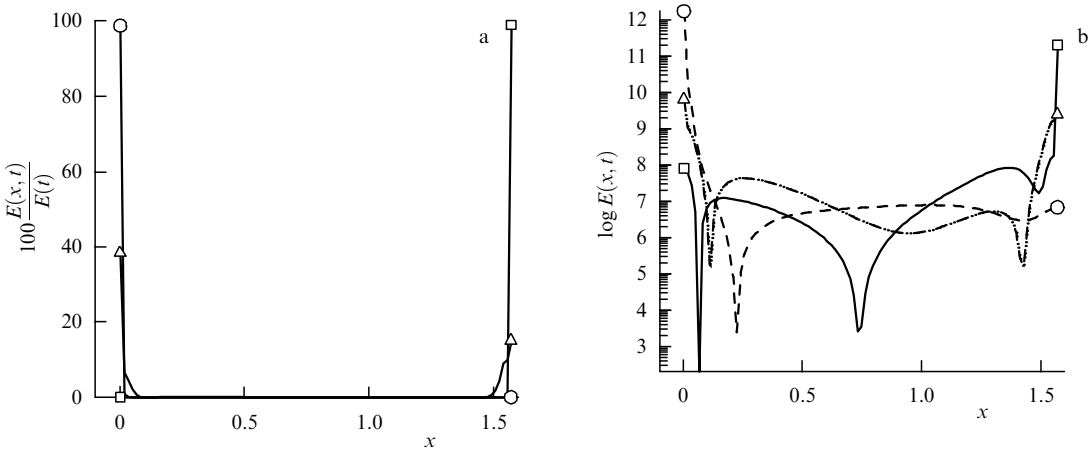

Figure 20. Dynamics of cluster disappearance at the point 0 and cluster emergence at the point $\pi/2$. The circle stands for the time instant $t = 10.4$, triangle for $t = 10.8$, and square for $t = 11.8$.

figure explicitly illustrates a gradual flow of the density field to the vicinities of points $x \approx 0$ and $x \approx \pi/2$, i.e., the formation of clusters at locations where the relative value of the density is large, whereas it is close to zero in the remaining space. Notice that at time instants t, such that $T(t) = 0$, the realization of the density field passes through the initial homogeneous state.

5.2 Turbulent dynamo model

For induction equation (20) and model (71) considered here, the x-component of the magnetic field is preserved, i.e., $H_x(\mathbf{r}, t) = H_{x0}$, and the transverse magnetic field component $\mathbf{H}_\perp(x, t)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} + v_x(t)\frac{\partial}{\partial x} f(x)\right)\mathbf{H}_\perp(x, t) = \mathbf{v}_\perp(t)\frac{\partial f(x)}{\partial x} H_{x0} \qquad (75)$$

with the initial condition $\mathbf{H}_\perp(x, 0) = \mathbf{H}_{\perp 0}$; its solution can be written in a statistically equivalent form (at $\mathbf{H}_{\perp 0} = 0$) as

$$\mathbf{H}_\perp(x, t) = 2kH_{x0} \times \int_0^t \mathrm{d}\tau \, \frac{\exp\left(T(\tau)\right)\cos^2(kx) - \exp\left(-T(\tau)\right)\sin^2(kx)}{\left[\exp\left(T(\tau)\right)\cos^2(kx) + \exp\left(-T(\tau)\right)\sin^2(kx)\right]^2} v_x(\tau)\mathbf{v}_\perp(\tau) \,. \qquad (76)$$

The expression on the right-hand side of formula (76) describes the generation of magnetic field $H_\perp(x, t)$ in the transverse plane (y, z) because of the presence of the initial field H_{x0}. And at $\mathbf{H}_{\perp 0} = 0$, field $H_\perp(x, t)$, proportional to the square of the random velocity field, defines the situation. The structure of field $H_\perp(x, t)$, similarly to that of density field, also experiences clustering, which is confirmed by the results of numerical simulations (see Refs [39, 57] and the monographs [17–20]), presented in dimensionless variables (73) in Fig. 20a, which plots the cluster share of the generated magnetic field energy with respect to the total energy in the layer at a given time instant, and in Fig. 20b, where we see the dynamics of how perturbations in magnetic energy flow from one domain boundary to the other one.

Let us point out a specific feature of equation (75). In this case, the parametric excitation of the magnetic field is accompanied by the *Gaussian generation* of the field itself. We now turn to the statistical analysis of this simple problem.

6. Parametrically excited dynamical systems with Gaussian pumping

6.1 Statistical analysis of simple turbulent dynamo problem with Gaussian pumping

The equation for the one-point in space and one-time probability density of magnetic field energy, written for dimensionless time (see Refs [39, 57] and monograph [17–20]), which corresponds to linear equation (75), is written as

$$\frac{\partial}{\partial \tau} P(\tau; E) = \left(\frac{\partial}{\partial E} E + 2\frac{\partial}{\partial E} E \frac{\partial}{\partial E} E + 2\frac{\partial}{\partial E} E \frac{\partial}{\partial E}\right) P(\tau; E) \,, \qquad (77)$$

with the initial condition $P(0; E) = \delta(E - \beta)$.

