Physics – Uspekhi **58** (9) 878 – 891 (2015)

CONFERENCES AND SYMPOSIA PACS numbers: **04.20. – q**, **04.60. – m**, **98.80. – k**

100th ANNIVERSARY OF THE BIRTH OF E M LIFSHITZ

Stochastic cosmology, perturbation theories, and Lifshitz gravity

I M Khalatnikov, A Yu Kamenshchik

DOI: 10.3367/UFNe.0185.201509f.0948

Contents

Abstract. **We review E M Lifshitz's work on gravity and cosmology and discuss later work by others who drew on his ideas. The major topics covered include the stochastic cosmology of an anisotropic universe and of an isotropic scalar field universe, the quasi-isotropic (gradient) expansion in cosmology, and Hořava–Lifshitz gravity and cosmology.**

Keywords: gravitation, cosmology, chaos, perturbation theory

1. Introduction

The name Evgenii Mikhailovich Lifshitz is known to physicists all over the world and even to a wider public in connection with the famous course of theoretical physics written by him in collaboration with his friend and teacher Lev Davidovich Landau. In this review, dedicated to the 100th anniversary of the birth of Lifshitz, we dwell on some areas of theoretical physics, in the fields of gravity and cosmology, developed by him and other members of the Landau school. In addition, we speak about some unexpected trends in modern physics connected with his name.

First of all, we discuss stochastic or chaotic cosmology. As is well known, the study of chaos has become very popular in modern physics, and chaotic phenomena have been found in different fields and sometimes in rather simple systems (see, e.g., [1–4]). It is amusing that one of the first examples of stochastic or chaotic behavior was discovered in the realm of cosmology. Initially, the study of chaos in cosmology was connected with the investigation of the so-called initial singularity problem. Later, it became clear that the stochastic properties of cosmological models are not necessarily related to the singularity and can arise in such simple systems as Friedmann universes.

We recall that Penrose and Hawking [5–7] proved the impossibility of an indefinite continuation of geodesics under certain conditions. This was interpreted as pointing to the existence of a singularity in the general solution of the Einstein equations. These theorems, however, did not allow finding the particular analytic structure of the singularity. The analytic behavior of the general solutions of the Einstein equations in the neighborhood of a singularity was investigated in papers by Lifshitz and Khalatnikov [8–11] and Belinsky, Lifshitz, and Khalatnikov [12–14]. These papers revealed the enigmatic phenomenon of an oscillatory approach to the singularity, which has also become known as the *Mixmaster Universe* [15]. The model of a closed homogeneous but anisotropic universe with three degrees of freedom (Bianchi IX cosmological model) was used to demonstrate that the universe approaches the singularity in such a way that its contraction along two axes is accompanied by expansion with respect to the third axis, and the axes change their roles according to a rather complicated law, which reveals chaotic behavior [13, 14, 16, 17].

The study of the dynamics of the universe in the vicinity of the cosmological singularity has become a rapidly developing field of modern theoretical and mathematical physics. First of all, we mention the generalization of the study of the oscillatory approach to the cosmological singularity to the case of multidimensional cosmological models. It was pointed out in [18–20] that the approach to the cosmological singularity in multidimensional (Kaluza–Klein) cosmological models has a chaotic character in spacetimes whose dimension is not higher than ten, while in the spacetimes of

I M Khalatnikov Landau Institute for Theoretical Physics, Russian Academy of Sciences, ul. Kosygina 2, 119334 Moscow, Russian Federation
E-mail: khalat@itp.ac.ru
A Yu Kamenshchik Landau Institute for Theoretical Physics, Russian Academy of Sciences, ul. Kosygina 2, 119334 Moscow, Russian Federation; Dipartimento di Fisica e Astronomia and Istituto Nazionale di Fisica Nucleare, via Irnerio 46, 40126 Bologna, Italy
E-mail: kamenshchik@bo.infn.it

Received 30 May 2015
Uspekhi Fizicheskikh Nauk **185** (9) 948 – 963 (2015)
DOI: 10.3367/UFNr.0185.201509f.0948
Translated by A Yu Kamenshchik; edited by A M Semikhatov

higher dimensions, a universe, after undergoing a finite number of oscillations, enters a monotonic Kasner-type contracting regime of oscillations [21].

The development of cosmological studies based on superstring models has revealed some new aspects of the dynamics in the vicinity of a singularity [22–24]. First, in these models, the mechanisms for changing Kasner epochs were shown to exist that are provoked not by gravitational interactions but by the influence of other fields present in these theories. Second, it was proved that cosmological models based on the six main superstring models plus the $D = 11$ supergravity model exhibit the chaotic oscillatory approach toward the singularity. Third, a connection between cosmological models manifesting the oscillatory approach toward the singularity and a special subclass of infinite-dimensional Lie algebras [25] — the so-called hyperbolic Kac–Moody algebras — has been discovered.

In the early 1980s, cosmology underwent a new revolution connected with the birth of so-called inflationary cosmology [26–28]. The theory of a hot universe that began its evolution from the singularity called the Big Bang [29] corresponded well with the observational data, especially after the discovery of cosmic microwave background radiation in the 1960s by Penzias and Wilson, but it suffered from some fundamental problems, like those of flatness, homogeneity, horizon, and some others. The theory of cosmological inflation, developed to resolve these problems, was based on the assumption that at the beginning of cosmological evolution, the universe underwent a period of quasi-exponential expansion. Such an expansion can be made possible by the presence of an effective cosmological constant. However, to exit the inflationary stage of cosmological evolution, which should be rather short, it is necessary to ensure the slow evolution of this effective constant, which is not constant. Thus, the models of Friedmann's homogeneous and isotropic universe filled with a scalar field, usually called the inflaton, became popular.

The dynamics of such models were studied in detail in [30, 31] by using the methods of the qualitative theory of differential equations. It was clear that a closed contracting Friedmann universe filled with a massive scalar field can sometimes escape falling into the singularity and undergoes a bounce. In [32], Page analyzed the suggestion that the infinitely bouncing trajectories in a closed Friedmann universe filled with a scalar field constitute a fractal set. In such a way, the stochastic behavior was investigated in the framework of very simple cosmological models. Page's hypothesis was further studied in [33, 34] and further arguments in favor of the stochasticity of the dynamics of Friedmann models were found.

There was also another question, discussed in the context of studying the stochastic behavior in cosmology: the search for some invariant characteristics of chaoticity. As is well known, general relativity is a reparameterization-invariant theory and it is important to have quantities that are independent of the chosen system of coordinates when studying gravitational phenomena. One of the invariant tools useful in studying chaoticity in cosmology is so-called topological entropy. In [35], this tool was applied to the study of the simplest model of a closed Friedmann universe filled with a scalar field whose potential includes only the massive term. In [36], topological entropy and some other characteristics were applied to the study of the Mixmaster Universe, while in [37], the topological entropy was calculated for a family of more complicated Friedmann models.

If one is unwilling to consider cosmology as a purely mathematical application of general relativity and has the intention to compare its predictions with observational data, one cannot limit oneself to studying the Friedmann or Bianchi universes. Treating the cosmological solutions of the Einstein equations requires developing some perturbative methods. It is remarkable that the first paper devoted to the development of the theory of linear cosmological perturbations on Friedmann backgrounds was written by Lifshitz as early as 1946 [38]. This theory was then further developed in [8, 9]. Later, a huge amount of work was done in this field. In particular, a gauge-invariant theory of cosmological perturbations should be noted [39–41].

While the theory of linear cosmological perturbations is basically applied to Friedmann backgrounds, another asymptotic tool developed by Lifshitz and Khalatnikov in [42]: the so-called quasi-isotropic expansion for solutions of the Einstein equations near a singularity, which can take a larger class of backgrounds as a zeroth approximation. This expansion has also been extensively studied, sometimes under the name "gradient expansion" [43–45]. In some sense, this expansion can also be generalized for the entire universe, without limitation to the vicinity of the singularity.

Recently, a new phenomenon of cosmic acceleration was discovered [46, 47]. This discovery has stimulated the search for so-called dark energy, which can explain this phenomenon [48, 49]. An alternative or complementary way of explaining cosmic acceleration is the construction of some modified theory of gravity. Besides, the quantum theory of gravity is nonrenormalizable, and this represents another stimulus for the modification of gravity. Among the different theories of modified gravity, there is a rather unexpected apparition: the Hořava–Lifshitz theory of gravity [50], which is inspired by a couple of old papers by Lifshitz devoted to phase transition theory [51, 52]. The main idea consists in the hypothesis that the graviton propagator can be Lorentz-noninvariant at large values of the momentum and its spatial part behaves differently than the temporal part. In such a way, we can obtain the renormalizability of the theory without encountering unpleasant objects such as tachyons or ghosts. Although breaking the Lorentz invariance is too radical a step for many researchers, it is amusing how the old papers by Lifshitz devoted to an object that stayed far away from gravitational thematics have given rise to the creation of a new trend in such a modern field as quantum gravity.

The structure of this paper is as follows: in Section 2, we recall the main features of the oscillatory approach to the singularity in relativistic cosmology and dwell on its stochastic nature; Section 3 is devoted to the study of stochasticity in the Friedmann cosmology; in Section 4, we present some new developments in the study of quasi-isotropic expansions; in Section 5, we give a brief review of the Hořava–Lifshitz gravity, and the last section is devoted to the conclusions.

2. Oscillatory approach to the singularity and stochastic cosmology

One of the first exact solutions found in the framework of general relativity was the Kasner solution [21] for the Bianchi-I cosmological model representing a gravitational field in an empty space with a Euclidean metric depending on time according to the formula

$$ds^2 = dt^2 - t^{2p_1}\,dx^2 - t^{2p_2}\,dy^2 - t^{2p_3}\,dz^2\,, \tag{1}$$

where the exponents p_1, p_2, and p_3 satisfy the relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1\,. \tag{2}$$

Choosing the ordering of the exponents as

$$p_1 < p_2 < p_3\,, \tag{3}$$

we can parameterize them as [3]

$$p_1 = -\frac{u}{1+u+u^2}\,, \quad p_2 = \frac{1+u}{1+u+u^2}\,, \quad p_3 = \frac{u(1+u)}{1+u+u^2}\,. \tag{4}$$

As the parameter u varies in the range $u \geqslant 1$, p_1, p_2, and p_3 take all their permissible values:

$$-\frac{1}{3} \leqslant p_1 \leqslant 0\,, \qquad 0 \leqslant p_2 \leqslant \frac{2}{3}\,, \qquad \frac{2}{3} \leqslant p_3 \leqslant 1\,. \tag{5}$$

The values $u < 1$ lead to the same range of values of p_1, p_2, p_3 because

$$p_1\left(\frac{1}{u}\right) = p_1(u)\,, \quad p_2\left(\frac{1}{u}\right) = p_3(u)\,, \quad p_3\left(\frac{1}{u}\right) = p_2(u)\,. \tag{6}$$

The parameter u, introduced in the early 1960s, has turned out to be very useful, and its properties are attracting the attention of researchers in various fields of study. For example, in recent paper [53], a connection was established between the Lifshitz–Khalatnikov parameter u and the invariants arising in the context of Petrov's classification of Einstein spaces [54].

