REVIEWS OF TOPICAL PROBLEMS

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Analytic models of relativistic accretion disks

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Contents

Abstract. We present not a literature review but a description, as detailed and consistent as possible, of two analytic models of disk accretion onto a rotating black hole: a standard relativistic disk and a twisted relativistic disk. Although one of these models is older than the other, both are of topical interest for black hole studies. The treatment is such that the reader with only a limited knowledge of general relativity and relativistic hydrodynamics, with little or no use of additional sources, can gain insight into many technical details lacking in the original papers.

Keywords: accretion, accretion discs, black hole physics, hydrodynamics

1. Relativistic standard accretion disk

In the first part of this paper, the model of a standard accretion disk around a rotating black hole is presented with general relativity effects rigorously taken into account. This model was first described in [1] and since then has been used in many studies to obtain convincing evidence of the existence of black holes in both stellar binary systems and active galactic nuclei. It remains topical because a full account of general relativistic properties of matter motion in the disk and the generation of its emission allows inferring the position of the inner disk edge and hence the black hole spin from observations. Observational appearances of relativistic disks were modeled for the first time in [2] (see [3, 4] and the references therein for representative examples). In addition, the standard accretion disk underlies the construction of more complicated theories of warped (twisted) accretion disks, which are formed when the accreting matter moves outside the equatorial plane of a rotating black hole. Such a theory is presented in [5] and is discussed in more detail in Section 2.

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Everywhere below, the natural units $G = c = 1$ are used. If the mass is measured in units of the black hole mass M, the unit of length is half the Schwarzschild gravitational radius $R_g/2 = GM/c^2 = 1$, and the unit of time is the time light crosses the unit of length.

In addition, Latin indices $i, j, k, \ldots$ taking values from 0 to 3 are used to denote components of vectors, with the zeroth component standing for the time coordinate. Also, wherever needed, the Einstein summation convention is used.

1.1 Space–time near rotating black holes

1.1.1 Kerr metric. Properties of space–time near a rotating black hole are described by an axially symmetric and stationary metric of the form (see, e.g., [6], paragraph 4.2):

$$ds^2 = \frac{\varrho^2 \varDelta}{\Sigma^2}\, dt^2 - \frac{\Sigma^2 \sin^2\theta}{\varrho^2}(d\phi - \omega\, dt)^2 - \frac{\varrho^2}{\varDelta}\, dR^2 - \varrho^2\, d\theta^2 \,, \tag{1.1}$$

where the signature $(1, -1, -1, -1)$ is chosen and the coefficients are

$$\Sigma^2 = (R^2 + a^2)^2 - a^2 \varDelta \sin^2\theta \,,$$

$$\varrho^2 = R^2 + a^2\cos^2\theta \,, \quad \varDelta = R^2 - 2R + a^2 \,.$$

The coordinates $\{t, \phi, R, \theta\}$ are called the Boyer–Lindquist coordinates. Far away from the gravitating body, the spatial part of these coordinates in the limit of a zero black hole spin parameter a transits into the usual spherical coordinates, where ϕ is the azimuthal angle. For nonzero a, the Boyer–Lindquist coordinates are transformed into generalized spherical coordinates in which the surfaces of constant radial distance $R = \text{const}$ are spheroids with the aspect ratio $R/(R^2 + a^2)^{1/2}$.

The space–time described by (1.1) is axially symmetric with respect to the line $\theta = 0$, called the black hole rotation axis; the plane corresponding to $\theta = \pi/2$ is called the black hole equatorial axis.

In (1.1), an important quantity appears:

$$\omega = \frac{2aR}{\Sigma^2}\,, \tag{1.2}$$

which has the dimension of frequency. This is the angular velocity that every freely moving observer, without exception, acquires in the direction of the black hole rotation.

As is described in the literature on the structure of rotating black holes (see, e.g., [7], paragraph 58), metric (1.1) has several special hypersurfaces, including the event horizon and the ergosphere. However, as we show below, of most importance for the astrophysical problem under consideration is the dynamics of free circular motion of particles in the equatorial plane of a gravitating body. This motion has additional features in comparison to the corresponding Newtonian problem. We also note that weakly elliptical orbits slightly tilted to the equatorial plane are considered in Section 2.

We consider a standard, and hence geometrically thin, accretion disk. Such a disk is basically flat. By definition, this is a stationary flow of matter with mirror symmetry with respect to its middle plane and axial symmetry with respect to the line perpendicular to this plane. Clearly, such a model flow can be described by dynamical equations in an axially symmetric metric only if the disk symmetry plane coincides with the equatorial plane of the black hole. To tackle the problem, the form of the metric near the plane $\theta = \pi/2$ is sufficient. Passing to cylindrical coordinates via the standard transformation

$$r = R\sin\theta\,,\quad z = R\cos\theta\,,$$

we can expand all metric coefficients g_{ik} in (1.1) in power series in the small ratio $z/r \ll 1$. For geometrically thin disks, the corrections to g_{ik} due to nonequatorial motion up to $(z/r)^2$ are sufficient. Indeed, one of the basic equations describing the disk, namely, the projection of the relativistic analog of the Euler equation onto the direction normal to the disk plane, must be odd under the coordinate reflection $z \to -z$ due to the mirror symmetry of the disk. This means that only odd powers of z/r must be present in the series expansion in z/r. By the main assumption on the smallness of z/r, only the first term in this expansion should be kept. This, in turn, corresponds to a series expansion of g_{ik} up to quadratic terms, because only first derivatives of g_{ik}, characterizing the 'strength' of the gravitational field, enter the dynamical equations.

We note, however, that hydrodynamic equations also contain a second covariant derivative of the velocity field (see below), and hence the final expressions can involve second derivatives of g_{ik} with respect to z, which may seem to require that we keep terms of the order of $(z/r)^3$ in g_{ik}. But this is not required, because, as follows from the explicit form of the stress–energy tensor, such terms can appear only when multiplied by some of the viscous coefficients, which in turn cannot be greater than approximately z/r, being proportional to the characteristic mixing length in the fluid, which is initially assumed to be less than the disk thickness.

As regards other equations (see below) — two projections of the relativistic analog of the Euler equation onto the disk plane, the energy balance equation, and the rest-energy conservation law — the same symmetry considerations imply that they are even under the coordinate reflection $z \to -z$; therefore, the leading term is of the zeroth order in z/r in the metric expansion.

Using these expansions and expressions for the coordinate differentials

$$\mathrm{d}R = \left(1 - \frac{1}{2}\frac{z^2}{r^2}\right)\mathrm{d}r + \frac{z}{r}\,\mathrm{d}z\,,$$
$$\mathrm{d}\theta = \frac{z}{r}\frac{\mathrm{d}r}{r} - \left(1 - \frac{z^2}{r^2}\right)\frac{\mathrm{d}z}{r}\,,$$

we find the metric in the form (see also [8]):

$$\begin{aligned}
\mathrm{d}s^2 = {} & \left[1 - \frac{2}{r} + \frac{z^2}{r^3}\left(1 + \frac{2a^2}{r^2}\right)\right]\mathrm{d}t^2 \\
& - \left[r^2 + a^2 + \frac{2a^2}{r} - \frac{a^2z^2}{r^2}\left(1 + \frac{5}{r} + \frac{2a^2}{r^3}\right)\right]\mathrm{d}\phi^2 \\
& + \frac{2a}{r}\left[2 - \frac{z^2}{r^2}\left(3 + \frac{2a^2}{r^2}\right)\right]\mathrm{d}t\,\mathrm{d}\phi \\
& - \left\{1 - \frac{z^2}{r^2D}\left[\frac{3}{r} - \frac{4}{r^2} - \frac{a^2}{r^2}\left(3 - \frac{6}{r} + \frac{2a^2}{r^2}\right)\right]\right\}\frac{\mathrm{d}r^2}{D} \\
& - \frac{2z}{rD}\left(\frac{2}{r} - \frac{a^2}{r^2}\right)\mathrm{d}r\,\mathrm{d}z \\
& - \left[1 + \frac{z^2}{r^2D}\left(\frac{2}{r} - \frac{2a^2}{r^3} + \frac{a^4}{r^4}\right)\right]\mathrm{d}z^2\,,
\end{aligned} \tag{1.3}$$

where we introduce the notation

$$D = 1 - \frac{2}{r} + \frac{a^2}{r^2}\,.$$

Below, we also use (with a few exceptions) the notation introduced in the original paper by Novikov and Thorne [1] for the relativistic correction coefficients.

Finally, the inverse of the matrix g^{ik} corresponding to the double-contravariant tensor has the form

$$g^{ik} = \left|\begin{matrix} (g_{tt}g_{\phi\phi} - g_{t\phi}^2)^{-1}\begin{vmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & -g_{tt}\end{vmatrix} & 0 \\ 0 & (g_{rr}g_{zz} - g_{rz}^2)^{-1}\begin{vmatrix} g_{zz} & -g_{rz} \\ -g_{rz} & -g_{rr}\end{vmatrix}\end{matrix}\right|. \tag{1.4}$$

1.1.2 Circular equatorial geodesics. The expression for circular equatorial geodesics can be conveniently found from the extremum condition for the distance along them. Here, we follow the presentation in [9, paragraphs 13.10 and 13.13]. Indeed, for time-like trajectories, the functional

$$S = \int L\,\mathrm{d}s = \int g_{ik}\,\frac{\mathrm{d}x^i}{\mathrm{d}s}\,\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\mathrm{d}s$$

should be minimal, which is equivalent to the Euler–Lagrange equations for L:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0\,, \tag{1.5}$$

where $U_g^i \equiv \mathrm{d}x^i/\mathrm{d}s \equiv \dot{x}^i$ is the four-velocity in the Boyer–Lindquist coordinates. Because L has no manifest dependence on t and ϕ, we have the conserved quantities

$$g_{ti}U_g^i = k\,,$$
$$g_{\phi i}U_g^i = -h\,,$$

where k and h have the respective meaning of the time and azimuthal covariant velocity components.

In explicit form, using the components g_{ik} from (1.3) at $z = 0$, we find

$$\left(1 - \frac{2}{r}\right)\dot{t} + \frac{2a}{r}\,\dot{\phi} = k\,, \tag{1.6}$$

$$\frac{2a}{r}\,\dot{t} - \left(r^2 + a^2 + \frac{2a^2}{r}\right)\dot{\phi} = -h\,. \tag{1.7}$$

We temporarily assume that the motion is not necessarily circular and $U_g^r \neq 0$. Instead of the r-component of the Euler–Lagrange equations, it is more convenient to use the condition of the normalization of the four-velocity of particles with a nonzero mass:

$$g^{tt}k^2 - 2g^{t\phi}kh + g^{\phi\phi}h^2 + g^{rr}(U_r)^2 = 1\,. \tag{1.8}$$

This yields an equation for k and h,

$$\frac{\dot{r}^2}{2} + V_{\mathrm{eff}}(r) = \frac{k^2 - 1}{2}\,, \tag{1.9}$$

where we introduce the effective potential

$$V_{\mathrm{eff}} = -\frac{1}{r} + \frac{h^2 - a^2(k^2-1)}{2r^2} - \frac{(h - ak)^2}{r^3}\,. \tag{1.10}$$

The conditions for circular motion include, first, $\dot{r} = 0$, and, second, $\ddot{r} = 0$ (for the particle to stay in a circular orbit). The latter condition is equivalent to the vanishing of the derivative of V_{eff} with respect to r:

$$1 + \frac{a^2(k^2-1) - h^2}{r} + \frac{3(h - ak)^2}{r^2} = 0\,. \tag{1.11}$$

Equation (1.9) with $\dot{r} = 0$ and Eqn (1.11) allow us to determine k and h as functions of r and then, using (1.6) and (1.7), to find U_g^t and U_g^ϕ.

To solve the first problem, we introduce the new variable $\mu \equiv h - ak$ and, to facilitate manipulations, make the change $u \equiv 1/r$. Then Eqn (1.9) yields an equation for μ:

$$u^2\left[(3u-1)^2 - 4a^2u^3\right]\mu^4 - 2u\left[(3u-1)(a^2u-1) - 2ua^2(u-1)\right]\mu^2 + (au-1)^2 = 0\,. \tag{1.12}$$

The solution of (1.12) for a stable circular prograde orbit has the form

$$\mu = -\frac{a\sqrt{u} - 1}{\left[u(1 - 3u + 2au^{3/2})\right]^{1/2}}\,. \tag{1.13}$$

Using (1.13) and Eqn (1.9) taken at $\dot{r} = 0$, we find the constants k and h, as well as the components U_g^i:

$$U_g^t = C^{-1/2}B\,, \quad U_g^\phi = (r^3C)^{-1/2}\,, \quad U_g^r = U_g^z = 0\,, \tag{1.14}$$

where

$$B = 1 + \frac{a}{r^{3/2}}\,, \quad C = 1 - \frac{3}{r} + \frac{2a}{r^{3/2}}\,. \tag{1.15}$$

It easy to verify that the modulus of this vector is equal to unity:

$$g_{ik}\,U_g^iU_g^k = 1\,.$$

The angular velocity measured by the clock of an infinite observer (who measures the coordinate time t), corresponding to such motion, is

$$\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t} = r^{-3/2}B^{-1}\,. \tag{1.16}$$

It follows that in the Schwarzschild case, this quantity exactly coincides with the Keplerian angular velocity.

1.1.3 Radius of the innermost (marginally) stable orbit. This is determined by the condition that the stable circular motion is no longer possible when the minimum of the function $V_{\mathrm{eff}}(r, h(r_{\mathrm{c}}), k(r_{\mathrm{c}}))$ disappears at $r = r_{\mathrm{c}}$, where r_{c} is the radius of a circular orbit. This is equivalent to the condition

$$\left.\frac{\mathrm{d}^2V_{\mathrm{eff}}}{\mathrm{d}r^2}\right|_{r=r_{\mathrm{c}}} = 0\,,$$

which leads to the quartic equation

$$z^4 - 6z^2 + 8az - 3a^2 = 0\,, \tag{1.17}$$

where $z \equiv r^{1/2}$.

Using the Ferrari method (see, e.g., [10]), we write the corresponding auxiliary cubic equation:

$$y^3 - 12y^2 + 12(3 + a^2)\,y - 64a^2 = 0\,. \tag{1.18}$$

The real root of Eqn (1.18) is related to the Cardano solution to the corresponding incomplete cubic equation and is given by

$$y_1 = -2(1-a^2)^{1/3}\left[(1+a)^{1/3} + (1-a)^{1/3}\right] + 4\,. \tag{1.19}$$

Next, having y_1, it is possible to use the Ferrari solution to write the quadratic equation that gives two real roots of (1.17):

$$p^2 + \sqrt{y_1}p + \frac{1}{2}\left(-6 + y_1 - \frac{8a}{\sqrt{y_1}}\right) = 0\,. \tag{1.20}$$

The larger root of (1.20), p_1, determines the boundary of the stable circular motion of test particles in the equatorial plane, for which we use the notation $r = r_{\mathrm{ms}}$. Thus,

$$r_{\mathrm{ms}} = p_1^2 = 3 + \frac{4a}{\sqrt{y_1}} - \left(-\frac{y_1^2}{4} + 4a\sqrt{y_1} + 3y_1\right)^{1/2}. \tag{1.21}$$

It is easy to verify that result (1.21) coincides with the expression presented in [11] [formula (15k)], taking into account that the auxiliary quantities $Z_{1,2}$ in [11] take the form $Z_1 \equiv 3 - y_1/2$ and $Z_2 \equiv 4a/\sqrt{y_1}$ for $a \geqslant 0$ in our notation.

In the case of the Schwarzschild metric, $a = 0$, we recover the well-known result that the circular motion becomes unstable for $r < 6$, i.e., at distances smaller than three gravitational radii from the black hole. For slow rotation, $1 \gg a > 0$, we have $r_{\rm ms} \approx 6 - 4\sqrt{6}a/3$, and hence the zone of stable motion shifts closer to the event horizon. In the limit case $a = 1$, we find $r_{\rm ms} = 1$, i.e., the marginally stable circular orbit coincides with the gravitational radius of the extreme-spin black hole.

During accretion, gas elements in the disk slowly approach $r_{\rm ms}$ by losing their angular momentum due to the action of viscous forces. Once the gas elements fall into the region with $r < r_{\rm ms}$, due to the instability of the circular motion, they need not lose the angular momentum any more to approach the black hole. This means that matter falls freely inside $r_{\rm ms}$, and the standard accretion disk model assumes that $r_{\rm ms}$ is the inner disk radius.

1.2 Choice of the reference frame

1.2.1 Bases in general relativity. Mechanical laws formulated in the form of vector equations can be written in the symbolic form independent of any observer or reference frame. But to represent some physical quantity describing a natural phenomenon in the form of a set of numerical values, the measurement procedure should be specified. In Newtonian mechanics, this means that the observer introduces a coordinate system, and then at each point of space he/she arbitrarily constructs three basis vectors. The coordinate system and the basis vectors can evolve in time, which is the same at all points. A tool measuring time, together with the coordinate grid and a vector basis, form a reference frame, in which any physical quantity (scalar, vector, or tensor) can be measured, i.e., can be represented by a number or a collection of numbers.

The situation in relativistic mechanics is different: because it is not possible to consider the time independently, it becomes the fourth component of the space–time continuum. Therefore, the choice of the reference frame reduces to the construction of a coordinate system and four basis vectors defined at each space–time point. In general, this set of basis vectors is usually referred to as a tetrad. There is no universal observer any more; instead, a set of observers moving along a certain family of world lines is considered. If one of the tetrad basis vectors, conventionally corresponding to the time direction, is tangent to these world lines at each point, the tetrad is said to be transported by the observers. The last statement can be easily understood because in such a basis, the four-velocity of each observer at any time has a nonzero projection only on the 'time' basis vector; in other words, the observers are at rest in this basis, i.e., transport it with them.

Coordinate representation. It follows that the choice of the coordinate system and of the tetrad are independent procedures. Nevertheless, if there is a coordinate system x^i, the tetrad is frequently chosen such that each basis vector $\mathbf{e}_i$ is tangent to the corresponding coordinate line. Here, the moduli of the basis vectors of these so-called coordinate bases are chosen such that their pairwise scalar products are equal to the corresponding metric coefficients:

$$(\mathbf{e}_i\,\mathbf{e}_k) = g_{ik}\,. \tag{1.22}$$

We recall that in differential geometry (see paragraphs 3.1–3.4 in [9]), such coordinate basis vectors are introduced as objects isomorphic to the partial derivatives of an arbitrary scalar function on the manifold with respect to coordinates,

$$\mathbf{e}_i \equiv \frac{\partial}{\partial x^i}\,. \tag{1.23}$$

Any (tangent) vector is a linear combination of the coordinate basis vectors, and the components of this linear combination are called contravariant components of the vector.

In addition to $\mathbf{e}_i$, the so-called dual basis $\mathbf{e}^i$ is introduced with the basis vectors defined as

$$(\mathbf{e}_i\,\mathbf{e}^j) = \delta_i^j\,, \tag{1.24}$$

where δ_i^j is the Kronecker symbol. Condition (1.24) implies that each basis vector of the dual basis has a unit projection on the corresponding basis vector of the coordinate basis and is orthogonal to all other basis vectors of the coordinate basis.

The dual coordinate basis vectors, in turn, are introduced as objects isomorphic to the coordinate differentials,

$$\mathbf{e}^j \equiv \mathrm{d}x^i\,. \tag{1.25}$$

Next, if we use the fact that any tangent vector $\mathbf{A}$ can be alternatively represented as a linear combination of dual coordinate basis vectors, whose coefficients are referred to as covariant vector components, we obtain the well-known rule for lowering vector indices:

$$\begin{aligned} A_k &= A_i(\mathbf{e}^i\,\mathbf{e}_k) = (A_i\,\mathbf{e}^i)\,\mathbf{e}_k = (A^i\mathbf{e}_i)\,\mathbf{e}_k \\ &= A^i(\mathbf{e}_i\,\mathbf{e}_k) = A^i g_{ik}\,. \end{aligned} \tag{1.26}$$

In a similar way, it is easy to show that if we introduce the notation $g^{ik} \equiv (\mathbf{e}^i\,\mathbf{e}^k)$, then due to the duality of bases, the matrix g^{ik} is inverse to the matrix g_{ik}, and the rule for raising vector indices holds. A similar representation in coordinate bases can be extended to the general case of tensors.