For the last equation, the large-time asymptotics for the moments of magnetic field energy were obtained in the form

$$\left\langle E^n(\tau)\right\rangle \sim A_n \exp\left[n(2n-1)\tau\right] \,,$$

which corresponds to the lognormal law for the probability density with a correction that accounts for the Gaussian field generation. Also, the expression for the Lyapunov exponent

$$\exp\left(\left\langle \ln E(\tau)\right\rangle\right) = \beta \exp(-\tau)$$

was derived, which indicates that the magnetic field energy decays at any point in space, i.e., that the clustering proceeds.

The last term in equation (77) is responsible for the generation of a Gaussian magnetic field, which dominates the magnetic field energy generation at small times. We present below the related equation and solutions for these times, bounded to the case of two dimensions which is of interest to us [58].

The equation for the probability density of the two-dimensional Gaussian vector field $\mathbf{H}_\perp(\mathbf{R}, t)$ for the spatially homogeneous case takes the form

$$\frac{\partial}{\partial \tau} P(\tau; \mathbf{H}_\perp) = \frac{1}{2}\frac{\partial^2}{\partial \mathbf{H}_\perp^2} P(\tau; \mathbf{H}_\perp) \,,$$

and its solution is written as

$$P(\tau; \mathbf{H}_\perp) = \frac{1}{2\pi\tau} \exp\left(-\frac{\mathbf{H}_\perp^2}{2\tau}\right) .$$

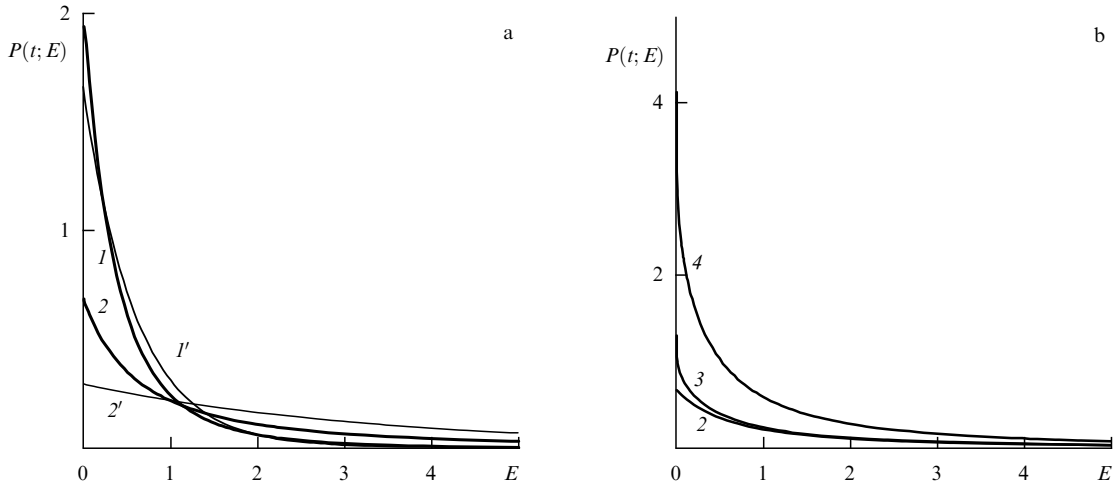

Figure 21. Probability density (77) for the time (a) $\tau = 0.3$ (curve *1*) and $\tau = 1.7$ (curve *2*). The thin lines correspond to the Gaussian distribution (78) for $\tau = 0.3$ (curve *1′*) and $\tau = 1.7$ (curve *2′*). (b) The same as in panel a, but for the time $\tau = 1.7$ (curve *2*), $\tau = 5.0$ (curve *3*), and $\tau = 8.3$ (curve *4*).

Respectively, on this short time interval, the probability density for the transverse energy $E = \mathbf{H}_{\perp}^{2}(\mathbf{R}, t)$, defined as

$$P(\tau; E) = \frac{1}{2\tau} \exp\left(-\frac{E}{2\tau}\right), \tag{78}$$

is described by the equation

$$\frac{\partial}{\partial\tau} P(\tau; E) = 2 \frac{\partial}{\partial E} E \frac{\partial}{\partial E} P(\tau; E).$$

As a consequence, the integral probability distribution function is governed by the equation

$$\frac{\partial}{\partial\tau} \Phi(\tau; E) = 2E \frac{\partial^2}{\partial E^2} \Phi(\tau; E),$$

the solution of which is written out as

$$\Phi(\tau; E) = 1 - \exp\left(-\frac{E}{2\tau}\right). \tag{79}$$

This relationship leads directly to the following expression for the typical realization curve:

$$E^*(\tau) = (2 \ln 2)\tau.$$

Clearly, for clustering to occur, the typical realization curve of a respective process must decay, in contrast to linear growth for a Gaussian process. From Fig. 21, which displays the results of a numerical solution of equation (77), it can be seen that at the initial stage the probability density of the magnetic field energy decays approximately as the Gaussian distribution, and the decay rate decreases with time, i.e., the process of Gaussian field generation prevails. At times of about $\tau = 1.7$, the situation changes: at larger times, clustering begins to play an important role, i.e., the rate of decrease in probability density begins to increase with time as the energy increases, just like its value at zero.