In the case of Bianchi-VIII or Bianchi-IX cosmological models, Kasner regime (1), (2) is no longer an exact solution of the Einstein equations; however, generalized Kasner solutions can be constructed [10–14]. It is possible to construct some kind of perturbation theory where exact Kasner solution (1), (2) plays the role of the zeroth-order approximation, while the terms in the Einstein equations that depend on spatial curvature tensors play the role of perturbations (apparently, such terms are absent in the Bianchi-I cosmology). This perturbation theory is effective in the vicinity of a singularity or, in other words, as $t \to 0$. A remarkable feature of these perturbations is that they imply the transition from the Kasner regime with one set of parameters to the Kasner regime with another one.

The metric of the generalized Kasner solution in a synchronous reference system can be written in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - (a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta)\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta\,, \tag{7}$$

where

$$a = t^{p_l}\,,\; b = t^{p_m}\,,\; c = t^{p_n}\,. \tag{8}$$

The three-dimensional vectors **l**, **m**, **n** define the directions along which the spatial distances vary with time according to power laws (8). For $p_l = p_1, p_m = p_2, p_n = p_3$, we have

$$a \sim t^{p_1}\,,\; b \sim t^{p_2}\,,\; c \sim t^{p_3}\,, \tag{9}$$

i.e., the universe is contracting in directions given by the vectors **m** and **n** and is expanding along **l**. It was shown that the perturbations caused by spatial curvature terms make the variables a, b, and c undergo a transition to another Kasner regime characterized by the formulas

$$a \sim t^{p'_l}\,,\; b \sim t^{p'_2}\,,\; c \sim t^{p'_3}\,, \tag{10}$$

where

$$p'_l = \frac{|p_1|}{1-2|p_1|}\,, \quad p'_m = -\frac{2|p_1| - p_2}{1-2|p_1|}\,, \quad p'_n = -\frac{p_3 - 2|p_1|}{1-2|p_1|}\,. \tag{11}$$

Thus, the effect of the perturbation is to replace one "Kasner epoch" by another such that the negative power of t is shifted from the **l** to the **m** direction. During the transition, the function $a(t)$ reaches a maximum and $b(t)$ a minimum. Hence, the previously decreasing quantity b now increases, a decreases, and $c(t)$ remains a decreasing function. The previously increasing perturbation that caused the transition from regime (9) to (10) is damped and eventually vanishes. Then another perturbation begins to grow, which leads to a new replacement of one Kasner epoch by another, and so on.

We emphasize that precisely the fact that a perturbation implies such a change in the dynamics that suppresses the perturbation allows us to use the perturbation theory so successfully. We note that the effect of changing the Kasner regime already exists in simpler cosmological models than those of Bianchi IX and Bianchi VIII types. As a matter of fact, in a Bianchi II universe, there is only one type of perturbation connected with spatial curvature, and this perturbation makes one change in the Kasner regime (one bounce). This fact was known to Lifshitz and Khalatnikov in the early 1960s, and they discussed this topic with Landau (just before the tragic accident), who greatly appreciated it. The results describing the dynamics of the Bianchi IX model were reported by Khalatnikov in his talk given in January 1968 at the Henri Poincaré Seminar in Paris. John A Wheeler, who was present there, pointed out that the dynamics of the Bianchi IX universe represent a nontrivial example of a chaotic dynamical system. Later, Kip Thorn distributed a preprint with the text of this talk.

Returning to the rules governing the bouncing of the negative power of time from one direction to another, it can be shown that they can be conveniently expressed by means of parameterization (4):

$$p_l = p_1(u)\,, \quad p_m = p_2(u)\,, \quad p_n = p_3(u)\,, \tag{12}$$

and then

$$p'_l = p_2(u-1)\,, \quad p'_m = p_1(u-1)\,, \quad p'_n = p_3(u-1)\,. \tag{13}$$

The greater of the two positive powers remains positive.

Successive changes (13), accompanied by a bouncing of the negative power between the directions **l** and **m**, continue as long as the integral part of u is not exhausted, i.e., until u becomes less than unity. Then, according to Eqn (6), the value $u < 1$ transforms into $u > 1$; at this moment, either the exponent p_l or p_m is negative, and p_n becomes the smaller of the two positive numbers ($p_n = p_2$). The next sequence of changes bounces the negative power between the directions **n** and **l** or **n** and **m**. We emphasize that the Lifshitz–Khalatnikov parameter u is useful because it allows encoding the rather complicated laws of transitions between different Kasner regimes (11) in the simple rules $u \to u - 1$ and $u \to 1/u$.

Consequently, the evolution of our model toward a singular point consists of successive periods (called eras) in which expansions and contractions of scale factors along two axes oscillate while the scale factor along the third axis decreases monotonically, and the volume decreases according to a law that is nearly $\sim t$. In the transition from one era to another, the axes along which the distances decrease monotonically are interchanged. The order in which the pairs of axes are interchanged and the order in which the eras of different lengths follow each other acquire a stochastic character.

A decreasing sequence of values of the parameter u corresponds to every (sth) era. This sequence has the form $u^{(s)}_{\max}, u^{(s)}_{\max} - 1, \ldots, u^{(s)}_{\min}$, where $u^{(s)}_{\min} < 1$. We introduce the notation

$$u^{(s)}_{\min} = x^{(s)}, \quad u^{(s)}_{\max} = k^{(s)} + x^{(s)}, \tag{14}$$

i.e., $k^{(s)} = [u^{(s)}_{\max}]$ (the square brackets denote the greatest integer $\leqslant u^{(s)}_{\max}$). The number $k^{(s)}$ defines the era length. For the next era, we obtain

$$u^{(s+1)}_{\max} = \frac{1}{x^{(s)}}, \quad k^{(s+1)} = \left[\frac{1}{x^{(s)}}\right]. \tag{15}$$

The ordering with respect to the length of $k^{(s)}$ of the successive eras (measured by the number of Kasner epochs contained in them) acquires a stochastic character asymptotically. The random nature of this process arises because of rules (14) and (15), which define transitions from one era to another in the infinite sequence of values of u. If all these infinite sequences begin with some initial value $u^{(0)}_{\max} = k^{(0)} + x^{(0)}$, then the lengths of the sequence $k^{(0)}, k^{(1)} \ldots$ are numbers involved in the expansion of a continuous fraction:

$$k^{(0)} + x^{(0)} = k^{(0)} + \cfrac{1}{k^{(1)} + \cfrac{1}{k^{(2)} + \cdots}}. \tag{16}$$

We can describe this sequence of eras statistically if instead of a given initial value $u^{(0)}_{\max} = k^{(0)} + x^{(0)}$ we consider a distribution of $x^{(0)}$ over the interval $[0, 1]$ governed by some probability law. Then we also obtain some distributions of the values of $x^{(s)}$ that terminate every sth series of numbers. It can be shown that with increasing s, these distributions tend to a stationary (independent of s) probability distribution $w(x)$ in which the initial value $x^{(s)}$ is completely "forgotten":

$$w(x) = \frac{1}{(1+x)\ln 2}. \tag{17}$$

It follows from Eqn (17) that the probability distribution of the lengths of sequences k is given by

$$W(k) = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}. \tag{18}$$

Moreover, it is possible to exactly calculate the probability distributions for other parameters describing successive eras, such as the parameter δ giving the relation between the amplitudes of logarithms of the functions a, b, c and the logarithmic time [17]. We also briefly describe this further development of the description of cosmological stochasticity presented in [17].

First, it was noted that from a formal standpoint, we here have a deterministic dynamical model governed by a system of three ordinary differential equations (the space–space components of the Einstein equations) plus an additional condition (the time–time component of the Einstein equations): thus, the phase space of this system is actually not six- but five-dimensional. Therefore, apart from the actual profound cosmological significance of this system, we have encountered a specific model of spontaneous stochastization of a deterministic system.

It was then underlined that the transition from one Kasner era to another can be described by mapping the interval $[0, 1]$ into itself by the formula

$$Tx = \left\{\frac{1}{x}\right\}, \quad x_{s+1} = \left\{\frac{1}{x_s}\right\}, \tag{19}$$

where curly brackets are for the fractional part of a number. This transformation belongs to the so-called expanding transformations of the interval $[0, 1]$, i.e., transformations $x \sim f(x)$ with $|f'(x)| > 1$. Such transformations have the property of exponential instability: if we initially take two close points, their mutual distance increases exponentially under iterations of the transformation. It is well known that the exponential instability leads to the appearance of strong stochastic properties.

To study the stochastic properties of the transitions between the Kasner eras quantitatively, it is convenient to introduce some new notation. The logarithmic time is

$$\Omega = -\ln t. \tag{20}$$

The parameters α, β, and γ are the logarithms of the scale factors:

$$\alpha = \ln a, \quad \beta = \ln b, \quad \gamma = \ln c. \tag{21}$$

In what follows, we discuss the statistical properties of the sequence of eras. The index s numbers eras beginning from an arbitrarily chosen initial one ($s = 0$). The symbol Ω_s denotes the initial instant of the sth era (defined as the instant when the scale function, which monotonically decreases during the preceding era, begins to increase). The initial amplitudes of the pair from among the functions α, β, γ, that experiences oscillations in a given era, is denoted as $\delta_s \Omega_s$; the quantities δ_s (which lake values between 0 and 1) measure these amplitudes in units of the corresponding Ω_s. The recurrence formulas, which determine the rules of transition from one era to the next one, are

$$\frac{\Omega_{s+1}}{\Omega_s} = 1 + \delta_s k_s \left(k_s + x_s + \frac{1}{x_s}\right) \equiv \exp \xi_s, \tag{22}$$

$$\delta_{s+1} = 1 - \frac{(k_s/x_s + 1)\,\delta_s}{1 + \delta_s k_s (k_s + x_s + 1/x_s)}. \tag{23}$$

Iterating this formula gives

$$\frac{\Omega_s}{\Omega_0} = \exp \sum_{p=1}^{s} \xi_p. \tag{24}$$

The quantities δ_s have a stable stationary statistical distribution $P(\delta)$ and a stable mean (with small relative fluctuations). This distribution $P(\delta)$ can be found exactly by an analytic method. Because we are interested in statistical properties in the stationary limit, it is reasonable to introduce the so-called natural extension of transformation (19) by continuing it without bound to negative indices. In other words, we pass from a one-sided infinite sequence of numbers $(x_0, x_1, x_2, \ldots)$ connected by equalities (19) to a "doubly infinite" sequence $X = (\ldots, x_i, x_0, x_1, x_2, \ldots)$ of numbers that are connected by the same equalities for all $-\infty < s < \infty$.