Tetrad representation. In this and subsequent sections, we mostly follow the exposition given in paragraph 7 of [12]. We suppose that we now want to project the same vectors and tensors on an arbitrary tetrad defined by the relations

$$\mathbf{e}_{(a)} = e_{(a)}{}^i\,\frac{\partial}{\partial x^i}\,, \tag{1.27}$$

where $e_{(a)}{}^i$ are some functions of coordinates, and the indices labeling the tetrad basis vectors are in parentheses.

Using duality condition (1.24), we can introduce the dual tetrad:

$$\mathbf{e}^{(a)} = e^{(a)}{}_i\,\mathrm{d}x^i\,, \tag{1.28}$$

where $e^{(a)}{}_i$ is the matrix inverse to $e_{(a)}{}^i$.

These matrices contain two kinds of indices: coordinate and tetrad. The coordinate indices can be lowered or raised using metric (1.1). We can impose an additional constraint on the tetrad:

$$e_{(a)}{}^i e_{(b)i} = \eta_{(a)(b)}\,, \quad e^{(a)i} e^{(b)}{}_i = \eta^{(a)(b)}\,, \tag{1.29}$$

where

$$\eta_{(a)(c)}\,\eta^{(c)(b)} = \delta_{(a)}{}^{(b)} \tag{1.30}$$

are mutually inverse matrices and $\eta_{(c)(b)} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric. In other words, we require that the original and dual tetrads be orthonormal in a four-dimensional pseudo-Euclidean space.

Using the above relations, it is straightforward to show that

$$e^{(a)}{}_i \, e_{(a)j} = g_{ij} \,, \tag{1.31}$$

and therefore the following alternative expression for the interval squared holds:

$$\mathrm{d}s^2 = \eta_{(a)(b)} \left(e^{(a)}{}_i \, \mathrm{d}x^i\right) \left(e^{(b)}{}_k \, \mathrm{d}x^k\right) = \eta_{(a)(b)} \, \mathbf{e}^{(a)} \mathbf{e}^{(b)} \,, \tag{1.32}$$

which is useful below.

We note that the indices in parentheses in the right-hand side of (1.32) can be considered infinitesimal shifts along the corresponding basis vectors of the tetrad; therefore, in the introduced tetrad representation with an orthonormal tetrad, the square of the interval takes the same form as in the Minkowski space–time of special relativity. Similarly, expressions (1.27) can be regarded as directional derivatives along the tetrad basis vectors, and these have exactly the form that the usual partial derivatives with respect to coordinates in the coordinate basis take when changing from the coordinate to the tetrad basis.

Using the definitions and relations given above, it is easy to see how the tetrad components of vectors are expressed in terms of coordinate components. Tetrad components of a vector are

$$A_{(a)} = e^i_{(a)} A_i \,, \quad A^{(a)} = e^{(a)}_i A^i = \eta^{ab} A_{(b)} \,. \tag{1.33}$$

Conversely,

$$A_i = e^{(a)}_i A_{(a)} \,, \quad A^i = e^i_{(a)} A^{(a)} \,.$$

Similar expressions can be written for the tensor of any valence. For example, for a two-covariant tensor, we have

$$T_{(a)(b)} = e^i_{(a)} e^j_{(b)} T_{ij} = e^i_{(a)} T_{i(b)} \,,$$

and conversely,

$$T_{ij} = e^{(a)}_i e^{(b)}_j T_{(a)(b)} = e^{(a)}_i T_{(a)j} \,. \tag{1.34}$$

We note in conclusion that relations (1.33) and isomorphism (1.25) can be used to find contarvariant components of the four-velocity in the tetrad representation:

$$U^{(a)} = \frac{\mathbf{e}^{(a)}}{\mathrm{d}s} \,. \tag{1.35}$$

This is again a unit tangent vector along the world line, but now its components are given by small shifts along the corresponding dual basis vectors. Using (1.33), it is easy to find the relation between the conventional coordinate components of the four-velocity, $U^i = \mathrm{d}x^i/\mathrm{d}s$, and its tetrad components. Covariant tetrad components are derived from contravariant ones using the standard rule in special relativity: lowering a spatial index is equivalent to changing the sign of the corresponding component.

Covariant derivative in the tetrad representation. We calculate the directional derivative along a tetrad basis vector from a contravariant component of a vector:

$$\begin{aligned} A_{(a),(b)} &= e^i_{(b)} \frac{\partial}{\partial x^i} A_{(a)} = e^i_{(b)} \frac{\partial}{\partial x^i} e^j_{(a)} A_j \\ &= e^i_{(b)} \left[e^j_{(a)} A_{j;\,i} + A_k e^k_{(a);\,i} \right] , \end{aligned} \tag{1.36}$$

where the semicolon denotes the usual covariant derivative in the coordinate basis.

Expression (1.36) can be rewritten as

$$A_{(a),(b)} = e^j_{(a)} A_{j;\,i} \, e^i_{(b)} + e_{(a)k;\,i} \, e^i_{(b)} e^k_{(c)} A^{(c)} \,, \tag{1.37}$$

whence

$$e^j_{(a)} A_{j;\,i} \, e^i_{(b)} = A_{(a),(b)} - \gamma_{(c)(a)(b)} A^{(c)} \,, \tag{1.38}$$

where

$$\gamma_{(a)(b)(c)} = e_{(b)k;\,i} \, e^i_{(c)} e^k_{(a)} \tag{1.39}$$

are the so-called Ricci rotation coefficients. An important point is that for orthonormal bases satisfying (1.29), the coefficients $\gamma_{(a)(b)(c)}$ are antisymmetric in the first two indices. Indeed,

$$\begin{aligned} 0 = \left(\eta_{(b)(a)}\right)_{,\,i} &= \left(e_{(b)k} \, e^k_{(a)}\right)_{;\,i} = e_{(b)\,k;\,i} \, e^k_{(a)} + e_{(b)k} e^k_{(a);\,i} \\ &= e_{(b)\,k;\,i} \, e^k_{(a)} + e^k_{(b)} e_{(a)\,k;\,i} \,. \end{aligned}$$

Comparing this relation with (1.39) proves the stated property of the Ricci coefficients.

We finally discuss one more useful property of coefficients (1.39): to calculate these coefficients, only partial derivatives of the components of the tetrad basis are needed, and therefore the Christoffel symbols are not required. Indeed, we consider auxiliary combinations

$$\lambda_{(a)(b)(c)} = e_{(b)i,j} \left[e^i_{(a)} e^j_{(c)} - e^j_{(a)} e^i_{(c)} \right] , \tag{1.40}$$

and rewrite them in the form

$$\lambda_{(a)(b)(c)} = e^i_{(a)} e^j_{(c)} \left[e_{(b)\,i,j} - e_{(b)j,\,i} \right] . \tag{1.41}$$

In the last expression, the ordinary partial derivatives can be replaced by covariant ones, because the additional terms with Christollel symbols are symmetric in i, j. Then expression (1.41) is equal to the difference $\gamma_{(a)(b)(c)} - \gamma_{(c)(b)(a)}$. But in such a case,

$$\gamma_{(a)(b)(c)} = \frac{1}{2} \left[\lambda_{(a)(b)(c)} + \lambda_{(c)(a)(b)} - \lambda_{(b)(c)(a)} \right] , \tag{1.42}$$

and, using (1.40), it is possible to calculate the Ricci rotation coefficients by taking partial derivatives of the components of the tetrad basis vectors.

We now consider formula (1.38). The left-hand side represents the projection on the tetrad basis of a rank-2 covariant tensor obtained by taking the derivative of some vector field. Therefore, this combination has the meaning of the covariant derivative of a vector taken in a noncoordinate basis.

Next, the right-hand side of (1.38) has exactly the same form as the covariant derivative in a coordinate basis, with the only difference that it involves tetrad indices (which can be raised or lowered, including for $\gamma_{(a)(b)(c)}$, using the Minkowski metric). It can be shown that the same holds for contravariant components of a vector field and for tensor fields in general.

Thus, because the components of a covariant derivative in a tetrad basis have the same form as in a coordinate basis, it is convenient to use the same notation and terminology that are used in the coordinate basis. In particular, the Ricci rotation coefficients are simply referred to as connection symbols in a given basis. We emphasize once again that they should not be confused with the Christoffel symbols, which represent another limit case of connection coefficients in a coordinate basis and have a different symmetry of the indices.

1.2.2 Tetrad transported by rotating observers. We construct a tetrad basis related at each point of space–time to observers moving around a black hole in equatorial circular orbits with an angular velocity Ω. At $z = 0$, strictly speaking, this is the free motion along geodesics found in Section 1.1.2. However, for a small deviation from the equatorial plane, such a motion, corresponding to a constant z, is possible only if there is some external supporting force; in the case of a gas disk, for example, this force is due to the pressure gradient.

To start the construction, we direct the time basis vector of the tetrad along the world line under discussion. Using the four-vector of the geodesic found in Section 1.1.2, we write it in the form

$$\mathbf{e}_{(t)} = (U_g^t + Z_0)\,\frac{\partial}{\partial t} + U_g^{\phi}\,\frac{\partial}{\partial \phi}\,,$$

where we add a correction factor $Z_0(z/r)$ to the time coordinate component, because the modulus of the vector $\mathbf{e}_{(t)}$ should be equal to unity away from the equatorial plane, whereas the vector $\mathbf{U}_g$ itself is unitary only at $z = 0$. Clearly, with this correction, $\mathbf{e}_{(t)}$ would correspond to the four-velocity of the real motion. Calculating the modulus of the vector $\mathbf{e}_{(t)}$ in metric (1.3) shows that it is equal to unity under the condition

$$Z_0 = -\left(\frac{z}{r}\right)^2 \frac{H}{2rGC^{1/2}}\,,$$

where we introduce the relativistic correction coefficients

$$G = 1 - \frac{2}{r} + \frac{a}{r^{3/2}}\,, \qquad (1.43)$$

$$H = 1 - \frac{4a}{r^{3/2}} + \frac{3a^2}{r^2}\,. \qquad (1.44)$$

Thus, the basis vector $\mathbf{e}_{(t)}$ is transported by the observer rotating around the black hole with a frequency equal to the ϕ-component of $\mathbf{e}_{(t)}$, which is independent of z. This frequency corresponds to the free circular motion in the equatorial plane of the black hole, and rotation occurs in planes of constant z.

We now calculate the time basis vector of the dual basis. According to the convention for raising and lowering coordinate indices, we have

$$\mathbf{e}^{(t)} = (U_g^t g_{tt} + U_g^{\phi} g_{t\phi})\,\mathrm{d}t + (U_g^t g_{\phi t} + U_g^{\phi} g_{\phi\phi})\,\mathrm{d}\phi\,.$$

We next consider the part of metric (1.3) containing the differentials $\mathrm{d}r$ and $\mathrm{d}z$. It can be rewritten in the form (see result (1.32) in Section 1.2.1):

$$\mathrm{d}s_{rz}^2 = -\left[\mathbf{e}^{(r)}\right]^2 - \left[\mathbf{e}^{(z)}\right]^2\,,$$

where

$$\mathbf{e}^{(r)} = |g_{rr}|^{1/2}\,\mathrm{d}r - \frac{g_{rz}}{|g_{rr}|^{1/2}}\,\mathrm{d}z\,,$$

$$\mathbf{e}^{(z)} = \left(|g_{zz}| - \frac{{g_{rz}}^2}{|g_{rr}|}\right)^{1/2} \mathrm{d}z$$

are the radial and vertical vectors of the dual basis. The coordinate components of the vectors $\mathbf{e}^{(t)}$, $\mathbf{e}^{(r)}$, and $\mathbf{e}^{(z)}$ satisfy orthonormality condition (1.29), as can be easily verified by direct substitution.

The orthonormality condition for a tetrad can now be used to determine the fourth basis vector corresponding to the azimuthal direction.

From three orthonormality conditions for three already known vectors, we conclude that their consistency relation must hold in the form

$$e^{(\phi)}{}_r = e^{(\phi)}{}_z = 0\,,$$

and the time and azimuthal components should be related as

$$e^{(\phi)}{}_{\phi} = -e^{(\phi)}{}_t\,\frac{e_{(t)}{}^t}{e_{(t)}{}^{\phi}}\,.$$

Finally, the normalization condition for $\mathbf{e}^{(\phi)}$ yields a quadratic equation for $e^{(\phi)}{}_t$, and the sign of the solution is dictated by the additional requirement of the choice of a right-hand triple of spatial vectors of the tetrad.

We thus obtain the dual tetrad basis with the leading corrections in z/r due to out-of-equatorial-plane motion in the form

$$\begin{aligned}\mathbf{e}^{(t)} = {} & C^{-1/2}\left\{G + \left(\frac{z}{r}\right)^2 \frac{1}{2rG}\right.\\ & \left.\times\left[D + \frac{2a}{r^{3/2}}\left(F - \frac{a}{r^{3/2}} + \frac{a^2}{r^2}\right)\right]\right\}\mathrm{d}t\\ & - C^{-1/2}\left\{r^{1/2}F + \left(\frac{z}{r}\right)^2 \frac{a}{rG}\,Z_1\right\}\mathrm{d}\phi\,,\end{aligned} \qquad (1.45)$$

$$\begin{aligned}\mathbf{e}^{(\phi)} = {} & -\left\{\left(\frac{D}{rC}\right)^{1/2} + \frac{1}{2}\left(\frac{z}{r}\right)^2 \frac{1 - a/r}{r^{3/2}}\,(DC)^{-1/2}\right\}\mathrm{d}t\\ & + \left\{rB\left(\frac{D}{C}\right)^{1/2} + \frac{1}{2}\left(\frac{z}{r}\right)^2\right.\\ & \left.\times\left[\left(1 - \frac{a}{r}\right)\frac{B}{(DC)^{1/2}} - \frac{H}{G}\left(\frac{D}{C}\right)^{1/2}\right]\right\}\mathrm{d}\phi\,,\end{aligned} \qquad (1.46)$$

$$\begin{aligned}\mathbf{e}^{(r)} = {} & D^{-1/2}\left\{1 - \frac{1}{2D}\left(\frac{z}{r}\right)^2 Z_2\right\}\mathrm{d}r\\ & + \frac{z}{r}\,D^{-1/2}\left(\frac{2}{r} - \frac{a^2}{r^2}\right)\mathrm{d}z\,,\end{aligned} \qquad (1.47)$$

$$\mathbf{e}^{(z)} = \left(1 + \frac{z^2}{r^3}\right)\mathrm{d}z\,. \qquad (1.48)$$

To obtain the original basis, which we use to write equations of motion, it suffices to calculate the inverse to the matrix $e^{(a)}{}_i$, which yields

$$\mathbf{e}_{(t)} = C^{-1/2}\left[B - \left(\frac{z}{r}\right)^2 \frac{H}{2rG}\right]\frac{\partial}{\partial t} + (r^3 C)^{-1/2}\,\frac{\partial}{\partial \phi}\,, \qquad (1.49)$$

$$\mathbf{e}_{(\phi)} = \left[\frac{F}{(rCD)^{1/2}} + O\left(\frac{z^2}{r^2}\right) \right] \frac{\partial}{\partial t} + \left[\frac{G}{r(DC)^{1/2}} + O\left(\frac{z^2}{r^2}\right) \right] \frac{\partial}{\partial \phi}\,, \tag{1.50}$$

$$\mathbf{e}_{(r)} = \left[D^{1/2} + \frac{1}{2}\left(\frac{z}{r}\right)^2 \frac{Z_2}{D^{1/2}} \right] \frac{\partial}{\partial r}\,, \tag{1.51}$$

$$\mathbf{e}_{(z)} = -\frac{z}{r^2}\left(2 - \frac{a^2}{r}\right)\frac{\partial}{\partial r} + \left(1 - \frac{z^2}{r^3}\right)\frac{\partial}{\partial z}\,. \tag{1.52}$$

The following notation for relativistic correction coefficients is introduced in the expressions for the original and dual bases:

$$F = 1 - \frac{2a}{r^{3/2}} + \frac{a^2}{r^2}\,, \tag{1.53}$$

$$Z_1 = 3 - \frac{5}{r} - \frac{a}{r^{1/2}} + \frac{3a}{r^{3/2}} - \frac{3a^2}{r^3} + \frac{a^2}{r^2} + \frac{2a^3}{r^{7/2}}\,, \tag{1.54}$$

$$Z_2 = \frac{3}{r} - \frac{4}{r^2} - \frac{a^2}{r^2}\left(3 - \frac{6}{r} + \frac{2a^2}{r^2}\right). \tag{1.55}$$

Here, we omit terms $\sim O(z^2/r^2)$ in the expression for the azimuthal vector of the original basis due to their complexity; in addition, as we see below, these terms are not required in the standard accretion disk model.

For the reader's convenience, we here preserve the notation introduced in paper [1] for the coefficients B, C, D, F, G, but use the standard style of Latin letters, which is more familiar to the reader. In addition, the coefficient H is equivalent to the coefficient C introduced in [8]. We also note that two other coefficients introduced in the same paper, A and B, are respectively equivalent to our coefficients D and C. It can be verified that the original and dual bases presented in [1] coincide with the bases derived here at $z = 0$.

Using formulas (1.45)–(1.48) and (1.35), it is easy to deduce that solution (1.14) indeed yields $U^{(a)} = (1, 0, 0, 0)$ in the equatorial plane.

Connection coefficients. Using (1.40) and then (1.42) and knowing the matrices of the original and dual bases given above, we can calculate the connection coefficients $\gamma_{(a)(b)(c)}$.

Of the 64 coefficients, 16 are equal to zero due to the antisymmetry of $\gamma_{(a)(b)(c)}$ in the first two indices. For the same reason, of the other coefficients, only half (i.e., 24) have to be found. Because we are interested in the region near the equatorial plane of the black hole, it makes sense to separate these coefficients into two groups: those that are $\sim (z/r)^0$ in the leading order, and those proportional to the first power of z/r. As mentioned in Section 1.1.1, the latter coefficients must appear in the vertical projection of the relativistic Euler equation, while the former emerge in other equations.

It can be shown that

(1) if there is no index (z) among the indices of $\gamma_{(a)(b)(c)}$, then $\gamma_{(a)(b)(c)} \sim (z/r)^0$,

(2) if only one such index is present, then $\gamma_{(a)(b)(c)} \sim z/r$, and, finally,

(3) if two indices (z) appear in $\gamma_{(a)(b)(c)}$, then this coefficient is of the second order in z/r.

Indeed, we examine formula (1.40). Here, the square brackets contain the original basis components, which are summed with the coordinate derivatives of the dual basis components (raising a tetrad index can only change the sign of the component).

In case (1), $(a), (b), (c) \neq (z)$. Because the (t), (ϕ), and (r) basis vectors of the original basis have no z-component, only terms that do not contain derivatives with respect to z of the dual basis components and have no z-component of the dual (r) basis vector make a nonzero contribution to $\gamma_{(a)(b)(c)}$. Only in these two cases can the contribution $\sim z/r$ appear, and hence we prove statement (1).

Now, in (1.40), let $(b) = (z)$. Then the nonzero contribution can be due to only those terms with the z-component of the (t)-, (ϕ)-, and (r) basis vectors of the initial basis that are absent. Therefore, to prove statement (2), we must consider only the variant where $(a) = (z)$ or $(c) = (z)$ in (1.40). Here, the terms containing separately either r- or z-components of the (z) basis vector of the original basis contribute. In the first variant, the proportionality to z/r is due to exactly the component $e^r_{(z)}$, while in the second, it is due to the derivative with respect to z of one of the dual basis components, which is always an even function of z, as can be easily verified.

We leave it to the reader to prove statement (3).