Consider now the integral probability function for the magnetic field energy. From equation (77), we obtain in a regular way the equation for its evolution:

$$\frac{\partial}{\partial\tau} \Phi(\tau; E) = \left[E + 2E \frac{\partial}{\partial E}(E + 1)\right] \frac{\partial}{\partial E} \Phi(\tau; E). \tag{80}$$

A numerical solution to the last equation is given in Fig. 22a; it indicates that at relatively small times, approximately until $\tau = 1.7$, the rate of increase in the integral function decreases with time, which is characteristic of the Gaussian distribution. At larger times, the rate of increasing begins to increase, which is characteristic of the lognormal distribution.

This observation is illustrated more transparently by the typical realization curve. We see in Fig. 22b that for $\tau \geqslant 3.0$ the typical realization curve decays for the process described by equation (77), indicating the presence of clustering. It is noteworthy that the process of clustering manifests itself even earlier, but it prevails over the generation from approximately this time instant.

Figure 23a plots the time dependence of the specific area of the regions where the magnetic field energy exceeds the maximum level of the typical realization curve, i.e., $E > 2$. In the case of Gaussian distribution, this area tends to unity, which testifies to a lack of clustering; however, in the presence of a magnetic field, the area begins to decrease, which points to the beginning of clustering at times $\tau \approx 1.7$. Figure 23b shows the time dynamics of the specific energy (normalized to the total energy at the respective time instant) confined in these regions, thus confirming the phenomenon of clustering.

6.2 Anomalous sea surface structures

Recent decades have seen an increased interest in such phenomena as anomalously large waves, also called rogue or freak waves (see, for example, Refs [59, 60]). There are many ideas on the mechanisms and methods to describe this phenomenon [59–62]. We presume that different mechanisms may exist, and not all of them deserve to be called anomalously high waves. For example, the development of such waves against the background of sufficiently high sea waves is apparently related to nonlinear effects (see, for example, Refs [63–66] and references cited therein). Different dynamic models are being studied based on numerical modeling, as well as on analytical results related to the nonlinear Schrödinger equation. The statistical properties of observed anomalous waves are also intensely being discussed (see, for example, the review [61]).

We propose a possible mechanism for the emergence of anomalous structures on the water surface, which may

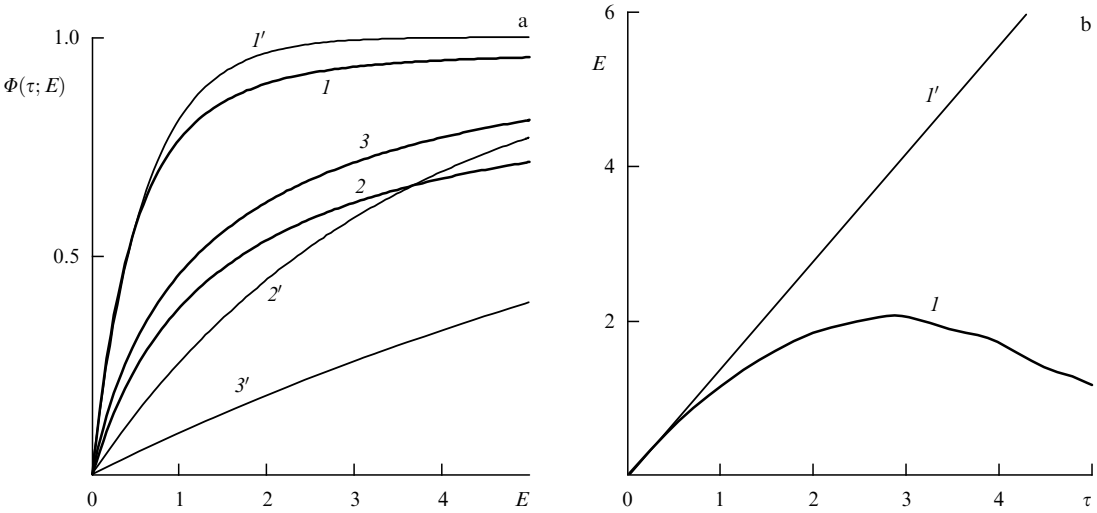

Figure 22. (a) Integral probability function (80) for three moments of dimensionless time: $\tau = 0.3$ (curve *1*), $\tau = 1.7$ (curve *2*), and $\tau = 5.0$ (curve *3*). The thin curves $1'-3'$ correspond to the Gaussian distribution for the same time instants. (b) Typical realization curves for magnetic field energy (*1*) and a Gaussian process (*1′*).