A sequence X is equivalent to the sequence of integers $K = (\ldots, k_1, k_0, \ldots)$ constructed by the rule $k_s = [1/x_{s-1}]$. Conversely, every number of X is determined by the integers of K as an infinite continuous fraction

$$x_s = \cfrac{1}{k_{s+1} + \cfrac{1}{k_{s+2} + \cdots}} \equiv x_{s+1}^+ . \tag{25}$$

The definition of x_s^+ can be written as

$$x_s^+ = [k_s, k_{s+1}, \ldots] . \tag{26}$$

We also introduce quantities defined by a continuous fraction with a retrograde sequence of denominators (in the direction of decreasing indices):

$$x_s^- = [k_{s-1}, k_{s-2}, \ldots] . \tag{27}$$

We now transform recurrence relation (23) by introducing the notation $\eta_s = (1 - \delta_s)/\delta_s$. Then (23) can be rewritten as

$$\eta_{s+1} x_s = \frac{1}{\eta_s x_{s-1} + k_s} . \tag{28}$$

By iteration, we arrive at an infinite continuous fraction,

$$\eta_{s+1} x_s = [k_s, k_{s-1}, \ldots] = x_{s+1}^- . \tag{29}$$

Hence, $\eta_s = x_s^-/x_s^+$, and finally

$$\delta_s = \frac{x_s^+}{x_s^+ + x_s^-} . \tag{30}$$

This expression for δ_s contains only two (instead of three) random quantities, x_s^+ and x_s^-, each of which takes values in the interval [0, 1].

It follows from definition (27) that $1/x_{s+1}^- = x_s^- + k_s = x_s^- + [1/x_s^+]$. Hence, the shift of the entire sequence by one step to the right means a joint transformation of the x_s^+ and x_s^- in accordance with

$$x_{s+1}^+ = \left\{ \frac{1}{x_s^+} \right\} , \quad x_{s+1}^- = \frac{1}{\left[\dfrac{1}{x_s^+} \right] + x_s^-} . \tag{31}$$

This is a one-to-one map of the unit square. Thus, we now have a one-to-one transformation of two quantities instead of transformation (85) of one quantity that is not one-to-one. The quantities x_s^+ and x_s^- have a joint stationary distribution $P(x^+, x^-)$. Because (31) is a one-to-one transformation, the condition for the distribution to be stationary is expressed by the equation

$$P(x_s^+, x_s^-) = P(x_{s+1}^+, x_{s+1}^-)\, J , \tag{32}$$

where J is the Jacobian of the transformation. The normalized solution of this equation is

$$P(x^+, x^-) = \frac{1}{(1 + x^+ x^-)^2 \ln 2} , \tag{33}$$

and its integration over x^+ or x^- yields the function $w(x)$ in (17).

By (30), δ_s is expressed in terms of the random quantities x_s^+ and x_s^-, and therefore knowledge of their joint distribution allows calculating the statistical distribution $P(\delta)$ by integrating $P(x^+, x^-)$ over one of three variables at a constant value of δ.

Due to the symmetry of function (33) with respect to the variables x^+ and x^-, we have $P(\delta) = P(1 - \delta)$, i.e., the function $P(\delta)$ is symmetric with respect to the point $\delta = 1/2$. We have

$$P(\delta)\,\mathrm{d}\delta = \mathrm{d}\delta \int_0^1 P\left(x^+, \frac{x^+ \delta}{1 - \delta} \right) \left(\frac{\partial x^-}{\partial \delta} \right)_{x^+} \mathrm{d}x^+ . \tag{34}$$

Calculating this integral, we finally obtain

$$P(\delta) = \frac{1}{\big(|1 - 2\delta| + 1\big) \ln 2} . \tag{35}$$

The mean value $\langle \delta \rangle = 1/2$ is a result of the symmetry of the function $P(\delta)$. Thus, in every era, the mean value of the initial amplitude of oscillations of the functions α, β, γ increases as $\Omega/2$.

We thus see from the results of a statistical analysis of evolution in the neighborhood of a singularity [16, 17] that the stochasticity and probability distributions of parameters already arise in classical general relativity.

At the end of this section, a historical remark is in order. Continuous fraction (16) was shown in 1968 to I M Lifshitz (Landau had already passed away), and he immediately noticed that the formula for a stationary distribution of x, Eqn (17), can be derived. It later became known that this formula was derived in the nineteenth century by Gauss, who had not published it but had described it in a letter to one of his colleagues.

3. Stochastic Friedmann cosmology

In the preceding section, we discussed stochasticity in the Mixmaster Universe. However, simple isotropic closed Friedmann–Robertson–Walker models manifest some elements of chaotic behavior that should be taken into account to correctly construct quantum cosmological theories [35].

The study of the classical dynamics of a closed isotropic cosmological model has a long history. First, it was noticed that in such a model with a minimally coupled massive scalar field, there is the possibility of escaping a singularity in contraction [55, 56]. Then, the periodic trajectories escaping the singularity were studied [57]. In [32], it was argued that the set of infinitely bouncing aperiodical trajectories has a fractal nature. Later, this result was reproduced in other terms in our papers [33, 34, 58, 59].

Here, we briefly describe the approach presented in [33]. The main idea amounts to the fact that in the closed isotropic model with a minimally coupled massive scalar field, all the trajectories have a point of maximum expansion. The localization of the points of maximum expansion on the configuration plane (a, ϕ), where a is a cosmological radius and ϕ is a scalar field, can be found analytically. The trajectories can then be classified according to the localization of their points of maximum expansion. The domain of the points of maximum expansion is located inside the so-called Euclidean or "classically forbidden" region. Numerical investigations show that this area has a quasi-periodic structure, with zones corresponding to the fall into the singularity intermingled with zones containing points of maximum expansion of trajectories having the so-called "bounce" or points of minimum contraction. Studying the substructure of these zones from the standpoint of the possibility of having two bounces, we can see that on the qualitative level, this substructure repeats the structure of the whole region of possible points of maximum expansion. Continuing this procedure *ad infinitum*, we can see that, as a result, we have a fractal set of infinitely bouncing trajectories.

The same scheme allows us to see that there is also a set of periodic trajectories. All these periodic trajectories contain bounces intermingled with series of oscillations of the scalar field ϕ. It is important that there are no restrictions on the lengths of series of oscillations in this case. In [35], the topological entropy was calculated in this case and was shown to be positive. In [35], the calculations in Ref. [37] were reproduced, and it was shown how they can be generalized to more complicated cases. Here, we briefly review the results in [37].

We first write the action for the simplest cosmological model with a scalar field:

$$S=\int \mathrm{d}^4x\sqrt{-g}\left\{\frac{m_{\mathrm{P}}^2}{16\pi}\,(R-2\Lambda)+\frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi-\frac{1}{2}\,m^2\phi^2\right\}, \tag{36}$$

where m_P is the Planck mass and Λ is the cosmological constant. The equations of motion for a closed isotropic universe are

$$\frac{m_{\mathrm{P}}^2}{16\pi}\left(\ddot a+\frac{\dot a^2}{2a}+\frac{1}{2a}\right)+\frac{a\dot\phi^2}{8}-\frac{m^2\phi^2 a}{8}-\frac{m_{\mathrm{P}}^2}{8\pi}\,\Lambda a=0\,, \tag{37}$$

$$\ddot\phi+\frac{3\dot\phi\dot a}{a}+m^2\phi=0\,. \tag{38}$$

A first integral of motion of our system is given by

$$-\frac{3}{8\pi}\,m_{\mathrm{P}}^2(\dot a^2+1)+\frac{a^2}{2}\left(\dot\phi^2+m^2\phi^2+\frac{m_{\mathrm{P}}^2}{8\pi}\,\Lambda\right)=0\,. \tag{39}$$

When the cosmological constant is equal to zero, the form of the boundary of the Euclidean region is given by an equation that can be easily obtained from Eqn (39):

$$m^2a^2\phi^2=\frac{3}{4\pi}\,m_{\mathrm{P}}^2\,. \tag{40}$$

In this model, investigated in many papers [32, 33, 35, 57], there are periodic trajectories with an arbitrary number of oscillations of the scalar field.

The inclusion of a positive cosmological constant, investigated in [34], gives rise to two possibilites: if Λ is small compared with the mass of the scalar field m, the qualitative behavior is the same as in the model without the cosmological constant; if the cosmological constant is of the order of m^2, the chaotic dynamics disappears in a jump-like manner [34].

If we include hydrodynamic matter with the equation of state

$$p=\gamma\epsilon\,, \tag{41}$$

where p is the pressure, ϵ is the energy density, and γ is a constant ($\gamma=0$ corresponds to dust matter, $\gamma=1/3$ corresponds to radiation, and $\gamma=1$ describes a massless scalar field), the form of the boundary of the Euclidean region is given by the equation

$$m^2a^2\phi^2=\frac{3}{4\pi}\,m_{\mathrm{P}}^2-\frac{D}{a^q}\,, \tag{42}$$

where D is a constant characterizing the quantity of the given type of matter in the universe and $q=3(\gamma+1)-2$. Numerical calculations show that in this case, again, only a restricted number of oscillations is possible. Moreover, the structure of periodic trajectories is much more complicated. Indeed, the law is as follows: a long series of oscillations can occur after a bounce followed by a short series of oscillations, while the behavior of a trajectory after a short series of oscillations is less restricted. The concrete laws governing the structure of trajectories depend on the parameters of the model under consideration. However, in this case, it is also possible to calculate the topological entropy, which we demonstrate below.

We calculate the topological entropy for the considered cases. Topological entropy measures the increase in the number of periodic orbits as their period increases. We define $N(k)$ as the number of periodic orbits of length k. The topological entropy is defined as

$$H_{\mathrm{T}}=\lim_{k\to\infty}\frac{1}{k}\ln N(k)\,. \tag{43}$$

If $H_{\mathrm{T}}>0$, it can be concluded that the dynamics are chaotic.

We can quantify the length of an orbit by the number of symbols. We first reproduce the calculations in Ref. [35] in some detail. It was suggested that the discrete coding of orbits in terms of two symbols be used:

A for a bounce of the trajectory,

B for a crossing of the line $\phi=0$.

For the simplest model with the scalar field, there is a single exclusion rule: two letters A cannot stay together, which means that it is impossible to have two bounces, one after another, without oscillations between them.

We let $Q(k)$ denote the number of "words" (trajectories) of length k satisfying this rule that begin with A and end with A and let $P(k)$ denote the number of words that begin with A and end with B. We can then easily write the recurrence relations

$$Q(k+1)=P(k);$$
$$P(k+1)=Q(k)+P(k)\,. \tag{44}$$

From Eqns (44), it is easy to deduce the following relation for $P(k)$:

$$P(k+1)=P(k)+P(k-1)\,. \tag{45}$$

It can easily be calculated that

$$P(2)=1,\quad P(3)=1\,, \tag{46}$$

and that Eqn (45) defines the series of Fibonacci numbers. We recall how to find the formula for the general term of the Fibonacci series. We seek $P(k)$ as a linear combination of terms λ^k, where λ is the solution of equation

$$\lambda^{k+1} = \lambda^k + \lambda^{k-1}$$

or, equivalently, because we are interested in only the nonzero roots,

$$\lambda^2 - \lambda - 1 = 0\,, \tag{47}$$

Thus, we seek $P(k)$ in the form

$$P(k) = c_1\lambda_1^k + c_2\lambda_2^k\,, \tag{48}$$

where λ_1 and λ_2 are the roots of Eqn (47) and, solving conditions (46), we obtain

$$P(k)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}+(-1)^{k-2}\left(\frac{\sqrt{5}-1}{2}\right)^{k-1}\right]. \tag{49}$$

Substituting (49) in the definition of topological entropy (43), gives

$$H_{\mathrm{T}} = \ln\left(\frac{1+\sqrt{5}}{2}\right) > 0\,, \tag{50}$$

where $(1+\sqrt{5})/2$ is the famous golden ratio. It is clear that only the largest root of Eqn (47) is essential for the calculation of the topological entropy.