The counting shows that there must be 9 connection coefficients without the index (z), and hence even functions of z, and 12 coefficients with the index (z), and hence odd functions of z. The calculation shows that only four coefficients of the first type are nonzero:

$$\gamma_{(t)(\phi)(r)} = -\frac{1}{2}\,\frac{H}{r^{3/2}C}\,, \quad \gamma_{(t)(r)(\phi)} = -r^{-3/2}\,, \tag{1.56}$$

$$\gamma_{(\phi)(r)(t)} = -r^{-3/2}\,, \quad \gamma_{(\phi)(r)(\phi)} = -\frac{1}{r}\,\frac{\mathrm{d}}{\mathrm{d}r}\left(rD^{1/2}\right). \tag{1.57}$$

To compute coefficients (1.56) and (1.57), it suffices to use bases taken without corrections in z.

In constructing the standard disk model, the following facts are also important. First, direct calculation shows that another five connection coefficients of this type are zero through the correction order $\sim (z/r)^2$. This is a rigorous result, because the coefficients $\gamma_{(a)(b)(c)}$ under discussion have no derivatives of the basis components with respect to z, and therefore the possible unaccounted for corrections due to terms $\sim (z/r)^3$ in $e_z^{(r)}$ and $e^r_{(z)}$ cannot contribute. Second, direct calculation similarly shows that $\gamma_{(t)(z)(z)} = 0$ through the order $\sim (z/r)^2$.

Calculating all nonzero coefficients of the second type is a much more cumbersome task. But as we see below, the only coefficient of this type that is needed has the form

$$\gamma_{(z)(t)(t)} = \frac{z}{r^3}\,\frac{H}{C}\,.$$

We note that all connection coefficients of the type $\gamma_{(a)(t)(t)}$ vanish in the equatorial plane $z = 0$. This is consistent with the requirement that the four-velocity $U^{(a)} = (1, 0, 0, 0)$ must satisfy the geodesic equation at $z = 0$:

$$\frac{\mathrm{D}U^a}{\mathrm{D}s} = U^b\,\mathbf{e}_{(b)}\left(U^{(a)}\right) + \eta^{(a)(c)}\gamma_{(c)(b)(d)}U^{(b)}U^{(d)} = \gamma_{(a)(t)(t)} = 0\,. \tag{1.58}$$

1.2.3 Relativistic hydrodynamic equations. Everywhere below, we only use the tetrad components of vectors, tensors, and covariant derivatives. Therefore, starting from this section, we substitute the tetrad notation by the standard one, familiar when using the coordinate basis. This means that we do not put tetrad indices in parentheses and let them be denoted by Latin letters i, j, k, and let the connection coefficients be denoted by Γ.[1]

The stress–energy tensor of a viscous fluid with energy flux has the form (see, e.g., paragraph 4.3 in [13] or paragraph 22.3 in [14])

$$T^{ik} = (\rho + \epsilon + p)\, U^i U^k - p\eta^{ik} + 2\eta\sigma^{ik} + \zeta\Theta P^{ik} - U^i q^k - U^k q^i\,, \qquad (1.59)$$

where ρ, ϵ, p, and η and ζ are the respective rest-energy density, internal energy density, pressure, and two viscosity coefficients measured in the local comoving fluid volume; $\mathbf{q}$ is the energy flux inside the fluid as measured by the local comoving observer.

The shear tensor is

$$\sigma^{ik} = \frac{1}{2}\left(U^i{}_{;j}\, P^{jk} + U^k{}_{;j}\, P^{ji}\right) - \frac{1}{3}\, U^j{}_{;j}\, P^{ik}\,, \qquad (1.60)$$

with the projection operator

$$P^{ik} = \eta^{ik} - U^i U^k\,. \qquad (1.61)$$

The divergence of the four-velocity is

$$\Theta = U^i{}_{;i}\,. \qquad (1.62)$$

The relativistic Euler equation is written as

$$P_{is}\, T^{sk}{}_{;k} = 0\,. \qquad (1.63)$$

The energy conservation law has the form

$$U_s\, T^{sk}{}_{;k} = 0 \qquad (1.64)$$

and the rest-energy conservation law is expressed as

$$(\rho\, U^k)_{;k} = 0\,. \qquad (1.65)$$

The covariant derivative in a noncoordinate basis is

$$A^i{}_{;j} = \mathbf{e}_j(A^i) + \Gamma^i_{kj} A^k\,,$$

while the divergence of a rank-2 contravariant tensor is

$$A^{ij}{}_{;j} = \mathbf{e}_j(A^{ij}) + \Gamma^i_{kj} A^{kj} + \Gamma^j_{kj} A^{ik}\,.$$

The energy flux vector and the shear tensor (the deformation tensor free from pure scaling) are purely space-like objects:

$$U_i q^i = 0\,, \quad U_i \sigma^{ik} = 0\,, \quad \sigma^i_i = 0\,. \qquad (1.66)$$

1.3 Construction of the standard accretion disk model

1.3.1 Basic assumptions and the vertical balance equation. We consider a disk from the standpoint of local observers rotating around a black hole near its equatorial plane with a relativistic Keplerian velocity. Before writing the dynamic equations in the projection onto tetrad (1.49)–(1.52), we discuss basic assumptions of the model and their consequences. In addition to obvious assumptions about axial symmetry and the stationarity of the flow (meaning that the derivatives ∂_t and ∂_ϕ are zero), the main hypothesis, which we have already used, is the small disk thickness, $\delta = h(r)/r \ll 1$, where $h(r)$ is the characteristic height of the disk along the z axis (more precisely, the disk half-thickness).

The disk symmetry with respect to the plane $z = 0$ implies that U^t, U^ϕ, U^r, q^t, q^ϕ, q^r, $\rho, p, \eta, \zeta, \epsilon$ are even functions of z, and U^z and q^z are odd functions.

We also assume that the characteristic scale of variations of these quantities in the radial direction is much larger than in the vertical direction,[2] that is, their ratio is greater than $\sim \delta^{-1}$.

Next, kinematic arguments suggest that

$$U^z \sim \delta U^r\,. \qquad (1.67)$$

If the energy flux determined by the vector $\mathbf{q}$ is proportional to the internal energy gradient ϵ, then $q^t_{\text{loc}} = 0$, and q^ϕ_{loc}, $q^r_{\text{loc}} \sim \delta q^z_{\text{loc}}$ for the local comoving observer. Taking (1.67) into account implies that the projection of $\mathbf{q}$ onto the four-velocity of circular equatorial motion is also small, i.e., of the order of $\sim \delta q^z_{\text{loc}}$. From the standard Lorentz transformations, we obtain that q^t, q^ϕ, $q^r \sim \delta q^z$, i.e., the energy flux relative to the tetrad should be directed mostly normally to the disk plane.

Now, taking all the above into account, we consider the projection of the relativistic analog of Euler equation (1.63) onto the basis vector $\mathbf{e}_z$ in more detail:

$$T^{zk}{}_{;k} + U_z U_s\, T^{sk}{}_{;k} = 0\,. \qquad (1.68)$$

Using the symmetry of physical quantities discussed above and symmetry properties of the basis vectors and connection coefficients (which become odd functions of z if they have at least one index z), discussed in Section 1.2.2, it is easy to verify that Eqn (1.68) is an odd function of z. Further, we see that the first term in (1.68) separately yields the term $\partial_z p$, and other terms containing p are smaller due to the smallness of U^z. All other terms together can always be written as $\sim z\rho f(r)(1 + g(r,z))$ with a function $g(r,z) \sim O(\delta^0)$.

Thus, we arrive at the important conclusion that necessarily

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \sim \delta \ll 1\,. \qquad (1.69)$$

This means that in a thin disk, $p, \partial_r p \sim \delta^2$, i.e., these variables are small relative to the dominant action of the gravitational force in this direction. Therefore, particles of the disk must move in trajectories close to geodesic ones. Clearly, in a steady-state and axially symmetric flow, this can be realized only in two cases: when the matter moves almost radially toward the gravitating center (and the specific angular momentum in the disk is everywhere close to zero) or when the matter moves in almost circular orbits (and the specific angular momentum is maximal). We note that both cases are consistent with the general assumptions discussed above and the result in (1.69). However, in the last case, the strict vertical hydrostatic equilibrium holds in the disk in the first order in δ;

[1] If one of the symbols t, ϕ, r, z appears among the indices, it means that the corresponding index takes this value.

[2] We note that the assumption about the velocity components in the disk plane, U^r and U^ϕ, should also be made such that they can substantially change in the vertical direction only on scales $\sim r$; otherwise, terms in the shear tensor could arise that strongly and dynamically contribute to the vertical balance condition, which would lead to a disk totally different from the basic case of interest here.

in other words, (1.68) can be rewritten in the form

$$\frac{1}{\rho}\frac{\partial p}{\partial z} \sim zf(r)(1+\delta^2+\ldots)\,. \tag{1.70}$$

When the flow is almost radial, the corrections in the parentheses in (1.70) are not small, and their value is determined by the contribution from the prevailing radial motion, when, due to the change in the disk thickness at each radius, the particles are accelerated in the z direction.

Thus, the standard disk model includes one more independent assumption on the closeness of the fluid particle trajectories to equatorial circular orbits around the central black hole. Therefore, we additionally suppose that in our reference frame, $U^\phi, U^r \sim sU^t$ with $s \ll 1$ and later we can see how this second small parameter is related to δ.

Consequently, we first write equations not only in the leading order in δ but also under the assumption that $s=0$, i.e., that the flow moves along geodesic orbits and $U^i=(1,0,0,0)$. Wherever needed, we then additionally evaluate the contribution from the terms in the leading order in s.

Deformation of the velocity field. We first find the nonzero components of the shear tensor in the leading order. The velocity divergence vanishes:

$$\Theta = U^j{}_{;j} = \Gamma^j_{kj}U^k = \Gamma^j_{0j} = 0\,. \tag{1.71}$$

We also have

$$U^i{}_{;j}P^{jk} = \Gamma^i_{tk}\eta^{kk} - \Gamma^i_{tt}$$

and, in view of the symmetry in i and k, we see that the only nonzero components of the shear tensor are

$$\sigma^{r\phi} = -\frac{1}{2}(\Gamma^\phi_{tr}+\Gamma^r_{t\phi}) = \frac{1}{2}\left(\frac{1}{2}\frac{H}{r^{3/2}C}+r^{3/2}\right) = \frac{3}{4}\frac{D}{r^{3/2}C}\,, \tag{1.72}$$

$$\sigma^{rz} = -\frac{1}{2}\Gamma^z_{t\phi} = O(z)\,. \tag{1.73}$$

Equation of hydrostatic equilibrium. Substituting $U^i=(1,0,0,0)$ in (1.68) and taking the smallness (due to the low sound velocity in the flow) of several nonzero terms containing η and components of $\mathbf{q}$ into account, we obtain

$$\frac{\partial p}{\partial z} = \rho\Gamma^z_{tt} = -\rho\,\frac{z}{r^3}\frac{H}{C}\,. \tag{1.74}$$

Radial direction. The radial projection of the relativistic Euler equation with $s=0$ is given by

$$T^{rk}{}_{;k} = 0\,, \tag{1.75}$$

and, eliminating terms $\sim\delta^4$ containing the connection coefficients and components of $\mathbf{q}$, we have only one nonzero term of the order δ^2, which has the form [3]

$$-[p\eta^{rk}]_{;k} = D^{1/2}\,\frac{\partial p}{\partial r}\,.$$

Clearly, this term should be balanced by the leading terms $\sim s$. Evidently, the contribution from

$$[\rho U^r U^k]_{;k}$$

should be considered first, and here it can be only due to terms containing one of the connection coefficients of zeroth order in z and the time velocity component. There is only one such term: $2\Gamma^r_{t\phi}U^tU^\phi = 2r^{-3/2}U^\phi$.

Hence, we reach an important conclusion that $s\sim\delta^2$, i.e., the velocity components in the disk plane are

$$U^r, U^\phi \sim \delta^2\,, \tag{1.76}$$

which is used below when determining the force balance in the azimuthal direction.

1.3.2 Azimuthal direction. We consider the last projection of the relativistic Euler equation, its component along the azimuthal basis vector. We proceed in the same way as above and first write terms that are present in the case $s=0$. Again, we take $U^i=(1,0,0,0)$ and see that

$$\left[(\rho+\epsilon+p)\,U^\phi U^k\right]_{;k} = 0\,,$$

because $\Gamma^\phi_{tt}=0$ through the order $\sim\delta^2$ (see the discussion at the end of Section 1.2.2). Next, the term with pressure is absent by virtue of the axial symmetry, and terms with q^i cannot contribute to an order higher than δ^4.

It remains to consider the contribution

$$\begin{aligned}[2\eta\sigma^{\phi k}]_{;k} &= D^{1/2}(2\eta\sigma^{r\phi})_{,r} + (2\eta\sigma^{\phi z})_{,z} + 4\eta\Gamma^\phi_{r\phi}\sigma^{r\phi} + \eta O(\delta^2)\\ &= -\frac{3}{2}D^{1/2}\left(\eta\frac{D}{r^{3/2}C}\right)_{,r} + (\eta\Gamma^t_{z\phi})_{,z}\\ &\quad + 3\eta(rD^{1/2})_{,r}\,\frac{D}{r^{5/2}C} + \eta O(\delta^2)\,.\end{aligned} \tag{1.77}$$

Here, we are also dealing with terms of the second order in δ^2; therefore, it is necessary to find the leading contribution from terms $\sim s$. Again, we consider only the prevailing part due to the ideal fluid term:

$$(\eta_{\phi i} - U_\phi U_i)[\rho U^i U^k]_{;k}\,.$$

The second part, which is proportional to U_ϕ, can be neglected because the term in square brackets cannot contribute to the zeroth order in δ^0, because there are no connection coefficients of the form $\Gamma^i_{tt}\sim\delta^0$, as was discussed at the end of Section 1.2.2.

As a result, we obtain

$$\begin{aligned}[\rho U^\phi U^k]_{;k} &= \rho\Gamma^\phi_{lk}U^lU^k + \rho\Gamma^k_{lk}U^\phi U^l = \rho(\Gamma^\phi_{tr}+\Gamma^\phi_{rt})\,U^r\\ &= -\rho\,\frac{U^r}{r^{3/2}}\left(\frac{1}{2}\frac{H}{C}-1\right) \equiv \rho\,\frac{U^r}{2r^{3/2}}\frac{E}{C}\,,\end{aligned} \tag{1.78}$$

where

$$E = 1 - \frac{6}{r} + \frac{8a}{r^{3/2}} - \frac{3a^2}{r^2}\,. \tag{1.79}$$

We now introduce the notation

$$T_v \equiv \int_{-h}^{+h} T_v^{r\phi}\,\mathrm{d}z = 2\sigma^{r\phi}\int_{-h}^{+h}\eta\,\mathrm{d}z\,, \tag{1.80}$$

[3] The order of components q^i can be estimated as follows. In the stationary case, the divergence of the energy flux must be of the order of the power generated due to viscous dissipation, which is in turn proportional to some scalar characterizing the degree of velocity shear and the viscosity coefficient η. In our case, the viscosity coefficient $\eta < \rho h c_s \sim \delta^2$. The divergence is mainly due to the term $\partial_z q^z$. This immediately implies that $q^z\sim\delta^3$ and $q^{t,\phi,r}\sim\delta^4$.

where T_v is the vertically integrated density of the flux of the ϕ-component of momentum in the radial direction. Then, by also integrating Eqns (1.77) and (1.78) over the disk thickness and combining them into one equation, we have

$$\frac{\partial T_v}{\partial r} + \frac{2T_v}{rD}\left(1 - \frac{1}{r}\right) + \frac{\Sigma U^r}{2r^{3/2}} \frac{E}{CD^{1/2}} = 0\,, \qquad (1.81)$$

where the contribution from $\sigma^{\phi z}$ vanishes because it is an odd function of z, and we neglect the dependence of U^r on z, which gives rise to a higher-order correction (see footnote 2). In formula (1.81), we have introduced the surface density of the disk,

$$\Sigma \equiv \int_{-h}^{+h} \rho\,\mathrm{d}z\,. \qquad (1.82)$$

Important Eqn (1.81), with known boundary conditions at the inner disk radius, allows calculating the profile $T_v(r)$ for the disk if the radial velocity is specified. We note that the equation for T_v can also be derived from the angular momentum conservation law, which was used in the original paper [1] [see Eqns (5.6.3)–(5.6.6) therein].

1.3.3 Rest-energy conservation law. Radial momentum transfer. To solve Eqn (1.81), the radial velocity profile should be specified. It can be obtained from the rest-energy conservation law (1.65):

$$\mathbf{e}_r(\rho U^r) + \mathbf{e}_z(\rho U^z) + \Gamma^i_{ki}\rho U^k = 0\,. \qquad (1.83)$$

Clearly, the substitution of $U^i = (1, 0, 0, 0)$ does not yield nonzero terms up to the order $\sim \delta^2$ (see the discussion at the end of Section 1.2.2). In our reference frame, this fact can be easily understood: the circular axially symmetric motion corresponds to zero velocity divergence. It is straightforward to verify that the following terms $\sim s$ appear in the continuity equation:

$$D^{1/2}(\rho U^r)_{,r} + (\rho U^z)_{,z} - \frac{(r D^{1/2})_{,r}}{r}\,\rho U^r = 0\,, \qquad (1.84)$$

where the last term arises due to the contribution from $\Gamma^{\phi}_{r\phi}\rho U^r$, and similar terms with other velocity components, even if they appear, have an order higher than $\sim \delta^4$.

After integrating over z, the contribution from the second term in (1.84) vanishes because $\rho \to 0$ far from the equatorial disk plane, and we obtain

$$(\Sigma U^r r D^{1/2})_{,r} = 0\,. \qquad (1.85)$$

The combination whose derivative is calculated in (1.85) is a constant, which must be identified with the radial flux of matter. After additionally integrating over ϕ, we obtain

$$2\pi \Sigma U^r r D^{1/2} = -\dot{M}\,, \qquad (1.86)$$

where $\dot{M} > 0$ is the rate of matter inflow into the disk at infinity, i.e., the mass accretion rate.

After substituting (1.86) in (1.81), we finally obtain

$$\frac{\mathrm{d}T_v}{\mathrm{d}r} + P_1 T_v + P_2 = 0\,, \qquad (1.87)$$

where

$$P_1 = \frac{2}{rD}\left(1 - \frac{1}{r}\right),$$

$$P_2 = -\frac{\dot{M}}{4\pi}\,\frac{E}{r^{5/2}CD}\,.$$

The solution of (1.87) with the boundary condition $T|_{r_{\mathrm{ms}}} = 0$ can be written in the form

$$T_v = \frac{1}{F(r)} \int_{r_{\mathrm{ms}}}^{r} P_2(x)\,F(x)\,\mathrm{d}x\,, \qquad (1.88)$$

$$F(r) = \exp\left(\int_{r_{\mathrm{ms}}}^{r} P_1(x)\,\mathrm{d}x\right). \qquad (1.89)$$

Integral (1.89) is elementary, and as a result we obtain

$$T_v = \frac{\dot{M}}{4\pi r^2 D} \int_{r_{\mathrm{ms}}}^{r} \frac{E}{r^{1/2}C}\,\mathrm{d}r\,. \qquad (1.90)$$

1.3.4 Energy balance. Here, we consider Eqn (1.64). As above, we set $U^i = (1, 0, 0, 0)$ and find terms of the leading order in δ. As in the case of the azimuthal projection of the relativistic Euler equation, 'ideal' terms $[(\rho + \epsilon + p)\,U^t U^k]_{;k}$ and $p\eta^{0k}{}_{;k}$ do not contribute here. From the shear term, we have

$$[2\eta\sigma^{tk}]_{;k} = \Gamma^t_{lk}\sigma^{lk} = 2\eta\left[(\Gamma_{t\phi r} + \Gamma_{tr\phi})\,\sigma^{r\phi} + O(\delta^2)\right]$$
$$= 2\eta\left[4\sigma^{r\phi} + O(\delta^2)\right].$$

Terms with q^i contribute due to the rapid change in the energy flux component normal to the disk:

$$(U^t q^k)_{;k} = \frac{\partial q^z}{\partial z} + O(\delta^4)\,.$$

Summing all terms, from the energy balance equation, we obtain

$$\frac{\partial q^z}{\partial z} = 4\eta(\sigma^{r\phi})^2 = \frac{3}{2}\,T_v^{r\phi}\,\frac{D}{r^{3/2}C}\,, \qquad (1.91)$$

whence, after integrating over the disk thickness, we derive the important relation

$$Q = \frac{3}{4}\,\frac{D}{r^{3/2}C}\,T_v\,, \qquad (1.92)$$

where $Q = q^z(z = h)$ is the vertical energy flux escaping from the disk. After specifying Q, we can calculate the radial profile of the effective temperature of the disk surface, because by definition $Q = \sigma T_{\mathrm{eff}}^4$. This is a universal result of the standard accretion disk theory: T_{eff} does not depend on the specific nature of the dissipation of the kinetic energy of matter or on the mechanism of thermal energy transfer toward the disk surface, and is proportional to $\dot{M}$ times some universal known function of r.