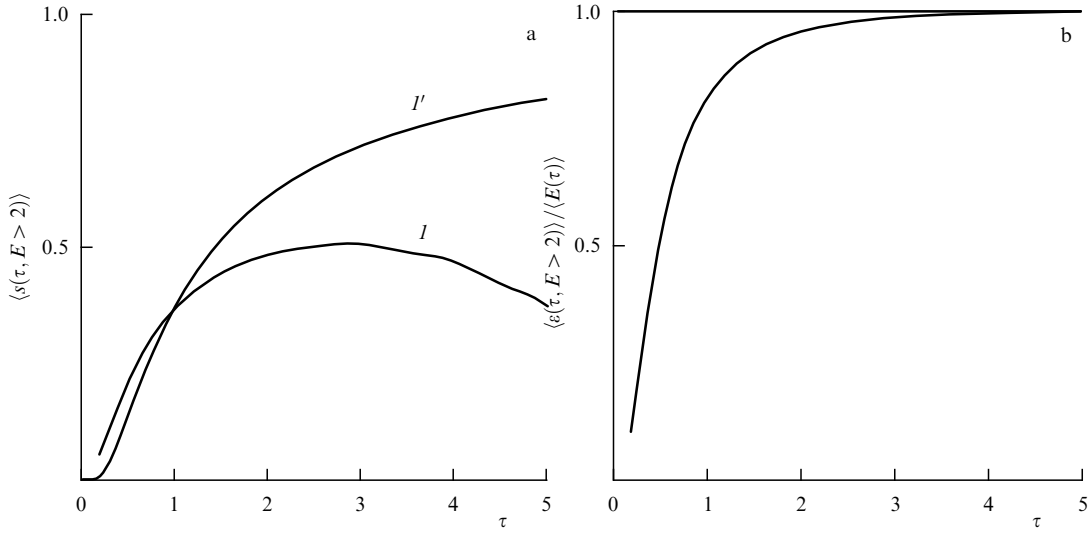

Figure 23. (a) Specific area of regions for distribution (77), where the level of magnetic field energy $E > 2$. (b) Specific energy confined in regions whose evolution is shown in panel a.

Figure 24. View on the water structure from the side (a). The front view: the beginning (b), and the middle (c) of the structure.

correspond to the appearance of such structures against the background of very weak waviness. Figure 24 presents three photos of an unusually narrow and elongated structure 4–5 m in height observed on 11 June 2006 in the vicinity of the Kamchatka Pacific coast, 1–1.5 km offshore [20, 67]. The photographer, M M Sokolovsky, describes the phenomenon documented by him in the following way: "It was, surely, a strange wave, for it was repeated several times, each time disappearing. There were no waves around, a completely still surface."

Figure 25. Example of a water structure on the sea surface.

Figure 25 also demonstrates the structure of the sea surface observed close to the coast of Ile de Ré (France) [68].[2]

From these examples, it is clear that if we construct a topographic map of the magnitude of the sea surface elevation gradient for them, we obtain a typical cluster structure for which a *positive field* is concentrated within a small area, being simply absent elsewhere.

It should be emphasized once again that the structures discussed above are notably different from the anomalous waves commonly considered. First, such structures can be both standing and moving, and second, they substantially exceed the background; here, we are not dealing with outliers that exceed the background just twofold [61]. In the examples above, one may rather speculate about the presence of high structures against a practically vanishing background. From these examples, it is also seen that the clustering of the positive field in this case is equivalent to such physical phenomena as the focusing of a wave field on passing through a random medium [61, 69] or of the intensity of laser radiation traversing a random medium (see Section 3.2.1).

6.2.1 Problem statement. The statement of the statistical problem on the emergence of anomalous structures on the sea surface in the kinematic approximation is presented in Ref. [69] and monograph [20].

We denote the three-dimensional spatial coordinate as $\mathbf{r} = \{r_i\}$, where $i = 1, 2, 3$, labeling the vertical coordinate as $z = r_3$, and use R_α ($\alpha = 1, 2$) to mark the coordinates in the horizontal plane perpendicular to the z-axis. In this notation, $\mathbf{r} = \{\mathbf{R}, z\}$. Accordingly, we represent the three-dimensional hydrodynamic velocity field $\mathbf{u}(\mathbf{r}; t)$ through its horizontal and vertical components, i.e., in the form $u_i(\mathbf{r}; t) = \{u_\alpha(\mathbf{R}, z; t), w(\mathbf{R}, z; t)\}$, where $i = 1, 2, 3$ and the subscript $\alpha = 1, 2$.

The sea surface elevation (displacement) $z = \xi(\mathbf{R}, t)$ is described through the kinematic boundary condition (Fig. 26), which is expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\xi(\mathbf{R}, t) = w_z(\mathbf{R}, z; t)\Big|_{z=\xi(\mathbf{R},t)} . \tag{81}$$

Here, $\mathrm{d}\xi(\mathbf{R}, t)/\mathrm{d}t$ is the full derivative of a sea surface elevation.

[2] Impressive photos of other anomalous structures on the sea surface can be found at the site http://imgur.com/a/4Y2Oo.

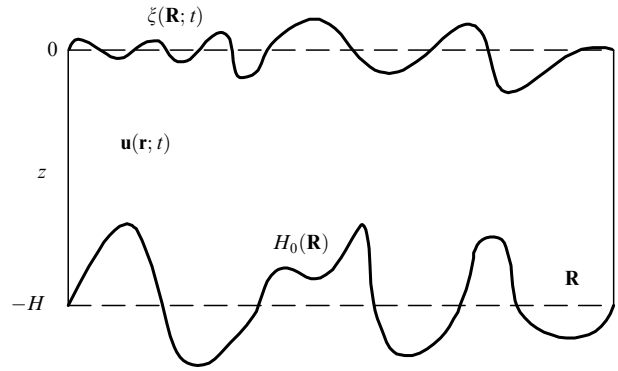

Figure 26. Perturbation of the sea surface.