Now, we can pass to the more involved case of the cosmological model with the scalar field and a positive cosmological constant. As was described above, the periodic trajectories in this model can have only a restricted number of oscillations of the scalar field ϕ. This rule can be encoded in forbidding more than n letters B to stay together, where the number n depends on the parameters of the model.

The recurrence relations now become

$$Q(k+1) = P(k)\,,$$

$$P(k+1) = Q(k) + P(k) - Q(k-n)\,\theta(k-n)\,, \tag{51}$$

where the θ function is defined in the usual manner. We are interested in the limit $k \to \infty$ and can substitute the number $\theta(k-n)$ instead of 1. We can now write the recurrence relation

$$P(k+1) = P(k) + P(k-1) - P(k-n-1)\,, \tag{52}$$

which in turn implies the following equation for the topological entropy:

$$\lambda^{n+2} - \lambda^{n+1} - \lambda^n + 1 = 0\,, \tag{53}$$

with the topological entropy being equal to the logarithm of the largest root:

$$H_{\mathrm{T}} = \ln\lambda\,. \tag{54}$$

For small values of n, the largest root of Eqn (53) can be found analytically:

— For $n = 1$, we have $\lambda = 1$, the topological entropy is equal to zero, and chaotic behavior is absent, which is clear from the physical standpoint;

— For $n = 2$, we obtain

$$\lambda = \frac{1}{3}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{1/3} + \frac{\left(1/2\,(9+\sqrt{69})\right)^{1/2}}{3^{2/3}} \approx 1.32\,; \tag{55}$$

— For $n = 3$,

$$\lambda = \frac{1}{3} + \frac{1}{3}\left(\frac{29}{2} - \frac{3\sqrt{93}}{2}\right)^{1/3} + \frac{1}{3}\left(\frac{29}{2} + \frac{3\sqrt{93}}{2}\right)^{1/2} \approx 1.47\,. \tag{56}$$

For higher values of n, we can find λ numerically, for example, for $n = 4$, $\lambda \approx 1.53$.

For large values of n, we can find an asymptotic value for the largest root λ:

$$\lambda = \frac{1+\sqrt{5}}{2} - \frac{1}{\sqrt{5}\{(1+\sqrt{5})/2\}^n}\,. \tag{57}$$

As has already been mentioned, in the model with a scalar field and matter or in the model with a complex scalar field and nonzero classical charge [60–64], the rules governing the structure are rather complicated. Nevertheless, in this case it is also possible to calculate the topological entropy. Here, we consider one particular example, but the algorithm that we present can be used for different sets of rules as well.

We formulate the rules for our model:

1. It is impossible to have more than 19 letters B.
2. After a series with 19 letters B (and the letter A), we can have the next series only with one letter B.
3. After a series with 18 letters B, we can have the next series with one or two letters B.
4. After a series with 17 letters B, we can have the next series with one, two, or three letters B.

. . .

5. After a series with 1 letter B, we can have a series with n letters B, where $0 \leqslant n \leqslant 19$.

We note that the system of rules has a remarkable symmetry with respect to the number $n_{\mathrm{C}} = 10$, which simplifies the calculations. These symmetric rules give us a good approximation for the description of the real physical situation. The value n_{C} is apparently a function of the parameters of the model under investigation. A more detailed numerical investigation implies a more complicated system of rules; however, it can also be formalized in a system of recurrence relations. Below, we see that the symmetric system of rules gives a rather cumbersome equation for the topological entropy.

We now introduce the following notation:

$Q(k)$ is the number of words that begin with the letter A and end with the letter A,

$Q_1(k)$ is the number of words that begin with the letter A and end with a series of one letter B,

$Q_2(k)$ is the number of words that begin with the letter A and end with a series of two letters B,

. . .,

$Q_{19}(k)$ is the number of words that begin with the letter A and end with a series of 19 letters B.

The system of recurrence relations for these quantities is

$$Q(k+1) = Q_1(k) + Q_2(k) + \cdots + Q_{19}(k)\,,$$

$$Q_1(k) = Q(k-1)\,,$$

$$Q_{19}(k) = Q_1(k-20)\,,$$

$$Q_d(k) = Q_{d-1}(k-1) - Q_{21-d}(k-d-1)\,,\ 2 \leqslant d \leqslant 10\,,$$

$$Q_d(k) = Q_{d+1}(k+1) + Q_{20-d}(k-d-1)\,,\ 11 \leqslant d \leqslant 18\,. \tag{58}$$

Solving this system for $Q(k)$, we obtain the recurrence relation:

$$
\begin{aligned}
Q(k+1) = {} & Q(k-1)+Q(k-2)+Q(k-3)+Q(k-4) \\
& + Q(k-5)+Q(k-6)+Q(k-7)+Q(k-8) \\
& + Q(k-9)+Q(k-10)+Q(k-13)+2Q(k-14) \\
& + 3Q(k-15)+4Q(k-16)+5Q(k-17)+6Q(k-18) \\
& + 7Q(k-19)+8Q(k-20)+9Q(k-21)-9Q(k-24) \\
& - 16Q(k-25)-21Q(k-26)-24Q(k-27) \\
& - 25Q(k-28)-24Q(k-29)-21Q(k-30) \\
& - 16Q(k-31)-9Q(k-32)-8Q(k-36) \\
& - 21Q(k-37)-36Q(k-38)-50Q(k-39) \\
& - 60Q(k-40)-63Q(k-41)-56Q(k-42) \\
& - 36Q(k-43)+36Q(k-47)+84Q(k-48) \\
& + 126Q(k-49)+150Q(k-50)+150Q(k-51) \\
& + 126Q(k-52)+84Q(k-53)+36Q(k-54) \\
& + 28Q(k-59)+84Q(k-60)+150Q(k-61) \\
& + 200Q(k-62)+210Q(k-63)+168Q(k-64) \\
& + 84Q(k-65)-84Q(k-70)-224Q(k-71) \\
& - 350Q(k-72)-400Q(k-73)-350Q(k-74) \\
& - 224Q(k-75)-84Q(k-76)-56Q(k-82) \\
& - 175Q(k-83)-300Q(k-84)-350Q(k-85) \\
& - 280Q(k-86)-126Q(k-87)+126Q(k-93) \\
& + 350Q(k-94)+525Q(k-95)+525Q(k-96) \\
& + 350Q(k-97)+126Q(k-98)+70Q(k-105) \\
& + 210Q(k-106)+315Q(k-107)+280Q(k-108) \\
& + 126Q(k-109)-126Q(k-116)-336Q(k-117) \\
& - 441Q(k-118)-336Q(k-119)-126Q(k-120) \\
& - 56Q(k-128)-147Q(k-129)-168Q(k-130) \\
& - 84Q(k-131)+84Q(k-139)+196Q(k-140) \\
& + 196Q(k-141)+84Q(k-142)+28Q(k-151) \\
& + 56Q(k-152)+36Q(k-153)-36Q(k-162) \\
& - 64Q(k-163)-36Q(k-164)-8Q(k-174) \\
& - 9Q(k-175)+9Q(k-185)+9Q(k-186) \\
& + Q(k-197)-Q(k-208)\,,
\end{aligned} \tag{59}
$$

which, in turn, give the following equation for the topological entropy:

$$
\begin{aligned}
& x^{209}-x^{207}-x^{206}-x^{205}-x^{204}-x^{203}-x^{202}-x^{201} \\
& - x^{200}-x^{199}-x^{198}-x^{195}-2x^{194}-3x^{193} \\
& - 4x^{192}-5x^{191}-6x^{190}-7x^{189}-8x^{188}-9x^{187} \\
& + 9x^{184}+16x^{183}+21x^{182}+24x^{181}+25x^{180} \\
& + 24x^{179}+21x^{178}+16x^{177}+9x^{176}+8x^{172} \\
& + 21x^{171}+36x^{170}+50x^{169}+60x^{168}+63x^{167} \\
& + 56x^{166}+36x^{165}-36x^{161}-84x^{160}-126x^{159} \\
& - 150x^{158}-150x^{157}-126x^{156}-84x^{155}-36x^{154} \\
& - 28x^{149}-84x^{148}-150x^{147}-200x^{146}-210x^{145} \\
& - 168x^{144}-84x^{143}+84x^{142}+224x^{137} \\
& + 350x^{136}+400x^{135}+350x^{134}+224x^{133} \\
& + 84x^{132}+56x^{126}+175x^{125}+300x^{124}+350x^{123} \\
& + 280x^{122}+126x^{121}-126x^{115}-350x^{114} \\
& - 525x^{113}-525x^{112}-350x^{111}-126x^{110} \\
& - 70x^{103}-210x^{102}-315x^{101}-280x^{100}-126x^{99} \\
& + 126x^{92}+336x^{91}+441x^{90}+336x^{89}+126x^{88} \\
& + 56x^{80}+147x^{79}+168x^{78}+84x^{77}-84x^{69} \\
& - 196x^{68}-196x^{67}-84x^{66}-28x^{57} \\
& - 56x^{56}-36x^{55}+36x^{46}+64x^{45}+36x^{44} \\
& + 8x^{34}+9x^{33}-9x^{23}-9x^{22}-x^{11}+1=0\,.
\end{aligned} \tag{60}
$$

Solving Eqn (60) numerically, we can find the largest root, which is equal to $\lambda \approx 1.61771$.

Accordingly, the topological entropy is given by the logarithm of the largest root.

The scheme described above can be applied to many different physical models, obeying different sets of exclusion rules governing the structure of periodic trajectories.

4. Quasi-isotropic expansion in cosmology

The quasi-isotropic solution of the Einstein equations near a cosmological singularity was found by Lifshitz and Khalatnikov [42] for a universe filled with radiation with the equation of state $p = \varepsilon/3$ in the early 1960s. In [43], a generalization was presented of the quasi-isotropic solution of the Einstein equations near a cosmological singularity to the case of an arbitrary one-fluid cosmological model. This solution was then generalized further to the case of a universe filled with two ideal barotropic fluids [44, 45].

