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Model of stretching vortex filaments and foundations of the statistical theory of turbulence

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Abstract. Although the statistical properties of small-scale velocity perturbations in homogeneous and isotropic hydrodynamic turbulence have been thoroughly studied experimentally and numerically, no definite theoretical explanation is available yet. The concept of breaking vortices, commonly accepted as the primary turbulent mechanism, not only fails to account for a number of facts but also is self-contradictory. In this review, we discuss an alternative concept according to which the stretching of vortices rather than their decay is the determining process. The evolution of stretching vortex filaments and their properties are derived directly from the Navier–Stokes equation. The model of stretching vortex filaments explains the power-law behavior of velocity structure functions and the intermittency of their exponents, thus imparting physical meaning to multifractal theory, which is based on dimensional considerations. The model of vortex filaments is the only theory that explains the observed differences between the scaling exponents of longitudinal and transverse structure functions.

Keywords: hydrodynamics, turbulence, statistical theory

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1. Introduction

Turbulence is a complex natural phenomenon which over the past half century has become a research topic attracting intensive effort of physicists, mathematicians, and engineers alike. The spectrum of problems arising in studies of turbulence is very broad, and in this review we limit ourselves to considering only one particular aspect: the theoretical description of small-scale perturbations in homogeneous isotropic hydrodynamical turbulence. The formulation of a theory that would enable computing all possible correlations in a random fluctuating flow of a fluid from 'first principles' is still far from completion, in spite of obvious successes achieved over more than seven decades following the publication of the pioneering work of Kolmogorov [1, 2] in 1941. Substantial progress in experimental techniques and computational facilities in recent years has considerably broadened the set of measured quantities and respective limitations in the theory. Moreover, in our opinion, modern results of both experiments and numerical simulations notably modify the traditional physical view on the structure of a turbulent flow.

For these reasons, we here do not pursue the goal of offering an all-embracing account of the existing literature; instead, we briefly sketch the emerging concept and concentrate on the difficulties it faces. We then present a model that in all probability describes the experimental data more adequately.

1.1 Brief overview of the problem

We discuss the properties of a turbulent flow on scales much smaller than the characteristic size of the entire flow (for example, a pipe radius); it is plausible to assume that the flow becomes homogeneous and isotropic on such scales. Most of the energy resides in large eddies (vortices). The range of sizes much smaller than the energy-carrying scales, and yet sufficiently large to neglect viscosity, is called the inertial range.

Because describing a turbulent flow by dynamical means is impossible, a statistical description is used: correlators of velocity, specific energy dissipation, etc., are computed. In particular, the longitudinal and transverse velocity structure functions are most commonly used:

$$S_n^{\parallel}(l) = \left\langle \left(\left[\mathbf{v}(\mathbf{r}+\mathbf{l},t) - \mathbf{v}(\mathbf{r},t)\right] \frac{\mathbf{l}}{l} \right)^n \right\rangle , \qquad S_n^{\perp}(l) = \left\langle \left| \left[\mathbf{v}(\mathbf{r}+\mathbf{l}) - \mathbf{v}(\mathbf{r})\right] \times \frac{\mathbf{l}}{l} \right|^n \right\rangle . \tag{1}$$

They are the nth-order moments of velocity increments between two close points separated by a vector $\mathbf{l}$; the averaging is over all pairs of points located at a distance l (i.e., over all $\mathbf{r}$ and over all directions of $\mathbf{l}$).[1]

Assuming the finiteness of the mean energy dissipation rate ε in the limit of vanishing viscosity ν, Kolmogorov in 1941 derived an exact expression for the third-order longitudinal structure function in the limit as $\nu \to 0$ and $l \to 0$ [1]:

$$S_3^{\parallel}(l) = -\frac{4}{5}\,\varepsilon l . \tag{2}$$

For the second-order structure function, relying on dimensional considerations, Kolmogorov obtained the power law [2]

$$S_2^{\perp}(l) \propto S_2^{\parallel}(l) \propto \varepsilon^{2/3} l^{2/3} . \tag{3}$$

A detailed discussion of the derivation of relation (3) is presented in books [3, 4].

This famous 'law of two-thirds' agrees well with experimental data, not only in hydrodynamical media [5, 6] but also in many other media, for example, laboratory and cosmic plasmas [7].

What can be said about higher correlators, $n > 3$? Generalizing Kolmogorov's dimensional analysis to this case (which Kolmogorov did not do), we can readily obtain a power-law dependence of structure functions on l, with the power-law exponent linearly dependent on the order n. Indeed, the Kolmogorov theory assumes the existence of a single parameter, the energy dissipation rate; it suffices to suppose, for example, that the distribution function $F(\delta v_{\parallel}; l)$ depends only on one argument (the index $\parallel$ is suppressed below for brevity),

$$F(\delta v; l) = F\left(\frac{\delta v}{\sqrt{\langle \delta v^2(l) \rangle}} \right) .$$

[1] By virtue of ergodicity, the volume mean is assumed to be equal to the ensemble mean.

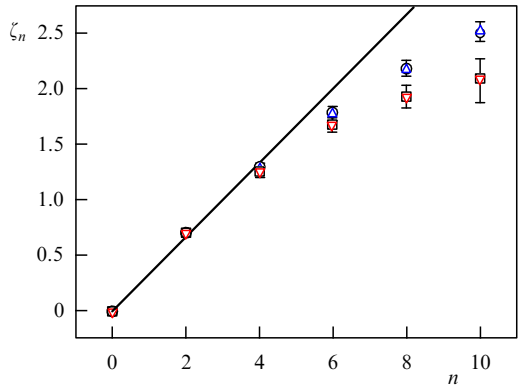

Figure 1. Power-law exponents of longitudinal and transverse structure functions of velocity: the results of numerical simulations in [8] (circles and squares) and [9] (triangles pointing up and down). The straight line with the slope 1/3 corresponds to Eqn (4).

Taking (3) into account, we then find

$$\langle \delta v^n(l) \rangle = \int F(\delta v; l)(\delta v)^n \, \mathrm{d}(\delta v) \propto \sqrt{\langle \delta v^2(l) \rangle}^{\,n} \propto l^{n/3} . \tag{4}$$

However, subsequent research has indicated that the problem is more complicated than it seems at first glance. As experimental techniques, computational power, and numerical modeling methods have evolved, it has become clear that the structure functions in the inertial range satisfy the power laws

$$S_n^{\parallel}(l) \propto l^{\zeta_n^{\parallel}} , \qquad S_n^{\perp}(l) \propto l^{\zeta_n^{\perp}} .$$

The power-law exponents $\zeta_n^{\parallel}$ and $\zeta_n^{\perp}$ computed and measured for the order n ranging from zero to 10 (Fig. 1) are substantially different from the exponents in (4). This distinction is the reflection of turbulence intermittency. We note that a process is said to be intermittent if the square of the nth correlator is much less than the $2n$th correlator, $S_n^2 \ll S_{2n}$. Intermittency comes from strong but rare fluctuations and, as can be seen from Eqn (4), this implies the existence of more than one parameter.

Why does relation (4) become incorrect? It turns out that the assumption about the homogeneity of ε is violated: the energy dissipation, as witnessed by measurements, is spread over space in a demonstrably nonuniform way [5]. Thus, the hypothesis that ε is a single dimensional parameter is invalid, and hence the conclusion on the power-law dependence of structure functions hinges on incorrect assumptions. (We note that the mere existence of a power-law dependence is nevertheless confirmed experimentally.)

The relation between the longitudinal and transverse structure functions of the same order is also vague. The theory predicts that $S_2^{\perp} \propto S_2^{\parallel}$ and $S_3^{\perp} \propto S_3^{\parallel}$ [3], and the opinion shared by theorists is that power-law exponents should coincide in all orders. However, a significant difference between the exponents $\zeta_n^{\parallel}$ and $\zeta_n^{\perp}$ for $n > 3$ was found in a number of recent papers [10–13] (see Fig. 1).

The following questions therefore arise.

- Why do the higher-order structure functions nevertheless satisfy a power law?
- Why does the power-law exponent show a nonlinear dependence on the order n (and what is this dependence)?

• Do the power-law exponents $\zeta_n^{\parallel}$ and $\zeta_n^{\perp}$ coincide for $n > 3$? If not, why are they different and how different are they?

Before embarking on a discussion of answers to these questions, we mention that different approaches to the description of homogeneous isotropic turbulence are known. Experimental data and results of numerical simulations of the Navier–Stokes equations for close values of the Reynolds number agree well with each other. Nobody doubts that the Navier–Stokes equation describes the phenomenon correctly. From the engineering standpoint, the problem is thus practically solved. From the mathematical standpoint, to reach perfect harmony, one only needs to prove the theorem on the existence and uniqueness of solutions of the Navier–Stokes equation [14]. The problem is very complex, yet there have been significant achievements (see Section 1.2 for singularities in solutions of the Navier–Stokes equation). It is noteworthy that the existence of solutions is proved, at least, for the Navier–Stokes equation in which the 'viscous' term $\nu\Delta\mathbf{v}$ is augmented with a term containing a higher-order derivative [it suffices to have Δ^{α} with $\alpha \geqslant 5/4$ (see Refs [15, 16])]. Because $\nu\Delta\mathbf{v}$ is only the first term of the expansion (in small velocity gradients) of the forces acting in a fluid, the full equation, describing results of arbitrary experiments, always contains higher-order corrections, and hence its solution exists without a doubt.

In our opinion, the problem should be given a different focus: it is necessary to understand how and from where these nontrivial properties of the correlations follow and, if possible, identify the objects responsible for the appearance of these properties in a turbulent flow.

This review attempts to propose a constructive answer to these questions.

1.2 Cascade model

Turbulent motion implies the coexistence of a large number of eddies with differing scales. Since the early 20th century, the main physical concept associated with turbulence is the concept of a cascade of eddies splitting into smaller ones, introduced by Richardson. Large eddies are assumed to successively break into smaller ones in a turbulent flow, forming a turbulent cascade (Fig. 2), in analogy with crushing stones in a landslide or the generation of elementary particles in an avalanche caused by a cosmic particle in collisions with air molecules.

Each of these processes is characterized by some conserved quantity: in the case of crushing stones, it is the total mass of the parts; for a cosmic particle, it is the total energy. In

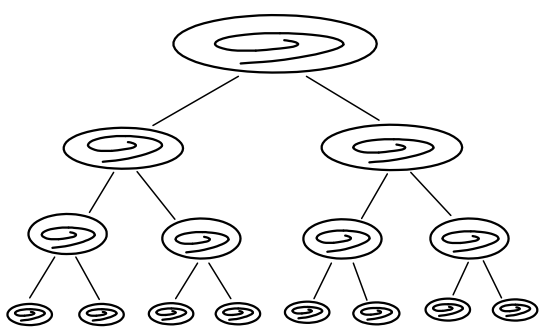

Figure 2. Cascade of breaking vortices. The view associated with Kolmogorov's theory (see Ref. [4]).

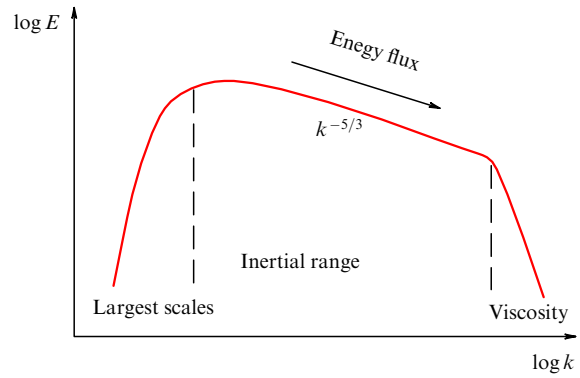

Figure 3. Kolmogorov–Richardson spectrum. The energy is pumped into the system on scales of the largest vortices and is carried to ever smaller scales through vortex breaking, passing through the entire inertial range and dissipating on 'viscid' scales. In the Kolmogorov theory, it is supposed that the energy flux at large Reynolds numbers is defined by pumping only and is independent of the Reynolds number.

the turbulent cascade, the conserved quantity is the energy flux moving from large to small scales. Energy is pumped at large scales (commonly as a consequence of flow instability), and at small scales the 'avalanche' dissipates due to the presence of viscosity. The eddies at intermittent scales form the inertial range where energy is transferred from large to small vortices (Fig. 3). It is worth noting that such a picture of wave breakup is indeed observed in the theory of weak turbulence dealing with waves [17].

In his work on turbulence theory, Kolmogorov, as could be expected, relied on the concept of cascading eddies. This created a picture of a turbulent flow in which the energy flux is independent of the scale (in the limit of viscosity tending to zero), which is the physical basis of the Kolmogorov theory.

The concept of cascading eddies, its immense illustrative and historical role notwithstanding, faces certain difficulties.