Boundary condition (81) can be considered a closed stochastic quasilinear equation in the framework of the kinematic approximation, i.e., for the given statistical characteristics of the velocity fields $\mathbf{u}(\mathbf{R}, z; t)$ and $w(\mathbf{R}, z; t)$:

$$\frac{\partial \xi(\mathbf{R}, t)}{\partial t} + u_\alpha\big(\mathbf{R}, \xi(\mathbf{R}, t), t\big)\,\frac{\partial \xi(\mathbf{R}, t)}{\partial R_\alpha} = w_z\big(\mathbf{R}, \xi(\mathbf{R}, t); t\big)\,, \tag{82}$$

with the initial condition $\xi(\mathbf{R}, 0) = \xi_0(\mathbf{R})$. Equation (82) describes the generation of waves on the sea surface excited by a Gaussian vertical component of a hydrodynamic velocity field.

Differentiating equation (82) over $\mathbf{R}$, we arrive at the equation for the gradient of sea surface elevation $p_\beta(\mathbf{R}, t) = \partial\xi(\mathbf{R}, t)/\partial R_\beta$, characterizing the slope of the sea surface:

$$\begin{aligned}
&\frac{\partial p_\beta(\mathbf{R}, t)}{\partial t} + \left(\frac{\partial u_\alpha(\mathbf{R}, z; t)}{\partial R_\beta} + \frac{\partial u_\alpha(\mathbf{R}, z; t)}{\partial z}\,p_\beta(\mathbf{R}, t)\right)_{z=\xi(\mathbf{R},t)} p_\alpha(\mathbf{R}, t) \\
&\quad + u_\alpha\big(\mathbf{R}, \xi(\mathbf{R}, t), t\big)\,\frac{\partial p_\alpha(\mathbf{R}, t)}{\partial R_\beta} \\
&\quad = \left(\frac{\partial w(\mathbf{R}, z; t)}{\partial R_\beta} + \frac{\partial w(\mathbf{R}, z; t)}{\partial z}\,p_\beta(\mathbf{R}, t)\right)_{z=\xi(\mathbf{R},t)} p_\alpha(\mathbf{R}, t)\,,
\end{aligned} \tag{83}$$

with the boundary condition $\mathbf{p}(\mathbf{R}, 0) = \mathbf{p}_0(\mathbf{R}) = \partial\xi_0(\mathbf{R})/\partial\mathbf{R}$.

Notice that one more boundary condition, related to the inhomogeneities in bottom topography (see Fig. 26), exists for the problem considered. In the framework of the kinematic approximation, this boundary condition is manifested in a functional form, namely, for variational derivatives of problem solutions $\xi(\mathbf{R}, t)$ and $\mathbf{p}(\mathbf{R}, t)$, the following relationships are valid:

$$\begin{aligned}
&\frac{\delta\xi(\mathbf{R}, t)}{\delta\mathbf{u}(\mathbf{R}', z', t')} \sim \theta\big(z' + H_0 - H(\mathbf{R})\big)\theta(t - t')\,, \\
&\frac{\delta\mathbf{p}(\mathbf{R}, t)}{\delta\mathbf{u}(\mathbf{R}', z', t')} \sim \theta\big(z' + H_0 - H(\mathbf{R})\big)\theta(t - t')\,,
\end{aligned} \tag{84}$$

where $\theta(z)$ is the Heaviside theta-function. Condition (84) corresponds to an impermeable sea bottom.

Thus, equations (82) and (83) together with boundary condition (84) represent a closed system in the kinematic approximation. The solution of this system of equations should provide an answer to the question of whether or not its equations contain information on the presence of anomalous structures on the sea surface with a probability one, i.e.,

for almost any realization of the random velocity field. But we put aside here the physical mechanisms underlying this structure formation.

6.2.2 Statistical analysis of the problem. A statistical analysis of the problem given by equations (82)–(84) was carried out in Refs [58, 70]. An equation for the joint probability density of the sea surface elevation $\xi(\mathbf{R}, t)$ and its spatial gradient $\mathbf{p}(\mathbf{R}, t)$ has been obtained. The result proved to be rather curious: for an infinitely deep sea and a spatially statistically homogeneous problem that corresponds to initial conditions $\xi_0(\mathbf{R})$, $p(\mathbf{R}, 0) = 0$, the situation looks as follows.

(1) The sea surface elevation $\xi(\mathbf{R}, t)$ does not correlate with its gradient and obeys the Gaussian distribution

$$P(t;\xi) = \frac{1}{\sqrt{4\pi B_{ww}(\mathbf{0},0)t}} \exp\left(-\frac{\xi^2}{4B_{ww}(\mathbf{0},0)t}\right), \tag{85}$$

which does not depend on the nonlinearity of the input equation (82). The variance of the sea surface elevation is given by

$$\sigma_\xi^2(t) = \left\langle \xi^2(\mathbf{R}, t)\right\rangle_\xi = \int_{-\infty}^{\infty} \xi^2 P(t;\xi)\,\mathrm{d}\xi = 2B_{ww}(\mathbf{0},0)t\,.$$