Thus, we have obtained the explicit form of the viscous stress integrated over the disk thickness T_v and the explicit form of the radiation energy flux from its surface Q. At the same time, we know only the combination ΣU^r, and not each of these variables separately. In addition, we should determine the disk half-thickness profile $h(r)$ and the temperature, pressure, and density distributions inside it. To do this, the vertical disk structure should be calculated.

1.3.5 Energy transfer equation and the vertical disk structure. The vertical disk structure is determined by three equations. Two of them have already been obtained above: vertical hydrostatic balance equation (1.74) and thermal energy generation equation (1.91).

The remaining equation is the transfer equation for the energy dissipating in the disk. In the simplest case, the energy transfer is due to the photon diffusion in heated matter. Strictly speaking, we should write a relativistic analog of the radiation heat conductivity equation, which is a variant of the kinetic Boltzmann equation for photons when their mean free path is much smaller than the characteristic spatial length of the problem. This equation was relativistically generalized in Section 2.6 in [1]. The standard transition to the diffusion approximation yields the following equation (see expression 2.6.43 in [1]):

$$q^i = \frac{1}{\tilde{\kappa}\rho}\,\frac{4}{3}\,bT^3 P^{ik}\left(\mathbf{e}_k(T) + a_k T\right), \tag{1.93}$$

where $\tilde{\kappa}$ is the Rosseland mean opacity of matter, T is the temperature, b is the radiation constant, and $a_k \equiv U_{k;j}U^j$ is the four-acceleration. The discussion of Eqn (1.93) can also be found in [13, p. 165].

As regards (1.93), we note that the four-acceleration never exceeds the order δ^2 because the four-velocity itself differs from the geodesic value (free circular equatorial motion) only in the second order in δ. At the same time, the derivative in the first term in parentheses in the right-hand side of (1.93) for $k = z$, in contrast, raises the order in δ, because T, as well as ϵ, varies significantly across the disk thickness. As a result, as already discussed in Section 1.3.1, we see that $\mathbf{q}$ is the leading component of the vector q^z and is determined by the equation

$$q^z = -\frac{1}{3\tilde{\kappa}\rho}\,\frac{\partial(bT^4)}{\partial z}, \tag{1.94}$$

which is identical to the Newtonian form for a thin disk.

Equations (1.74), (1.91), and (1.94) must be supplemented with the equation of state of matter

$$p(\rho, T),$$

the opacity law

$$\tilde{\kappa}(\rho, T),$$

and the explicit form of

$$\eta(\rho, T) \quad \text{or} \quad T_v^{r\phi}(\rho, T),$$

depending on the type of parameterization of the turbulent viscosity in the disk.

In addition, it is necessary to set boundary conditions at the integration interval $z \in [0, h]$. In the simplest case, we assume that the disk has no atmosphere and

$$\rho|_{z=h} = T|_{z=h} = 0.$$

Furthermore, the energy flow is absent in the equatorial disk plane:

$$q^z|_{z=0} = 0.$$

Finally, we set

$$2\int_0^h T_v^{r\phi}\,\mathrm{d}z = T_v.$$

We note that the above equations and boundary conditions for the vertical disk structure automatically guarantee the validity of Eqns (1.86), (1.90), and (1.92) for the radial disk structure.

After calculating the vertical structure, we can specify the surface density distribution using (1.82) and then U^r using (1.86).

1.3.6 Parameterization of turbulent viscosity and the explicit disk structure. The estimates already carried out in [1, 15] in accordance with the algorithm described in Section 1.3.5 show that at a sufficiently high accretion rate $\dot{M}$, which is the free parameter of the problem, the radiation energy becomes dominant in the inner parts of the disk. The estimate of the threshold value of $\dot{M}$ can be found, for example, in [15] [see formula (2.18) therein]. It turns out that the disk thickness far away from its inner radius is independent of r, and for $\dot{M}$ of the order of and above the critical value $\dot{M}_{\rm cr}$, when the disk luminosity reaches the Eddington value in the inner parts of the disk, $\delta > 1$, corresponding to the spherization of the flow (see expression (7.1) and its discussion in [15]). In addition, later studies showed that the radiation-dominated region is both thermally unstable [16] and convectively unstable [17].

This means that for the correct description of the inner parts of accretion disks at high accretion rates, when δ increases, terms of higher orders in δ should be taken into account. These include the radial pressure gradient $\sim \delta^2$ in the radial force balance and the advection term $U^r T\partial S/\partial r \sim \delta^4$, which arises in the energy balance and accounts for the radial heat transfer. The latter, in fact, implies that the heat diffusion time in the vertical direction is comparable to the radial advection time due to radial transfer of matter. In other words, the main property of the standard accretion disk model considered here is violated: the local energy balance in the disk, when the heat generated due to turbulent energy dissipation is locally released from the disk surface. It was found that accounting for the new terms also allows correctly describing the region near $r_{\rm ms}$, where $U^r \to \infty$ in the standard model, and constructing a stationary solution with $\delta < 1$ for $\dot{M}$ of the order of and above $\dot{M}_{\rm cr}$, which is stable under thermal perturbations (so-called 'slim disks'; see [18, 19] and the references therein, and, e.g., [20]). Later, these results were confirmed by numerical simulations (see, e.g., [21, 22]). We add that the transition from a standard disk to a slim disk with increasing $\dot{M}$ in the relativistic model around a rotating black hole should occur even earlier due to a higher accretion efficiency (which is due to both decreasing $r_{\rm ms}$ and the additional angular momentum loss from the disk surface by radiation).

Assuming that $\dot{M} \ll \dot{M}_{\rm cr}$, we calculate the disk vertical profile, which is to be useful in the next part of the paper, in the simplest case where the pressure is mainly determined by the fully ionized hydrogen plasma, i.e.,

$$p = \frac{2\rho k_{\rm B} T}{m_{\rm p}}, \tag{1.95}$$

where $m_{\rm p}$ is the mass of the proton, $k_{\rm B}$ is the Boltzmann constant, and the opacity is determined by Thomson scattering, $\tilde{\kappa} = \kappa_{\rm T} = 0.4\ \mathrm{cm}^2\ \mathrm{g}^{-1}$.

We also assume that the kinematic viscosity ν is independent of z and can be parameterized in the form

$$\nu = \alpha c_s h \,, \tag{1.96}$$

where $0 < \alpha < 1$ is the Shakura parameter determining the turbulent viscosity in the disk (see [15, 23]), and c_s is the speed of sound in the equatorial disk plane. Here, due to (1.95),

$$c_s^2 = \frac{2k_B T_c}{m_p} \,, \tag{1.97}$$

where $T_c = T(z = 0)$.

Equation (1.94) yields

$$\int_0^h \mathrm{d}z\, q^z \rho = -\frac{1}{3\kappa_T}\, bT^4 \bigg|_0^h = \frac{1}{3\kappa_T}\, bT_c^4 \,.$$

On the other hand,

$$\int_0^h \mathrm{d}z\, q^z \rho = C_q Q \int_0^h \rho\, \mathrm{d}z = \frac{1}{2}\, C_q \Sigma Q \,,$$

where C_q is some correction factor of the order of unity corresponding to the difference between the escaping radiation flux Q and its mean value inside the disk thickness. As a result, we have

$$T_c = \left(\frac{3\kappa_T}{2} \frac{C_q}{b}\, \Sigma Q \right)^{1/4} . \tag{1.98}$$

Next, in the left-hand side of (1.74), assuming for simplicity that the entropy is constant along z, we can divide by ρ, introduce the enthalpy $\mathrm{d}w = \mathrm{d}p/\rho$, and, after integrating (1.74), obtain the central value of w, $w_c \equiv w(z = 0)$:

$$w_c = -\int_0^h \mathrm{d}w = \int_0^h \frac{z}{r^3} \frac{H}{C} = \frac{h^2}{2r^3} \frac{H}{C} \,.$$

Hence, assuming that $w_c = nc_s^2$, where n is the polytrope index, we obtain

$$c_s^2 = \frac{h^2}{2nr^3} \frac{H}{C} \,. \tag{1.99}$$

Finally, due to definition (1.80), parameterization (1.96), and Eqn (1.90), we find

$$T_\nu = \frac{3}{2} \frac{D}{r^{3/2} C}\, \alpha \Sigma c_s h = \frac{\dot{M}}{2\pi} \frac{Y}{r^{3/2} D} \,, \tag{1.100}$$

where in the second equality we introduce the new variable

$$Y \equiv (2r)^{-1/2} \int_{r_{ms}}^{r} \frac{E}{r^{1/2} C}\, \mathrm{d}r \,, \tag{1.101}$$

which in the Newtonian limit, far away from the inner edge of the disk, tends to unity.

Equations (1.92), (1.97)–(1.99) are sufficient to eliminate all unknowns except Σ and the free parameters $\dot{M}$ and α from (1.100). We thus obtain the surface density profile

$$\Sigma = \Sigma_0 \alpha^{-4/5} \dot{M}^{3/5} r^{-3/5} C^{3/5} D^{-8/5} H^{2/5} Y^{3/5} \,, \tag{1.102}$$

where the dimensional constant Σ_0 combines all relevant physical constants and numerical coefficients. Its explicit form and numerical value (which depends on the black hole mass to which we normalize all quantities) can be found by the reader.

Now, using formulas (1.97)–(1.99) and (1.102), it is possible to derive the profile $h(r)$. The resulting disk aspect ratio is

$$\delta(r) = \delta_* r^{1/20} C^{9/20} D^{-1/5} H^{-9/20} Y^{1/5} \,, \tag{1.103}$$

where δ_* is a constant that determines the characteristic disk thickness δ.

2. Relativistic twisted accretion disk

2.1 Introductory remarks

In Section 1, we described a flat disk in the equatorial plane around a rotating black hole. Its axially symmetric structure was evident and consistent with the symmetry of space near the black hole. If we now relax the main assumption that the flow of matter at all distances lies in the equatorial plane, the question arises: what can the dynamics of this more complicated flow, both stationary and nonstationary, be? Is this configuration similar to a disk in any way? For thin disks considered here, the answer to this question proves to be positive under some restrictions.

The main reason for the deformation of the disk (for example, an initially flat one) is that the black hole spin gives rise to an additional off-center gravitational interaction with the gas elements of the flow. It can be shown that far away from the event horizon but close to the equatorial plane of the black hole, this interaction is represented by an axially symmetric field of force directed to the black hole spin axis in the planes parallel to the equatorial one (see [24], chapter 3, paragraph A). This force is called the gravitomagnetic force and is given in this case by the expression

$$F_{GM} = \frac{4a\Omega}{r^2} \frac{\partial}{\partial r} \,, \tag{2.1}$$

where Ω is the Keplerian frequency and $\partial/\partial r$ is the radial coordinate basis vector of the cylinder coordinate system. Clearly, this external force can change the proper angular momentum of the disk elements (and hence deform the disk), if these are moving away from the equatorial plane of the black hole. Here, only the projection of the gravitomagnetic force onto the angular momentum direction matters, which is proportional to the sine of the angle between the angular momentum vector and the black-hole spin axis. As we see shortly, the restriction that allows treating the new configuration as a disk (both stationary and nonstationary) requires that the gravitomagnetic force be smaller than the central gravitational attraction force, i.e., requires the smallness of the parameter a ($a \ll 1$). In addition, one more restriction can be formulated for the disk to be hydrodynamically stable: the noncomplanarity of the disk with the orbital plane, as well as the degree of its deviation from the planar form (i.e., twist, warp) not exceeding some small values (see [25], paragraph 7 and [5], paragraph 4.2.4).

We split a thin planar disk into rings of small widths. In each ring, the motion of gas elements is mainly due to the gravitational attraction force from the central body. The characteristic time of this motion is $t_d \sim \Omega^{-1}$. In addition, t_d

determines the time it takes for the disk to restore the hydrodynamic equilibrium across the ring, because the disk aspect ratio (the ratio of the disk thickness to the radial distance) is of the order of the ratio of the sound velocity to the orbital velocity. This conclusion can also be arrived at by noticing that the acceleration of a unit mass gas element is δ^{-1} times greater than the vertical pressure gradient, i.e., the gradient is exactly as small as the ratio of the vertical size of the ring to its radius. Therefore, we can conclude that if other forces acting on a given ring from the adjacent rings or from the black hole lead to the dynamics with a characteristic time $t_{\rm ev} \gg t_{\rm d}$, the hydrostatic equilibrium is maintained in the ring; in other words, the ring remains flat, and the entire flow preserves a disk-like shape. This is undoubtedly so in a flat disk, because in this case equally oriented rings interact by the viscous force acting in the azimuthal direction and the angular momentum changes due to the inflow and outflow of matter accreting through the ring, with both processes occurring on the diffusion time scale $t_v \sim \Omega^{-1}\delta^{-2} \gg t_{\rm d}$.

We now let the disk be tilted to the equatorial plane of the black hole by a small angle $\beta \ll 1$. In a flat disk, the gravitomagnetic force contributes only to the modulus of the acceleration of gas elements moving in circular orbits, but now, due to a nonzero projection of this force ($\propto \beta$) onto the angular momentum of gas elements, this force makes the orbits precess around the black-hole spin axis. For free particles, this effect is described in detail in the second part of the next section in terms of the difference between the frequencies of circular and vertical motions. We also show in what follows that the precession frequency is much smaller than the circular frequency for $a \ll 1$ [see formula (2.12)], which is equivalent to the condition $t_{\rm ev} \gg t_{\rm d}$ for whole rings composed of gas elements.

Equation (2.12) suggests that the precession of the rings is differential, i.e., depends on the distance to the center. As a result, the relative orientation of initially coaxial rings changes, and the disk is no longer flat. However, we keep in mind that under the condition $t_{\rm ev} \gg t_{\rm d}$, each of the rings behaves 'rigidly' in its vertical direction, which is now also a function of r. The new configuration is similar to a twisted (or warped) disk, i.e., a flow symmetric relative to some (now not planar) surface, which can be called the equatorial surface of the twisted disk. Here, the cross section of the equatorial surface by a plane passing through the center is a circle — the instantaneous shape of orbits of gas elements rotating at a given radial distance r. The disk turns into a set of rings tilted to the black hole equatorial plane by a constant angle β but depending on r node lines (the line formed by the intersection of the ring planes with the black hole equatorial plane). The node line is now determined by the position angle $\gamma(r)$ measured in the equatorial plane in the positive direction from a fixed direction to the ascending node of a given ring. The key point here is that the pressure gradient in the twisted disk, directed (as in any thin disk in general) almost normal to its warped surface, is not normal to the planes of the rings composing the disk. Therefore, we conclude that the pressure gradient has two projections. The main projection is coaxial with the rotational axis of each ring. We conventionally let it be denoted by $(\nabla p)_\xi$, where ξ is the distance from the equatorial surface of the twisted disk measured along the direction of rotation of the ring (ξ reduces to z in the case of a flat disk). We note from the beginning that $(\nabla p)_\xi \propto \xi$ due to the hydrostatic equilibrium maintained across the ring. The second projection of the pressure gradient, conventionally denoted as $(\nabla p)_r$, lies in the ring plane along the radial direction connecting the disk center and a given gas element of the ring. The ratio of these two projections is a small value proportional to the rate of change of orientations of rings in the disk, which in turn depends on the radial direction chosen in the given ring plane. From purely geometrical considerations, we rigorously show in what follows that for a disk with $\beta = \text{const}$, $(\nabla p)_r/(\nabla p)_\xi \propto \beta\, {\rm d}\gamma/{\rm d}r \cos\psi$, where ψ is the angle measured in the azimuthal direction for a given ring from its ascending node to the given gas element. We note that the normal to the twisted disk surface is orthogonal to the ring plane only at two diametrically opposite points — where its plane intersects the planes of the adjacent rings. At $\beta = \text{const}$, these points are characterized by $\psi = \pm\pi/2$. At the same time, at the other pair of points with $\psi = 0,\ \pi$, the value $(\nabla p)_r$ reaches a positive (negative) maximum.

Thus, in the case of a flat disk, the dynamics in the radial direction is controlled in the leading order in $\delta \ll 1$ by the gravitation force and the corrections $\sim \delta^2$ are neglected, but in a twisted disk, the radial projection of the pressure gradient starts additionally contributing to the radial balance. This addition, on the one hand, depends on the degree of the twist, and on the other hand, increases proportionally to the distance ξ from the equatorial disk surface. Because it also depends harmonically on the azimuthal direction, the gas elements (for $\xi \neq 0$) are subjected to periodic disturbance by this force with the orbital period, and their orbits become ellipses with a small eccentricity. As is well known, the eigenfrequency of small oscillations of free particles in eccentric orbits is equal to the epicyclic frequency κ. Because the pressure gradient projection considered here excites exactly such oscillations, the radial profile of the epicyclic frequency $\kappa(r)$ is an important characteristic that determines the shape of both stationary and nonstationary twisted configurations. In the next section, we derive the required relativistic profile $\kappa(r)$ for equatorial circular orbits in the Kerr metric [see Eqn (2.8)]. We note from the very beginning that in the special case of Newtonian gravity, $\kappa = \Omega$, and hence the action of the external exciting force on gas elements with the same frequency results in a resonance: the perturbed motion amplitude, characterized by perturbation of the orbital velocity $\mathbf{v}$, must increase without a bound. This increase, however, is always limited by turbulent viscosity in the disk. Indeed, because the exciting force amplitude $\propto \xi$, so is the amplitude $\mathbf{v}$. But this would mean the presence of a vertical velocity shear $\partial_\xi \mathbf{v}$ in each ring. Together with the vertical density gradient in the disk (and hence the vertical gradient of the dynamic viscosity), this gives rise to a volume viscous force that damps the driving of individual layers of each disk rings by the resonance force. We note that near the black hole, where the frequency κ deviates from Ω, the amplitude $\mathbf{v}$ remains bounded even in the absence of viscous forces. This allows the existence of low-viscosity stationary twisted disks around black holes, in which $\beta(r)$ takes an oscillatory form (see [25]).