First, it is not clear how eddies break up into smaller ones. For this to happen, an eddy should first be stretched and then twisted to become a figure eight; finally, its streamlines should be reconnected (Fig. 4). But the reconnection of streamlines is only possible by virtue of viscosity ν; in this case, the reconnection time is $\sim l^2/\nu$, where l is the characteristic size of the reconnection domain. For the eddy reconnection time to be substantially smaller than its viscous dissipation time (i.e., for eddies to undergo many reconnections to create a cascade), it is necessary to assume that eddies are very thin at the reconnection sites (singular in the limit $\nu \to 0$).

The question of whether singularities may evolve in an incompressible fluid obeying the Euler equation has been the topic of numerous studies [18–20], but is still open. A negative answer to it would leave the mechanism of eddy breakup indeterminate. If singularities may emerge, this alone is already sufficient for explaining the scaling of structure functions (see below). Thus, in this case, the concept of successive breakups of large-scale eddies would appear redundant.

Second, there are current reports claiming observations of long-lived coherent structures — vortex filaments [21] — in experiments and numerical simulations. The process of the emergence of vortex structures is suggested by recent acoustic measurements [22].

The fact that the velocity curl — the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$ — is distributed highly inhomogeneously in space and

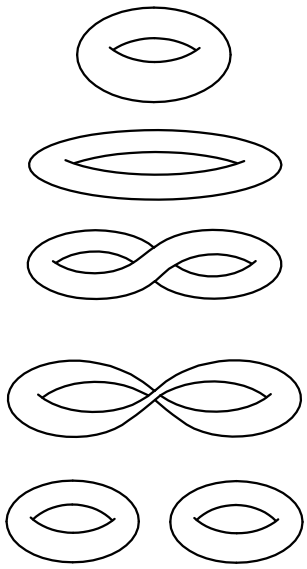

Figure 4. Vortex breakup. A schematic representation of the evolution of a typical vortex in the cascade model. The vortex lines lie on a torus. For a reconnection to occur, the vortex (torus) should be very thin in the vicinity of the reconnection point. In the limit of zero viscosity, a singularity must occur for the reconnection to happen.

that domains with high vorticity form a complex web of interlacing thin filaments has been known for a long time [23, 24]. However, it was demonstrated in Refs [21, 25] that these filaments are in reality stable dynamical formations. Using modern powerful computers and methods of wavelet analysis, it has become possible to isolate two components: noise and coherent (nonrandom) structures in a flow evolving with time. It turns out that just the coherent structures ensure the functioning of the two-thirds law (3): the correlators computed over the domains occupied by them (more precisely, computed by the velocity field reconstructed from the 'coherent' Fourier transform) practically coincide with the real ones, and, if the coherent structures are 'detached' from the velocity field, the structure functions computed with respect to the remaining 'noise' prove to be trivial, $\langle(\delta v)^2\rangle \propto l^2$, and much smaller than the original ones. The prevailing part of the energy of oscillations and enstrophy (dissipation) also resides in the coherent structures.

The life span of coherent structures is several times longer than the correlation time — the large-eddy turnover time t_0. Thus, the mere existence of such structures is at variance with the cascade concept. Indeed, in the Kolmogorov theory, $\delta v(\varepsilon, l) \sim l^{1/3}$, implying that the time associated with an eddy of a scale l is $t_l(\varepsilon, l) \sim l^{2/3} \ll t_0$, because there are no other parameters in the inertial range and the breakup time coincides by an order of magnitude with the eddy turnover time. Hence, according to the cascade model, all eddies should successively break up over the correlation time, from the largest to those of the scale η where dissipation becomes essential.

Finally, because dissipation occurs in narrow regions, their Fourier transform has a broad spectrum (in contrast to dissipation at $k \approx 1/\eta$, as in the case of cascades). Moreover, in realistic conditions, the forcing of turbulence is linked to the instability in narrow boundary layers; their Fourier representation also contributes to a broad spectrum. It thus turns out that the meaning of the inertial range as the scale interval void of forcing and energy dissipation becomes lost.

Is there an alternative to the idea of a cascade? A hypothesis about the formation of singularities in the vorticity field might be considered as such. If we assume a power-law dependence of the velocity on any coordinate, it is then not difficult to 'arrange' the power-law behavior of structure functions. Admittedly, the nonlinear dependence of the exponents on the function order would require considering an infinite set of singularities of various degrees.

Whether the singularities may in reality develop in a finite time (for smooth boundary conditions) is currently not known for both the Euler and the Navier–Stokes equations. Intensive studies are being carried out in this regard, and certain constraints have been found. For example, it has been shown that the Hausdorff dimension of the set of singularities of a viscous flow in four-dimensional space–time does not exceed unity [26], i.e., even if singularities are present, they are rather rare (see also Ref. [27]). On the other hand, as we have already mentioned in Section 1.1, it is proved that the flow preserves regularity if the viscous term $\nu\Delta\mathbf{v}$ is replaced with the expression $-\nu(-\Delta)^{\alpha}\mathbf{v}$, where $\alpha \geqslant 5/4$ [15, 16].[2] It is probable that the Navier–Stokes equation does not allow singularities altogether, but no one has succeeded in proving this. For the Euler equation, in contrast, it would be natural to expect singularities to form, because a smoothing mechanism is absent. However, neither numerical simulations nor theoretical analysis can confirm or refute the last statement. It is possible that just the formation of stable structures (filaments, 'pancakes', and so on) prevents singularities from being formed [4, 20, 28].

1.3 Multifractal theory

An important property of the Euler equation is its scale invariance, i.e., invariance under the transformations

$$r \to r' = \gamma r\,, \qquad v \to v' = \gamma^h v\,, \qquad t \to t' = \gamma^{1-h} t \tag{5}$$

with arbitrary γ and h.

This circumstance laid the basis for various methods of turbulence research using dimensional analysis, with renormalization group methods [29, 30] and models relying on perturbation theory [31, 32] among them. The multifractal model proposed in Ref. [33] is the most widely disseminated, richest in results, and most well recognized. (For an up-to-date review, see Ref. [34].) We discuss this model in more detail.

Having a solution, we can use transformation (5) to obtain an infinite set of solutions for any given h. Thus, the entire ensemble of solutions can be split into classes indexed by different values of h. The proposal of the model is that within a particular h-class, the leading contribution to the structure functions comes from solutions that satisfy the condition

$$\delta\mathbf{v}(\mathbf{l}, \mathbf{r}) = \mathbf{v}(\mathbf{r} + \mathbf{l}) - \mathbf{v}(\mathbf{r}) \propto l^h\,.$$

inside some domain $S_h \in \mathbb{R}^3$. The fractal dimension of the domain S_h, denoted as $D(h)$, is assumed to be independent of the concrete form of the flow.

[2] This term looks much more natural in the Fourier representation: $-\nu k^{2\alpha}\mathbf{v}$.

In the Kolmogorov theory, the global scale invariance with a single universal parameter $h = 1/3$ is assumed. The multifractal model is therefore a generalization of the Kolmogorov theory to the case of local scale invariance. Different intermittent scaling characteristics (probability densities, correlators of velocity, acceleration, dissipation, and so on) are expressed in the multifractal model in terms of a single function $D(h)$.

The introduction of the 'fractal dimension' $D(h)$ is based on the mathematical theory of large deviations. It is natural to expect that for $l \ll L$, where L is the correlation scale of turbulence, structure functions and other characteristics are determined by rare events (otherwise we would arrive at the trivial result $\langle \Delta v^n \rangle \sim l^n$). The large deviation theory asserts that the probability P of the velocity increment with a scaling h, i.e., the probability of a pair of points $\mathbf{r}, \mathbf{r} + \mathbf{l}$ landing in the region S_h, is given (simply because of the absence of a scale) by the power-law function

$$P = l^{3-D(h)} \,. \tag{6}$$

If the function $D(h)$ is known, then all structure functions are expressed as

$$\langle \Delta v^n \rangle = \int l^{nh} l^{3-D(h)} \, \mathrm{d}\mu(h) \,,$$

where $\mu(h)$ defines the relative weights of different h; up to logarithmic corrections, we can take $\mu(h) = h$.

All approximations discussed here are justified in the limit $l \to 0$, and in this case the integral can be readily estimated by the steepest descent method:

$$\lim_{l \to 0} \frac{\ln \langle \Delta v^n \rangle(l)}{\ln l} = \zeta_n \,,$$
$$\zeta_n = \min_h \big(nh + 3 - D(h) \big) \,. \tag{7}$$

As we see from Eqn (7), the exponent ζ_n is related to $D(h)$ by the Legendre transformation. Figure 5 illustrates the simple geometrical meaning of this relation. Without losing generality, $D(h)$ can be considered to be concave, i.e., $D'' < 0$. Then the inverse relation

$$D(h) = \min_n (nh + 3 - \zeta_n) \tag{8}$$

exists.

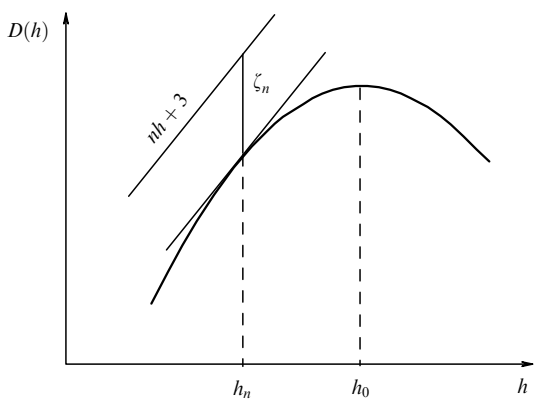

Figure 5. Determining ζ_n from the dependence $D(h)$. The quantity h_n satisfies the equation $D'(h_n) = n$; at this point, the distance from the plot of $D(h)$ to the straight line $nh + 3$ reaches a minimum.

Hence, any set of power-law exponents ζ_n corresponds to a function $D(h)$. The multifractal model allows relating different types of correlators, expressed in terms of $D(h)$, but does not answer the question of what $D(h)$ is, and, accordingly, does not allow computing ζ_n without additional assumptions.

Study [33] introducing the notion of multi-fractality has the title "On the singularity structure of fully developed turbulence." Indeed, the phenomenological description given above assumes the existence of singularities in a turbulent flow. This can be illustrated with the following example.

We consider a pair of points from the domain S_h; then

$$\delta v_{\parallel}(l) = C l^h \,.$$

The distance l between the points evolves with time according to the law

$$\frac{\mathrm{d}l}{\mathrm{d}t} = \delta v_{\parallel}(l) \,;$$

hence follows the differential equation

$$\frac{\mathrm{d}l}{\mathrm{d}t} = C l^h$$

with the solution

$$l = \big(l_0^{1-h} + (1-h) C t \big)^{1/(1-h)} \,.$$

A nontrivial scaling occurs only if $h < 1$. For negative C, the solution exists only for the finite time

$$t_0 = \frac{l_0^{1-h}}{|C|(1-h)} \,.$$

This implies that the multifractal model assumes the emergence of a singularity in solutions of the Euler equation at the instant t_0. At different space locations (related to different S_h), singularities evolve at different times; moreover, the degree of the singularity $1/(1-h)$ is also space dependent.

All this resembles the solution of the three-dimensional Riemann–Hopf equation [3, 35]. But the emergence of singularities in the Riemann–Hopf equation is related to gas compressibility. Moreover, the Riemann–Hopf equation describes a limit opposite to that for an incompressible fluid: it corresponds to the speed of sound $c_s \to 0$, whereas incompressibility implies $c_s \to \infty$. In the Riemann–Hopf equation, moreover, only singularities of several different orders are possible and their number is always finite, while in the Euler equation, according to the multifractal theory, they can be practically arbitrary ($h > 0$) and their orders form a continuous spectrum.

Because the existence of any singularities in the Euler equation is not proved, the presence of an infinite set of various singularities in the model is highly undesirable. To alleviate this difficulty, a probabilistic interpretation of the multifractal model has been proposed. Instead of individual realizations of the velocity field, functions of the probability density $P(\delta v, l)$ are discussed:

$$\langle \delta v^p \rangle = \int \delta v^p P(\delta v, l) \, \mathrm{d}\delta v \,, \tag{9}$$

where the probability $P(\delta v, l)$ of finding δv in a flow is determined by the multifractal probability $P(h, l) \propto l^{3-D(h)}$

[Eqn (6)] and the scaling $\delta v \propto l^h$:

$$P(\delta v, l) \propto \frac{l^{3-D(\ln \delta v/\ln l)}}{\delta v \ln l} .$$

Expression (9) has all scaling properties, but does not require interpreting the multifractal as a set of structures or singularities in space and allows generalizing the notion of dimension D to arbitrary negative values. But in the statistical interpretation, we refrain from discussing any structures.

To summarize, although the multifractal theory is (from a formal standpoint) capable of correctly describing the observed correlation functions of developed turbulence, it does not answer the question of how such a solution can appear in a flow.