Here, the subscript ξ labels averaging over an ensemble of realizations of field $\xi(\mathbf{R}, t)$. The diffusion coefficient $B_{ww}(\mathbf{0}, 0)$ in expression (85) is linked to the correlation function of the vertical velocity $w(\mathbf{R}, z; t)$ by the relationship

$$B_{ww}(\mathbf{0},0) = \int_0^\infty \mathrm{d}\tau\, B_{ww}(\mathbf{0},0;\tau)\,,$$

where $B_{ww}(\mathbf{R}, z; t)$ is the correlation function of field $w(\mathbf{R}, z; t)$. The coefficient $B_{ww}(\mathbf{0}, 0)$, in turn, is linked with the variance of random three-dimensional velocity field $\mathbf{u}(\mathbf{r}, t)$ (23):

$$B_{ww}(\mathbf{0},0) = \frac{1}{3}\int \mathrm{d}\mathbf{k}\,\left[2E^{\mathrm{s}}(k) + E^{\mathrm{p}}(k)\right] = \frac{1}{3}\,\sigma_{\mathbf{u}}^2\tau_0\,, \tag{86}$$

where τ_0 is the characteristic temporal correlation radius of a random velocity field $\mathbf{u}(\mathbf{r}, t)$ [see equality (25)].

As a consequence, the complex structure of the velocity field in the lower subspace cannot be the *direct reason* for stochastic structure formation for the sea surface elevation.

Note that expressions for conditional means follow from formula (85):

$$\left\langle \xi(\mathbf{R},t)|\xi>0\right\rangle_\xi = \int_0^\infty \mathrm{d}\xi\;\xi P(t;\xi) = \sqrt{\frac{1}{\pi}\,B_{ww}(\mathbf{0},0)t}\,, \tag{87}$$

$$\left\langle \xi(\mathbf{R},t)|\xi<0\right\rangle_\xi = \int_{-\infty}^0 \mathrm{d}\xi\;\xi P(t;\xi) = -\sqrt{\frac{1}{\pi}\,B_{ww}(\mathbf{0},0)t}\,. \tag{88}$$

Here, certainly, $\langle \xi(\mathbf{R}, t)\rangle_\xi = 0$.

(2) The probability density $P(t; I)$ for the modulus squared of the sea surface elevation gradient, $I(\mathbf{R}, t) = \mathbf{p}^2(\mathbf{R}, t)$, is described by an equation of universal form:

$$\frac{\partial}{\partial\tau}\,P(\tau;I) = \frac{\partial}{\partial I}\,I(1+I)P(\tau;I) + 2\,\frac{\partial}{\partial I}\,I\,\frac{\partial}{\partial I}(1+I)^2P(\tau;I)\,, \tag{89}$$

where the dimensionless time was introduced:

$$\tau = \frac{2}{15}\,(4D^{\mathrm{s}} + D^{\mathrm{p}})t\,, \tag{90}$$

and the quantities D^{s} and D^{p} are described by equalities (27) with $d = 3$.

We note, first, that for any fixed point in space $\tilde{\mathbf{R}}$, the function $I(\tau; \tilde{\mathbf{R}})$ is a random process with respect to time, with the one-time probability density, independent of $\tilde{\mathbf{R}}$ and described by the equation obtained.

Alongside this, in physical space $\{\mathbf{R}\}$, the process of structure formation in the field $I(\tau; \mathbf{R}) = |\mathbf{p}(\tau; \mathbf{R})|^2$, considered as a physical object, may take place in the form of closed regions with an augmented gradient concentration — clustering, which is also described by equation (89).

A qualitative analysis and estimates presented in Refs [58, 70] have shown the following. Equation (89) is rather complex and comprises two effects. For one thing, field $I(\tau; \mathbf{R})$ is generated by the random Gaussian velocity field. But, on the other hand, by virtue of the dynamics of the input stochastic equations, it is parametrically excited. The random field $I(\tau; \mathbf{R})$ decays with a probability one (i.e., in almost all its realizations) at a sufficiently large time at any point in space and, hence, should undergo clustering in small spatial regions.

The integral probability density distribution function $P(\tau; I)$, defined as

$$\Phi(\tau,I) = \int_0^I \mathrm{d}I'\,P(\tau;I') = \left\langle \theta\big(I - I(\tau;\mathbf{R})\big)\right\rangle_{\mathbf{u}}\,,$$

is the probability of event $\mathcal{P}\,(I(\tau;\mathbf{R}) < I)$. The function $\Phi(\tau, I)$ satisfies the equation

$$\frac{\partial}{\partial\tau}\,\Phi(\tau,I) = I(1+I)\,\frac{\partial}{\partial I}\,\Phi(\tau,I) + 2I\,\frac{\partial}{\partial I}(1+I)^2\,\frac{\partial}{\partial I}\,\Phi(\tau,I)\,, \tag{91}$$

which follows from equation (89). Relatedly, the function

$$\tilde{\Phi}(\tau,I) = \int_I^\infty \mathrm{d}I'\,P(\tau;I') = \left\langle \theta\big(I(\tau;\mathbf{R}) - I\big)\right\rangle_{\mathbf{u}} = 1 - \Phi(\tau,I)$$

is the probability of event $\mathcal{P}\,(I(\tau;\mathbf{R}) > I)$. Because of parametric excitation, function $\Phi(\tau, I)$ rapidly approaches unity with time, while function $\tilde{\Phi}(\tau, I)$ tends to zero [58, 70].