Thus, we see that the twist of the disk caused by the gravitomagnetic force necessarily results in a perturbation of the circular motion of gas elements in the disk rings. The velocity field $\mathbf{v}$ of this perturbation depends on r (in addition to its being proportional to ξ, as explained above) and is determined by the current shape of the disk. By virtue of the continuity of the flow, this gives rise to density inhomogeneities outside the disk equatorial surface, $\rho_1 \propto \xi$. Because $(\nabla p)_r \propto \cos\psi$, these inhomogeneities take opposite signs at the diametrically opposite points of any given ring. But this implies that the ring is subjected to the total torque of the central gravitational force acting on the density excesses outside the equatorial plane of the ring (i.e., outside $\xi = 0$). We let $\mathbf{T}_g$ denote this torque. Because the disk is thin and the gravitational acceleration along the ring axis is itself $\propto \xi$, the corresponding component of the gravitational force, and $\mathbf{T}_g$ as well, is quadratic in ξ. In addition, we recall that the torque $\mathbf{T}_g$ is proportional to the small warp magnitude, $\mathbf{T}_g \propto \beta\, \mathrm{d}\gamma/\mathrm{d}r$. Thus, we arrive at the conclusion that the dynamics of the twisted disk rings is controlled by $\mathbf{T}_g$, together with the torque due to the gravitomagnetic force discussed above in this introductory section. We note that in the case $\beta = \mathrm{const}$ considered here, $(\nabla p)_r$ and, correspondingly, ρ_1 take the maximum absolute value (but with the opposite sign) at $\psi = 0, \pi$, i.e., at the node line of each ring.[4] But this implies that $\mathbf{T}_g$ lies in the plane made by the angular momentum of each ring and the black-hole spin axis. By virtue of the symmetry of the problem, the total contribution to $\mathbf{T}_g$ from other azimuths does not alter its direction. Therefore, immediately after the gravitomagnetic force turns the imaginary tilted plane disk into a twisted configuration with $\beta = \mathrm{const}$, the gravitational force acting on the matter of the disk located asymmetrically relative to the surface $\xi = 0$ tends to change the tilt angle of the rings: either to align them with the equatorial plane of the black hole or, conversely, to displace them from it. On the other hand, once β becomes dependent on r, the maximum of the absolute values of $(\nabla p)_r$ are shifted from the node line of each ring to some new ψ, which gives rise to a component in $\mathbf{T}_g$ that also contributes to the precession motion of the disk rings, as the gravitomagnetic torque does.

The dynamics of twisted disks sketched above is complicated by the presence of nonzero viscosity in the disk. First of all, each ring of the disk is subjected to the action of the viscous force arising due to the difference between the direction of the tangential velocity of the ring and that of the adjacent rings. This difference is maximal in the directions where the ring planes intersect, i.e., exactly where $(\nabla p)_r$ vanishes. In the above example of the configuration with $\beta = \mathrm{const}$, this corresponds to $\psi = \pm\pi/2$, i.e., perpendicular to the node line of the rings. The viscous force, being proportional to the difference in tangential velocities, is directed at these points perpendicular to the ring plane and has different signs on different sides of the node line. Therefore, the corresponding torque $\mathbf{T}_\nu$ is perpendicular to the plane made by the ring angular momentum and the black-hole spin axis. In other words, the viscous interaction between the disk rings leads only to their precession around the black hole spin. We also note that the viscous torque is proportional to the difference between the tangential velocities of adjacent rings, $\mathbf{T}_\nu \propto \beta\, \mathrm{d}\gamma/\mathrm{d}r$, and, due to the viscosity coefficient, $\mathbf{T}_\nu \propto \xi^2$. It is important to note that as soon as the profile $\beta(r)$ is formed due to the gravitational torque $\mathbf{T}_g$, $\mathbf{T}_\nu$ also causes the alignment/misalignment of the ring with the equatorial plane of the black hole. This happens for the same reasons by which $\mathbf{T}_g$ also starts contributing to the precession motion, as discussed above.

In addition to giving rise to $\mathbf{T}_\nu$, the viscosity in a twisted disk, as in a flat accretion disk, leads to the radial diffusion transfer of the angular momentum component parallel to the equatorial plane of the black hole (which is nonzero exactly for a tilted/twisted disk) toward the disk center due to simple transport of the accreting matter, and toward its periphery due to the corresponding angular momentum outflow. In the case of a relativistic disk, an additional loss of this angular momentum component occurs due to the thermal energy outflow by radiation from the disk surface (see Eqn (C6) in [5]).

All forces participating in the dynamics of twisted disks appear in the so-called 'twist' equation — the principal equation of the twisted disk theory. This equation is derived and analyzed in what follows.

[4] To make the description as rigorous as possible, it is also important to add that the noted coincidence of the azimuthal location of maxima of $(\nabla p)_r$ and ρ_1 occurs only when the action of viscosity on the gas elements of the ring is neglected.

2.1.1 Weakly perturbed circular equatorial motion: epicyclic frequency and the frequency of vertical oscillations. In a twisted disk, the motion of matter outside the equatorial plane of the Kerr metric is assumed; this motion is not necessarily circular in the projection onto that plane. Therefore, we first analyze the properties of free particles moving in orbits slightly different from circular ones.

We first assume that particles move exactly in the equatorial plane but in slightly noncircular orbits. The problem can be solved using relativistic hydrodynamic equations with zero pressure and by assuming that there is a small addition to the purely circular velocity. Then, instead of Eqns (1.63) and (1.64), it is better to use the original equations in the form

$$T^{ik}{}_{;k} = 0\,, \tag{2.2}$$

where in the considered case of free motion, $T^{ik} = \rho U^i U^k$ and $\rho = \mathrm{const}$. Under the last assumption, the velocity field, as follows from rest-energy conservation law (1.65), is divergence-free, and Eqn (2.2) is equivalent to the equation

$$U^i{}_{;k} U^k = 0\,. \tag{2.3}$$

From the four-velocity field, we now segregate a small addition to the main circular equatorial motion and let v^i denote it. The unperturbed motion corresponds to rest in the projection onto tetrad (1.49)–(1.52) used to construct the flat accretion disk model, i.e., is given by the four-velocity $U_0^i = \{1, 0, 0, 0\}$. Substituting the sum $U_0^i + v_i$ in (2.3), we obtain linear equations for small perturbations of the four-velocity v_i, which is assumed to be a function of t only:

$$v^i{}_{;k} U_0^k + U_{0;k}^i v^k = 0\,. \tag{2.4}$$

Taking into account that $U_{0;k}^i = \Gamma_{tk}^i$ for $i = 1, 2$, we obtain the system of equations

$$v^r{}_{;t} + \Gamma_{t\phi}^r v^\phi = C^{-1/2} B\, \frac{\mathrm{d}v^r}{\mathrm{d}t} - 2r^{-3/2} v^\phi = 0\,, \tag{2.5}$$

$$v^\phi{}_{;t} + \Gamma_{tr}^\phi v^r = C^{-1/2} B\, \frac{\mathrm{d}v^\phi}{\mathrm{d}t} + r^{-3/2}\left(1 - \frac{1}{2}\frac{H}{C}\right) v^r = 0\,. \tag{2.6}$$

It follows that small perturbations of the four-velocity components in the equatorial plane of the rotating black hole oscillate in time. For example, v^r satisfies the equation

$$\frac{\mathrm{d}^2 v^r}{\mathrm{d}t^2} + \frac{2C}{r^3 B^2}\left(1 - \frac{H}{2C}\right) v^r = 0\,, \tag{2.7}$$

which implies that the square of the frequency of these oscillations, which is the epicyclic frequency by definition, has the form

$$\kappa^2 = r^{-3}B^{-2}(2C - H)$$
$$= r^{-3}\left(1+\frac{a}{r^{3/2}}\right)^{-2}\left(1-\frac{6}{r}+\frac{8a}{r^{3/2}}-\frac{3a^2}{r^2}\right). \tag{2.8}$$

A somewhat different derivation of κ can be found in the appendix in [26]. It is important to note that (2.8) contains a derivative with respect to the coordinate time, and therefore the epicyclic frequency is determined by the clock of an infinitely remote observer, similarly to circular frequency (1.16) introduced above. By comparing Eqn (1.17), which defines the location of the innermost stable circular equatorial orbit in the Kerr metric, $r_{\rm ms}$, with (2.8), we infer that $\kappa^2(r_{\rm ms}) = 0$. For $r < r_{\rm ms}$, the epicyclic frequency becomes imaginary, and Eqn (2.7) has exponentially growing solutions. This must be so because the free circular motion around a rotating black hole becomes unstable in this region. In Section 1.1.3, this result was obtained from the analysis of the form of the effective centrifugal potential in which a test particle moves in an equatorial circular orbit. Nevertheless, we see that $r_{\rm ms}$ can be determined alternatively from the calculation of the profile $\kappa^2(r)$ in the Kerr metric.

It is well known that for Newtonian motion, so-called Keplerian degeneration occurs when $\kappa = \Omega$ for a noncircular motion, which causes nonrelativistic orbits to be closed. But this symmetry is broken for relativistic free motion, and the epicyclic frequency κ differs from Ω already near a nonrotating ($a = 0$) black hole, where its square is

$$\kappa^2 = r^{-3}\left(1-\frac{6}{r}\right) = \Omega^2\left(1-\frac{6}{r}\right) < \Omega^2. \tag{2.9}$$

The difference between the epicyclic and circular frequencies results in the well-known effect of the precession of the major axis of an elliptical orbit. Far away from the horizon of a Schwarzschild black hole, i.e., for $r \gg 1$, the frequency of the orbit rotation, called the Einstein precession frequency, is $\Omega_{\rm p} \approx 3/r^{5/2}$.

We now suppose that we rotate together with the test particle at some radius. When considering the problem in the projection onto tetrad (1.49)–(1.52), this particle remains at rest. We now impart a small velocity to it in the direction perpendicular to the equatorial plane. Equation (1.74) of hydrostatic equilibrium for a flat disk implies that in our reference frame, the particle, being in free motion, is subjected to acceleration that is proportional to z and tends to return it to the position $z = 0$. As a result, the test particle harmonically oscillates with a frequency whose square is

$$\Omega_{\rm v}^{l\,2} = \frac{H}{r^3 C}, \tag{2.10}$$

where the superscript l reminds us that the frequency is measured in the reference frame comoving with the particle in its main circular equatorial motion. To re-express this frequency as measured by the clock of an infinite observer, as has been done for both circular and epicyclic frequencies, the frequency $\Omega_{\rm v}^{l}$ must be divided by the time dilation factor (comparing the proper time of the particle and the time at infinity), i.e., by the t-component of four-velocity (1.14). Then the square of the frequency of vertical oscillations is

$$\Omega_{\rm v}^2 = r^{-3}B^{-2}H = r^{-3}\left(1+\frac{a}{r^{3/2}}\right)^{-2}\left(1-\frac{4a}{r^{3/2}}+\frac{3a^2}{r^2}\right), \tag{2.11}$$

which coincides, for example, with the expression presented in [27] (see also [28]). Equation (2.11) implies that $\Omega_{\rm v} = \Omega$ around a nonrotating black hole. This means that the vertical and circular motions have the same period, and the total motion of the particle is again the circular motion in a closed orbit whose plane, however, is now slightly tilted to the initial equatorial plane. The situation changes for $a \neq 0$, because for $\Omega_{\rm v} \neq \Omega$, the orbit is not closed any more, and the orbital plane starts precessing around the spin axis of the black hole. The frequency of the orbital precession is equal to the difference between the circular and vertical frequencies. For a slowly rotating black hole with $a \ll 1$, the precession frequency of a slightly tilted orbit is

$$\Omega_{\rm LT} = \Omega - \Omega_{\rm v} \approx r^{3/2}\left(1-\frac{a}{r^{3/2}}\right)$$
$$- r^{3/2}\left(1-\frac{3a}{r^{3/2}}\right) = \frac{2a}{r^3} \ll \Omega. \tag{2.12}$$

This is simply the angular velocity of frame dragging by the rotating black hole [see Eqn (1.2)] in the limit $a \ll 1$. The frequency $\Omega_{\rm LT}$ is also referred to as the Lense–Thirring frequency.

In the most general case, where the test particle deviates from circular motion simultaneously in the vertical and horizontal directions, the particle motion in space can be described by a slightly elliptical orbit, with both plane and apse line turning with an angular velocity respectively proportional to the difference between the circular and vertical frequency and the difference between the circular and epicyclic frequency. For $a \ll 1$, the precession of the orbital plane then occurs on a timescale much longer than the dynamical time, $t_{\rm LT} \gg t_{\rm d}$, where $t_{\rm LT} \sim \Omega_{\rm LT}^{-1}$ (see the discussion in the preceding section).

2.2 Choice of the reference frame

2.2.1 The metric. Taking the general conclusions in Section 2.1 into account, we consider slowly rotating black holes, $a \ll 1$. In this case, the linear expansion of the Kerr metric in the parameter a is sufficient. Then formula (1.1) takes the form

$$ds^2 = \left(1-\frac{2}{R}\right)dt^2 - \left(1-\frac{2}{R}\right)^{-1}dR^2$$
$$- R^2(d\theta^2 + \sin^2\theta\, d\phi^2) + 4\frac{a}{R}\sin^2\theta\, d\phi\, dt. \tag{2.13}$$

Metric (2.13) is identical to that of a nonrotating black hole written in Schwarzschild coordinates, except for one nondiagonal term responsible for the Lense–Thirring precession.

Our main purpose in this section is to introduce the relativistic reference frame that follows the disk twist. The symmetry of the problem implies that the equations of motion should have the simplest form in such a frame. As in a flat disk, it is convenient to use some orthonormal noncoordinate basis. For this basis to follow the disk shape, its two spatial basis vector should be tangent to the disk symmetry plane. At each spatial point, we take the basis vectors of the 'flat' basis, which are determined, say, by the equatorial plane of the black hole, and turn them by the angles β and γ defining the disk shape. This is done in the simplest way by using a Cartesian coordinate system with the z axis parallel to the black hole spin. However, we should first understand which four-dimensional basis (whose dual tetrad must transform metric (2.13) into the Minkowski metric) in the flat-space limit would produce the spatial part described by a Cartesian coordinate system.

This can be done by changing the radial variable in (2.13), namely, by passing from R to the so-called 'isotropic' radial coordinate R_{I}:

$$R = R_{\mathrm{I}}\left(1+\frac{1}{2R_{\mathrm{I}}}\right)^2 . \tag{2.14}$$

Substituting (2.14) in (2.13) yields

$$\begin{aligned} \mathrm{d}s^2 &= \left(\frac{1-1/(2R_I)}{1+1/(2R_I)}\right)^2 \mathrm{d}t^2 - \left(1+\frac{1}{2R_{\mathrm{I}}}\right)^4 \\ &\quad \times (\mathrm{d}R_{\mathrm{I}}^2 + R_{\mathrm{I}}^2\,\mathrm{d}\theta^2 + R_{\mathrm{I}}^2 \sin^2\theta\,\mathrm{d}\phi^2) \\ &\quad + 4\frac{a\sin^2\theta}{R_{\mathrm{I}}(1+1/(2R_{\mathrm{I}}))^2}\,\mathrm{d}t\,\mathrm{d}\phi\,, \end{aligned} \tag{2.15}$$

where the second term represents the elementary spherical volume. Now, it is easy to pass to the Cartesian coordinates by the change $\{x = R_{\mathrm{I}}\cos\phi\sin\theta,\ y = R_{\mathrm{I}}\sin\phi\sin\theta,\ z = R_{\mathrm{I}}\cos\theta\}$. Because $R_{\mathrm{I}}^2\sin^2\theta\,\mathrm{d}\phi = x\,\mathrm{d}y - y\,\mathrm{d}x$, we have

$$\begin{aligned} \mathrm{d}s^2 &= K_1^2\,\mathrm{d}t^2 + 2aK_1K_3(x\,\mathrm{d}y - y\,\mathrm{d}x)\,\mathrm{d}t \\ &\quad - K_2^2(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)\,, \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} K_1 &= \frac{1-1/(2R_{\mathrm{I}})}{1+1/(2R_{\mathrm{I}})}\,, \quad K_2 = \left(1+\frac{1}{2R_{\mathrm{I}}}\right)^2, \\ K_3 &= \frac{2}{R_{\mathrm{I}}^3}\,\frac{1}{1-1/(2R_{\mathrm{I}})^2}\,, \end{aligned} \tag{2.17}$$

which shows that $K_{1,2,3}$ are functions of $R_{\mathrm{I}} = (x^2+y^2+z^2)^{1/2}$ only.

Metric (2.16) generates the dual basis

$$\begin{aligned} &\mathbf{e}^t = K_1\,\mathrm{d}t + aK_3(x\,\mathrm{d}y - y\,\mathrm{d}x)\,, \quad \mathbf{e}^x = K_2\,\mathrm{d}x\,, \\ &\mathbf{e}^y = K_2\,\mathrm{d}y\,, \quad \mathbf{e}^z = K_2\,\mathrm{d}z\,. \end{aligned} \tag{2.18}$$

We note that basis (2.18) corresponds to observers at rest in the Schwarzschild coordinates, because their world lines defined by the condition $U^i = \mathbf{e}^i/\mathrm{d}s = \{1,0,0,0\}$ correspond to the equalities $\mathrm{d}x = \mathrm{d}y = \mathrm{d}z = 0$. Their identical clocks are synchronized in such a way that in equal time intervals determined by the vector $\mathbf{e}^t$, light travels an equal distance in any direction defined by a combination of the $\mathbf{e}^{x,y,z}$. If the observers used the coordinate time t, they would discover, for example, that the light signal in the azimuthal direction prograde with the black hole spin travels a larger distance than in the opposite (retrograde) direction. This follows from the frame-dragging effect of a rotating black hole and is equivalent to the well-known tilt of light cones in the azimuthal direction. Finally, we note that another choice of the orthonormal basis is possible in principle, which also compensates the space-dragging effect. Such a basis is called the frame of locally nonrotating observers, launched in the azimuthal direction with an angular velocity equal to (1.2); mathematically, this corresponds to the correction of the azimuthal basis vector instead of the time one (see [29]).

Below, we need to rotate the spatial part of (2.18) so as to obtain the dual twisted basis and then the original basis, which, as we recall, is needed to write the projection of the hydrodynamic equation. For this, we first introduce twisted cylindrical coordinates.

2.2.2 Twisted coordinates. We define twisted cylindrical coordinates $\{\tau, r, \psi, \xi\}$ such that the condition $\xi = 0$ determines a coordinate plane coincident with the equatorial plane of a twisted disk. Here, τ, r, ψ, and ξ are the respective new time variable and twisted analogs of the radial, azimuthal and vertical cylindrical coordinates.[5] These coordinates were first introduced in [30, 31]. At each fixed $r = \mathrm{const}$, the angle ψ is measured in the positive direction from the ascending node of the circle $\xi = 0$ crossing the equatorial plane of the black hole. The relation between $\{\tau, r, \psi, \xi\}$ and $\{t, x, y, z\}$ can be obtained by a sequence of rotations at each radial distance by the angles $\beta(r,\tau)$ and $\gamma(r,\tau)$.

We take the radius vector with coordinates

$$\begin{bmatrix} \tau \\ r\cos\psi \\ r\sin\psi \\ \xi \end{bmatrix}, \tag{2.19}$$

where three spatial Cartesian coordinates are defined in a frame with the z axis tilted by the angle $\beta(r,\tau)$ toward the black hole spin and the x axis lying in the black hole equatorial plane and rotated through the angle $\gamma(r)$ relative to some direction common to all r.

We next perform consecutive rotations of this coordinate system through the angle $\beta(r,\tau)$ about its x axis in the negative direction and then through the angle $\gamma(r,\tau)$ about its z axis in the negative direction. After these two rotations, this coordinate system transforms into a 'flat' Cartesian system common to all r, with the xy plane coinciding with the equatorial plane of the black hole. The new coordinates of the radius vector are then obtained by multiplying (2.19) first by the matrix

$$A_1(\beta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{bmatrix}, \tag{2.20}$$

and then by the matrix

$$A_2(\gamma) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma & 0 \\ 0 & \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.21}$$

As a result, we obtain the following relation between the twisted cylindrical and 'flat' Cartesian coordinates taken in the linear approximation in small β:

$$\begin{aligned} t &= \tau\,, \\ x &= r\cos\gamma\cos\psi - \sin\gamma(r\sin\psi - \xi\beta)\,, \\ y &= r\sin\gamma\cos\psi + \cos\gamma(r\sin\psi - \xi\beta)\,, \\ z &= r\beta\sin\psi + \xi\,. \end{aligned} \tag{2.22}$$

[5] Here and hereafter, r denotes the twisted radial coordinate.