To explain the physical meaning of the quasi-isotropic solution, we recall that it represents the most generic spatially inhomogeneous generalization of the Friedmann spacetime: the spacetime is locally Friedmann-like near the cosmological singularity $t = 0$ (in particular, its Weyl tensor is much less than its Riemann tensor). On the other hand, generically it is very inhomogeneous globally and may have a very complicated spatial topology. As was shown in [42, 43] (see also [65, 66]), such a solution contains three arbitrary functions of spatial coordinates. From the Friedmann–Robertson–Walker (FRW) standpoint, these three degrees of freedom represent the increasing (nondecreasing in terms of metric perturbations) mode of adiabatic perturbations and the nondecreasing mode of gravitational waves (with two polarizations) when deviations of the spacetime metric from the FRW one are not small. Hence, the quasi-isotropic solution is not a general solution of the Einstein equations with a barotropic fluid. Therefore, we should not expect this solution to arise in the course of generic gravitational collapse (in particular, inside a black-hole event horizon). The generic solution near a space-like curvature singularity (for $p < \varepsilon$) has a completely different structure consisting of an infinite sequence of anisotropic vacuum Kasner-like eras with space-dependent Kasner exponents (see Section 2).

For this reason, the quasi-isotropic solution did not attract much interest for about twenty years. Its new life began after the development of successful inflationary models (i.e., with the 'graceful exit' from inflation) and the theory of the generation of perturbations during inflation, because it immediately became clear that generically (without a fine tuning of the initial conditions), the scalar metric perturbations after the end of inflation remained small in a finite region of space that was much less than the whole causally connected spatial volume produced by inflation. It appears that the quasi-isotropic solution can be used for a global description of a part of spacetime after inflation, which belongs to "one post-inflationary universe." This is defined as a connected part of spacetime, where the hypersurface $t = t_f(\mathbf{r})$ describing the instant when inflation ends is spacelike and can therefore be made the surface of constant (zero) synchronous time by a coordinate transformation. This directly follows from the derivation of perturbations generated during inflation given in [67] [see Eqn (177) in that paper], which is also valid in the case of large perturbations. When used in this context, the quasi-isotropic solution represents an *intermediate* asymptotic regime during the expansion of the universe after inflation. The synchronous time t appearing in it is the proper time since the *end* of inflation, and the region of validity of the solution is from $t = 0$ up to an instant in the future when spatial gradients become important. For sufficiently large scales, this instant may be rather late, even around or more than the present age of the Universe. We also note an analogue of the quasi-isotropic solution *before* the end of inflation, the generic quasi-de Sitter solution found in [68]. Both solutions can be smoothly matched across the hypersurface of the end of inflation.

A slightly different version of quasi-isotropic expansion has been developed in recent decades, which is known under the names of long-wave expansion or gradient expansion [69–74].

Originally, quasi-isotropic expansion was developed as a technique to generate some kind of perturbative expansion in the vicinity of the cosmological singularity, where the cosmic time parameter served as a perturbative one. However, a more general treatment of the quasi-isotropic expansion is possible if we note that the next order of the quasi-isotropic expansion contains higher orders of the spatial derivatives of the metric coefficients. Thus, it is possible to construct a natural generalization of the quasi-isotropic solution of the Einstein equations that would be valid not only in the vicinity of the cosmological singularity but also in the full time range. In this case, the simple algebraic equations that have to be solved for higher orders of the quasi-isotropic approximation in the vicinity of the singularity are substituted by differential equations, where the time dependence of the metric can be rather complicated, in contrast to the power-law behavior of the coefficients of the original quasi-isotropic expansion.

We now show how this works for a universe filled with one barotropic fluid [45]. We consider a universe with a fluid with the equation of state $p = w\varepsilon$. The spatial metric is now

$$\gamma_{\alpha\beta} = a_{\alpha\beta} t^{\kappa} + c_{\alpha\beta}\,, \tag{61}$$

where

$$\kappa = \frac{4}{3(1+w)}\,. \tag{62}$$

The inverse metric is

$$\gamma^{\alpha\beta} = a^{\alpha\beta} t^{-\kappa} - c^{\alpha\beta} t^{-2\kappa}\,. \tag{63}$$

We then have the following formulas for the extrinsic curvature:

$$K_{\alpha\beta} = a_{\alpha\beta}\kappa t^{\kappa-1} + \dot{c}_{\alpha\beta}\,, \tag{64}$$

$$K_{\alpha}^{\beta} = \delta_{\alpha}^{\beta}\kappa + \dot{c}_{\alpha}^{\beta} t^{-\kappa} - c_{\alpha}^{\beta}\kappa t^{-\kappa-1}\,, \tag{65}$$

$$K = \frac{3\kappa}{t} + \dot{c} t^{-\kappa} - c\kappa t^{-\kappa-1}\,, \tag{66}$$

$$\frac{\partial K}{\partial t} = -\frac{3\kappa}{t^2} + \ddot{c} t^{-\kappa} - 2\dot{c}\kappa t^{-\kappa-1} + c\kappa(\kappa+1) t^{-\kappa-2}\,, \tag{67}$$

$$K_{\alpha}^{\beta} K_{\beta}^{\alpha} = \frac{3\kappa^2}{t^2} + 2\dot{c}\kappa t^{-\kappa-1} - 2c\kappa^2 t^{-\kappa-2}\,. \tag{68}$$

Using formulas (67) and (68), we have

$$R_0^0 = \frac{3\kappa(2-\kappa)}{4t^2} - \frac{\ddot{c} t^{-\kappa}}{2} + \frac{\dot{c}\kappa t^{-\kappa-1}}{2} - \frac{c\kappa t^{-\kappa-2}}{2}\,. \tag{69}$$

Now, using the Einstein equation in the lowest order of the approximation, we obtain the energy density of the fluid under consideration in the form

$$\varepsilon^{(0)} = \frac{4}{3(1+w)^2 t^2}\,. \tag{70}$$

Using the zeroth component of the energy–momentum conservation law, we can find a relation between the first correction to the energy density $\varepsilon^{(1)}$ and the trace of the first correction to the metric c:

$$\varepsilon^{(1)} = -\frac{c\kappa t^{-\kappa-2}}{2}\,. \tag{71}$$

Now, using the standard expression for the scalar curvature R and the 00 component of the Einstein equation in the form $R_0^0 - (1/2)\,R = T_0^0$, in the first quasi-isotropic order, we obtain the equation

$$\frac{P}{2} + \frac{1}{4}\,K^{(0)}K^{(1)} - \frac{1}{8}\,(K_{\alpha}^{\beta}K_{\beta}^{\alpha})^{(1)} = \varepsilon^{(0)}\,. \tag{72}$$

Combining (72) and (71), we obtain a differential equation for c,

$$\dot{c} = \frac{c(\kappa-1)}{t} - \frac{\bar{P}t}{\kappa}\,. \tag{73}$$

Integrating (73) gives

$$c = \frac{\bar{P}t^2}{\kappa(\kappa-3)} = -\frac{9(1+w)^2\bar{O}t^2}{4(5+9w)}\,. \tag{74}$$

Now, to find the traceless part of the first correction to the metric $\tilde{c}_{\alpha\beta}$, we use the traceless part of the spatial–spatial component of the Einstein equations, which in the case of one fluid and in the first-order approximation has a particularly simple form:

$$\tilde{R}_{\alpha}^{\beta} = -\tilde{P}_{\alpha}^{\beta} - \frac{1}{2}\frac{\partial\tilde{K}_{\alpha}^{\beta}}{\partial t} - \frac{3\kappa}{4t}\,\tilde{K}_{\alpha}^{\beta} = 0\,. \tag{75}$$

(We note that the traceless part of the extrinsic curvature $\tilde{K}_{\alpha}^{\beta}$ does not have zeroth-order terms). Equation (75) can be rewritten as

$$\frac{\partial\tilde{K}_{\alpha}^{\beta}}{\partial t} + \frac{3\kappa}{2t}\,\tilde{K}_{\alpha}^{\beta} = -2\tilde{\bar{P}}_{\alpha}^{\beta} t^{-\kappa}\,. \tag{76}$$

Integrating (76), we obtain

$$\tilde{K}_{\alpha}^{\beta} = -\frac{4}{\kappa+2}\,\tilde{\bar{P}}_{\alpha}^{\beta} t^{-\kappa+1}\,. \tag{77}$$

Using the relation

$$\tilde{c}_{\alpha}^{\beta} = t^{\kappa} \int \tilde{K}_{\alpha}^{\beta} \, \mathrm{d}t \,, \tag{78}$$

we arrive at

$$\tilde{c}_{\alpha}^{\beta} = \frac{4}{\kappa^2 - 4} \, \tilde{P}_{\alpha}^{\beta} t^2 = -\frac{9(1+w)^2}{(3w+1)(3w+5)} \, \tilde{P}_{\alpha}^{\beta} t^2 \,. \tag{79}$$

We can see that results (74) and (79), valid in the full range of time, coincide with those valid in the vicinity of the initial cosmological singularity ($t = 0$) [43] obtained by the algebraic method [42]. The general expression for the first correction to the metric for the one-fluid case is given in formula (37) in [43]. The metric $b_{\alpha\beta}$ in [43] corresponds to $c_{\alpha\beta}$ in this paper, while for the equation of state parameter, the symbol k is used instead of w. Formula (37) contains a misprint: in front of the second term in the brackets in the right-hand side of this equation, the factor $1/4$ should stay. At first glance, the first correction to metric (37) contains a pole at $3k + 1 = 0$; however, calculating the trace of this metric, it can be seen that this pole is canceled and is present only in its anisotropic part.

Thus, for a universe filled with string gas, $w = -1/3$, the quasi-isotropic expansion does not work because the expression for $\tilde{c}_{\alpha\beta}$ becomes singular.

To conclude, we consider a special case where the metric $a_{\alpha\beta}$ has a conformally flat form:

$$a_{\alpha\beta} = \exp\left[\rho(x)\right] \delta_{\alpha\beta} \,. \tag{80}$$

In this case, the spatial Ricci tensor is

$$\bar{P}_{\alpha\beta} = \frac{1}{4}\,(\rho_{,\alpha}\rho_{\beta} - 2\rho_{,\alpha\beta}) - \frac{1}{4}\,\delta_{\alpha\beta}(\rho_{,}^{\;\mu}\rho_{,\mu} + 2\rho_{,\mu}^{\;\mu}) \tag{81}$$

or

$$\bar{P}_{\alpha}^{\beta} = \frac{1}{4}\,(\rho_{,\alpha}\rho_{,}^{\;\beta} - 2\rho_{,\alpha}^{\;\beta}) - \frac{1}{4}\,\delta_{\alpha}^{\beta}(\rho_{,}^{\;\mu}\rho_{,}^{\;\mu} + 2\rho_{,\mu}^{\;\mu}) \,. \tag{82}$$

Hence,

$$\bar{P} = -2\rho_{,\mu}^{\;\mu} - \frac{1}{2}\,\rho_{,\mu}^{\;\mu} \,, \tag{83}$$

and the traceless part of the Ricci tensor is

$$\tilde{\bar{P}}_{\alpha}^{\beta} = \frac{1}{4}\,(\rho_{,\alpha}\rho_{,}^{\;\beta} - 2\rho_{,\alpha}^{\;\beta}) + \frac{1}{12}\,\delta_{\alpha}^{\beta}(2\rho_{,\mu}^{\;\mu} - \rho_{,\mu}\rho_{,}^{\;\mu}) \,. \tag{84}$$

If

$$\rho = A_{\mu\nu} x^{\mu} x^{\nu} \,, \tag{85}$$

then

$$\bar{P} = -4A_{\mu}^{\mu} - 2A_{\mu\nu}A_{\alpha}^{\mu} x^{\nu} x^{\alpha} \,, \tag{86}$$

$$\tilde{\bar{P}}_{\alpha}^{\beta} = \frac{1}{3}\, x^{\gamma} x^{\nu} (3A_{\alpha\gamma}A_{\nu}^{\beta} - \delta_{\alpha}^{\beta} A_{\mu\gamma}A_{\nu}^{\mu}) \,. \tag{87}$$

Thus, it is easy to see that if the metric in the lowest order of the quasi-isotropic expansion has the Gaussian form determined by Eqns (80) and (85), then its first correction $c_{\alpha\beta}$ determined by the curvature tensors (86) and (87) has a non-Gaussian form due to the presence of the terms quadratic in x^{μ} in front of the Gaussian exponential.