1.4 Vortex filaments as an alternative concept of turbulence

We see that the concept of breaking eddies contradicts some observations, and that the development of singularities in a finite time remains a hypothesis. Meanwhile, many authors point at the possible important role of vortex filaments in the general picture of turbulence (see, e.g., Ref. [36]). In [37, 38], the entangled net of vortex filaments is considered as a fractal that corresponds to the trajectory of self-avoiding random walk. Kolmogorov's law of two-thirds (3) is derived from the properties of random walk (albeit under assumptions very distant from reality), although no dissipation is present in such an intermittent model.

In [39], a phenomenological model is constructed based on the idea of vortex filaments, which allows obtaining realistic relations between the power-law exponents of structure functions of different orders.

However, special attention is paid in the above papers to the mutual location of vortex filaments and to the geometry of the network formed by them, while the structure elements (the filaments) are considered as given. The evolution of individual filaments is left behind the scene. In reality, just this evolution, as we believe, may furnish a physical mechanism alternative to the idea of a cascade.

We tried to fill this gap in Refs [40–43]. In the next sections, we describe the results obtained there in detail. Here, we sketch the general framework for these results.

The evolution of a domain around a local vorticity maximum looks like a sequence of rotations and deformations. Although forces acting on this fluid domain are random, their net effect happens to contain a systematic component: on the background of random rotations, there is exponential stretching (and because of the incompressibility, the associated transverse contraction), which corresponds to filament formation. The vorticity becomes strongly intermittent, and the ensemble mean of ω^n depends on n nonlinearly, $\langle \omega^{2n} \rangle \gg \langle \omega^n \rangle^2$.

As a filament is elongated, the velocity profile is 'adjusted' to the power-law behavior everywhere except a narrow nonstationary domain just in its center. Thus, despite there being no singularity at the center, the structure functions attain a power-law form. The intermittency in their power-law exponents comes from the difference in the geometry of filaments contributing most to the correlators of different orders (or, in terms of the multifractal theory, to various h-classes). In this manner, the multifractal model acquires a rigorous and transparent substantiation.

What happens in the center of the filament? In the case of a small but finite viscosity, the increase in vorticity stops when the transverse contraction reaches scales making viscosity essential. In the absence of viscosity, a singularity takes an *infinite* time to develop in the center of the filament. But at any finite time instant, the vorticity and velocity distributions remain smooth. In this case, the solution differs from the steady, 'viscid' one only in a narrow transient domain in the center: the unsteadiness takes the role of viscosity. Although the initial perturbations are not smoothed out in the 'inviscid' solution, they are transferred to ever smaller scales. Likewise, solutions of the Euler and Navier–Stokes equations behave similarly in the limit of large time and vanishing viscosity.

We see that the model of a stretching vortex filament has all the advantages of 'singular' models, but there is no singularity in it; it warrants energy transfer across scales, albeit endowing it with strong intermittency; finally, it may elucidate the nature of the dissipative anomaly — the independence of dissipation from viscosity in the limit $\nu \to 0$. Indeed, even in the absence of viscosity, when the dissipation is removed altogether, the energy flux over scales is preserved owing to ever lasting contraction of vortex cores.

A discussion and substantiation of the model of stretching vortex filaments is the topic of this review.

2. Stochastic equation for small-scale fluctuations

2.1 Traditional formulation: adding an external force

The famous paper by Kolmogorov [2] begins with the words "In considering the turbulence it is natural to assume (that) the components of velocity ... are random quantities." Further, averaging the Navier–Stokes equation with different weights (and using the results in Ref. [44]), Kolmogorov [1] arrives at the relation between the correlators $S_2^{\parallel}$ and $S_3^{\parallel}$ (see Ref. [3] for more details):

$$-\frac{4}{5}\,\varepsilon l = S_3 - 6\nu\,\frac{\partial S_2}{\partial l} \tag{10}$$

(in our notation). Here, the quantity

$$\varepsilon = \frac{1}{2}\,\frac{\partial}{\partial t}\,\langle v^2 \rangle \tag{11}$$

is introduced, called 'the average dispersion rate of energy' by Kolmogorov.[3] In the limit $\nu \to 0$, it leads to relation (3) for the third structure function.

Later, relation (10) was derived in a somewhat different way [4]. In fact, this new derivation was needed because, viewed from a mathematical standpoint, Kolmogorov's argument looks insufficiently rigorous. Indeed, the Navier–Stokes equation is a dynamical equation, but what is then implied in the first phrase that the velocity is a random quantity? Further, Kolmogorov's definition of ε in (11) invites associations with decaying turbulence. Modern literature typically deals with stationary turbulence, in which case $\partial\langle v^2 \rangle/\partial t = 0$ and the quantity ε requires a different definition.

[3] Here, following Kolmogorov, we discard the term $l\,\partial S_2/\partial t$, which is small compared with ε for small l.

To alleviate these difficulties, a random force $\mathbf{F}(\mathbf{r}, t)$ is added to the right-hand side of the Navier–Stokes equation:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} P + \mathbf{F}(\mathbf{r}, t) + \nu\Delta\mathbf{v}\,, \qquad \boldsymbol{\nabla}\mathbf{v} = 0\,. \tag{12}$$

After this step, the equation becomes stochastic and the velocity $\mathbf{v}$ becomes random. The force $\mathbf{F}$ is assumed to be a stationary, homogeneous, and isotropic random process, usually Gaussian. It is expected that the statistical properties of small-scale pulsations are independent of the way $\mathbf{F}$ is introduced, implying that the force $\mathbf{F}$ should at least be large-scale. This means not only that the correlator decays rapidly at scales smaller than the size of large-scale pulsations L but also that any realization of the random process $\mathbf{F}$ contains only large-scale harmonics.

It is presumed that the force thus introduced not only adds stochasticity to the Navier–Stokes equation but also 'pumps' energy into the flow. Then, under the assumptions of stationarity and vanishing viscosity, ε is defined as $\varepsilon = \langle \mathbf{F}\mathbf{v} \rangle$.

It is unlikely that Kolmogorov did not see the theoretical objections outlined above and failed to realize that everything can be 'mended' by adding a random force to the right-hand side. He did not do it, however, possibly because the generation of a random flow by a random force seemed to him to be a somewhat different problem. This is indeed so — in particular, in this approach, in order to warrant stationarity, one needs dissipation; it furnishes stationarity but drops from the final result.[4] In Kolmogorov's approach, relation (2) can be obtained exactly also for the Euler equation (its derivation hinges only on spatial averaging). We note that the manifestation of the Kolmogorov law is often encountered in nature when dissipation is not essential (for example, in a cosmic plasma [7]).

On the other hand, the external force, generally speaking, is not necessarily pumping energy into the system. The fact that the sign of $\langle \mathbf{F}\mathbf{v} \rangle$ is positive is an attribute of faith and does not follow from anywhere. A situation is possible where $\langle \mathbf{F}\mathbf{v} \rangle = 0$, there is no energy pumping, and the flow is stabilized because of nonlinearity.[5]

Finally, in a realistic turbulent flow, external body forces do not act. Turbulence and stochasticity are generated by instabilities of large-scale eddies of the basic energy-containing scale. In this sense, Kolmogorov's definition of ε is 'more rigorous' and agrees better with the process physics. To formalize the appearance of randomness at small scales, it would in all probability be more natural to introduce random initial conditions; strictly speaking, a proof of the equivalence of different formulations is needed.

In any case, large-scale forces that generate stochasticity at small scales should be related in some way to large-scale velocity fluctuations. In other words, these random quantities should not be independent: the mechanism generating randomness is already inherent in some sense in the Navier–Stokes equations. In the ideal perspective, we would like to separate large-scale random fluctuations and then explore the dynamics of small scales.

To realize this program, we suggest another approach: large-scale velocity pulsations instead of a large-scale force F. In light of the discussions presented above, this approach seems to be closer to the physics of the process. Furthermore, this approach, in contrast to the 'standard' method, allows simplifying the problem substantially. The extent to which it is equivalent to the traditional approach involving a random external force is discussed in Section 2.2.

2.2 Introduction of large-scale velocity pulsations

We introduce a random field $\tilde{\mathbf{U}}$ such that $\boldsymbol{\nabla}\tilde{\mathbf{U}} = 0$. To make it large-scale at distances $l \ll L$, we smooth it according to the relation

$$\mathbf{U}_i(\mathbf{r}, t) = \frac{1}{L^3}\int \tilde{U}_i(\mathbf{r} + \boldsymbol{\rho}, t)\exp\left(-\frac{\rho^2}{L^2}\right)\mathrm{d}\boldsymbol{\rho}\,, \tag{13}$$

whence $\boldsymbol{\nabla}\mathbf{U} = 0$.

We now *define* the random large-scale force $\mathbf{F}$ by the relation

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U}\boldsymbol{\nabla})\mathbf{U} = -\nabla\pi + \mathbf{F}(\mathbf{r}, t) + \nu\Delta\mathbf{U}\,, \qquad \boldsymbol{\nabla}\mathbf{F} = 0\,. \tag{14}$$

Equation (14) is simultaneously the definition of the function $\pi(\mathbf{r}, t)$. By construction, it is obvious that $\mathbf{F}$ is a large-scale force satisfying the conditions formulated in Section 2.1. (The time derivative does not introduce additional complications. Moreover, as we see in Section 3, the final results contain only the function $\mathbf{U}$ and its integrals, but not derivatives.)

We now substitute this random force $\mathbf{F}$ in the right-hand side of Eqn (12) and seek its solutions in the form

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{U} + \mathbf{u}\,, \qquad P = p + \pi\,.$$

We then find the equation

$$\begin{gathered}\frac{\partial}{\partial t}u_i + (\mathbf{U}\boldsymbol{\nabla})u_i + (\mathbf{u}\boldsymbol{\nabla})U_i + (\mathbf{u}\boldsymbol{\nabla})u_i = -\nabla_i p + \nu\Delta u_i\,,\\ \nabla_i u_i = 0\,.\end{gathered} \tag{15}$$

Equation (15) is a stochastic version of the Navier–Stokes equation, with stochasticity implemented through the imposed large-scale random flow, and not a large-scale force. Pulsations of the velocity $\mathbf{U}$ stand here as parametric noise. Similarly prescribed velocity pulsations, playing the role of parametric noise, were previously explored in the framework of the linear problem of passive scalar transport in Refs [45–47], where they were assumed to be Gaussian. It was shown that the scalar correlation functions exhibit intermittency.

We note that Eqn (15) is a reformulation of Eqn (12), but the force $\mathbf{F}$ is absent from the final equation, and all randomness is incorporated in terms of $\mathbf{U}$. Certainly, this way of implementing $\mathbf{F}$ somewhat narrows its range. But this is not a more voluntary action than that inherent in the mere assumption of the existence of a random force acting in the entire fluid volume.

2.3 Small-scale limit

We now simplify Eqn (15). We first note that the drift part of the velocity $\mathbf{U}(0, t)$ drops from the final result because all correlators in homogeneous isotropic turbulence depend on the difference of coordinates. For this reason, the drift part of the velocity $\mathbf{U}(0, t)$ can be set to zero without any loss of generality by an appropriate choice of the reference frame (i.e., the transformation $\mathbf{U} \to \mathbf{U} - \mathbf{U}(0, t)$, $\mathbf{r} \to \mathbf{r} - \int \mathbf{U}(0, t)\,\mathrm{d}t$).

[4] This is manifested in the necessity of keeping the order of limits: first $t \to \infty$ and then $\nu \to 0$.

[5] An example of a dynamical system with such behavior is a nonlinear oscillator. The appearance of an external force leads to a phase shift, such that $\langle \mathbf{F}\mathbf{v} \rangle = 0$, instead of systematic energy growth.

In what follows, we are interested in domains of a size $r \ll L$; a smooth function $\mathbf{U}$ inside such a domain can be expanded in the Taylor series in r/L,

$$U_i(\mathbf{r}, t) = A_{ij}(t) r_j + A_{ijk}(t)\,\frac{r_j r_k}{L} + A_{ijkl}(t)\,\frac{r_j r_k r_l}{L^2} + \dots\,, \qquad A_{ii} = 0\,. \tag{16}$$

The coefficients $A_{ijk}(t)$, $A_{ijkl}(t), \dots$ have the dimension of inverse time and are determined by the characteristic turnover time T of large-scale eddies induced by U.

We now pass to the limit $L \to \infty$ while keeping the time T fixed. All terms in the right-hand side of Eqn (16), except the first, then vanish.[6] As a result, from Eqn (15), we obtain

$$\frac{\partial}{\partial t}\, u_i + (A_{kj} r_j \nabla_k) u_i + A_{ik} u_k + (\mathbf{u}\boldsymbol{\nabla}) u_i = -\nabla_i\, p + \nu \Delta u_i\,, \qquad \nabla_i\, u_i = 0\,. \tag{17}$$

This is the main equation of our theory. It is an exact consequence of the Navier–Stokes equation in the limit $r \ll L$. The large-scale strain rate tensor A_{ij} replaces the external force. With the statistics of this tensor specified, the statistical properties of small-scale pulsations can be computed.