2.2.3 Tetrad transported by the twist-following observers. We now pass from 'flat' basis (2.18) to the twisted one by rotating its spatial basis vectors by the twisting angles at each spatial point. First, we need to perform the rotation strictly opposite to what we did in Section 2.2.2. This means that we should take basis (2.18) as a column and first multiply it by the matrix $A_2(-\gamma)$ and then by the matrix $A_1(-\beta)$. After that, because we wish to obtain the basis corresponding to the (twisted) cylindrical coordinate system, it is necessary to additionally 'advance' the three spatial basis vectors by a further azimuthal angle ψ, which is achieved by additional multiplication of the basis by the matrix $A_2(-\psi)$.

As a result, we obtain the twisted dual basis that contains some linear combinations of the 'flat' coordinate basis vectors, $\{\mathrm{d}t, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z\}$. It remains to express it as linear combinations of coordinate basis vectors of the twisted coordinate frame, $\{\mathrm{d}\tau, \mathrm{d}r, \mathrm{d}\psi, \mathrm{d}\xi\}$. For this, it suffices to take differentials of the coordinate transformation [given by (2.22) in the linear approximation in β] and to substitute them in the twisted dual basis obtained after the rotations. It can be verified that in the approximation linear in β and a, we have

$$\mathbf{e}^\tau = (K_1 - ar\xi K_3 \partial_\varphi U)\,\mathrm{d}\tau + a\xi K_3 \partial_\varphi (Z - rW)\,\mathrm{d}r + arK_3(r - \xi Z)\,\mathrm{d}\varphi - arK_3\partial_\varphi Z\,\mathrm{d}\xi\,, \tag{2.23}$$

$$\mathbf{e}^r = -\xi K_2 U\,\mathrm{d}\tau + K_2(1 - \xi W)\,\mathrm{d}r\,, \tag{2.24}$$

$$\mathbf{e}^\varphi = -\xi K_2 \partial_\varphi U\,\mathrm{d}\tau - \xi K_2 \partial_\varphi W\,\mathrm{d}r + rK_2\,\mathrm{d}\varphi\,, \tag{2.25}$$

$$\mathbf{e}^\xi = rK_2 U\,\mathrm{d}\tau + rK_2 W\,\mathrm{d}r + K_2\,\mathrm{d}\xi\,, \tag{2.26}$$

where we introduce the new azimuthal variable $\varphi = \psi + \gamma(r, \tau)$ and pass to partial derivatives with respect to the corresponding new coordinates.

We also introduce new variables characterizing the disk geometry:

$$\Psi_1 = \beta\cos\gamma\,, \qquad \Psi_2 = \beta\sin\gamma\,, \tag{2.27}$$

and from now on use them instead of the angles β and γ. Additionally,

$$Z = \beta\sin\psi = \Psi_1\sin\varphi - \Psi_2\cos\varphi\,, \quad U = \dot{Z}\,, \quad W = Z'\,, \tag{2.28}$$

where partial derivatives with respect to τ and r are denoted by the dot and the prime.

It follows that for $\beta = \gamma = 0$ and after passing to the Cartesian coordinates, basis (2.23)–(2.26) is transformed into the flat basis (2.18).

As discussed above, observers transporting basis (2.18) are at rest in the Schwarzschild coordinates. On the contrary, observers corresponding to basis (2.23)–(2.26) move in space by following the changing shape of the twisted disk (in the nonstationary dynamics).

As we have seen in Section 1, the original basis onto which hydrodynamic equations are projected is obtained by inverting the dual basis matrix. Using (2.23)–(2.26), in the approximation linear in β and a, we have

$$\mathbf{e}_\tau = \frac{1}{K_1}\left(\partial_\tau + \xi U\,\partial_r + \frac{\xi}{r}\,\partial_\varphi U\,\partial_\varphi - rU\,\partial_\xi\right), \tag{2.29}$$

$$\mathbf{e}_r = \frac{1}{K_2}\left(-\,a\xi\frac{K_3}{K_1}\,\partial_\varphi Z\,\partial_\tau + (1 + \xi W)\,\partial_r + \frac{\xi}{r}\,\partial_\varphi W\,\partial_\varphi - rW\,\partial_\xi\right), \tag{2.30}$$

$$\mathbf{e}_\varphi = \frac{1}{K_2}\left(-\,a\frac{K_3}{K_1}(r - \xi Z)\,\partial_\tau - a\xi\frac{K_3}{K_1}\,rU\,\partial_r + \left(\frac{1}{r} - a\xi\frac{K_3}{K_1}\,\partial_\varphi U\right)\partial_\varphi + ar\frac{K_3}{K_1}\,rU\,\partial_\xi\right), \tag{2.31}$$

$$\mathbf{e}_\xi = \frac{1}{K_2}\left(ar\frac{K_3}{K_1}\,\partial_\varphi Z\,\partial_\tau + \partial_\xi\right). \tag{2.32}$$

With the original and dual bases now available, using the algorithm presented in Section 1.2.1, we can calculate the connection coefficients. This very cumbersome but straightforward procedure yields the following nonzero connection coefficients in the linear approximation in β and a:

$$\begin{gathered}
\Gamma_{\tau r\tau} = \frac{K_1'}{K_1K_2}\,, \quad \Gamma_{\tau r\varphi} = \frac{K_3}{K_2^2}\left(1 - \frac{1}{2}(r - \xi Z)\,K_4\right),\\
\Gamma_{\tau r\xi} = -a\frac{K_3}{K_2^2}\,\partial_\varphi Z\left(1 - \frac{1}{2r}(r^2 + \xi^2)\,K_4\right), \quad \Gamma_{\tau\varphi r} = -\Gamma_{\tau r\varphi}\,,\\
\Gamma_{\tau\varphi\xi} = a\frac{K_3}{K_2^2}\left(Z + \frac{\xi}{2r}(r - \xi Z)\,K_4\right), \quad \Gamma_{\tau\xi\tau} = \frac{\xi}{r}\,\frac{K_1'}{K_1K_2}\,,\\
\Gamma_{\tau\xi r} = -\Gamma_{\tau r\xi}\,, \quad \Gamma_{\tau\xi\varphi} = -\Gamma_{\tau\varphi\xi}\,,\\
\Gamma_{r\varphi\tau} = \frac{\xi}{r}\,\frac{1}{K_1}\,\partial_\varphi U - \Gamma_{\tau r\varphi}\,, \quad \Gamma_{r\varphi r} = \frac{\xi}{r}\,\frac{1}{K_2}\,\partial_\varphi W\,,\\
\Gamma_{r\varphi\varphi} = \frac{(rK_2)'}{rK_2^2} - a\xi\frac{K_3}{K_1K_2}\,\partial_\varphi U\,, \quad \Gamma_{r\xi\tau} = \frac{U}{K_1} - \Gamma_{\tau r\xi}\,,\\
\Gamma_{r\xi r} = \frac{W}{K_2} - \frac{\xi}{r}\,\frac{K_2'}{K_2^2}\,, \quad \Gamma_{r\xi\varphi} = -ar\frac{K_3}{K_1K_2}\,U\,,\\
\Gamma_{r\xi\xi} = \frac{K_2'}{K_2^2}\,, \quad \Gamma_{\varphi\xi\tau} = \frac{1}{K_1}\,\partial_\varphi U - \Gamma_{\tau\varphi\xi}\,,\\
\Gamma_{\varphi\xi r} = \frac{1}{K_2}\,\partial_\varphi W\,, \quad \Gamma_{\varphi\xi\varphi} = -\frac{\xi}{r}\,\frac{K_2'}{K_2^2} - ar\frac{K_3}{K_1K_2}\,\partial_\varphi U\,,
\end{gathered} \tag{2.33}$$

where $K_4 \equiv (K_3/K_1)(K_1/K_3)'$. The other nonzero Γ_{ijk}, as usual, can be obtained by taking their antisymmetry in the first two indices into account.

It follows that basis (2.29)–(2.32), together with connection coefficients (2.33), is the sum of two parts: the main one that persists at $\beta = 0$ and a small additional one proportional to β. In what follows, we conventionally let B_0 and B_1 denote these parts.

2.3 System of twist equations

2.3.1 Projection of dynamical equations onto the twisted basis for a thin disk.

Separation of equations into two systems describing a flat disk and a twisted disk. We take the relativistic hydrodynamic equations in the original form

$$T^{ik}_{\ \ ;k} = 0\,, \tag{2.34}$$

where the stress–energy tensor and its components are presented in Section 1.2.3. Equations (2.34) should now be projected onto twisted basis (2.29)–(2.32). To perform this, we assume that $\beta \ll 1$. In other words, mathematically we regard the twist of the disk as a small perturbation of its 'ground' state, i.e., of the model of a flat disk, also referred to as the background. It is important that the appearance of a twist gives rise to new terms in the equations not only due to the bending of the basis but also due to the appearance of additional perturbations of physical quantities themselves that enter the stress–energy tensor, including the density, pressure, and four-velocity.

For a twisted disk, instead of (2.34), we can write

$$\left((T_0{}^{ik} + T_1{}^{ik})_{;k}\right)_0 + \left((T_0{}^{ik} + T_1{}^{ik})_{;k}\right)_1 = 0\,, \tag{2.35}$$

where $T_0{}^{ik}$ corresponds to the background state and $T_1{}^{ik}$ is a small Eulerian perturbation of the stress–energy tensor. The indices 0 and 1 that follow the notation of the covariant derivative mean that the derivative is taken in the bases B_0 and B_1.

The action of the covariant derivative with the index 0, evidently, yields 0, because these are equations for the background:

$$\left(T_0{}^{ik}{}_{;k}\right)_0 = 0\,. \tag{2.36}$$

In the linear approximation in β, we find the twist equations

$$\left(T_1{}^{ik}{}_{;k}\right)_0 + \left(T_0{}^{ik}{}_{;k}\right)_1 = 0\,. \tag{2.37}$$

We assume that in a twisted disk, the four-velocity, pressure, rest-mass energy density, internal energy, viscosity coefficient, and energy flux density, defined in accordance with their standard meaning (see Section 1.2.3), are given by

$$U^i = U_0^i + v^i\,, \quad p = p_0 + p_1\,, \quad \rho = \rho_0 + \rho_1\,, \quad \epsilon = \epsilon_0 + \epsilon_1\,,$$

$$\eta = \eta_0 + \eta_1\,, \quad q^i = q_0^i + q_1^i,$$

where the indices 0 and 1 denote respective quantities related to the background and perturbations; v^i are perturbations of the four-velocity. [6]

Thus, T_0^{ik} is the stress–energy tensor that contains only unperturbed values in accordance with definition (1.59), and its perturbation has the form

$$\begin{aligned} T_1^{ik} &= w_1 U_0^i U_0^k + w_0 (v^i U_0^k + U_0^i v^k) - p_1 \eta^{ik} + 2\eta_1 \sigma_0^{ik} \\ &\quad + 2\eta_0 \sigma_1^{ik} - U_0^i q_1^k - U_0^k q_1^i - v^i q_0^k - v^k q_0^i\,, \end{aligned} \tag{2.38}$$

where $w_0 = \rho_0 + \epsilon_0 + p_0$ is the background enthalpy and $w_1 = \rho_1 + p_1 + \epsilon_1$ is its perturbation.

In addition, σ_0^{ik} is the shear tensor that contains only unperturbed quantities in accordance with definition (1.60), and σ_1^{ik} is its perturbed part of the form

$$\begin{aligned} \sigma_1^{ik} &= \frac{1}{2}\left[(v^i_{;j})_0 P_0^{jk} + (v^k_{;j})_0 P_0^{ji}\right] - \frac{1}{3}\,(v^j_{;j})_0 P_0^{ik} \\ &\quad + \frac{1}{2}\left[(U_0{}^i_{;j})_0 P_1^{jk} + (U_0{}^k_{;j})_0 P_1^{ji}\right] - \frac{1}{3}(U_0{}^j_{;j})_0 P_1^{ik} \\ &\quad + \frac{1}{2}\left[(U_0{}^i_{;j})_1 P_0^{jk} + (U_0{}^k_{;j})_1 P_0^{ji}\right] - \frac{1}{3}\,(U_0{}^j_{;j})_1 P_0^{ik}\,, \end{aligned} \tag{2.39}$$

[6] To shorten the equations, we omit the term with the second viscosity ζ: using the analysis given below, it can be shown that this term does not contribute to the final equations in the leading order in the small parameters of the problem.

where P_0^{ik} is the projection tensor that contains only unperturbed quantities in accordance with definition (1.61), and its perturbation is written as $P_1^{ik} = -U_0^i v^k - U_0^k v^i$.

Everywhere below, we omit the index 0 for the unperturbed variables. In addition, the viscous part of the stress–energy tensor in the disk is marked with v wherever necessary: $T_v^{ik} = 2\eta\sigma^{ik}$.

Additional relations used to write the equations. The relations given below are valid through terms of the order $\propto \delta^2$, which is sufficient for the theory of twisted disks in the leading order in the small parameter δ. In deriving these relations, such a simplification enables us to assume that in the background solution, only U^τ and U^φ are nonzero, while $U^r \propto \delta^2$, and U^r can be temporarily set equal to zero.

We first note a relation between the components U^τ and U^φ to be used below,

$$(U^\tau)^2 = (U^\varphi)^2 + 1\,, \tag{2.40}$$

which follows from the expression for the norm of the four-velocity in an orthonormal basis. Constraint (2.40) is also useful in the differential form:

$$U^\tau\,\mathrm{d}U^\tau = U^\varphi\,\mathrm{d}U^\varphi\,. \tag{2.41}$$

Next, because the normalization of the four-velocity is also valid in the twisted disk, and the four-velocity perturbations are small, in the linear approximation we have

$$\begin{aligned} &(U^\tau + v^\tau)^2 - (U^\varphi + v^\varphi)^2 \\ &\qquad = (U^\tau)^2 + 2U^\tau v^\tau - (U^\varphi)^2 - 2U^\varphi v^\varphi = 1\,, \end{aligned}$$

and hence, with (2.40), v^i is 'orthogonal' to U^i:

$$U^\tau v^\tau = U^\varphi v^\varphi\,. \tag{2.42}$$

Finally, from the condition that σ^{ik} is space-like, we have

$$\sigma^{r\tau} U^\tau = \sigma^{r\varphi} U^\varphi\,,$$

and thus, in the basis B_0 used in this section, in the flat disk model, not only $T_v^{r\varphi}$ but also $T_v^{r\tau}$ is nonzero in the order in δ that is of interest to us here:

$$T_v^{r\tau} = \frac{U^\varphi}{U^\tau}\,T_v^{r\varphi}\,. \tag{2.43}$$

We note that in basis (1.49)–(1.52) comoving with the azimuthal motion, only the component $T_v^{r\varphi}$ was nonzero [see (1.72)].

Equation of free azimuthal motion. The quantities corresponding to the background model and entering twist equations (2.37) should be obtained separately from Eqns (2.36). For this, it suffices to use the results in Section 1 taking only the transition from basis (1.49)–(1.52) to the basis B_0 into account.

Nevertheless, when deriving the twist equations, it is also necessary to use some of equations (2.36) written exactly in the basis B_0. We mean the r- and ξ-projections of these equations in the leading order in the small disk thickness, which, as we know, describe its azimuthal rotation in the equatorial plane of the black hole and its vertical hydrostatic equilibrium. We emphasize that these relations are valid for both stationary and nonstationary accretion flows for any viscosity parameterization, as well as for any specific vertical and radial structure of the flow. Only the condition $\delta \ll 1$ is important.

At the first stage of deriving the twist equation, we need only the r-projection of (2.36). Setting $T^{ik} = \rho U^i U^k$, we obtain $T^{rk}{}_{;k} = 0$, which yields

$$\frac{K_1'}{K_1}\,(U^\tau)^2 + a\,\frac{K_3}{K_2}\,(2 - rK_4)\,U^\tau U^\varphi - \frac{(rK_2)'}{rK_2}\,(U^\varphi)^2 = 0\,. \tag{2.44}$$

Exactly this combination (2.44) is used in the derivation; however, it can be verified that together with (2.40) in the approximation linear in a, it gives the solution

$$U^\varphi = (r_{\mathrm{S}} - 3)^{-1/2}\left(1 - a r_{\mathrm{S}}^{-1/2}(r_{\mathrm{S}} - 3)^{-1}\right), \tag{2.45}$$

where we have passed to the Schwarzschild radial coordinate r_{S} that is equivalent to r that we used in Section 1 in the expression for U_g^φ [see formula (1.14)]. It is easy to verify that $U_g^\varphi = U^\varphi / r_{\mathrm{S}}$, as must be the case with the transition from the coordinate basis to B_0 taken into account.

'Gauge' condition of the twisted frame. The principal kinematic constraint for the twisted reference frame requires a constant vertical position of fluid particles:

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = 0\,, \tag{2.46}$$

which is provided by fast establishment of hydrostatic equilibrium across the disk compared with the dynamical time of the twist change, as discussed in Section 2.1. However, as was already noted in [32], an important point is that this does not mean that the projection of the four-velocity of the fluid onto $\mathbf{e}_\xi$ is also zero, because our basis is noncoordinate and its basis vectors are not tangent to coordinate lines.

By definition,

$$v^\xi = \frac{\mathbf{e}^\xi}{\mathrm{d}s}\,.$$

Using (2.26), we have

$$v^\xi = rK_2 U\,\frac{\mathrm{d}\tau}{\mathrm{d}s} + rK_2 W\,\frac{\mathrm{d}r}{\mathrm{d}s}\,;$$

we should substitute $\mathrm{d}\tau/\mathrm{d}s$ and $\mathrm{d}r/\mathrm{d}s$ in this relation in the zeroth order in β, in other words, in the form of quantities corresponding to the flat disk dynamics. Expressions for $\mathbf{e}^\tau$, $\mathbf{e}^\varphi$, and $\mathbf{e}^r$ at $\beta = 0$ give

$$\frac{\mathrm{d}\tau}{\mathrm{d}s} = \frac{1}{K_1}\left(U^\tau - ar^2 K_3\,\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right), \quad \frac{\mathrm{d}r}{\mathrm{d}s} = \frac{U^r}{K_2}\,, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{U^\varphi}{rK_2}\,, \tag{2.47}$$

where by definition $U^i \equiv \mathbf{e}^i/\mathrm{d}s$. As a result, we obtain

$$v^\xi = rU^\tau K\,\frac{K_2}{K_1}\,U + rU^r W\,, \tag{2.48}$$

where

$$K = 1 - ar\,\frac{K_3}{K_2}\,\frac{U^\varphi}{U^\tau}\,.$$

In (2.48), the velocity components U^τ and U^r should be taken from the corresponding background solution for a flat disk.

Explicit form of the system of equations for a twisted disk. Now, using (2.40)–(2.48), we write Eqns (2.37) in explicit form by keeping only the terms of the leading order in the two small parameters δ and $u \equiv t_{\mathrm{d}}/t_{\mathrm{ev}}$ [7]. Here, we take into account that quantities of 'thermal' origin in the background solution are small, i.e., $p, e, \eta \propto \delta^2 \rho$, and $q^\xi \propto \delta^3 \rho$, $q^{r,\varphi} \propto \delta^4$ (see Section 1).

We postpone discussing the effects of the fluid nonideality for a while. We note that this not only corresponds to the vanishing of terms with a viscosity coefficient and energy flow density or their perturbations but also means the absence of contributions $\propto U^r$. To select the leading-order terms in the ideal fluid approximation, we start with considering second terms in the τ-, r-, and φ-projections of (2.37). It turns out that such terms are proportional to $\delta\beta$ here, and in the r-projection of (2.37), this contribution is due to the projection of the vertical pressure gradient onto the orbital plane of motion of matter in the twisted disk (see the analysis in Section 2.1, where this quantity was denoted by $(\nabla p)_r$). In addition, the τ- and φ-projections of (2.37) involve terms $\propto \delta^{-1} u\beta$, which should also be kept. On the other hand, the first terms in the τ-, r-, and φ-projections of (2.37) give rise to terms containing Eulerian velocity perturbations $v^{\tau,r,\varphi}$, as well as the Eulerian rest-mass energy density perturbation ρ_1. We hence conclude that

$$v^{\tau,r,\varphi} \propto \max\{\delta, \delta^{-1}u\}\,\beta, \quad \rho_1 \propto \max\{\delta, \delta^{-1}u\}\,\rho\beta\,. \tag{2.49}$$

In addition, for reasons that become clear below, we temporarily keep partial derivatives of v^i and ρ_1 with respect to time, despite their being u^{-1} times smaller that the quantities themselves. Finally, the first terms of the τ- and φ-projections of (2.37) also contain terms with the combination $\partial_\xi \rho v^\xi$, whose amplitudes are restricted by the order $\propto \max\{\delta, \delta^{-1}u\}\,\beta$ by Eqn (2.48).