In general, the area over which the random field $I(\tau; \mathbf{R})$ exceeds the fixed level $\bar{I}$ is described by the integral

$$S(\tau,\bar{I}) = \int \mathrm{d}\mathbf{R}\,\theta\big(I(\tau;\mathbf{R}) - \bar{I}\big)\,, \tag{92}$$

and the total 'mass' of field $I(\tau; \mathbf{R}) > \bar{I}$ confined within this area, is given by

$$I(\tau; I > \bar{I}) = \int \mathrm{d}\mathbf{R}\;I(\tau;\mathbf{R})\,\theta\big(I(\tau;\mathbf{R}) - \bar{I}\big)\,. \tag{93}$$

Averaging expressions (92) and (93) over an ensemble of realizations of random field $I(\tau; \mathbf{R})$, we obtain for the mean quantities:

$$\begin{aligned} \left\langle S(\tau,\bar{I})\right\rangle_I &= \int \mathrm{d}\mathbf{R}\;\left\langle \theta\big(I(\tau;\mathbf{R}) - \bar{I}\big)\right\rangle_I\,, \\ \left\langle I(\tau;I>\bar{I})\right\rangle_I &= \int \mathrm{d}\mathbf{R}\;\left\langle I(\tau;\mathbf{R})\,\theta\big(I(\tau;\mathbf{R}) - \bar{I}\big)\right\rangle_I\,. \end{aligned} \tag{94}$$

For a spatially homogeneous and isotropic problem, all one-point statistical means are independent of spatial coordinate $\mathbf{R}$, so that the first equality in Eqn (94) should be written for the specific quantity

$$s_{\text{hom}}(\tau, \bar{I}) = \left\langle \theta\big(I(\tau;\mathbf{R}) - \bar{I}\big)\right\rangle_I , \tag{95}$$

which is the probability of event $\mathcal{P}\,(I(\tau;\mathbf{R}) > \bar{I})$. According to the ideas of statistical topography of random fields, the function $s_{\text{hom}}(\tau, \bar{I})$ for a statistically homogeneous field has a geometric sense of the specific quantity (i.e., related to a unit area), over which field $I(\tau;\mathbf{R})$ exceeds any given value $\bar{I}$ (see, for example, Refs [17–20]). However, for a field decaying at almost all points in space, this probability tends to zero, which says that the basic statistical characteristics, such as moment functions, are concentrated on this small area. The specific gradient modulus squared i, concentrated in this region, is described by the equality

$$\begin{aligned}\left\langle i\,(\tau; I > \bar{I})\right\rangle_I &= \int_{\bar{I}}^{\infty} \mathrm{d}I\; IP(\tau; I)\\ &= \left\langle I(t)\right\rangle_I \left(1 - \frac{1}{\left\langle I(t)\right\rangle_I}\int_0^{\bar{I}} \mathrm{d}I\; IP(\tau; I)\right) .\end{aligned} \tag{96}$$

Hence, it follows that, as time progresses, the expression in Eqn (96) to the right of the second equality sign tends to $\langle I(t)\rangle_I$. Accordingly, in this small fraction of space, a strong gradient should in all probability simply 'extrude' the free surface upward (i.e., to form *tall narrow structures*), as well as downward (i.e., to create *deep narrow troughs*—*maelstroms* of free surface) on a decrease in area where it is confined. And this should correspond to rare large fluctuations of the Gaussian field of sea surface elevation $\xi(\mathbf{R}, t)$.

6.2.3 Statistical topography of the sea surface elevation field. Consider now how the process of stochastic structure formation can be described for the sea surface $\xi(\mathbf{R}, t)$ itself. We know that the values of the random field of gradient modulus squared $I(\tau;\mathbf{R})$ exceeding a fixed magnitude $\bar{I}$, i.e. $I(\tau;\mathbf{R}) > \bar{I}$, are confined within small spatial regions (92) in individual realizations. Namely within this small fraction of space, the process of structure formation of the sea surface elevation $\xi(\mathbf{R}, t)$ should take place. Consequently, for the field $\xi(\mathbf{R}, t)$, we should study the probability density in the region of space (92), i.e., the conditional probability density for $\xi > 0$, $I(\tau;\mathbf{R}) > \bar{I}$. In this case, the fields $\xi(\mathbf{R}, t)$ and $I(\mathbf{R}, t)$ are statistically independent.

Thus, the integral quantity of interest to us, viz.

$$\int \mathrm{d}\mathbf{R}\; \xi(\mathbf{R}, t)\, \theta\big(\xi(\mathbf{R}, t)\big)\, \theta\big(I\,(\tau;\mathbf{R}) - \bar{I}\big) ,$$

averaged over ensembles of realizations of fields $\xi(\mathbf{R}, t)$ and $I(\tau;\mathbf{R})$, will define the conditional mean which, for a spatially homogeneous and isotropic problem, according to formulas (87) and (95), takes the form

$$\left\langle \xi(t)|\xi > 0\right\rangle s_{\text{hom}}(\tau, \bar{I}) = \sqrt{\frac{1}{\pi}\, B_{ww}(\mathbf{0}, 0)t}\; s_{\text{hom}}(\tau, \bar{I}) . \tag{97}$$

Taking into account formula (90), we can express time t in terms of dimensionless time τ and, considering equalities (25), recast conditional mean (97) as

$$\begin{aligned}&\left\langle \xi(\mathbf{R}, t)\, \theta\big(\xi(\mathbf{R}, t)\big)\, \theta\big(I(\tau;\mathbf{R}) - \bar{I}\big)\right\rangle_{\xi, I}\\ &\quad = \sqrt{\frac{5\sigma_{\mathbf{u}}^2\tau_0}{2\pi(4D^{\text{s}} + D^{\text{p}})}}\, F(\tau, \bar{I}) ,\end{aligned} \tag{98}$$

where the dimensionless function

$$F(\tau; \bar{I}) = \sqrt{\tau}\; s_{\text{hom}}(\tau, \bar{I}) , \tag{99}$$

and the coefficients D^{s} and D^{p} in relationships (27) are expressed through statistical parameters of velocity fields by formulas (28).