3. Asymptotic analysis of the modified stochastic Navier–Stokes equation in the limit of vanishing viscosity

We now turn to the analysis of Eqn (17) in the limit of large time t. We do not immediately seek a stationary solution, as is done in the standard approach, where the limit $t \to \infty$ is necessarily taken at finite viscosity. We can therefore first take the limit $\nu \to 0$. The contribution of dissipation is discussed in Section 4.3.

We therefore consider the Euler equation instead of the Navier–Stokes one, which implies setting $\nu = 0$ in Eqn (17).

As mentioned in Section 1.4, a change in the velocity $\mathbf{u}$ under the action of the external field A is given by a superposition of random rotations and systematic deformations. Our first task is to separate these two components.

We first eliminate linear terms; for this, we introduce the variable transformation $\mathbf{r}, \mathbf{u} \to \mathbf{X}, \mathbf{w}$,

$$u_i(\mathbf{r}, t) = g_{i\mu}(t)\, w_\mu(\mathbf{X}_\nu, t)\,, \qquad X_\nu = q_{\nu j}(t) r_j\,, \tag{18}$$

where the matrices $g_{i\mu}(t)$ and $q_{\nu j}(t)$ satisfy the equations

$$\dot{g}_{i\alpha} + A_{ij}\, g_{j\alpha} = 0\,, \qquad g_{i\alpha}(0) = \delta_{i\alpha}\,, \qquad \dot{q}_{\gamma i} + q_{\gamma j} A_{ji} = 0\,, \qquad q_{\gamma j}(0) = \delta_{\gamma j}\,. \tag{19}$$

(Latin indices correspond to the old reference frame and the Greek correspond to the new one.)[7]

Substituting relations (18) in Eqns (17), we obtain

$$g_{i\mu}\left(\frac{\partial w_\mu}{\partial t} + q_{\kappa\gamma}\, g_{\gamma\alpha} w_\alpha\, \frac{\partial w_\mu}{\partial X_\kappa}\right) = -q_{\nu i}\, \frac{\partial p}{\partial X_\nu}\,, \qquad q_{\nu i} g_{i\mu}\, \frac{\partial w_\mu}{\partial X_\nu} = 0\,. \tag{20}$$

Because A_{ij} is a random process, the matrices $g_{i\mu}$ and $q_{\mu j}$ are also random. To analyze the solution as $t \to \infty$, we need to know their asymptotic behavior.

3.1 Asymptotic behavior of T-exponentials

It follows from Eqn (19) that the matrices $g_{i\mu}$ and $q_{\mu j}$ can be represented in the form of T-exponentials. When the matrix A is a random Gaussian process, these quantities are computed in Ref. [48]. However, as we see in Section 5.2, this limitation on A is not satisfactory in the turbulence theory: in this case, the energy flux transferred by large-scale pulsations into smaller scales would be equal to zero. To analyze the solution in the general case of arbitrary A, we consider a discrete approximation: a sequence of nth time instants separated by intervals Δt. Let the components $A_{ij}(t) = (A_n)_{ij}$ be constant within each interval. Then the solution of Eqn (19) takes the form

$$q_n = q_{n-1} \exp\left(-A_n \Delta t\right)\,,$$

whence

$$q_N = \exp\left(-A_1 \Delta t\right) \exp\left(-A_2 \Delta t\right) \dots \exp\left(-A_N \Delta t\right)\,. \tag{21}$$

To proceed, we need the Iwasawa decomposition, stating that any real-valued unimodular matrix q can be represented as

$$q = z(q)\, d(q)\, s(q)\,, \tag{22}$$

where z is an upper triangular matrix with diagonal elements equal to unity, d is a diagonal matrix with positive eigenvalues, and s is a rotation matrix.

According to Eqn (21), the matrix q_N is the product of N random real-valued unimodular matrices[8] realized by the same probability distribution. The asymptotic behavior of such objects is studied in detail in the mathematical literature (a brief review is available in Ref. [49]), and a set of important results has been derived. In particular, under plausible assumptions about the distribution of A, the following theorems have been proved:[9]

(1) the limit $\lim_{N\to\infty} (1/N) \ln d_i(q_N) = \lambda_i$ exists with probability one, where λ_i are some nonrandom values characterizing the random process A_n, which are the same for all its realizations; $\lambda_1 < \lambda_2 < \lambda_3$ (ordering arises due to the presence a of triangular matrix in the Iwasawa decomposition, which violates the equal status of the axes) and $\lambda_1 + \lambda_2 + \lambda_3 = 0$. The λ_i are known as the Lyapunov exponents [50];

(2) the distribution of $\eta_i = (\ln d_i(q_N) - \lambda_i N)/\sqrt{N}$ is asymptotically close to Gaussian and converges to it (weakly) as $N \to \infty$ [51, 52];

(3) $z(q_N)$ converges as $N \to \infty$: $z(q_N) \to z_\infty$ with probability one. Unlike the Lyapunov exponents, z_∞ depends on the individual realization of A_n [53];

[6] The limit transition done here implements a scale separation between $\mathbf{U}$ and $\mathbf{u}$. For the approximation to be valid, the spectrum of $\mathbf{u}$ should not 'drift' to the region of large scales as time progresses. As we see in Sections 3 and 4, this condition is satisfied for solutions that are of interest to us.

[7] We note that Eqns (18) define a transformation of the phase space, not the coordinate space, because velocities and coordinates transform differently.

[8] Unimodularity is a consequence of the zero trace of A_n, i.e., of the fluid incompressibility.

[9] We note that neither the symmetry nor the Gaussian character of the matrices A is needed to prove the theorems.

(4) asymptotically as $N \to \infty$, the random quantities $\eta_i(q_N)$ and $z(q_N)$ are independent [54].

For our purposes in what follows, theorems 1–4 can be briefly formulated as

$$z(q_N) \to z_\infty\,, \qquad \eta_i(q_N) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_i(n)\,,$$
$$d(q_N) = \operatorname{diag}\left(\mathrm{e}^{\lambda_1 N + \eta_1\sqrt{N}},\, \mathrm{e}^{\lambda_2 N + \eta_2\sqrt{N}},\, \mathrm{e}^{\lambda_3 N + \eta_3\sqrt{N}}\right), \tag{23}$$
$$\lambda_1 < \lambda_2 < \lambda_3\,, \qquad \lambda_1 + \lambda_2 + \lambda_3 = 0\,.$$

Here, $\eta_i(q_N)$ and ξ_i are stationary random processes with a zero mean. It should be remembered that the matrix z_∞ is a random quantity and that λ_i are constants depending only on the statistical characteristics of the random process A_n. The matrix $s(q_N)$ is a rapidly varying function of N, strongly dependent on a concrete realization.

The transition to the limit $N \to \infty$ corresponds to $t \to \infty$ with Δt kept fixed. To return from the discrete description to the continuous one, it is then necessary to also take the limit $\Delta t \to 0$. In that case, N is replaced by t, λ_i and ξ_i are renormalized, and the exponential factors in Eqns (23) takes the form $\lambda_i t + \int \xi_i \,\mathrm{d}t$.

3.2 Asymptotic form of the solution of Eqn (20)

We turn to Eqn (20). To simplify the presentation, we limit ourselves to considering symmetric matrices A_{ij}. We show in what follows that this limitation is not essential. From the symmetry $A_{ij} = A_{ji}$, it follows that

$$g_{i\mu}(t) = q_{\mu i}(t)\,.$$

Equation (20) then becomes

$$\frac{\partial w_\mu}{\partial t} + q_{\kappa i}\, g_{i\alpha} w_\alpha \frac{\partial w_\mu}{\partial X_\kappa} = -\frac{\partial p}{\partial X_\mu}\,,$$
$$q_{\nu i}\, g_{i\mu} \frac{\partial w_\mu}{\partial X_\nu} = 0\,. \tag{24}$$

We note that Eqns (24) involve only the matrix combination $q_{\mu i}\, g_{i\nu}$. Remarkably, the factor $s(q)$ drops out of this combination:

$$qg = qq^{\mathrm{T}} \simeq z_\infty d^2\big[q(t)\big] z_\infty^{\mathrm{T}}\,,$$
$$d^2\big[q(t)\big] = \exp\left(2\lambda_3 t + 2\eta_3\sqrt{t}\,\right) \operatorname{diag}(0,0,1) + \ldots\,.$$

We keep only the fastest growing component here. Because $\lambda_1 + \lambda_2 + \lambda_3 = 0$, it follows that $\lambda_3 > 0$. Neglecting the terms growing more slowly, we find

$$qg = C \exp\left(2\lambda_3 t + 2\eta_3\sqrt{t}\,\right) \equiv C \exp(2\chi t)\,,$$

where [up to terms $O(\exp[(\lambda_2 - \lambda_3)t])$] C is a constant (independent of t) random symmetric matrix, $C = z_\infty \operatorname{diag}(0,0,1)\, z_\infty^{\mathrm{T}}$.

We now introduce a new vector $\mathbf{W} = C\mathbf{w}$ instead of $\mathbf{w}$. From Eqns (24), we then obtain

$$\frac{\partial \mathbf{W}}{\partial t} + \exp(2\chi t)\left(\mathbf{W}\frac{\partial}{\partial \mathbf{X}}\right)\mathbf{W} = -C\,\frac{\partial P}{\partial \mathbf{X}}\,, \qquad \frac{\partial \mathbf{W}}{\partial \mathbf{X}} = 0\,,$$
$$\chi = \lambda_3 + \frac{\eta_3}{\sqrt{t}}\,.$$

The asymptotic form as $t \to \infty$ becomes

$$\left(\mathbf{W}\frac{\partial}{\partial \mathbf{X}}\right)\mathbf{W} = -C\,\frac{\partial}{\partial \mathbf{X}}\,\Pi\,, \quad \frac{\partial \mathbf{W}}{\partial \mathbf{X}} = 0\,,$$
$$P = \exp(2\chi t)\Pi\,. \tag{25}$$

The matrix C is symmetric and independent of time, and hence it can always be diagonalized by a suitable rotation of the coordinate system. Therefore, Eqn (25) is equivalent to the stationary Euler equation, while its solutions correspond to different stationary configurations of the hydrodynamic flow.

We note that the entire random process $A_{ij}(t)$ is reduced in Eqn (25) to six random quantities forming the matrix C. Certainly, this has happened owing to the special choice of the variables $(\mathbf{X}, \mathbf{W})$: all the randomness was 'absorbed' into the s-components of the matrices q and g, which vary rapidly with time.

Next, the right-hand side of Eqn (25) contains pressure, but situations in which pressure grows exponentially in some domain, becoming infinite in the end, are highly improbable. As we see in Section 6.2, this is only possible for special solutions that correspond to extreme vortices. Therefore, in general, it is natural to expect that $\Pi \to 0$ as $t \to \infty$. Thus, we arrive at the degeneration of the nonlinearity in Eqn (25):

$$\left(\mathbf{W}\frac{\partial}{\partial \mathbf{X}}\right)\mathbf{W} = 0\,. \tag{26}$$

This agrees with the results in Ref. [20] on the 'suppression' (disappearance) of nonlinearity. As we see in Section 3.3, Eqn (26) describes the stretching of a vortex filament. The pressure, although it plays a decisive role in the balance of forces in directions transverse to the filament [55], does not contribute to the vortex stretching. The absence of the effect of nonlinearity on the exponential growth of vorticity was first discussed in Ref. [40] and was substantiated there by physical considerations.

We now return to the general case $A \neq A^{\mathrm{T}}$. Equation (20) contains two random matrices, qg and $g^{-1}q^{\mathrm{T}}$. According to definition (19), the matrix g can be represented as

$$g(A) = \left(q(A^{\mathrm{T}})\right)^{\mathrm{T}}\,. \tag{27}$$

Here and hereafter, we let $q(A)$, $z(A)$, $d(A)$, $s(A)$ denote the functionals $q[A(t)]$, $z[A(t)]$, ... Taking Eqn (27) into account and using the Iwasawa decomposition (22) again, we obtain

$$qg = z(A)\, d(A)\, s(A)\, s^{\mathrm{T}}(A^{\mathrm{T}})\, d(A^{\mathrm{T}})\, z^{\mathrm{T}}(A^{\mathrm{T}})\,.$$

The two rotation matrices do not cancel this time, but give one more rotation matrix $\tilde{S}$ as their product. The fastest growing components d of all matrices are the matrix elements d_{33}; therefore, in the product $d(A)\tilde{S}d(A^{\mathrm{T}})$ in the limit $t \to \infty$, the $(3,3)$ matrix element proportional to their product plays the leading role. After left and right multiplication by triangular matrices z, we obtain the asymptotic from

$$qg \simeq \alpha(t) C_1 \exp\left[\left(\chi(A) + \chi(A^{\mathrm{T}})\right)t\right]\,, \tag{28}$$

where C_1 is a time-independent matrix and $\alpha(t) = \tilde{S}_{33}$ is a rapidly varying random factor.