5. Quantum field theory and quantum gravity at a Lifshitz point

It is generally recognized that the complete theory of elementary particles and fundamental interactions should include quantum gravity. However, the quantum gravity theory is nonrenormalizable (see, e.g., [75]). The main obstacle in trying to achieve perturbative renormalizability of general relativity in $3+1$ dimensions is the fact that the gravitational coupling constant (the Newton constant) is dimensional, with the negative dimension $[G_{\mathrm{N}}] = -2$ in mass units. The graviton propagator, like all propagators in quantum field theory, scales with the four-momentum $k_{\mu} = (\omega, \mathbf{k})$ as

$$\frac{1}{k^2} \,, \tag{88}$$

where $k = \sqrt{\omega^2 - \mathbf{k}^2}$. When we calculate the Feynman diagrams with an increasing number of loops, it is necessary to introduce more and more counterterms of an increasing degree in curvature.

An improved ultraviolet behavior of the theory can be obtained if some terms of higher orders in curvature are added to the Lagrangian. The terms quadratic in curvature not only yield new interactions (with a dimensionless coupling) but also modify the propagator. Omitting the tensor structure of this propagator, we can write it as

$$\frac{1}{k^2} + \frac{1}{k^2}\, G_{\mathrm{N}} k^4 \frac{1}{k^2} + \frac{1}{k^2}\, G_{\mathrm{N}} k^4 \frac{1}{k^2}\, G_{\mathrm{N}} k^4 \frac{1}{k^2} + \cdots = \frac{1}{k^2 - G_{\mathrm{N}} k^4} \,. \tag{89}$$

It is easy to see that at high energies, the propagator is dominated by the $1/k^4$ term. This solves the problem of ultraviolet divergences. However, a new problem arises: the resummed propagator (89) has two poles:

$$\frac{1}{k^2 - G_{\mathrm{N}} k^4} = \frac{1}{k^2} - \frac{1}{k^2 - 1/G_{\mathrm{N}}} \,. \tag{90}$$

One of these poles describes massless gravitons, while the other corresponds to ghost excitations and implies violations of unitarity.

Some years ago, Hořava introduced a new class of gravity models [50], which he called "Quantum gravity at a Lifshitz point". The creation of this approach was inspired by the papers by Lifshitz [51, 52] devoted to the theory of second-order phase transitions and of critical phenomena, published in 1941. The Hořava–Lifshitz gravity models exhibit scaling properties that are anisotropic between space and time. The degree of anisotropy between space and time is measured by the dynamical critical exponent z such that

$$\mathbf{x} \to b\mathbf{x} \,, \quad t \to b^{z} t \,. \tag{91}$$

Such an anisotropic scaling is common in condensed matter systems. The prototype of the class of the condensed matter models of this family is the theory of a Lifshitz scalar in $D+1$ dimensions, whose action is

$$S = \int \mathrm{d}t \, \mathrm{d}^{D} x \left\{ (\dot{\Phi})^2 - (\triangle \Phi)^2 \right\} , \tag{92}$$

where $\triangle$ is the spatial Laplacian. Here, the critical exponent is $z = 2$. To the action, we can add the term

$$-c^2 \int \mathrm{d}t\,\mathrm{d}^D x\,\partial_i \Phi\,\partial_i \Phi\,, \tag{93}$$

where we explicitly introduce the speed of light c. Under the influence of this deformation, the theory flows to the infrared region $z = 1$, with the Lorentz invariance emerging at long distances.

In the approach to quantum gravity suggested in [50], actions were considered such that scaling at short distances exhibited a strong anisotropy between space and time, with $z > 1$. This improves the short-distance behavior of the theory. Indeed, the propagator for such gravitons is proportional to

$$\frac{1}{\omega^2 - c^2\mathbf{k}^2 - G(\mathbf{k}^2)^z}\,. \tag{94}$$

At high energies, the propagator is dominated by the anisotropic term $1/(\omega^2 - G(\mathbf{k}^2)^z)$. For a suitably chosen z, this modification improves the short-distance behavior and the theory becomes power-counting renormalizable. The $c^2\mathbf{k}^2$ term becomes important at low energies, where the theory naturally flows to $z = 1$.

Unlike in relativistic higher-derivative theories mentioned above, higher-order time derivatives are not generated, and the problem with ghost excitations and nonunitarity is resolved. In $3+1$ gravity, renormalizability is achieved by the choice $z = 3$.

We now consider the form of the Lagrangian of the gravitational theory for symmetry (91). As usual, we introduce the $D+1$ split of the manifold under consideration with the spatial metric tensor g_{ij} and the lapse functions N and shift functions N_i [76]. We wish to construct the kinetic term of the Lagrangian that is quadratic in time derivatives $\dot g_{ij}$ and is invariant under the foliation-preserving diffeomorphisms, i.e., diffeomorphisms that respect the $D+1$ foliation. This kinetic term depends on the second fundamental form (extrinsic curvature):

$$K_{ij} = \frac{1}{2N}\,(\dot g_{ij} - \nabla_i N_j - \nabla_j N_i)\,, \tag{95}$$

where ∇_i is the covariant derivative involving the spatial metric. Then the kinetic part of the action is

$$S_K = \frac{2}{\kappa^2}\int \mathrm{d}t\,\mathrm{d}^D x\sqrt{g}\,N(K_{ij}K^{ij} - \lambda K^2)\,. \tag{96}$$

In general relativity, the requirement of invariance under all spacetime diffeomorphisms implies that $\lambda = 1$.

The potential term of the action should include only the spatial metric and its spatial derivatives, and it has the form

$$S_V = \int \mathrm{d}t\,\mathrm{d}^D x\sqrt{g}\,NV[g_{ij}]\,. \tag{97}$$

Considering the high-energy regime of the theory, we concentrate on the terms that have the dimension of the kinetic term. In the case $D = 3$ and $z = 3$, there are many terms of this type, some of which are quadratic in curvature,

$$\nabla_k R_{ij}\nabla^k R^{ij}, \quad \nabla_k R_{ij}\nabla^i R^{jk}, \quad R\triangle R\,, \quad R^{ij}\triangle R_{ij}\,, \tag{98}$$

and they modify the propagator. Other terms, such as

$$R^3\,, \quad R^i_j R^j_k R^k_i\,, \quad RR_{ij}R^{ij}\,, \tag{99}$$

are cubic in curvature and represent pure interacting terms. The list of independent operators is very large, implying a proliferation of coupling constants. In order to reduce the number of independent coupling constants, it is necessary to impose an additional symmetry on the theory. The way in which this restriction is implemented is very similar to that is used in the theory of critical phenomena.

We require the potential term to be of the special form

$$S_V = \frac{\kappa^2}{8}\int \mathrm{d}t\,\mathrm{d}^D x\sqrt{g}\,NE^{ij}G_{ijkl}E^{kl}\,, \tag{100}$$

where the tensor E^{ij} itself follows from a variation,

$$\sqrt{g}\,E^{ij} = \frac{\delta W[g_{kl}]}{\delta g_{ij}}\,, \tag{101}$$

of some action W, while the tensor G_{ijkl} is the inverse of the generalized DeWitt supermetric

$$G^{ijkl} = \frac{1}{2}\,(g^{ik}g^{jl} + g^{il}g^{jk}) - \lambda g^{ij}g^{kl}\,. \tag{102}$$

The theories whose potential is of form (100) for some W are said to satisfy the "detailed balance condition". Systems that satisfy the detailed balance condition have a simpler quantum behavior than general systems. Their renormalization can be reduced to the simpler renormalization of the associated theory described by W.

We are interested in constructing a theory that satisfies the detailed balance condition and exhibits short-distance scaling with $z = 3$, leading to power-counting renormalizability in $3+1$ dimensions. Therefore, E^{ij} must be of third order in spatial derivatives. It turns out that there is a unique tensor with the necessary properties, the Cotton tensor:

$$C^{ij} = \varepsilon^{ikl}\nabla_k\left(R^j_l - \frac{1}{4}\,R\delta^j_l\right), \tag{103}$$

which is a variation of the action

$$W = \int \mathrm{d}^3 x\varepsilon^{ijk}\left(\Gamma^m_{il}\partial_j\Gamma^l_{km} + \frac{2}{3}\,\Gamma^n_{il}\Gamma^l_{jm}\Gamma^m_{kn}\right). \tag{104}$$

We can now write the full action for $z = 3$ gravity theory in $3+1$ dimensions:

$$\begin{aligned} S &= \int \mathrm{d}t\,\mathrm{d}^3 x\sqrt{g}\,N\left\{\frac{2}{\kappa^2}\,(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4}\,C_{ij}C^{ij}\right\} \\ &= \int \mathrm{d}t\,\mathrm{d}^3 x\sqrt{g}\,N\left\{\frac{2}{\kappa^2}\,(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4}\right. \\ &\quad\left.\times\left(\nabla_i R_{jk}\nabla^i R^{jk} - \nabla_i R_{jk}\nabla^j R^{ik} - \frac{1}{8}\,\nabla_i R\nabla^i R\right)\right\}. \end{aligned} \tag{105}$$

This action depends on three constants (κ, w, and λ), and some relevant terms providing the correct infrared limit of the theory can be added to it. It has been shown that at the special values $\lambda = 1/3$ or $\lambda = 1$, the metric field has two degrees of freedom, just like the graviton in general relativity.

The relevant terms can respect the detailed balance condition. To the action W, we can add the term

$$\mu \int \mathrm{d}^3 x \sqrt{g}\,(R - 2\Lambda_W)\,. \tag{106}$$

Then the action becomes

$$S = \int \mathrm{d}t\, \mathrm{d}^3 x \sqrt{g}\, N \left\{ \frac{2}{\kappa^2}\,(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4}\, C_{ij}C^{ij} + \frac{\kappa^2 \mu}{2w^2}\, \varepsilon^{ijk} R_{il} \nabla_j R^l_k - \frac{\kappa^2 \mu^2}{8}\, R_{ij}R^{ij} + \frac{\kappa^2 \mu^2}{8(1-3\lambda)} \left(\frac{1-4\lambda}{4}\, R^2 + \Lambda_W R - 2\Lambda_W^2 \right) \right\}. \tag{107}$$

At long distances, the potential is dominated by the last two terms in (107): the spatial curvature scalar and the constant term. As a result, the theory flows in the infrared region to $z = 1$.