Now, using result (2.49), it is easy to select the leading terms entering the τ-, r-, and φ-projections of (2.37) due to the fluid nonideality. The most troublesome here is the contribution due to the shear tensor perturbations, $2\eta\sigma_1^{ik}$, which appears in T_1^{ik} [see (2.38) and (2.39)]. However, most of the terms from this contribution contain $\eta \propto \delta^2$ and $v^i \propto \delta\beta$ simultaneously; therefore, it is clear that only the terms in which the derivative with respect to ξ (lowering the order in δ) occurs twice must be kept. This fact strongly reduces the number of 'viscous' terms to be kept. Besides, by similar considerations, the final expressions does not contain terms with $\mathbf{q}$, $\mathbf{q}_1$, and η_1. Finally, we stress once again that in addition to the purely 'viscous' terms mentioned above, the contribution due to the radial advection that appears in the background solution with nonzero viscosity should not be forgotten. We are concerned with the terms that can appear in the 'nonviscous' part of the stress–energy tensor [see the first term in (1.59)] due to the nonzero value of $U^r \propto \delta^2$.

Taking all the above into account and using relations derived in three preceding sections, we obtain the τ-, r-, and

[7] As we discussed above, the smallness of $t_{\mathrm{d}}/t_{\mathrm{ev}}$ is necessary to ensure that the accretion flow outside the equatorial plane of the black hole can be regarded as a 'disk'. In turn, this is jointly ensured by the smallness of both δ and $t_{\mathrm{d}}/t_{\mathrm{LT}} \ll 1$ (see Section 2.1.1).

φ-projections of (2.37) in the form

$$K\,\frac{K_2}{K_1}\,(U^\tau)^2\dot\rho_1+\left(2U^\varphi-ar\,\frac{K_3}{K_2}\,\frac{(U^\varphi)^2+(U^\tau)^2}{U^\tau}\right)\frac{K_2}{K_1}\,\rho\dot v^\varphi$$
$$+\frac{1}{r}\,U^\tau U^\varphi\,\partial_\varphi\rho_1+\frac{1}{r}\,\frac{(U^\varphi)^2+(U^\tau)^2}{U^\tau}\,\rho\partial_\varphi v^\varphi$$
$$+\partial_r(\rho U^\tau v^r)+\partial_\xi\rho U^\tau v^\xi+\frac{(rK_1^2K_2^2)'}{rK_1^2K_2^2}\,\rho U^\tau v^r+F_v^\tau$$
$$=r\partial_\xi\rho(U^\tau)^2K\,\frac{K_2}{K_1}\,U+\frac{\xi}{r}\,\rho U^\tau U^\varphi\partial_\varphi W\,, \qquad (2.50)$$

$$K\,\frac{K_2}{K_1}\,U^\tau\dot v^r+\frac{U^\varphi}{r}\,\partial_\varphi v^r$$
$$-\left[2\,\frac{K_1'}{K_1U^\varphi}+a\,\frac{K_1}{rK_2U^\tau}\left(\frac{r^2K_3}{K_1}\right)'\right]v^\varphi+\frac{1}{\rho}\,F_v^r$$
$$=Wr\,\frac{\partial_\xi p}{\rho}-a\xi\,\frac{K_3^2}{K_1K_2}\left(\frac{K_1}{K_3}\right)'ZU^\tau U^\varphi\,, \qquad (2.51)$$

$$K\,\frac{K_2}{K_1}\,U^\tau U^\varphi\dot\rho_1+\left(\frac{(U^\varphi)^2+(U^\tau)^2}{U^\tau}-2ar\,\frac{K_3}{K_2}\,U^\varphi\right)\frac{K_2}{K_1}\,\rho\dot v^\varphi$$
$$+\frac{(U^\varphi)^2}{r}\,\partial_\varphi\rho_1+2\,\frac{U^\varphi}{r}\,\rho\partial_\varphi v^\varphi+\partial_r(\rho U^\varphi v^r)+\partial_\xi\rho U^\varphi v^\xi$$
$$+\frac{(r^2K_1K_2^3)'}{r^2K_1K_2^3}\,U^\varphi\rho v^r-a\,\frac{K_1}{rK_2}\left(\frac{r^2K_3}{K_1}\right)'U^\tau\rho v^r+F_v^\varphi$$
$$=K\,\frac{K_2}{K_1}\,r\partial_\xi\rho U^\tau U^\varphi U+\frac{\xi}{r}\,\rho(U^\varphi)^2\partial_\varphi W\,, \qquad (2.52)$$

where

$$K=1-ar\,\frac{K_3}{K_2}\,\frac{U^\varphi}{U^\tau}\,,$$

and $F_v^{\tau,r,\varphi}$ is the total contribution due to nonzero viscous forces and the radial advection of matter in the background solution $\propto U^r$. Explicitly,

$$F_v^\tau=\frac{U^\varphi}{U^\tau}\,(\partial_\xi T_v^{\varphi\xi}-rW\partial_\xi T_v^{r\varphi})-r\partial_\xi\rho U^\tau U^rW\,,\quad F_v^r=\partial_\xi T_v^{r\xi}\,,$$
$$F_v^\varphi=(\partial_\xi T_v^{\varphi\xi}-rW\partial_\xi T_v^{r\varphi})-r\partial_\xi\rho U^\varphi U^rW\,, \qquad (2.53)$$

where

$$T_v^{r\xi}=-\frac{\eta}{K_2}\,(\partial_\xi v^r+U^\varphi\partial_\varphi W)\,,$$
$$T_v^{\varphi\xi}=-\frac{\eta}{K_2}\left(\partial_\xi v^\varphi-2a\,\frac{K_3}{K_2}\,U^\tau(U^\varphi)^2Z\right), \qquad (2.54)$$
$$T_v^{r\varphi}=-\eta r\left(\frac{U^\varphi}{rK_2}\right)'.$$

We note that $T_v^{r\xi}$ and $T_v^{\varphi\xi}$ have the meaning of perturbations of the viscous stress tensor. In these expressions, the terms $\propto\beta$ contributing to the shear tensor perturbations appear due to the twisted basis. Conversely, $T_v^{r\varphi}$ relates to the background. Nevertheless, for brevity, we use the same notation with the index *v* for these two quantities. Finally, we assume in (2.50)–(2.54) that in the relativistic coefficients K_1, K_2, and K_3, the argument $R_{\rm I}$ is replaced by r, because $R_{\rm I}^2=r^2+\xi^2$ and accounting for the dependence on ξ here always gives rise to a small correction $\propto\delta^2$ only.

It remains to write the explicit form of the ξ-projection of (2.37). We start with the contribution of terms in the ideal fluid approximation, and first rearrange the first term in (2.37). The leading-order terms in δ here are, in particular, ρv^φ and ρ_1, but additionally multiplied by ξ. This means that their amplitudes are restricted by the order $\max\{\delta^2,u\}\,\beta$. Moreover, v^ξ now enters the term $U^\varphi\rho\partial_\varphi v^\xi$, which also implies an increase in the order of smallness by δ compared to (2.50)–(2.52) (it can be seen that v^ξ enters formulas (2.50) and (2.52) in combination with $\partial_\xi\rho$). In addition, of all terms of a 'thermal' origin, we must now keep the term with $\partial_\xi p_1$, because it is also of the order δ^2 due to $p_1\sim\delta^2\rho_1\propto\rho\delta^3\beta$.

Turning now to the second term in the ξ-projection of (2.37), we write all terms through the order $\max\{\delta^2,u\}\,\beta$. From similar considerations, the terms due to the fluid nonideality (including the 'advective' terms proportional to U^r) are also kept here, with their smallness increased by the coefficient δ compared to what we had in (2.50)–(2.52).

We thus obtain the equation

$$U^\varphi\partial_\varphi v^\xi+r\,\frac{\partial_\xi p_1}{\rho}+\xi\,\frac{(U^\varphi)^2}{r}\left(1-2ar\,\frac{K_3}{K_2}\,\frac{U^\tau}{U^\varphi}\right)\frac{\rho_1}{\rho}$$
$$+2\xi U^\varphi v^\varphi\left[\frac{K_1'}{K_1}-\frac{K_2'}{K_2}-\frac{ar}{2}\,\frac{K_3^2}{K_1K_2}\left(\frac{K_1}{K_3}\right)'\left(\frac{U^\tau}{U^\varphi}+\frac{U^\varphi}{U^\tau}\right)\right]$$
$$+\frac{r}{\rho}\,F_v^\xi=-\left[\frac{K_2}{K_1}\,\partial_\varphi U-2a\,\frac{K_3Z}{K_2}+a\,\frac{\xi^2}{r}\,\frac{K_3^2Z}{K_1K_2}\left(\frac{K_1}{K_3}\right)'\right]$$
$$\times\,rU^\tau U^\varphi+ar^2\,\frac{K_3}{K_1}\,(U^\varphi)^2\partial_\varphi U\,, \qquad (2.55)$$

where

$$F_v^\xi=\frac{1}{rK_1K_2^3}\,\partial_r(rK_1K_2^3T_v^{r\xi})+\partial_\xi T_v^{\xi\xi}+\frac{1}{r}\,\partial_\varphi T_v^{\varphi\xi}$$
$$+\partial_\varphi W(T_v^{r\varphi}+T_{\rm adv}^{r\varphi})+a\,\frac{K_1}{rK_2}\left(\frac{r^2K_3}{K_1}\right)'$$
$$\times\,\partial_\varphi Z\left(\frac{U^\varphi}{U^\tau}\,T_v^{r\varphi}+\frac{U^\tau}{U^\varphi}\,T_{\rm adv}^{r\varphi}\right) \qquad (2.56)$$

and $T_{\rm adv}^{r\varphi}=\rho U^\varphi U^r$. We do not provide the explicit form of $T_v^{\xi\xi}$ here because it is not required in the final form of the twist equations.

Everywhere in (2.55) and (2.56) except in the second term in square brackets in the right-hand side of (2.55), the argument $R_{\rm I}$ in the relativistic coefficients K_1, K_2, and K_3 is replaced by r. The mentioned term is an exception because this term alone has the zeroth order in the small parameters δ and u in Eqn (2.55). But because we have kept the terms $\propto\max\{\delta^2,u\}$ in (2.55), in the term under discussion it is necessary to take corrections $\propto\delta^2$ into account due to the dependence of the relativistic coefficients K_2 and K_3 on ξ. We did not do that for the reason discussed in the next paragraph.

2.3.2 Completing the derivation of twist equations. We have written the twist equations in the leading orders in small parameters δ and u. All corrections linear in the Kerr parameter a were taken into account. If we temporarily set $a=0$ and consider Eqn (2.55), we see that on the one hand, it contains terms proportional to the rate of change of the disk twist, $\propto U$, and on the other hand, it has terms containing perturbations of the physical quantities of the order $\propto \delta^2$. Thus, we can say that only due to the internal forces does a thin twisted disk evolve on a long timescale such that $u \sim \delta^2$. It then becomes totally clear that Eqn (2.50)–(2.52) are restricted by the order $\propto \delta$, and Eqn (2.55) is restricted by the order $\propto \delta^2$.

At the same time, when the parameter a is nonzero, a 'large' term of the zeroth order in δ and $\propto aZ$ arises in the right-hand side of (2.55). This term describes the gravitomagnetic interaction of the rotating black hole with the tilted/twisted disk. In order that all terms in (2.55) be balanced, we must assume that $a \sim \delta^2$. But it then becomes clear that all additional corrections $\sim a$ in Eqns (2.50)–(2.52) have the next order in δ and can be omitted. The same relates to all terms $\propto a\delta^2$ in Eqn (2.55), including the correction $\propto \delta^2$ due to the dependence of the relativistic coefficients on R_{I} in the gravitomagnetic term itself.

In fact, this means that when considering the dynamics of a twisted thin accretion disk near a rotating black hole, it suffices to use the background model, i.e., the corresponding flat disk in the Schwarzschild metric with $a = 0$. The assumption of the slow black hole rotation itself was needed because otherwise the accretion flow (including the nonstationary one) could not be regarded as a disk, since the vertical hydrostatic equilibrium there would be violated (see Section 2.1). Of course, these conclusions pertain to only slightly tilted/twisted and geometrically thin disks with $\beta \ll 1$ and $\delta \ll 1$.

In what follows, we therefore set $a = 0$ in all terms except the gravitomagnetic one. This significantly simplifies the calculations that are required for obtaining the twist equations in the final form. We first analyze Eqns (2.50) and (2.52). It is convenient to consider their combinations that contain either $\dot\rho$ or $\dot v^{\varphi}$.

Eliminating $\dot v^{\varphi}$ at $a = 0$, we obtain the equation

$$U^{\varphi}\partial_{\varphi}\rho_1 + \frac{1}{(U^{\tau})^2}\,\rho\partial_{\varphi}v^{\varphi} + \frac{U^{\tau}}{K_2^2}\frac{\partial}{\partial r}\left(rK_2^2\,\frac{\rho v^r}{U^{\tau}}\right)$$
$$= \xi U^{\varphi}\rho\partial_{\varphi}W + \frac{U^{\varphi}}{(U^{\tau})^2}\left(\partial_{\xi}T_v^{\varphi\xi} - rW\partial_{\xi}T_v^{r\varphi}\right), \qquad (2.57)$$

where we have omitted the term $\dot\rho_1$, which is of the next order in δ compared to the other terms. In the Newtonian limit $r \to \infty$, Eqn (2.57) reduces to the continuity equation for perturbations.

Next, eliminating $\dot\rho_1$ at $a = 0$,[8] we obtain the equation

$$\frac{K_2}{K_1}\,\dot v^{\varphi} + \frac{1}{r}\frac{U^{\varphi}}{U^{\tau}}\,\partial_{\varphi}v^{\varphi} + \left(\frac{\partial_r U^{\varphi}}{U^{\tau}} + \frac{K_1'}{K_1}\frac{U^{\tau}}{U^{\varphi}}\right)v^r$$
$$+ \frac{1}{\rho U^{\tau}}\left(\partial_{\xi}T_v^{\varphi\xi} - rW\partial_{\xi}T_v^{r\varphi}\right) = 0\,. \qquad (2.58)$$

In the Newtonian limit, (2.58) reduces to the azimuthal component of the Navier–Stokes equation for perturbations.

Finally, (2.51) with $a = 0$ takes the form

$$\frac{K_2}{K_1}\,U^{\tau}\dot v^r + \frac{U^{\varphi}}{r}\,\partial_{\varphi}v^r - 2\,\frac{K_1'}{K_1 U^{\varphi}}\,v^{\varphi} + \frac{1}{\rho}\,\partial_{\xi}T_v^{r\xi} = Wr\,\frac{\partial_{\xi}p}{\rho}\,. \qquad (2.59)$$

In the Newtonian limit, (2.59) reduces to the radial component of the Navier–Stokes equation for perturbations.

It is important to explain why we have kept terms with $\dot v^r$ and $\dot v^{\varphi}$ in Eqns (2.58) and (2.59) although they are of the next order in δ. As mentioned in Section 2.1, in the Newtonian limit, the epicyclic frequency becomes equal to the Keplerian circular frequency, which results in a resonance growth of the amplitude of velocity perturbations of gas elements in the disk under the action of the radial projection of the vertical pressure gradient $(\nabla p)_r$, which is limited only by the viscosity. Mathematically expressed, in the limit of an inviscid Keplerian disk, Eqn (2.58) yields in the leading order in the parameter u (with the terms $\propto \dot v^{\varphi}$ omitted) such a relation between v^r and v^{φ} that the sum of the second and third terms in (2.59) vanishes. But because there is a term $\propto \delta\beta$ in the right-hand side of (2.59), it follows that $\dot v^r$ (and hence $\dot v^{\varphi}$ as well) acquires the first order in δ in the considered case. Either viscosity or relativistic corrections eliminate the Keplerian resonance, and the amplitudes of $\dot v^r$ and $\dot v^{\varphi}$ decrease again to the third order in δ.

Now, from Eqn (2.55), we need to derive the so-called twist equation that plays the principal role in the twisted disk theory. For this, we need to explicitly determine $\partial_{\xi}p/\rho$, which is done in the next section. Although the Schwarzschild approximation is sufficient, we also take linear corrections in a into account. This is required below in obtaining an additional expression for the Lense–Thirring frequency in terms of the relativistic coefficients used in the twisted basis.

Equation of the vertical hydrostatic equilibrium. We write the ξ-projection of Eqn (2.36) in the basis B_0 in the leading order in δ, as we did in Section 1 using basis (1.49)–(1.52) [see Eqn (1.74)]. Taking into account that the four-velocity of the flow is $\{U^{\tau},\,0,\,U^{\varphi},\,0\}$ in the leading order in δ, we obtain the equation

$$\frac{\partial_{\xi}p}{\rho} = \frac{\xi}{r}\,(U^{\varphi})^2\left[\frac{K_2'}{K_2} - \left(\frac{U^{\tau}}{U^{\varphi}}\right)^2 + ar\,\frac{K_3K_4}{K_2}\,\frac{U^{\tau}}{U^{\varphi}}\right], \qquad (2.60)$$

where U^{τ} and U^{φ} satisfy normalization condition (2.40) and geodesic equation (2.44). With this in mind, we arrive at the final form of the hydrostatic equilibrium equation

$$\frac{\partial_{\xi}p}{\rho} = -\frac{\xi}{r}\,\frac{(U^{\varphi})^2}{r}\left(1 - 2ar\,\frac{K_3}{K_2}\,\frac{U^{\tau}}{U^{\varphi}}\right), \qquad (2.61)$$

where the Schwarzschild profiles of U^{τ} and U^{φ} are used in the term with the parameter a.

It can be verified that with the substitution $\xi \to z/K_2$, Eqn (2.61) is equivalent to (1.74) in the linear approximation in a. Here, we should only take into account that $r_{\mathrm{S}} = K_2 r$, where r_{S} is the Schwarzschild coordinate, equivalent to the coordinate r in (1.74).

Twist equation. Our goal is to rewrite (2.55) in divergent form. Without the gravitomagnetic term, Eqn (2.55), in which we also set $a = 0$, must respect the conservation law of the angular momentum projection of the twisted disk onto the equatorial plane of the black hole (the conservation of the disk angular momentum projection onto the black hole spin in our problem, linear in β, follows from equations for the background, because the corrections due to the small tilt are proportional to $1 - \cos\beta = \beta^2$), which reflects the spherical symmetry of the Schwarzschild metric.

[8] $a = 0$ also in the expression for $T_v^{\varphi\xi}$.

This requires eliminating v^φ and ρ_1 from the left-hand side of (2.55). Therefore, we use Eqns (2.57)–(2.59) with $\dot{v}^r = \dot{v}^\varphi = 0$ for our purposes, because we do not deal with resonance combinations of v^r and v^φ that vanish in the leading order in u in the Keplerian inviscid limit [see the comment on Eqns (2.58) and (2.59) above].