To find the asymptotic form of the matrix $g^{-1}q^{\mathrm{T}}$, we use the equality $g(A)q(-A) = I$, which obviously follows from

(21), and an analogous equation for g. Then

$$g^{-1}q^{\mathrm{T}} = q(-A)q^{\mathrm{T}}(A) = z(-A)d(-A)s(-A)s^{\mathrm{T}}(A)d(A)z^{\mathrm{T}}(A)\,.$$

From considerations fully analogous to those given above, it follows that

$$g^{-1}q^{\mathrm{T}} \simeq \beta(t)C_2 \exp\left[\left(\chi(-A)+\chi(A)\right)t\right], \tag{29}$$

where C_2 is a constant matrix and $\beta(t)$ is a random function.

To see which of the terms containing (28) and (29) increases faster, we need to compare nonrandom parts of their exponentials. Hence, the term with pressure in Eqn (20) can be neglected (with probability one) if $\lambda_3(A^{\mathrm{T}}) > \lambda_3(-A)$.

It can be shown that $\lambda_3(-A) = -\lambda_1(A^{\mathrm{T}})$ [43, 56]. Additionally, excluding some pathological cases, $\lambda_i(A^{\mathrm{T}}) = \lambda_i(A)$. For this reason, the condition written above is satisfied if and only if $\lambda_2(A) < 0$. As we see in Section 5.2, $\lambda_2 = 0$ corresponds to a flow symmetric under time reversal. If this degeneracy does not occur (and in the case of a nonzero energy flux toward small scales, the T-symmetry is undoubtedly broken), then (28) and (29) increase at different rates. In the general case, there has been no success in relating the sign of the last inequality to the direction of the energy flux; however, in the framework of the simplified model considered in Sections 4 and 5, the choice $\lambda_2(A) < 0$ corresponds to the correct sign of the energy flux. Thus, we once again arrive at the situation in Eqn (26).

3.3 Properties of the solution obtained

To clarify the general properties of solution (25), we return to the variables $(\mathbf{r}, \mathbf{u})$. We once again limit ourselves to the case of symmetric matrices $A = A^{\mathrm{T}}$. The general case is analyzed similarly and leads to the same results; some distinctions are identified below. Additionally, in this section, we neglect the random noise in the exponential of the matrix d. This approximation is justified in Section 6, while the contribution of η to the solution is discussed in Section 5.

Taking relations (18), (19), and (22) into account, we find

$$\mathbf{u} = g\mathbf{w} = gC^{-1}\,\mathbf{W}(\mathbf{X}) = q^{\mathrm{T}}C^{-1}\,\mathbf{W}(\mathbf{X}) = s^{\mathrm{T}}dz^{\mathrm{T}}C^{-1}\,\mathbf{W}(\mathbf{X})\,,$$

$$\mathbf{X} = q\mathbf{r} = zds\mathbf{r}\,.$$

To separate a random rotation, we perform one additional transformation of variables:

$$\mathbf{r}' = s\mathbf{r}\,, \qquad \mathbf{u}' = s\mathbf{u}\,. \tag{30}$$

This reference frame performs random rotations because the matrix s is a random function of time (in contrast to matrices z and d, which become more regular as time progresses). Instead of the vector $\mathbf{W}(\mathbf{X})$, we introduce the new vector function

$$\mathbf{V}(y) = z^{\mathrm{T}}C^{-1}\,\mathbf{W}(zy)\,.$$

Then

$$\mathbf{u}' = d\mathbf{V}(d\mathbf{r}')\,, \tag{31}$$

or in a more detailed form,

$$u_i' = \exp(\lambda_i t)\, V_i\big(\exp(\lambda_1 t)r_1', \exp(\lambda_2 t)r_2', \exp(\lambda_3 t)r_3'\big)$$

(we note the absence of summation here).

Thus, in the rotating reference frame $\mathbf{r}'$ in the asymptotic limit as $t \to \infty$, the solution ceases to be random. Indeed, the third velocity component, u_3', dominates, and the fluid element stretches exponentially (and contracts accordingly) with different coefficients along different axes. In this case, the dependence of the velocity $\mathbf{u}'$ on r_3' is the strongest. We note, however, that the condition $\nabla\mathbf{u} = 0$ implies that $\partial V_3/\partial r_3 = 0$.

We now compute the velocity curl:

$$\omega_k' = \varepsilon_{kji}\,\frac{\partial u_i'}{\partial r_j'} = \varepsilon_{kji}\exp(\lambda_i t)\,\frac{\partial V_i}{\partial y_j}\exp(\lambda_j t)\,.$$

Because $\sum_i \lambda_i = 0$, it follows that $\omega_k \propto \exp(-\lambda_k t)$. Accordingly, the vorticity is oriented mainly along the r_1' axis:

$$\omega' \simeq \omega_1' = \exp(-\lambda_1 t)\, f\big(\exp(\lambda_3 t)\, r_3'\big)\,, \tag{32}$$

where $f = \varepsilon_{1ji}\,\partial V_i/\partial y_j$. We also note that because $\omega' = s\omega$, the absolute value of vorticity is the same in the rotating frame and the rest frame, $\omega = \omega'$.

All this remains valid for arbitrary matrices $A \neq A^{\mathrm{T}}$, although transformation of variables (30) is generalized to the transformation $\mathbf{r}' = s\mathbf{r}$, $\mathbf{u}' = s(A^{\mathrm{T}})\mathbf{u}$, which is no longer a rotation and does not have a previous illustrative meaning. In general, the relation $\mathbf{u}' = d(A^{\mathrm{T}})\,\mathbf{V}(d(A)\,\mathbf{r}')$ is satisfied instead of Eqn (31). However, from the isotropy of the distribution of A, it follows that $d(A) = d(A^{\mathrm{T}})$ and the result proves to be identical.

Thus, vorticity is carried from the boundaries to the center, exponentially increasing with time. However, this behavior is impossible everywhere in the volume. Otherwise, the total energy flux and dissipation would exponentially grow with time.

To maintain a stationary state, we must require that at some point in the volume of the fluid $r_3' \approx L$, the vorticity remains approximately constant,

$$\omega(t, L) \sim 1\,. \tag{33}$$

In reality, the weaker condition that there be no exponential growth is sufficient. This condition is realizable at points with typical behavior. But as we see in what follows, the choice of the condition leads to the difference only in multiplicative factors, but do not affect the power-law exponents in structure functions. For this reason, we use condition (33) for simplicity.

With boundary condition (33), we have $f(\exp(\lambda_3 t')L) \sim \exp(\lambda_1 t')$ for any time instant t'. Selecting t' so as to satisfy the equality $\exp(\lambda_3 t)\, r_3' = \exp(\lambda_3 t')L$, we can represent solution (32) as

$$\omega(t, r_3') \propto \left(\frac{r_3'}{L}\right)^{\lambda_1/\lambda_3}. \tag{34}$$

Because the derivation of Eqn (32) relies on asymptotic approximations from Section 3.1, the time t is required to be sufficiently large, such that Eqn (23) is valid for some $t > t_0$. Then Eqn (34) is valid for all times $t > t^*(r_3') = (1/\lambda_3)\ln(r_3'/L) + t_0$, or for $r_3' > L\exp[\lambda_3(t_0 - t)]$. For smaller r_3', the effect of boundary conditions is not yet noticeable and ω is determined by the initial conditions.

Thus, Eqn (34) does not imply the occurrence of a singularity. For any finite time instant, the vorticity distribution is smoothed in the vicinity of the maximum, with the size

of the smooth part decreasing with time. This notwithstanding, expression (34) ensures a power-law dependence of the structure functions. We therefore derived the power-law, or scaling (but not yet multi-scaling) behavior of vorticity directly from stochastic Euler equation (17) [43].

In Section 4, we discuss a simplified model of the solution obtained, which helps to better illustrate its properties and also clarify the role of viscosity. We then discuss multifractal properties of the solution.

4. Simplified model: solution details and inclusion of viscosity

We see that the solution of stochastic equation (17) is quasi-one-dimensional and nonrandom (predominantly) at large times in a specially selected reference frame (30). The randomness is incorporated into the matrix s 'controlling' rotations of the reference frame. To better understand the solution obtained, we consider the simple example of a one-dimensional deterministic flow with similar properties.

The idea is to 'straighten' the random flow by excluding rotations of the reference frame. For this, we fix a matrix A_{ij}. Additionally, for simplicity, we limit ourselves to the consideration of a one-dimensional small-scale velocity $u = u_y(x, t)$; the general case of $\mathbf{u}(x, y, z, t)$ leads to analogous results, but is more cumbersome. We see below that the nonlinearity disappears in this shear flow similarly to the general case, and the solution turns out to be exactly the same as (32).

Based on this model, we discuss the influence of viscosity and the mechanism leading to the appearance of a multifractal solution.

4.1 Simplified model without randomness

We consider the velocity field

$$v_x = -ax\,, \qquad v_y = -by + u(x, t)\,, \qquad v_z = -cz\,.$$

The quantities a, b, and c are parameters of the large-scale matrix A_{ij}. From the incompressibility, it follows that

$$a + b + c = 0\,.$$

For simplicity of the analysis, we assume that a and b are constant [although a solution can be obtained for arbitrary functions $a(t)$ and $b(t)$]. We also let

$$a > b > c\,, \qquad b < 0\,.$$

The Euler equation becomes

$$\begin{aligned} &a^2 x = -\frac{\partial p}{\partial x}\,,\\ &\frac{\partial u(x,t)}{\partial t} - ax\,\frac{\partial u(x,t)}{\partial x} - bu(x,t) + b^2 y = -\frac{\partial p}{\partial y}\,,\\ &c^2 z = -\frac{\partial p}{\partial z}\,. \end{aligned} \tag{35}$$

Because the pressure gradient is linear in the variables x, y, z, it follows that

$$p(\mathbf{r}, t) = \frac{a^2}{2}\,x^2 + \frac{b^2}{2}\,y^2 + \frac{c^2}{2}\,z^2\,,$$

which corresponds to definition (14). We still have a y-independent part of the second equation in (35), which is written as

$$\frac{\partial u(t,x)}{\partial t} - ax\,\frac{\partial u(t,x)}{\partial x} - bu(t,x) = 0\,. \tag{36}$$

Equation (36), describing the evolution of the small-scale component of velocity, is analogous to Eqn (17). In the inertial range, the velocities are small (compared to the large-scale ones), but the vorticities are large, and we therefore proceed with discussing vorticity instead of velocity. Because $\omega(t, x) \equiv \omega_z = \partial u/\partial x$, the corresponding equation takes the form

$$\frac{\partial \omega}{\partial t} - ax\,\frac{\partial \omega}{\partial x} + c\omega = 0\,. \tag{37}$$

Equations (36) and (37) can be obtained from Eqn (17) in the particular case of a constant diagonal matrix A and a one-dimensional shear velocity $u(x, t) = u_y$. However, it can be easily seen that Eqn (37) is satisfied by any function for which condition (32) holds if we take

$$a = \lambda_3\,, \qquad c = \lambda_1\,, \qquad r_3' = x\,.$$

Thus, solutions of Eqn (37) show a dependence of the vorticity modulus on the variable r_3' in the general case.

We analyze the solution of Eqn (37) in the domain $x \in [0, 1]$ for $t \geqslant 0$. We specify the boundary condition as

$$\omega(t, 1) = 1\,, \tag{38}$$

which is an analog of Eqn (33). To satisfy boundary condition (38), the initial condition $\omega(0, x) = \omega_0(x)$ must be such that

$$\omega_0(1) = 1\,, \qquad a\,\frac{\partial \omega_0}{\partial x}\,(1) - c = 0$$

(assuming all quantities in this equality to be defined, i.e., $t \geqslant t' \geqslant 0$, $x \exp[a(t - t')] \leqslant 1$). As can be seen from Eqn (39), the vorticity, while increasing, 'propagates' from larger to smaller x.

For any $x > \exp(-at)$, with $t'(x, t)$: $x = \exp[-a(t - t')]$ we obtain

$$\begin{aligned} &\omega(t, x) = \exp\left[-c(t - t')\right]\omega(t', 1) = x^{c/a}\,,\\ &x > \bar{x}(t) = \exp(-at)\,. \end{aligned} \tag{40}$$

Thus, ω is determined in this domain by the boundary; it is a power-law function of x, independent of time.