To summarize, we can say that having been inspired by Lifshitz's old papers [51, 52], Hořava invented a new quantum gravity theory, where the Lorentz invariance arises only at relatively long distances but is absent in the ultraviolet limit. This theory is renormalizable and does not suffer from the presence of ghosts.

The idea to break the Lorentz invariance at very small distances to provide ultraviolet renormalizability of theories with higher-derivative terms was also used outside the gravity context. In [77], renormalizable field theories including the scalar fields and Yang–Mills fields with higher-derivative terms in the action were considered. The presence of higher-derivative terms changes the corresponding dispersion relations, and the effective speed of light can grow to infinity in the ultraviolet limit of the theory. This has permitted the authors of [77] to explain observable time delays in gamma-ray bursts.

A rather large amount of work was connected with applications of Hořava–Lifshitz gravity to cosmology (for a review, see [78]). There are a number of interesting cosmological implications of Hořava–Lifshitz gravity. We discuss some of them.

First, we note that the action of Hořava–Lifshitz gravity is invariant under space–time-dependent spatial diffeomorphisms and under only time-dependent time reparametrizations. This means that instead of four local first-class constraints, which we have in general relativity (corresponding to variation with respect to lapse and shift functions), we have three local constraints, corresponding to three components of the shift function and a global constraint corresponding to the lapse function. The absence of the local constraint means that the analog of the 00 component of the Einstein equations is not valid [78]. If we consider a Friedmann universe, that means that we cannot use the first Friedmann equation, and we should use the second Friedmann equation, including the time derivative of the Hubble parameter $H = \dot{a}/a$. This equation has a first integral, which gives us the first Friedmann equation, but with an additional constant included. The appearance of this constant is equivalent to the appearance of a substance which gravitationally behaves just like dust matter. Thus, we have some kind of dark matter without dark matter. This effect is already present in the flat Friedmann model.

Now, following [78], we write the general action for $z = 3$ Hořava–Lifshitz gravity, without thinking of the detailed balance condition and including the lower-derivative terms. This action has the form

$$\begin{aligned}
I &= I_{\mathrm{kin}} + I_{z=3} + I_{z=2} + I_{z=1} + I_{z=0} + I_m\,,\\
I_{\mathrm{kin}} &= \frac{1}{16\pi G} \int N\, \mathrm{d}t \sqrt{g}\, \mathrm{d}^3 x (K^{ij}K_{ij} - \lambda K^2)\,,\\
I_{z=3} &= \int N\, \mathrm{d}t \sqrt{g}\, \mathrm{d}^3 x (c_1 \nabla_i R_{jk} \nabla^i R^{jk} + c_2 \nabla_i R \nabla^i R \\
&\quad + c_3 R^j_i R^k_j R^i_k + c_4 R R^j_i R^i_j + c_5 R^3)\,,\\
I_{z=2} &= \int N\, \mathrm{d}t \sqrt{g}\, \mathrm{d}^3 x (c_6 R^j_i R^i_j + c_7 R^2)\,,\\
I_{z=1} &= c_8 \int N\, \mathrm{d}t \sqrt{g}\, \mathrm{d}^3 x R\,,\\
I_{z=0} &= c_9 \int N\, \mathrm{d}t \sqrt{g}\, \mathrm{d}^3 x\,,
\end{aligned} \tag{108}$$

where I_m is the matter action. For the Friedmann universe with an arbitrary curvature k, the second Friedmann equation for such a theory with action (108) is

$$-\frac{3\lambda - 1}{2}\,(2\dot{H} + 3H^2) = 8\pi G P - \frac{\alpha_3 k^3}{a^6} - \frac{\alpha_2 k^2}{a^4} + \frac{k}{a^2} - \Lambda\,, \tag{109}$$

where

$$\alpha_3 = 192\pi G (c_3 + 3c_4 + 9c_5)\,, \quad \alpha_2 = 32\pi G (c_6 + 3c_7)\,, \tag{110}$$

where P is the matter pressure. A first integral of this equation is given by

$$\frac{3(3\lambda - 1)}{2}\, H^2 = 8\pi G \left(\rho + \frac{C}{a^3} \right) - \frac{\alpha_3 k^3}{a^6} - \frac{3\alpha_2 k^2}{a^4} - \frac{3k}{a^2} + \Lambda\,, \tag{111}$$

where ρ is the energy density of matter and C is the effective dark matter integration constant, mentioned above.

Now, it is convenient to rewrite Eqn (111) as the energy conservation equation

$$\frac{\dot{a}^2}{a^2} + \frac{2}{3\lambda - 1}\, V(a) = 0\,, \tag{112}$$

where

$$V(a) = \frac{\alpha_3 k^3}{6a^4} + \frac{\alpha_2 k^2}{2a^2} + \frac{k}{2} - \frac{\Lambda}{6}\, a^2 - \frac{4\pi G}{3} \left(\rho a^2 + \frac{C}{a} \right). \tag{113}$$

The shape of the potential $V(a)$ completely determines the behavior of the system. When $\lambda > 1/3$, the universe can have only such values of a that the potential $V(a) \leqslant 0$. At those points where $V(a) = 0$, the universe has turning points, i.e., points of minimal contraction (bounce) or points of maximum expansion.

Thus, properly choosing the coefficients at the higher-order spatial curvature terms, we can have different cosmological regimes. We give some interesting examples.

1. If there is only one value of the cosmological radius $a = a_0$, where the potential $V(a)$ is equal to zero, and if

$V(a) > 0$ at $a < a_0$ and $V(a) < 0$ at $a > a_0$, then the universe has a bounce solution. Namely, at the beginning of evolution, the universe contracts and after the bounce it expands.

2. If there are two values a_1 and a_2 such that $V(a_1) = V(a_2) = 0$, and $V(a) < 0$ at $a_1 < a < a_2$ and $V(a) > 0$ at $a > a_2$ or $a < a_1$, then the universe has both a point of minimum contraction and a point of maximum expansion, and its evolution is periodic.

3. If $V(a_0) = 0$ and at this point the function $V(a)$ has a local maximum, we have a nonstable static universe.

4. If the function $V(a)$ has a local minimum and is equal to zero at the point a_0, we have a stable static universe.

The Hořava–Lifshitz cosmology has another interesting feature. In its framework, it is possible to predict scale-invariant cosmological perturbations without requiring the existence of an inflationary stage of its expansion [78]. We recall that the dispersion relation for the standard linear cosmological perturbations is

$$\omega^2 = c_s^2 \frac{k_c^2}{a^2}, \tag{114}$$

where c_s is the speed of sound and k_c is the comoving wave number.

If the mode under consideration satisfies $\omega^2 \gg H^2$, where $H = \dot{a}/a$ is the Hubble parameter, then the evolution of this model does not feel the expansion of the universe and the mode simply oscillates. When $\omega^2 \ll H^2$, the expansion of the universe is so rapid that the Hubble friction freezes the mode, which remains almost constant. Generation of cosmological perturbations from quantum fluctuations is the oscillation followed by a freeze-out. Therefore, the condition for the generation of cosmological perturbations is

$$\frac{d}{dt}\left(\frac{H^2}{\omega^2}\right) > 0. \tag{115}$$

If the standard dispersion relation (114) is valid, Eqn (115) implies that $\ddot{a} > 0$ for an expanding universe. Therefore, the generation of cosmological perturbations from quantum fluctuations requires accelerated expansion of the universe, i.e., inflation.

In Hořava–Lifshitz gravity with the critical index z, the dispersion relation for the perturbations in the ultraviolet region is

$$\omega^2 = M^2 \left(\frac{k_c^2}{M^2 a^2}\right)^z, \tag{116}$$

where M is some energy scale. Substituting Eqn (116) in condition (115), we obtain

$$\frac{d^2 a^z}{dt^2} > 0 \tag{117}$$

for an expanding universe. Obviously, for large values of z, for example, $z = 3$, condition (117) does not require an accelerating universe. Indeed, power-law expansion $a \sim t^p$, where $p > 1/z$ does the job. Thus, in the Hořava-Lifshitz cosmology, we can generate the spectrum of cosmological perturbations without inflation.

Concluding this section, we add that practically all traditional aspects of cosmology have also been studied in the context of Hořava–Lifshitz gravity. The theory of linear perturbations, dating back to the pioneering work of 1946 by Lifshitz [389], was studied, for example, in [79]. The quasi-isotropic (gradient) expansion for the Hořava–Lifshitz cosmology in the presence of a scalar field was considered in [80]. There was also some activity concerning the oscillatory approach to the singularity and the stochasticity phenomena in the Hořava–Lifshitz cosmology [81–83]. It seems that the question about the presence or absence of such stochasticity has not yet been resolved and requires further studies.

6. Conclusions

In this review, we have considered some new avenues of development of theoretical physics connected with gravitation and cosmology, which in one way or another stem from some old work by E M Lifshitz. Specifically, we have discussed the topics such as the oscillatory approach to the singularity and stochasticity in anisotropic and isotropic cosmologies, quasi-isotropic expansion, and so-called Hořava–Lifshitz gravity and cosmology.

Sometimes, these developments appear to be rather unexpected and amusing. Indeed, the study of the oscillatory approach to the cosmological singularity and stochasticity phenomena, being extended to cosmological models based on superstring theories, revealed connections between cosmological billiard dynamics and the properties of infinite-dimensional Kac–Moody algebras.

The ideas of some anisotropy between space and time, useful for describing second-order phase transitions and critical phenomena, as contained in the paper by Lifshitz written in 1941, inspired the creation of Hořava–Lifshitz gravity, which looks to be renormalizable at the quantum level and which, when applied to cosmology, shows many interesting effects and properties.

Now, coming to the end of our paper, we briefly mention one more paper by Lifshitz, which is connected with quantum field theory, gravity, and condensed matter physics simultaneously. We mean the theory of macroscopic Van der Waals forces between solids [84]. It is well known that in quantum field theory, one always has a vacuum energy, which diverges. Considering standard scattering processes, one can forget about this energy, subtracting it by means of the operation of normal ordering. In 1948, Casimir understood that if the electromagnetic field satisfies some special boundary conditions, then its vacuum energy is different from that in the Minkowski spacetime and the difference between the corresponding energy densities can be finite [85]. He considered two infinite conducting planes and showed that this difference was negative and that it implied the presence of an attracting force. This effect, called the Casimir effect, was also observed experimentally. Later, the role of Casimir energy in cosmology was studied by different authors (see, e.g., [86]).

In his paper published in 1956 [84], Lifshitz studied another aspect of the Casimir effect. Casimir had considered an ideal conductor ensuring absolute screening of the electromagnetic field. Hence, the tangential component of the electric field disappears on the conducting planes. Instead, Lifshitz noticed that ideal conductors do not exist and that the dielectric permittivity (which is infinite for the ideal conductor) actually depends on the frequency of the electromagnetic field, and for higher frequencies, this characteristic tends to unity, and the material becomes transparent for electromagnetic fields. The formulas taking this effect and the thermic fluctuations, together with quantum ones into account formed the basis for a new discipline in physics, which is sometimes called the theory of macroscopic Van der Waals forces [87]. This approach to the Casimir effect is also very popular now, and the corresponding forces that take their origin from quantum and thermic fluctuations are called Casimir–Lifshitz forces [88–90].