We note that in incompressible hydrodynamic turbulence ($D^{\text{p}} = 0$), formula (98) is substantially simplified and, with account for equalities (28), acquires the universal form

$$\left\langle \xi(\mathbf{R}, t)\, \theta\big(\xi(\mathbf{R}, t)\big)\, \theta\big(I(\tau;\mathbf{R}) - \bar{I}\big)\right\rangle_{\xi, I} = \sqrt{\frac{5}{4\pi}}\, l_{\boldsymbol{\omega}} F(\tau, \bar{I}) , \tag{100}$$

where $l_{\boldsymbol{\omega}} = \sigma_{\mathbf{u}}/\sigma_{\boldsymbol{\omega}}$ is the spatial correlation radius of the velocity/vorticity field.

For small times, this conditional mean grows with time, but then decays because of a rapid decrease in function $s_{\text{hom}}(\tau, \bar{I})$, passing through the maximum which characterizes the mean maximum amplitude of anomalous structure formation on the sea surface.

In a similar way, one may compute other conditional moment functions like

$$\langle \xi^n(\mathbf{R}, t)|\xi > 0; I(\tau;\mathbf{R}) > \bar{I}\rangle .$$

It is also possible to consider the integral quantity

$$\int \mathrm{d}\mathbf{R}\; \xi(\mathbf{R}, t)\theta(-\xi(\mathbf{R}, t))\theta(I(\tau;\mathbf{R}) - \bar{I}) ,$$

which, on averaging over ensembles of realizations of fields $\xi(\mathbf{R}, t)$ and $I(\tau;\mathbf{R})$, defines the conditional mean for the spatially homogeneous and isotropic case:

$$\left\langle \xi(t)|\xi < 0\right\rangle s_{\text{hom}}(\tau, \bar{I}) = -\sqrt{\frac{1}{\pi}\, B_{ww}(\mathbf{0}, 0)t}\; s_{\text{hom}}(\tau, \bar{I}) . \tag{101}$$

For small times, conditional mean (101) decays with time, but will increase further because of a rapid decrease in positive function $s_{\text{hom}}(\tau, \bar{I})$, passing through a minimum value which characterizes the mean minimum amplitude of anomalous negative structure formation on the sea surface (i.e., deep trough).

Thus, based on the equations obtained in Refs [58, 70], it is shown that in the process of structure formation of the gradient modulus, the field of sea surface elevation increases in small regions which comprise the entire gradient, reaching a maximum and then beginning to decrease. Thus, the structures studied may reach a substantial amplitude on the fluid surface, but are limited both in magnitude and lifetime.

Let us make a remark concerning the interpretation of statistical means from the standpoint of statistical topography. Formulas (97) and (101) show that sea surface elevations differing from zero are confined in the regions where the modulus of the gradient undergoes clustering, i.e., in such regions where this modulus is larger than any value of $\bar{I}$,

however small. Note that at $\bar{I} = 0$, formulas (97) and (101) lose their meaning since they cease to describe the distribution of the elevation in space and become, therefore, expressions for ordinary means, the interpretation of which from the standpoint of behavior in individual realizations does not convey useful information (see, for example, books [17–20]).

We also note that the estimates obtained in Refs [58, 70] and general features of the sea surface elevation amplitude only point to the existence of an effect. More detailed characteristics and additional verification can be furnished by numerically solving equations (89) and (91), and also by numerical simulations of the input equation (82).

7. Conclusion

We have considered the processes of stochastic structure formation in two-dimensional geophysical fluid dynamics based on statistical analysis of Gaussian random fields, as well as stochastic structure formation in dynamical systems with parametric excitation of positive random fields $f(\mathbf{r}, t)$ described by partial differential equations. We also considered two examples of stochastic structure formation in dynamical systems with parametric excitation in the presence of Gaussian pumping. Such structure formation in dynamical systems with parametric excitation in space and time either happens or not! However, if it occurs in space, then this almost always happens in individual realizations, i.e., with a unit probability (exponentially fast), and for the spatially homogeneous statistical case consists in the following:

(1) the field decays at almost all points in space with time (clearly with fluctuations superimposed);

(2) the regions of small volume where this field is concentrated (clustered) develop in space, and stochastic structure formation is caused by diffusion of random field $f(\mathbf{r}, t)$ in its phase space $\{f\}$.

In the case considered, clustering of the field $f(\mathbf{r}, t)$ of any nature is a general feature of dynamical fields, and one may claim that structure formation for such arbitrary random fields is the *law of nature*.

In this study, we clarified conditions under which such structure formation takes place. It is worth noting that these conditions have a transparent physical–mathematical sense and are described at a sufficiently elementary mathematical level by resorting to the ideas of statistical topography.

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