For small values of x, we cannot use this procedure because the corresponding t' would be negative. The choice $t' = 0$ gives

$$\omega(t, x) = \exp(-ct)\omega_0\big(x \exp(at)\big)\,, \qquad x < \bar{x}(t)\,. \tag{41}$$

The effect of boundary conditions does not propagate over this internal domain, and the profile of ω in it is still determined by the initial conditions. Therefore, everywhere except a narrowing internal domain, the function $\omega(x)$ is constant and follows a power law; but for small x, there is always a narrow region of transient behavior, and the vorticity at the center $(x = 0)$ increases unboundedly with

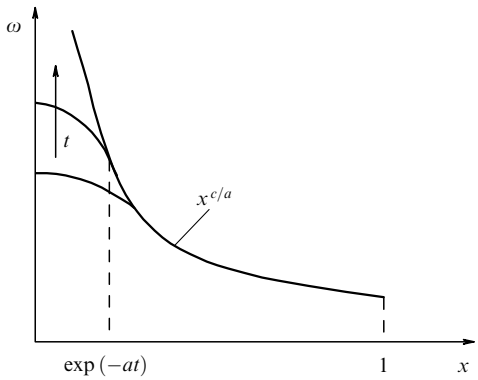

Figure 6. Illustration for Eqns (40) and (41): the evolution of the spatial distribution of vorticity with time.

time, but is finite at any time instant (Fig. 6). A quite similar picture occurs in the general case (32), (34).

What happens if we select different boundary conditions? Let the vorticity not be fixed at the boundary as in Eqn (38), but depend on time in an arbitrary way, $\omega(t,1) = f(t)$. Then, for a sufficiently large t, choosing $t'(t,x)$: $x = \exp[-a(t-t')]$, we obtain

$$\omega(t,x) = x^{c/a} f\left(t + \frac{1}{a}\ln x\right) \underset{t\to\infty}{\to} x^{c/a} f(t)$$

for any given x. Hence, any plausible functions f (increasing less rapidly than exponentials) do not modify the power law but only change the coefficient, which acquires a time dependence (Fig. 7). Because boundary conditions correspond to large scales, the characteristic time of variability agrees by the order of magnitude with the large-eddy turnover time. This is also valid in the general case (34).

4.2 Evolution of the spectrum

We now discuss solution (40), (41) in terms of the Fourier transform. This is useful because it allows relating the result obtained to the idea of a cascade, and, simultaneously, taking viscosity into account.

The Fourier transform of vorticity is [10]

$$\omega(t,k) = \exp(-ct)\int_0^{\bar{x}(t)} \exp(\mathrm{i}kx)\,\omega_0\big(x\exp(at)\big)\,\mathrm{d}x + \int_{\bar{x}(t)}^{\infty} \exp(\mathrm{i}kx)x^{c/a}\,\mathrm{d}x\,.$$

The first integral, which can be represented in the form

$$\exp(-ct)\exp(-at)\int_0^1 \exp\big[\mathrm{i}ky\exp(-at)\big]\omega_0(y)\,\mathrm{d}y \simeq \exp(bt)\,\omega_0\big(k\exp(-at)\big)\,,$$

weakly depends on k for all $k < \bar{k} = \exp(at)$ and decays with time exponentially. (It should be remembered that $a > 0$ and $\bar{k} \gg 1$ as $t \to \infty$.) The second integral, which can be written as

$$k^{b/a}\int_{k\bar{x}(t)}^{\infty} \exp(\mathrm{i}y)\,y^{c/a}\,\mathrm{d}y\,,$$

is a power-law function for $k < \exp(at)$ and rapidly decays for large k.

We see that the function $\omega(t,k)$ has a stepwise profile, with the step moving to the right (into the domain of large k) exponentially fast.

To visually illustrate this property of the solution, we consider a particular case where the Fourier transform can readily be computed analytically. We do not insist now that boundary condition (38) be satisfied, because we have verified that it suffices to require the absence of exponential growth of $\omega(t,1)$. Accordingly, if this is the case, we need not stay in the interval $0 \leqslant x \leqslant 1$.

For example, we take the initial distribution of vorticity in the form

$$\omega_0(x) = (1+\mathrm{i}x)^{c/a} + (1-\mathrm{i}x)^{c/a}\,.$$

In agreement with Eqns (37) and (39), the evolution of $\omega(t,x)$ becomes

$$\omega(x,t) = \exp(-ct)\Big[\big(1+\mathrm{i}\exp(at)x\big)^{c/a} + \big(1-\mathrm{i}\exp(at)x\big)^{c/a}\Big] = 2\exp(-ct)\big(1+x^2\exp(2at)\big)^{c/2a}\cos\frac{\phi c}{a}\,,$$

where $\phi = \arctan[x\exp(at)]$. (For $x \gg \exp(-at)$, we have $\phi \simeq \pi/2$, $\omega \propto x^{c/a}$, a power-law dependence once again.) The Fourier transform of this function has the simple form

$$\omega(k,t) = |k|^{b/a}\exp\big[-|k|\exp(-at)\big]\,. \tag{42}$$

The spectrum is exponentially suppressed at $k \sim \bar{x}^{-1} = \exp(at)$. The same tapering occurs for nonzero viscosity, but in the case of dissipation, the boundary of the cutoff does not depend on time; here, the spectral boundary continually moves toward large values of k.

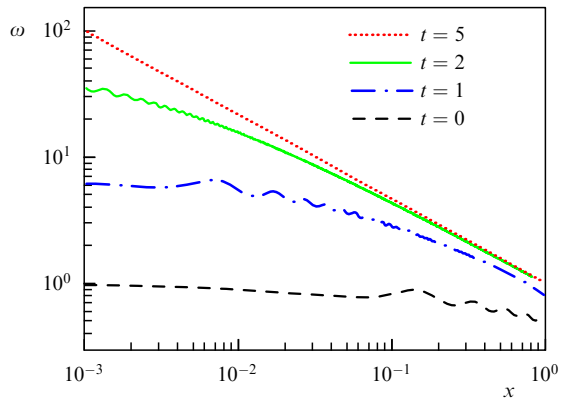

Figure 7. The evolution of the vorticity distribution in the particular case $a = 3$, $b = -1$,

$$\omega_0(x) = \big\{1 + \big[x + 0{,}1\sin(10\pi x)\big]^{2/3}\big\}^{-1},$$
$$\omega(t,1) = \big\{\exp(-2t) + \big[1 + 0{,}1\exp(-3t)\sin\big(10\pi\exp(3t)\big)\big]^{2/3}\big\}^{-1}.$$

The domain of strong oscillations displaces with time toward smaller x; in the inertial range, the fluctuations become negligibly small, and a power law sets in.

[10] In Section 4.1, we bounded x by the interval $[0,1]$, and therefore we needed in fact to discuss the Fourier series instead of the Fourier integral. However, $x = 1$ corresponds to the scale L, meaning that we are interested in the short-wave limit $k \gg 1$, which is equivalent to $L \to \infty$. Moreover, we extend the range of x to all positive values in what follows.

Such a stepwise function, propagating toward large k, is commonly interpreted as a cascade of reconnecting (breaking) eddies. We see that here such a solution occurred for rather different reasons. Nevertheless, energy is transferred toward the range of small scales. In Section 5.2, we revisit the discussion of the energy transfer process.

It is commonly believed that dissipation is indispensable to warrant a statistically stationary picture of turbulence. And yet, the model considered here indicates that the stationary spectrum (and stationary correlators of velocity and vorticity, likewise) can be obtained in a certain domain of scales, even in the absence of dissipation.

4.3 The influence of viscosity

Equation (35) can easily be generalized by including viscosity in the consideration. We add the term $\nu\Delta u$ to the right-hand side of Eqn (36); because the velocity depends only on one argument, this equation becomes

$$\frac{\partial u(x,t)}{\partial t} - ax\,\frac{\partial u(x,t)}{\partial x} - bu(x,t) = \nu\,\frac{\partial^2 u}{\partial x^2}\,.$$

The corresponding term with viscosity should also be added to the right-hand side of Eqn (37). Passing to the new variables $q = x\exp(at)$, we obtain

$$\frac{\partial\omega(q,t)}{\partial t} + c\omega(q,t) = \nu\exp(2at)\,\frac{\partial^2\omega}{\partial q^2}\,.$$

The last equation is easily solved with the help of the Fourier transformation:

$$\omega(q,t) = \int\tilde{\omega}(\tilde{k},t)\exp(\mathrm{i}\tilde{k}q)\,\mathrm{d}\tilde{k}\,,$$

$$\tilde{\omega}(\tilde{k},t) = C_1(\tilde{k})\exp\left\{-ct - \frac{\nu}{2a}\tilde{k}^2\left[\exp(2at) - 1\right]\right\}.$$

Returning to the Fourier transform $\omega(x,t)$, we find

$$\omega(x,t) = \int\omega(k,t)\exp(\mathrm{i}kx)\,\mathrm{d}k\,,$$

$$\omega(k,t) = \exp(bt)\,\omega_0\big(k\exp(-at)\big) \times\exp\left\{-\frac{\nu}{2a}\,k^2\left[1-\exp(-2at)\right]\right\}.$$

In particular, in the example considered in Section 4.2, with viscosity taken into account, we obtain

$$\omega(k,t) = |k|^{b/a}\exp\left[-|k|\exp(-at)\right] \times\exp\left\{-\frac{\nu}{2a}\,k^2\left[1-\exp(-2at)\right]\right\}.$$

It can be seen that the viscosity and the presence of a narrowing domain of exponential growth at small x give similar results in all cases: both effects lead to an exponentially sharp spectral cut-off. But nonsteadiness creates a 'moving step' for $k \sim \exp(at)$, while the drop-off created by viscosity is steeper; moreover, for times $t \gtrsim 1/2a$, it equilibrates for $k \sim \sqrt{2a/\nu}$.

We conclude that viscid and inviscid solutions do not differ in the range of wave numbers $|k| < \exp(at)$, $|k| < \sqrt{2a/\nu}$. Even if perturbations cannot be smoothed in the absence of viscosity, they are transferred toward smaller wave numbers. As a result, solutions of the Euler and Navier–Stokes equations coincide in the limit $t \to \infty$, $\nu \to 0$. In this sense, the Euler equation can be treated as an inviscid limit of the Navier–Stokes equation.

5. Fluctuations around the Lyapunov exponents and intermittency

5.1 Fluctuations of the exponents of the matrix *d*

In Section 4, we considered the model in which large-scale velocity fluctuations were nonrandom and the matrix A was constant. We have seen that this model leads to a power-law dependence of the vorticity on coordinate (40) and thus provides a scaling (power-law) behavior of correlators. However, in real turbulence, one encounters multi-scaling instead of scaling, because the power-law exponents of structure functions and correlators depend on their order nonlinearly (see Fig. 1). Such a picture can be obtained if the random character of the matrix A is taken into account.

Indeed, according to the theorems quoted in Section 3.1, the elements of the diagonal matrix $d(q_N)$ have the form

$$d_i = \exp\left(\lambda_i t + \int\xi_i(t)\,\mathrm{d}t\right),$$

where $\xi_i(t)$ are stationary random processes with zero means. If we set the variance of ξ_i to zero, as we have seen, the Lyapunov exponents ensure a power-law distribution of vorticity; the fluctuations of ξ_i can be the rationale of multi-scaling. A rigorous analysis of stochastic equation (17) with account for these fluctuations can, in our opinion, result in a full theory. However, we limit ourselves to the generalization of the model considered in Section 4 in order to show, using it as an example, how the fluctuations lead to intermittency.