This work was supported in part by the RFBR grant 14-02-00894.

References

1. Lorenz E N *J. Atmos. Sci.* **20** 130 (1963)
2. Feigenbaum M J *J. Stat. Phys.* **19** 25 (1978)
3. Mandelbrot B *The Fractal Geometry of Nature* (New York: W.H. Freeman, 1977)
4. Gutzwiller M *Chaos in Classical and Quantum Mechanics* (Berlin: Springer-Verlag, 1990)
5. Penrose R *Structure of Space-Time* (New York: W.A. Benjamin, 1968); Translated into Russian: *Struktura Prostranstva–Vremeni* (Moscow: Mir, 1972)
6. Hawking S W, Ellis G F R *The Large Scale Structure of Space-Time* (Cambridge: Univ. Press, 1973); Translated into Russian: *Krupnomasshtabnaya Struktura Prostranstva–Vremeni* (Moscow: Mir, 1977)
7. Hawking S W, Penrose R *Proc. R. Soc. London A* **314** 529 (1970)
8. Lifshitz E M, Khalatnikov I M *Sov. Phys. Usp.* **6** 495 (1964); *Usp. Fiz. Nauk* **80** 391 (1963)
9. Lifshitz E M, Khalatnikov I M *Adv. Phys.* **12** 185 (1963)
10. Khalatnikov I M, Lifshitz E M *Phys. Rev. Lett.* **24** 76 (1970)
11. Lifshitz E M, Khalatnikov I M *JETP Lett.* **11** 123 (1970); *Pis'ma Zh. Eksp. Teor. Fiz.* **11** 200 (1970)
12. Belinskii V A, Khalatnikov I M *Sov. Phys. JETP* **29** 911 (1969); *Zh. Eksp. Teor Fiz.* **56** 1701 (1969)
13. Belinskii V A, Lifshitz E M, Khalatnikov I M *Sov. Phys. Usp.* **13** 745 (1971); *Usp. Fiz. Nauk* **102** 463 (1970)
14. Belinskii V A, Khalatnikov I M, Lifshitz E M *Adv. Phys.* **31** 639 (1982)
15. Misner C W *Phys. Rev. Lett.* **22** 1071 (1969)
16. Lifshitz E M, Lifshitz I M, Khalatnikov I M *Sov. Phys. JETP* **32** 173 (1971); *Zh. Eksp. Teor. Fiz.* **59** 322 (1970)
17. Khalatnikov I M et al. *J. Stat. Phys.* **38** 97 (1985)
18. Demaret J, Henneaux M, Spindel P *Phys. Lett. B* **164** 27 (1985)
19. Demaret J et al. *Phys. Lett. B* **175** 129 (1986)
20. Demaret J, de Rop Y, Henneaux M *Int. J. Theor. Phys.* **28** 1067 (1989)
21. Kasner E *Am. J. Math.* **43** 217 (1921)
22. Damour T, Henneaux M *Phys. Rev. Lett.* **85** 920 (2000)
23. Damour T et al. *Phys. Lett. B* **509** 323 (2001)
24. Damour T, Henneaux M, Nicolai H *Class. Quantum Grav.* **20** R145 (2003)
25. Kac V G *Infinite Dimensional Lie Algebras* (Cambridge: Cambridge Univ. Press, 1990)
26. Starobinsky A A *Phys. Lett. B* **91** 99 (1980)
27. Guth A H *Phys. Rev. D* **23** 347 (1981)
28. Linde A D *Phys. Lett. B* **129** 177 (1983)
29. Gamow G *Phys. Rev.* **70** 572 (1946)
30. Belinskii V A, Grishchuk L P, Zel'dovich Ya B, Khalatnikov I M *Sov. Phys. JETP* **62** 195 (1985); *Zh. Eksp. Teor. Fiz.* **89** 346 (1985)
31. Belinskii V A, Khalatnikov I M *Sov. Phys. JETP* **66** 441 (1987); *Zh. Eksp. Teor. Fiz.* **93** 784 (1987)
32. Page D N *Class. Quantum Grav.* **1** 417 (1984)
33. Kamenshchik A Yu, Khalatnikov I M, Toporensky A V *Int. J. Mod. Phys. D* **06** 673 (1997)
34. Kamenshchik A Yu, Khalatnikov I M, Toporensky A V *Int. J. Mod. Phys. D* **07** 129 (1998)
35. Cornish N J, Shellard E P S *Phys. Rev. Lett.* **81** 3571 (1998)
36. Cornish N J, Levin J J *Phys. Rev. D* **55** 7489 (1997)
37. Kamenshchik A Yu, Khalatnikov I M, Savchenko S V, Toporensky A V *Phys. Rev. D* **59** 123516 (1999)
38. Lifshitz E M *Zh. Eksp. Teor. Fiz.* **16** 587 (1946)
39. Bardeen J M *Phys. Rev. D* **22** 1882 (1980)
40. Mukhanov V F *Sov. Phys. JETP* **67** 1297 (1988); *Zh. Eksp. Teor. Fiz.* **94** 1 (1988)
41. Sasaki M *Prog. Theor. Phys.* **76** 1036 (1986)
42. Lifshitz E M, Khalatnikov I M *Sov. Phys. JETP* **12** 108 (1961); *Zh. Eksp. Teor. Fiz.* **39** 149 (1960)
43. Khalatnikov I M, Kamenshchik A Yu, Starobinsky A A *Class. Quantum Grav.* **19** 3845 (2002)
44. Khalatnikov I M, Kamenshchik A Yu, Martellini M, Starobinsky A A *JCAP* (03) 001 (2003)
45. Khalatnikov I M, Kamenshchik A Yu, Starobinsky A A, arXiv: 1312.0237
46. Riess A G et al. *Astron. J.* **116** 1009 (1998)
47. Perlmutter S et al. *Astrophys. J.* **517** 565 (1999)
48. Sahni V, Starobinsky A *Int. J. Mod. Phys. D* **9** 373 (2000)
49. Sahni V, Starobinsky A *Int. J. Mod. Phys. D* **15** 2105 (2006)
50. Hořava P *Phys. Rev. D* **79** 084008 (2009)
51. Lifshitz E M *Zh. Eksp. Teor. Fiz.* **11** 255 (1941)
52. Lifshitz E M *Zh. Eksp. Teor. Fiz.* **11** 269 (1941)
53. Bini D, Cherubini C, Jantzen R T *Class. Quantum Grav.* **24** 5627 (2007)
54. Petrov A Z *Einstein Spaces* (Oxford: Pergamon Press, 1969); Translated from Russian: *Prostranstva Einsteina* (Moscow: Fizmatgiz, 1961)
55. Parker L, Fulling S A *Phys. Rev. D* **7** 2357 (1973)
56. Starobinskii A A *Sov. Astron. Lett.* **4** 82 (1978); *Pis'ma Astron. Zh.* **4** 155 (1978)
57. Hawking S W, in *Relativity, Groups and Topology II. Proc. Les Houches 1983, Session XL* (Eds B S De Witt, R Stora) (Amsterdam: North-Holland, 1984) pp. 333–379
58. Khalatnikov I M, Kamenshchik A Yu *Phys. Rep.* **288** 513 (1997)
59. Khalatnikov I M, Kamenshchik A Yu *Phys. Usp.* **41** 525 (1998); *Usp. Fiz. Nauk* **168** 593 (1998)
60. Khalatnikov I M, Mezhlumian A *Phys. Lett. A* **169** 308 (1992)
61. Khalatnikov I M, Schiller P *Phys. Lett. B* **302** 176 (1993)
62. Amendola L, Khalatnikov I M, Litterio M, Occhionero F *Phys. Rev. D* **49** 1881 (1994)
63. Kamenshchik A Yu, Khalatnikov I M, Toporensky A V *Phys. Lett. B* **357** 36 (1995)
64. Kamenshchik A Yu, Khalatnikov I M, Toporensky A V *Int. J. Mod. Phys. D* **6** 649 (1997)
65. Comer G L et al. *Phys. Rev. D* **49** 2759 (1994)
66. Deruelle N, Langlois D *Phys. Rev. D* **52** 2007 (1995)
67. Starobinsky A A *Phys. Lett. B* **117** 175 (1982)
68. Starobinskii A A *JETP Lett.* **37** 66 (1983); *Pis'ma Zh. Eksp. Teor. Fiz.* **37** 55 (1983)
69. Salopek D S, Bond J R *Phys. Rev. D* **42** 3936 (1990)
70. Salopek D S *Phys. Rev. D* **43** 3214 (1991)
71. Salopek D S, Stewart J M *Class. Quantum Grav.* **9** 1943 (1992)
72. Wands D et al. *Phys. Rev. D* **62** 043527 (2000)
73. Nambu Y, Taruya A *Class. Quantum Grav.* **13** 705 (1996)
74. Tanaka Y, Sasaki M *Prog. Theor. Phys.* **117** 633 (2007)
75. Weinberg S, in *General Relativity. An Einstein Centenary Survey* (Eds S W Hawking, W Israel) (Cambridge: Cambridge Univ. Press, 1979) p. 790
76. Arnowitt R, Deser S, Misner C W, in *Gravitation: an Introduction to Current Research* (Ed. L Witten) (New York: Wiley, 1962) Ch. 7, pp. 227–265; republished in: *Gen. Relat. Grav.* **40** 1997 (2008)
77. Chen B, Huang Q-G *Phys. Lett. B* **683** 108 (2010)
78. Mukohyama Sh *Class. Quantum Grav.* **27** 223101 (2010)
79. Gao X et al. *Phys. Rev. D* **81** 083508 (2010)
80. Gümrükçüoğlu A E, Mukohyama S, Wang A *Phys. Rev. D* **85** 064042 (2012)
81. Bakas I, Bourliot F, Lüst D, Petropoulos M *Class. Quantum Grav.* **27** 045013 (2010)
82. Myung Y S et al. *JHEP* (03) 085 (2010)
83. Maeda K, Misonoh Y, Kobayashi T *Phys. Rev. D* **82** 064024 (2010)
84. Lifshitz E M *Sov. Phys. JETP* **2** 73 (1956); *Zh. Eksp. Teor. Fiz.* **29** 94 (1955)
85. Casimir H B G *Proc. K. Ned. Akad. Wet.* **51** 793 (1948)
86. Ford L H *Phys. Rev. D* **11** 3370 (1975)
87. Dzyaloshinskii I E, Lifshitz E M, Pitaevskii L P *Sov. Phys. Usp.* **4** 153 (1961); *Usp. Fiz. Nauk* **73** 381 (1961)
88. Bordag M et al. *Phys. Rev. B* **74** 205431 (2006)
89. Antezza M et al. *Phys. Rev. A* **77** 022901 (2008)
90. Rosa F S S, Dalvit D A R, Milonni P W *Phys. Rev. Lett.* **100** 183602 (2008)