Before passing to the analysis of solutions, it is worth keeping in mind that according to the central limit theorem, if $t - t'$ is sufficiently large, the random quantities η_i are Gaussian:

$$\eta_i = \frac{1}{\sqrt{t-t'}}\int_{t'}^{t}\xi_i(t'')\,\mathrm{d}t''\,.$$

(The factor $\sqrt{t-t'}$ is introduced for normalization.) Because $\det d = 1$ due to the incompressibility, it follows that $\eta_1 + \eta_2 + \eta_3 = 0$. Therefore, it suffices to consider two (generally speaking, not independent) quantities η_1 and η_3 with the covariance matrix

$$\Gamma_{ij} = \langle\eta_i\eta_j\rangle = \begin{pmatrix} D_1 & D_{13} \\ D_{13} & D_3 \end{pmatrix}.$$

The characteristic function $K(\mathbf{y}) = \langle\exp(\mathrm{i}\mathbf{y}\boldsymbol{\eta})\rangle$ is expressed in terms of Γ_{ij} as

$$K(\mathbf{y}) = \exp\left(-\frac{1}{2}\,y_i\Gamma_{ij}y_j\right), \qquad i,j\in\{1,3\}\,.$$

Below, we need the moment of the form

$$\big\langle\exp(\alpha_i\eta_i)\big\rangle = K(-\mathrm{i}\alpha_1,-\mathrm{i}\alpha_3) = \exp\left(\frac{D_1}{2}\,\alpha_1^2 + D_{13}\alpha_1\alpha_3 + \frac{D_3}{2}\,\alpha_3^2\right).$$

Eventually, the stochastic generalization of model (37) takes the form

$$\frac{\partial\omega}{\partial t} - \big(a + \xi_3(t)\big)x\,\frac{\partial\omega}{\partial x} + \big(c + \xi_1(t)\big)\omega = 0\,. \tag{43}$$

(We recall that in the simplified model, $a = \lambda_3 > 0$ and $c = \lambda_1 < 0$.) All the relations in Section 4 can easily be generalized to this case: in particular, Eqn (39) transforms into

$$\omega(t,x) = \exp\left[-c(t-t') - \int_{t'}^{t} \xi_1\,\mathrm{d}t\right] \times \omega\left(t', x\exp\left[a(t-t') + \int_{t'}^{t}\xi_3\,\mathrm{d}t\right]\right). \tag{44}$$

We first consider what happens to the solution at the center, $x = 0$. Setting $t' = 0$, we find

$$\omega(t,0) = \exp\left(-ct - \int_0^t \xi_1\,\mathrm{d}t\right)\omega(0,0)\,.$$

The moments of this quantity are expressed as

$$\big\langle\omega^n(t,0)\big\rangle = \exp\left(-nct\right)\big\langle\exp\left(-n\eta_1\sqrt{t}\,\right)\big\rangle\omega^n(0,0) = \exp\left(-nct + \frac{D_1}{2}\,n^2 t\right). \tag{45}$$

The exponential divergence characterizes the solution inside an unsteady internal region with growing vorticity (41). The width of this domain $\bar{x}$ is determined by the condition

$$\bar{x} \simeq \exp\left(-at - \eta_3\sqrt{t}\,\right).$$

As $t \to \infty$, the value of $\bar{x}$ exponentially decreases with probability one. Hence, adding noise to the Lyapunov exponents a and c adds to the speed of the vorticity growth at the center of the filament profile [the more so, the stronger the vorticity because of intermittency (45)] and reduces the rate of narrowing of the evolving domain.

We now turn to the external domain $x > \bar{x}(t)$. In analogy with Eqn (40), we choose $t'(x,t)$ such that

$$x = \exp\left[-a(t-t') - \int_{t'}^{t}\xi_3\,\mathrm{d}t\right]. \tag{46}$$

Then

$$\omega(t,x) = \exp\left[-c(t-t') - \int_{t'}^{t}\xi_1\,\mathrm{d}t\right]\omega(t',1)\,.$$

Expressing $\exp(t-t')$ in terms of x and η, we obtain

$$\omega(t,x) = x^{c/a}\exp\left[\int_{t'}^{t}\left(\frac{c}{a}\,\xi_3 - \xi_1\right)\mathrm{d}t\right].$$

For sufficiently small x ($(-\ln x)/a \gg 1$), the last expression can be represented as

$$\omega(t,x) = x^{c/a}\exp\left[\left(\frac{c}{a}\,\eta_3 - \eta_1\right)\sqrt{t-t'}\,\right]. \tag{47}$$

The quantity $t'(x,t)$ is now random and is defined by Eqn (46). Computing the mean of (47) now proves to be a difficult task, but for small x, a good approximation of the correct result can be obtained by replacing $t - t'$ with its mean value $\langle t - t'\rangle = (-\ln x)/a$. Then

$$\langle\omega^n\rangle \simeq x^{nc/a}\left\langle\exp\left[n\left(\frac{c}{a}\,\eta_3 - \eta_1\right)\left(\frac{-\ln x}{a}\right)^{1/2}\right]\right\rangle = x^{n\frac{c}{a} - \frac{n^2}{a}\left(\frac{c^2}{a^2}\frac{D_3}{2} - \frac{c}{a}D_{13} + \frac{D_1}{2}\right)}\,.$$

We have thus obtained a power-law dependence of the vorticity moments on the coordinate with the power-law exponent nonlinear in n. This scaling for the vorticity moments is naturally equivalent to the scaling of velocity structure functions

$$\big\langle\Delta v^n(l)\big\rangle \sim \langle\omega^n\rangle l^n \sim l^{\zeta_n}\,,$$
$$\zeta_n = -\frac{b}{a}\,n - \frac{1}{a}\left(\frac{c^2}{a^2}\,\frac{D_3}{2} - \frac{c}{a}\,D_{13} + \frac{D_1}{2}\right)n^2\,. \tag{48}$$

A similar dependence was obtained previously in Ref. [57] in the framework of the shell model of the Navier–Stokes equation.

Relations (48) explain intermittency, i.e., a nonlinear dependence of the exponents of structure functions on their order [4]. In this case, the part linear in n is determined by the Lyapunov exponents, and the nonlinearity is set by noise.

The nonlinearity of the power-law exponents of structure functions is described by a multifractal model. The form of the dependence $\zeta_n(n)$ is uniquely related to the fractal dimension $D(h)$ in Eqns (7) and (8). For this reason, the quadratic dependence (in the first approximation) derived here for the exponents defines the form of $D(h)$ and in that way serves as justification for the multifractal description.

5.2 Time reversibility

Unfortunately, the numerical values of Lyapunov exponents λ_i [and, accordingly, the coefficients in Eqn (48)] are not fixed by the theorems — they depend on the properties of large-scale fluctuations, and hence a separate study is needed for computing λ_i. However, it is possible to obtain some general bounds on the properties of large-scale pulsations. For example, it is shown in Refs [45, 58] for the polar decomposition of the matrices q and g that the coefficient λ_2 vanishes if the random matrices A_{ij} obey the Gaussian statistics. This is also true for the Iwasawa decomposition. Moreover, $\lambda_2 = 0$ for any statistically isotropic distribution, i.e., the distribution whose probability density is invariant under rotations R,

$$P(A) = P(RAR^{-1})\,, \qquad \forall\, R \in \mathrm{SO}(3)\,,$$

if the additional condition [56]

$$P\big[A_{ij}(t)\big] = P\big[-A_{ij}(t)\big]$$

is fulfilled. It should be remembered that the quantity A_{ij} is defined by relation (16) as $\partial U_i/\partial r_j$, where $\mathbf{U}$ is the large-scale flow velocity. For this reason, the transformation $A \to -A$ represents the time reversal. As a consequence, the symmetry $\lambda_2 = 0$, $\lambda_1 = -\lambda_3$ corresponds to the invariance under time reversal.

However, large-scale turbulent pulsations must give rise to the energy flux into the turbulent flow. This must violate the symmetry with respect to time. Indeed, the mean energy flux of large-scale pulsations passing through a sphere of

radius r is expressed as

$$\langle \Phi \rangle = \left\langle \int U^2 \mathbf{U}\, \mathrm{d}\mathbf{s} \right\rangle = \left\langle A_{ij} A_{im} A_{kp} \int r_j r_m r_p \frac{r_k}{r} r^2 \, \mathrm{d}\Omega \right\rangle \propto \langle \mathrm{tr}\, A^3 \rangle .$$

This flux should be directed inside the sphere for any r; hence, $\langle \mathrm{tr}\, A^3 \rangle < 0$. Currently, it is not clear precisely which limitations are imposed by the last requirement on λ_i, but for the simple model without random rotations considered above, it gives $\lambda_2 = b < 0$.

6. Velocity structure functions

6.1 Longitudinal and transverse correlators

The increased accuracy of experiments and numerical computations allows measuring the exponents of longitudinal and transverse structure functions of velocity (1) separately. It turns out that for $n > 3$, these exponents are different, and $\zeta_n^{\perp} < \zeta_n^{\parallel}$ [10–13]. Many theoreticians are skeptical about these results: it was thought before that structure functions of the same order have the same exponents. The rationale for this proposition was provided by two arguments: first, according to Kolmogorov's assumption, all correlators in a homogeneous and isotropic medium can depend only on one argument — the energy dissipation rate ε; second, by integrating the Navier–Stokes equation, a chain of equations connecting all possible (longitudinal and transverse) velocity structure functions can be derived [55, 59], and different kinds of structure functions in this chain are strongly 'entangled', such that their 'separation' seems to be difficult.

These arguments, however, do not look insurmountable and in all probability can be reconsidered: indeed, the first of them is related more to amplitudes than to the exponents, which do not depend on ε. When it became clear that the power-law exponents ζ_n depend on n nonlinearly, the naive assumption of the Kolmogorov model was replaced by the multifractal theory. The values of ζ_n in the multifractal theory are defined by the function $D(h)$, which cannot be found from dimensional considerations. Analogously, the difference between $\zeta_n^{\perp}$ and $\zeta_n^{\parallel}$, from the standpoint of the multifractal model, simply implies the existence of two different functions $D^{\parallel}(h)$ and $D^{\perp}(h)$ [34]. As concerns the second argument, the relation $S_n \propto l^{\zeta_n}$ is only valid in the formal limit $l \to 0$. In reality, for finite l, a series $S_n = \alpha_n l^{\zeta_n} + \beta_n l^{\zeta_n^{(2)}} + \ldots$ arises, where the unimportant terms have higher powers. For $\zeta_n^{\perp} < \zeta_n^{\parallel}$, the leading term is $S_n^{\perp} \propto l^{\zeta_n^{\perp}}$, and the quantity $S_n^{\parallel}$ can in fact be compensated by the next terms in the series for $S_n^{\perp}$.

From the perspective of the model of stretching vortices, the asymmetry between longitudinal and transverse motions not only does not seem strange but is in fact rather natural. Indeed, a vortex is an object whose transverse velocities are much higher than longitudinal. As an illustration, we consider a flow composed of randomly oriented cylindrical vortices performing a solid-body rotation — a set of 'tops'; then the longitudinal structure function turns out to vanish, and strong anisotropy is observed at each point. Simultaneously, the flow is statistically isotropic because of the random (isotropic) orientation of the top axes.

Admittedly, a real flow is composed of different eddies contributing differently to both types of structure functions (we discuss this in more detail in Section 6.2). But the local asymmetry inside each eddy fixed by the direction of the vorticity is in general a natural cause for the correlators to be different from each other.

The main objections against the observed difference in the power-law exponents $\zeta_n^{\parallel}$ and $\zeta_n^{\perp}$ amount to statements that this difference stems from the finiteness of the Reynolds number [60] or not fully eliminated large-scale anisotropy [61] in real experiments or numerical simulations and will be reduced with an improvement in their quality.

The problem is also complicated by the fact that the longitudinal and transverse structure functions of various orders demonstrate different dependences on the Reynolds number. For example, it is shown in Refs [9, 62] that structure functions of different orders have cutoffs at different scales in the presence of viscosity. This fact makes it difficult to propose a universal definition for the inertial range within which power-law dependences should be expected. For this reason, numerous authors warn that the difference between exponents has to be considered with caution [61].

Recently, however, evidence has been presented favoring the statement that the difference between structure function exponents is independent of the Reynolds number [13]. Analogous conclusions were expressed by the authors of earlier studies [11, 12].

As concerns large-scale anisotropy, which may affect the results of measurements (or computations) at small scales, it is indeed such that its influence is very difficult to avoid. From the standpoint of the vortex model, the major contribution to structure functions is made by filaments — vortex lines — and the higher the order of a function is, the more elongated filaments contribute to it. The length of these filaments may easily reach a scale of the order of the inhomogeneity. Such a filament, 'hooked' at the fluid boundary, may create anisotropy observed in experiments and numerical simulations, particularly noticeable in higher-order correlators. However, the inhomogeneity is not expected to essentially distort the internal eddy structure and, consequently, to distort the power-law exponents of structure functions, affecting only the pre-exponential factor. In that case, the anisotropy reduces upon an increase in the inhomogeneity scale L, whereas the difference between the longitudinal and transverse components is preserved.

We therefore suppose that the observed difference between the exponents $\zeta_n^{\perp}$ and $\zeta_n^{\parallel}$ does not come from the imperfectness of experiments or simulations, but is a real physical effect coming from the local anisotropy of a turbulent medium — the random orientation of the vorticity vector.

6.2 Finding the exponents of structure functions

We now apply the theory presented in Sections 2–5 to determine the velocity structure functions. The model of stretching vortex filaments assumes that different orders of structure functions are defined by different kinds of eddies. Each particular eddy has a power-law velocity profile, with the power-law exponent fixed by a random realization of random noise $\xi(t)$ characterizing large-scale pulsations.

In Section 5, in the framework of the simplified model, we derived expression (48) describing the power-law behavior of velocity structure functions:

$$S_n(l) \propto l^{\zeta_n} , \qquad \zeta_n = \alpha n - \beta n^2 . \tag{49}$$

Because the velocity increments were obtained from the vorticity, we are dealing with transverse structure functions.

The quadratic dependence of the power-law exponents ζ_n looks plausible: it satisfies the condition of being concave, $\zeta_n'' < 0$ (this constraint on ζ_n follows from the Hölder inequality for moments of random quantities [4]). However, the condition that the velocity be bounded, $\zeta_n' \geqslant 0$, implies that the applicability of Eqn (49) is limited by some maximum value $n_{\max} \leqslant \alpha/(2\beta)$. We find what the cause of this limitation is and how the the power-law exponents are built for $n > n_{\max}$.

Let a vortex filament have the velocity distribution $v \sim r^\gamma$ (and, consequently, the vorticity distribution $\omega \sim r^{\gamma-1}$). The smaller γ is, i.e., the higher the singularity of this distribution, the larger the contribution of this filament to the structure functions for large n. However, because of the finiteness of velocity (and pressure), γ cannot be negative.[11] Is a filament with $\gamma = 0$ possible? Yes, this 'extreme vortex filament' has a cylindrical symmetry and the velocity profile

$$\mathbf{v} = \left[\mathbf{e}_z, \frac{\mathbf{r}}{r}\right]. \tag{50}$$

This velocity distribution is associated with a logarithmically diverging pressure. This implies that for an arbitrarily small $h > 0$, there is a configuration with a converging pressure.

Integrating over $r\,\mathrm{d}r$ in an axially symmetric filament gives l^2 (see below for a more accurate expression); because $\Delta v \sim l^0$, the structure function defined by this extreme filament does not depend on n: $S_n(l) \propto l^2$ for any n. Hence, dependence (49) should tend to the horizontal asymptote with $\zeta_{n\to\infty} = 2$. This condition implies a constraint on the coefficients α and β in Eqn (49):

$$\zeta(n \geqslant n_{\max}) = \frac{\alpha^2}{4\beta} = 2\,.$$

As the second relation, we take the Kolmogorov four-fifths law, $\zeta_3 = 1$. Then the coefficients in Eqn (49) become defined uniquely as

$$\zeta_n^\perp = \begin{cases} 0.391n - 1.91\times 10^{-2}n^2\,, & n \leqslant 10.2\,, \\ 2\,, & n > 10.2\,. \end{cases} \tag{51}$$

All the above considerations can be naturally restated in terms of the multifractal model, and then the notions it deals with become physically transparent. Indeed, different kinds of eddies correspond to space decomposition into h-classes, with velocity fluctuations in each of them obeying the scaling law $\delta v(l,h) \sim l^h$, $l \to 0$ (see Section 1.3).

According to Eqn (7) (see also Fig. 5), large values of n are associated with small h. From the condition that velocities be bounded (or that ζ_n be nondecreasing), it follows that $h \geqslant 0$. Extreme vortex (50) found by us belongs to the class $h = 0$, which implies that $\delta v \propto l^0 = \text{const}$. As can be seen from Eqn (7), the fact that $h_{\min} = 0$ corresponds exactly to the presence of the horizontal asymptote ζ_n. The axial symmetry of filament (50) fixes the dimension of the corresponding h-class:

$$\zeta_\infty^\perp = 2\,, \qquad D^\perp(0) = 1\,. \tag{52}$$

The power-law exponents in (51) correspond to the quadratic dependence $D(h) = 3 - 2(1 - h/\alpha)^2$ in the range from $h_{\min} = 0$ to $h_{\max} = \alpha$.

[11] In reality, infinite pressure would have led to a continuity break, i.e., to the formation of a cavity in the vortex center.

Thus far, we have considered the transverse structure function. We now turn to the longitudinal one. We assume that relation (49), even though it was derived for transverse functions, also holds (albeit with other coefficients) for the power-law exponents of longitudinal structure functions.

Computations of longitudinal and transverse structure functions for profile (50) in the limit $n \to \infty$ can be carried out directly [63], giving the result

$$\left\langle \left| \Delta\mathbf{v} \times \frac{\mathbf{l}}{l} \right|^n \right\rangle \propto \frac{2^n}{n}\, l^2 \ln\frac{L}{l}\,, \qquad \left\langle \left| \Delta\mathbf{v}\, \frac{\mathbf{l}}{l} \right|^n \right\rangle \propto n^{-5/2} l^2\,. \tag{53}$$

The power-law exponent equals two in both cases. However, the contribution of filament (50) to the transverse structure function increases with n, becoming dominant as $n \to \infty$, whereas the contribution to the longitudinal function tends to zero. This is natural: the main motion in eddies is orbital, and therefore longitudinal velocity increments are small compared to the transverse ones. Large contributions to the longitudinal structure functions come from locations where vortex filaments are bent. Accounting for the bending would add a term of the next order in l/L to correlators (53). It would be the leading term in the longitudinal structure function as $n \to \infty$,

$$\zeta_\infty^\parallel = 3\,, \qquad D^\parallel(0) = 0\,. \tag{54}$$

In [63], a limit velocity distribution corresponding to $h = 0$ was constructed that determines the behavior of $D^\parallel(0)$ in the same fashion as profile (50) determines $D^\perp(0)$. Such an extreme filament for the longitudinal structure function should be strongly ('extremely') bent. For example, in spherical coordinates, the 'limit' configuration is $\mathbf{v} = (v_r(\theta), v_\theta(\theta), 0)$. In order to satisfy the condition $h = 0$, the solution, just as in the case of cylindrical extreme filament (50), does not depend on r, but now r is the distance to the center of the sphere, and not to the axis. Averaging over $r^2\,\mathrm{d}r$, we find that the structure function is proportional to l^3.

The difference between two limit values, given by expressions (52) and (54), determines practically the full difference between the functions $\zeta_n^\perp$ and $\zeta_n^\parallel$ as $n \to \infty$.

Selecting expression (54) as the condition imposed on the coefficients α and β in Eqn (49), with $\zeta_3^\parallel = 1$, we obtain

$$\zeta_n^\parallel = \begin{cases} 0.367n - 1.12\times 10^{-2}n^2\,, & n \leqslant 16.3\,, \\ 3\,, & n > 16.3\,. \end{cases} \tag{55}$$

A comparison of theoretical predictions (51) and (55) with the results of numerical simulations [8, 9] is plotted in Fig. 8. The theoretical curve proves to lie very close to the results of numerical simulations, within the measurement errors. We stress that (51) and (55) neither involve fitting nor contain a tunable parameter.

We have to admit that our simple model encounters a certain difficulty: the two parabolas shown in Fig. 8 coincide at the points $n = 0$ and $n = 3$ and consequently fail to do so at $n = 2$. This contradicts the rigorous theoretical statement that $\zeta_2^\parallel = \zeta_2^\perp$. This difficulty arises from the approximation of ζ_n by a quadratic dependence. Certainly, it accounts for only the first terms of the expansions; there are terms in higher orders. However, the difference between $\zeta_2^\parallel$ and $\zeta_2^\perp$ in (51) and (55) is very small, amounting 1.6×10^{-2}. The coefficient at the next, cubic term of the expansion (and, likewise, the corrections to the first two) is very small, $\leqslant 10^{-4}$. Accordingly, the part of ζ_n

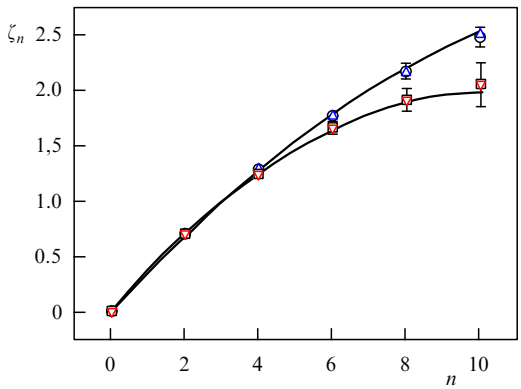

Figure 8. Power-law exponents of longitudinal (upper curve) and transverse (lower curve) structure functions of velocity: the results of numerical simulations [8] (circles and squares, respectively) and [9] (triangles pointing up and down); the solid lines correspond to theories (51) and (52).

not accounted for here would modify the theoretical curve in Fig. 8 only slightly, in a manner practically indistinguishable to one's eye. The neglected part of ζ_n can only change the rate at which it approaches the asymptotic limit as $n \to \infty$. The contribution of these terms is essential for $10 < n < 15$. The mere fact that the power-law exponents of structure functions tend to a plateau is an exact consequence of the theory (namely, the existence of 'extreme' vortices that correspond to $h = 0$). The tendency of the exponents of structure functions to the horizontal asymptote as $n \to \infty$ was discussed in Ref. [64]; saturation also exists in the passive scalar theory [58].

We see that the concept of stretching vortices not only offers an explanation of the nature of intermittency in developed turbulence, suggesting a clear interpretation of the idea of multifractality, but also provides additional information owing to which the power-law exponents of structure functions can be computed with good accuracy.

We also note that because the leading contribution to structure functions comes from vortex filaments, it follows that in the limit $l \to 0$, transverse, not longitudinal, structure functions become most essential, the former being more difficult to measure. More 'common' longitudinal structure functions turn out to be related to rather rare phenomena ('twisted' filaments); in the chains of equations discussed in [55, 59], they are secondary terms. This asymmetry is in correspondence with the sign of the inequality observed in simulations and experiments: the power-law exponents of longitudinal structure functions are larger than for the transverse ones. For this reason, the theory proposed here describes the leading term (the transverse structure functions) well and is less reliable for the longitudinal structure functions.

7. Conclusions

We have discussed the observed statistical characteristics of homogeneous and isotropic hydrodynamical turbulence and approaches to explaining them. We described the paradigm based on the idea of vortices breaking down and pointed out its drawbacks: the impossibility of explaining the existence of long-lived coherent structures, the lack of a mechanism for the breakup of vortices, the need to invoke the hypothesis on the occurrence of singularities, difficulties in explaining the power-law character of higher-order structure functions and their intermittency, and the observed disparity between the longitudinal and transverse components.

We have also discussed an alternative concept — the model of stretching vortex filaments. The existence of vortex filaments is an experimental fact, and their configuration is the subject of an immense number of papers. Our dynamical model differs in its focus on the description of the evolution of an individual vortex. In our opinion, the stretching of filaments is the basic process determining the statistics of small-scale motions in a turbulent flow.

The cause of the intensification of vorticity and further stretching of the vortex filament is that random forces, while deforming the vortex filament in various directions, contribute to its systematic stretching. To separate this systematic tendency, we abandoned the traditional way of introducing stochasticity into the Navier–Stokes equation by the addition of a large-scale force. As an independent random quantity, we took large-scale velocity pulsations, segregating them in the Navier–Stokes equation as parametric noise. This way of introducing randomness is physically motivated and natural, because the mechanism generating randomness is already present in the equation. Moreover, the resulting stochastic formulation of Navier–Stokes equation (15), (17) enables a substantial simplification of the solution.

It turns out that for a certain transformation of variables, a degeneration of the nonlinearity occurs in the limit $t \to \infty$: the pressure, despite its role in the balance of forces keeping the filaments from disintegration, does not influence their stretching. This is why Eqn (26), governing the evolution of vorticity, proves to be rather simple. An analysis of its solution (32) shows that a power-law distribution of vorticity builds up inside the filament, but the singularity does not develop even in the limit $\nu = 0$, because the center of the filament contains a domain of unsteadiness exponentially narrowing with time, which ensures the smoothness of the solution. The solution thus obtained offers all the advantages of a power-law distribution of the vorticity, but does not share the drawbacks of 'singular' models.

In the framework of the solution obtained, the randomness in Eqn (17) is split into two parts: random rotations of coordinates and noise around the Lyapunov exponents. With the simplified model where random rotations are excluded, we could give a clear interpretation to the properties of the solution, discuss the details of the limit transition $\nu \to 0$, and investigate the effect of the remaining noise — random inhomogeneity in the filament stretching — decoupled from rotations. It turns out that this inhomogeneity leads to a nonlinear dependence of the power-law exponents of velocity structure functions on their order, Eqn (48), i.e., to intermittency.

It is thus shown that by exploring the evolution of vortex filaments, we succeed in explaining both the power-law character of structure functions and their intermittency. In doing so, we explicitly identify objects 'populating' different h-classes of the multifractal theory, which helps to recover its original meaning without resorting to a probabilistic formulation and at the same time without assuming the existence of singularities.

The model of vortex filaments is the only theory at present that explains the difference in the power-law exponents for longitudinal and transverse velocity structure functions observed both in experiment and in numerical simulations. According to this model, the difference found does not stem

from the finiteness of the Reynolds number, but is a fundamental property of turbulence. Thus, if the difference between $\zeta_n^{\perp}$ and $\zeta_n^{\parallel}$ is reliably confirmed in the future, this will also be a confirmation of the theory presented here.

Power-law exponents (51) and (55) computed in the model framework coincide with the results of numerical simulations within the error bars (see Fig. 8). The theory predicts saturation of $\zeta_n^{\perp}$ with the limit value $\zeta_{\infty}^{\perp} = 2$. As concerns longitudinal structure functions, they are secondary with respect to the transverse ones because $\zeta_n^{\perp} < \zeta_n^{\parallel}$. Thus, although result (55) describes the experimental data, from the standpoint of theory, its reliability is lower than that for the transverse component.

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