First, in the right-hand side of (2.57), we rewrite the term with $\partial_\varphi W$ through v^r and v^φ using (2.59) and (2.61) with $a = 0$. In the resulting expression for ρ_1, we replace v^φ using (2.58). Here, the derivative with respect to φ can be eliminated using the harmonic dependence on φ [see (2.28)]. In other words, $\partial_{\varphi\varphi} = -1$. Substituting the obtained expressions for ρ_1 and v^φ in (2.55), integrating over ξ, and performing integration by parts wherever necessary using the fact that the corresponding surface terms vanish as $\rho \to 0$, we arrive at the compact equation

$$\Sigma U^\tau U^\varphi \left\{ \partial_\varphi U - a\,\frac{K_1 K_3}{K_2^2}\, Z \right\} + \partial_\varphi W\, \frac{K_1}{K_2} \{ \Sigma U^\varphi U^r + \bar{T}_v^{r\varphi} \}$$
$$= -\frac{1}{2r^2 K_2^4} \int \mathrm{d}\xi \left\{ \partial_r (\xi r K_1 K_2^3 U^\varphi \rho \partial_\varphi v^r + r^2 K_1 K_2^3 T^{r\xi}) \right\}, \tag{2.62}$$

where, as usual, $\Sigma = \int \rho\, \mathrm{d}\xi$ is the surface density of the disk and the bar over $T^{r\varphi}$ means that it is integrated over ξ. In the appendix of [5], it is shown that (2.62) can be used to obtain the angular momentum conservation for the twisted disk.

Equations (2.58), (2.59), and (2.62) represent a closed system of equations describing the dynamics of twisted configurations as long as the corresponding model background is specified. Unknown variables in this system include the velocity perturbations v^r and v^φ and the quantity Z characterizing the disk geometry. We emphasize that in deriving these equations, we essentially used only three main assumptions: $a \ll 1$, $\delta \ll 1$, and $\beta \ll 1$. This means that the equations describe the dynamics of any geometrically thin accretion flow (disk) with any parameterization of viscosity and any radial and vertical structure in both the stationary and nonstationary cases. In the second case, we mean the nonstationary background: the equations determine not only the dynamics of twisted perturbations propagating in a stationary flat disk but also the dynamics of twisted rings/tori, when the evolution of the geometrical shape occurs in parallel with its expansion in the radial direction due to turbulent viscosity, which also results in the evolution of the background itself.

Once again about the characteristic frequencies of the problem. In Section 2.1.1, we already obtained relativistic expressions for the characteristic frequencies of the problem. These include the circular and epicyclic frequencies of free equatorial motion, as well as the frequency of vertical oscillations and the precession frequency of tilted orbits. Here, we wish to obtain expressions for these frequencies, but now in terms of the values used above to construct the theory of twisted disks, i.e. in the basis B_0. These expressions are needed in writing the twist equations in a more compact form.

The circular frequency of the free equatorial motion as measured by the clock of an infinitely remote observer, which we already presented in Eqn (1.16), can be obtained simply by dividing $\mathrm{d}\varphi/\mathrm{d}s$ by $\mathrm{d}\tau/\mathrm{d}s$ given in (2.47). We obtain

$$\Omega = \frac{K_1}{K K_2}\, \frac{U^\varphi}{r U^\tau}\,. \tag{2.63}$$

Using (2.45) and (2.40), and also remembering that $r_\mathrm{S} = r K_2$, we verify that (2.63) coincides with (1.16) in the linear approximation in a.

We now consider small vertical deviations from the circular equatorial motion. We discussed in Section 2.1.1 that the frequency of vertical oscillations measured by an infinitely remote observer, Ω_v, is the locally measured frequency Ω_l divided by the t-component of the four-velocity of circular motion, U_g^t. The frequency $_\mathrm{l}$ explicitly enters the equation of hydrostatic equilibrium [see Eqn (1.74) or equivalent equation (2.61) with the substitution $\xi \to z/K_2$]. Using relations (2.47), we express $U_g^t \equiv \mathrm{d}\tau/\mathrm{d}s$ in terms of U^τ:

$$U_g^t = K K_1^{-1} U^\tau\,,$$

whence

$$\Omega_\mathrm{l} = \Omega_\mathrm{v}\, \frac{K U^\tau}{K_1} = \frac{U^\varphi}{r K_2}\, \frac{\Omega_\mathrm{v}}{\Omega}\,, \tag{2.64}$$

where the final expression was obtained using (2.63). But then, from a comparison of (2.64) and (2.61), we see that

$$\Omega_\mathrm{v} = \Omega \left(1 - a r\, \frac{K_3}{K_2}\, \frac{U^\tau}{U^\varphi} \right), \tag{2.65}$$

where the Schwarzschild profiles for U^τ and U^φ are used in the term with the parameter a. Using (2.12), we now obtain the Lense–Thirring frequency

$$\Omega_\mathrm{LT} = a\, \frac{K_1 K_3}{K_2^2}\,. \tag{2.66}$$

It is sufficient for our purposes to know the epicyclic frequency in the Schwarzschild case with $a = 0$. This expression can be most easily derived directly from the twist equations, more precisely, from that part that describes the dynamics in the plane of disk rings, i.e., from (2.58) and (2.59). Setting the 'viscous' terms and radial projection of the pressure gradient in the right-hand side of (2.59) equal to zero, as well as omitting the dependence of v^r and v^φ on φ, we obtain equations for Eulerian perturbations that describe a free motion of gas elements slightly deviating from the circular motion. Clearly, these equations are equivalent to (2.5) and (2.6), which were written in bases (1.49)–(1.52). From these equations, we obtain the equation for v^r:

$$\ddot{v}^r + 2\, \frac{K_1 K_1'}{K_2^2 (U^\tau)^2} \left(\frac{\partial_r U^\varphi}{U^\varphi} + \frac{K_1'}{K_1}\, \frac{(U^\tau)^2}{(U^\varphi)^2} \right) v^r = 0\,, \tag{2.67}$$

where the expression before v^r is equal to κ^2. It can be rewritten in a more compact form

$$\kappa^2 = 2\, \frac{K_1' (K_1 U^\tau)'}{K_2^2 U^\tau (U^\varphi)^2} \tag{2.68}$$

to ensure that it coincides with (2.9), considering that the radial Schwarzschild coordinate $r_\mathrm{S} = r K_2$ enters the last equation.

Finally, for convenience, we introduce another quantity with the dimension of frequency that appears in our problem. In the Schwarzschild case $a = 0$,

$$\tilde{\Omega} = \frac{K_1'}{K_2}\, \frac{1}{U^\tau U^\varphi} = \frac{r_\mathrm{S} - 3}{r_\mathrm{S}^2 (r_\mathrm{S} - 2)^{1/2}}\,, \tag{2.69}$$

which tends to the Keplerian value in the Newtonian limit.

Using (2.63), (2.68), and (2.69) allows us to write Eqns (2.58) and (2.59) in a more compact form. Lense–Thirring frequency (2.66), evidently, enters the gravitomagnetic term in (2.62). However, we deal with this rewriting in the next section when considering a specific background model.

2.3.3 Twist equations in the particular case of a stationary vertically isothermal α-disc. We now consider the form the twist equations take in the specific background of a stationary α-disk, which we discussed in Section 1. This does not mean, however, that only stationary twisted solutions are to be considered. In other words, the equations that we obtain are also applicable to arbitrary nonstationary dynamics of the corresponding twisted disk. For example, they enable us to calculate the evolution of the shape of an (infinite) initially flat disk momentarily tilted to the equatorial plane of a rotating black hole. The initial stage of the evolution of such a disk was qualitatively described in Section 2.1. In addition, these equations describe the wave-like (for a disk with sufficiently small $\alpha < \delta$; see also [33]) or diffusion-like (for a disk with sufficiently large $\alpha > \delta$; see also [34]) dynamics of some twisted perturbation imposed on the disk lying initially in the equatorial plane of the black hole.

Explicit form of the required background profiles. The twist equations contain the quantity $\bar{T}_\nu^{r\varphi}$ (as well as $\bar{\eta}$) related to the corresponding flat disk model. We could obtain the explicit form of these quantities by integrating the τ- and φ-projections of Eqn (2.36). But it is simpler to use the results in Section 1, where we have already obtained this quantity, denoted by T_ν there [see Eqn (1.90)]. We should only take into account that now we are working in another basis than that used for the flat disk, and therefore the transition from T_ν to $\bar{T}_\nu^{r\varphi}$ should be specified. First, using the orthogonality of the shear tensor and hence of viscous stress tensor (1.66), we see that only one component of the viscous stress tensor, $T_\nu^{r\varphi\prime}$, is nonzero in basis (1.49)–(1.52), because the four-velocity there has only a nonzero time component up to the terms $\propto \delta^2$. The prime here marks basis (1.49)–(1.52). Further, the (orthonormal) bases are different only in that the observers associated with basis (1.49)–(1.52) move in the azimuthal direction with the velocity of the free equatorial circular motion, whereas the basis B_0 corresponds to observers at rest. Therefore, the transformations of vectors and tensors must be equivalent to the usual Lorentz transformations. Using [35] (see exercise 1, paragraph 6 there), we see that $T_\nu^{r\varphi} = U^\tau T_\nu^{r\varphi\prime}$, where U^τ is the Lorentz factor of azimuthal motion. Finally, we must additionally take into account that the integration over ξ differs from that over z by the coefficient K_2. As a result, we obtain

$$\bar{T}_\nu^{r\varphi} = \frac{U^\tau}{K_2}\, T_\nu\,. \tag{2.70}$$

We note that it is possible to pass from $T_\nu^{r\varphi\prime}$ to $T_\nu^{r\varphi}$ using relation (1.34) by writing it for two bases, equating the right-hand sides, and then multiplying one of the sides of the obtained equalities by matrices inverse to the basis matrices there. Here, we should only take into account that in the basis B_0, the radial coordinate was changed in (2.14), i.e., that $r_S = rK_2$ in the notation in this part of the paper.

Next, in the case $a = 0$, which is sufficient here, it is easy to express T_ν in terms of elementary functions. Indeed, the integral in (1.90) can be taken by the substitution $y \equiv \sqrt{r_S}$:

$$\int \frac{E}{r_S^{1/2} C}\, \mathrm{d}r_S = \int \frac{y^2-6}{y^2-3}\, \mathrm{d}y = y + \frac{\sqrt{3}}{2} \ln \frac{y+\sqrt{3}}{y-\sqrt{3}}\,.$$

For $\bar{T}^{r\varphi}$, we then use (2.70) to finally obtain

$$\bar{T}^{r\varphi} = \frac{\dot{M}}{2\pi}\, U^\tau r^{-3/2}\, \frac{L(r)}{K_2^{5/2} K_1^2}\,, \tag{2.71}$$

where

$$L = 1 - \frac{\sqrt{6}}{y} - \frac{\sqrt{3}}{2y} \ln \frac{(y-\sqrt{3})(3+2\sqrt{2})}{y+\sqrt{3}}\,. \tag{2.72}$$

As it must be, $L = 0$ at $r_S = 6$. We note that $L = Y(a = 0)$, where Y was defined in (1.101).

On the other hand, the expression for $\sigma^{r\varphi}$ in (1.72), in our case $a = 0$ in the basis B_0, can be rewritten in the form

$$\sigma^{r\varphi} = \frac{3}{4}\, \frac{D}{r_S^{3/2} C}\, U^\tau = \frac{3}{4}\, K_1^2 U_g^\varphi U_g^\tau U^\tau = \frac{3}{4}\, \frac{K_1}{rK_2}\, U^\varphi (U^\tau)^2\,,$$

where, as usual, we use relations (2.47). Then

$$\bar{T}^{r\varphi} = \frac{3}{2}\, \bar{\eta}\, \frac{K_1}{rK_2}\, (U^\tau)^2 U^\varphi\,. \tag{2.73}$$

As in Section 1.3.6, equating expressions (2.71) and (2.73), we obtain

$$\bar{\eta} = \frac{\dot{M}}{3\pi} \left(\frac{r^{-1/2}}{U^\tau U^\varphi}\, \frac{L}{K_1^3 K_2^{3/2}} \right). \tag{2.74}$$

In the Newtonian limit, far away from the inner edge of the disk, Eqn (2.74) gives the well-known result $\bar{\eta} = \dot{M}/(3\pi)$.

We assume that the kinematic viscosity is proportional to the characteristic disk half-thickness times the sound velocity in the disk:

$$\nu \sim \alpha c_s h_p\,, \tag{2.75}$$

where h_p is the proper characteristic half-thickness of the disk, which in our coordinate system is $h_{\mathrm{proper}} = K_2 h$, and α is the Shakura parameter, which is assumed to be constant. Because (2.61) implies that $c_s \sim \sqrt{P/\rho} \sim U^\varphi h/r$, we finally determine α from the equality

$$\nu = \frac{\alpha K_2 U^\varphi h^2}{r}\,. \tag{2.76}$$

Using (2.74) and (2.76), we obtain the relation

$$\Sigma h^2 = \frac{\dot{M}}{3\pi\alpha} \left(\frac{r^{1/2}}{U^\tau (U^\varphi)^2}\, \frac{L}{K_1^3 K_2^{5/2}} \right). \tag{2.77}$$

To find U^r in the advective term in (2.62), we use the rest-energy conservation law in the basis B_0 for the stationary disk. Again, we use result (1.86). Recalling the transition to the isotropic radial coordinate, the relation between the coordinate and physical velocities (1.47) and (2.47), and the difference in the definitions of Σ, we obtain

$$-\frac{\dot{M}}{2\pi} = \Sigma K_1 K_2^2 r U^r \,. \tag{2.78}$$

Then U^r can be derived from (2.78) and (2.77) as

$$U^r = -\frac{3\alpha}{2}\frac{\delta^2}{L} K_1^2 U^\tau (U^\varphi)^2 \sqrt{K_2 r}\,. \tag{2.79}$$

Finally, we need to know the profile $\delta(r)$. We note that this quantity is invariant under the transition between the bases (1.49)–(1.52) and B_0, since the change from h_p to h and from r_S to r is scaled with the same coefficient K_2.

In a gas-pressure-dominated disk with the Thomson scattering opacity, it follows from (1.103) with $a = 0$ that

$$\delta(r) = \delta_* K_1^{1/2} K_2^{1/20} (U^\tau)^{-9/10} L^{1/5} r^{1/20}\,. \tag{2.80}$$

To derive a simpler form of the twist equations, we need to specify the vertical profile of the rest-energy density. Here, we use its simplest form in an isothermal disk:

$$\rho = \rho_\mathrm{c} \exp\left(-\frac{\xi^2}{2h^2}\right), \tag{2.81}$$

where $\rho_\mathrm{c}(r)$ is the equatorial density.

Transition to complex amplitudes. In the case of an isothermal disk, the velocity perturbations v^r and v^φ taken in the form

$$v^\varphi = \xi(A_1 \sin\varphi + A_2 \cos\varphi), \quad v^r = \xi(B_1 \sin\varphi + B_2 \cos\varphi) \tag{2.82}$$

satisfy Eqns (2.58) and (2.59) if v does not change with the height, and the amplitudes A_1, A_2, B_1, and B_2 are functions of r and τ. Indeed, in this case, all 'thermal' terms are $\propto \xi$, and the dependence on ξ with ansatz (2.82) is identically satisfied in the considered equations.

We introduce the complex amplitudes

$$\mathbf{A} = A_2 + \mathrm{i}A_1\,, \quad \mathbf{B} = B_2 + \mathrm{i}B_1\,,$$

$$\mathbf{W} = \Psi_1 + \mathrm{i}\Psi_2 = \beta \exp(\mathrm{i}\gamma)\,. \tag{2.83}$$

By composing two combinations, $(2.58) + \mathrm{i}\,\partial_\varphi(2.58)$ and $(2.59) + \mathrm{i}\,\partial_\varphi(2.59)$, we see that all terms in these combinations are $\propto \exp(-\mathrm{i}\varphi)$. In particular, the terms containing W and $\partial_\varphi W$ pass into the terms respectively containing $-\mathrm{i}\mathbf{W}'\exp(-\mathrm{i}\varphi)$ and $\mathbf{W}'\exp(-\mathrm{i}\varphi)$.

As a result, we obtain the complex equations

$$\dot{\mathbf{A}} - (\mathrm{i} - \alpha)\,\Omega\mathbf{A} + \frac{\kappa^2}{2\tilde{\Omega}}\,\mathbf{B} = -\frac{3}{2}\,\mathrm{i}\alpha K_1 (U^\tau)^2 U^\varphi \Omega \mathbf{W}'\,, \tag{2.84}$$

$$\dot{\mathbf{B}} - (\mathrm{i} - \alpha)\,\Omega\mathbf{B} - 2\tilde{\Omega}\mathbf{A} = -(\mathrm{i} + \alpha)\, U^\varphi \Omega\, \mathbf{W}'\,, \tag{2.85}$$

where we have used Eqn (2.76) as well as expressions for frequencies (2.63), (2.68), and (2.69) obtained in Section 2.3.2.

In a similar way, by using (2.82) and (2.83) and composing the combination $(2.62) + \mathrm{i}\,\partial_\varphi(2.62)$, we derive an equation for complex amplitudes. In the right-hand side of this equation, the integration over ξ should be performed under the derivative with respect to r. For an isothermal disk with density distribution (2.81), the equality $\int \rho\xi^2\,\mathrm{d}\xi = \Sigma h^2$ holds. Hence, the derivative with respect to r acts on terms proportional to Σh^2 or $\bar{\eta}$. Instead of the combinations mentioned above, we substitute Eqns (2.77) and (2.74) there and group common constant factors before the derivative with respect to r. Additionally, instead of U^r and $\bar{T}^{r\varphi}$, we substitute expressions (2.79) and (2.71) in the left-hand side of the discussed equation and then divide the equation by Σ. The obtained equation contains $\dot{M}$ and Σ only in the combination $\dot{M}/\Sigma$, which we express through δ^2 and other known quantities using (2.77). Also using the expression for Lense–Thirring frequency (2.66), we finally arrive at the equation

$$\dot{\mathbf{W}} - \mathrm{i}\Omega_\mathrm{LT}\mathbf{W} + \frac{3}{2}\,\alpha\delta^2\,\frac{K_1^2}{K_2}\,U^\varphi\left(U^\tau - K_1(rK_2)^{1/2}\,\frac{U^\varphi}{L}\right)\mathbf{W}'$$
$$= \frac{\delta^2 K_1^3 U^\varphi}{2r^{1/2} K_2^{3/2} L}\,\frac{\partial}{\partial r}\left\{r^{3/2} K_2^{1/2}\,\frac{L}{K_1^2 U^\tau U^\varphi}\left[(\mathrm{i}+\alpha)\,\mathbf{B} + \alpha U^\varphi \mathbf{W}'\right]\right\}. \tag{2.86}$$

Equations (2.84)–(2.86) form a closed system of equations for $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{W}$ as functions of r and τ. In the weak gravity limit, they reduce to Eqns (30), (31), and (33) in [36].

2.4 Stationary twisted disk

2.4.1 Main equation and boundary condition. We now consider stationary solutions of the system of equations (2.84)–(2.86). The main goal of this section is to calculate the shape of a stationary twisted disk.

We set $\dot{\mathbf{A}} = \dot{\mathbf{B}} = \dot{\mathbf{W}} = 0$. After eliminating $\mathbf{A}$ from (2.84), (2.85), we obtain

$$\left[1 + \frac{\kappa^2}{(\mathrm{i} - \alpha)^2\,\Omega^2}\right](\mathrm{i} - \alpha)\,\Omega\mathbf{B}$$
$$= \left[(\mathrm{i} + \alpha)\,U^\varphi\Omega - \frac{3\mathrm{i}\alpha}{\mathrm{i} - \alpha}\,K_1 (U^\tau)^2 U^\varphi \tilde{\Omega}\right]\mathbf{W}'\,, \tag{2.87}$$

whence we express $\mathbf{B}$ through $\mathbf{W}'$ and substitute it in Eqn (2.86). We thus obtain the equation

$$\frac{K_1}{r_\mathrm{S}^{1/2} L}\,\frac{\mathrm{d}}{\mathrm{d}r_\mathrm{S}}\left(\frac{r_\mathrm{S}^{3/2} L}{K_1 U^\tau}\,f^*(\alpha, r_\mathrm{S})\,\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}r_\mathrm{S}}\right)$$
$$- 3\alpha U^\tau (1 - L^{-1})\,\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}r_\mathrm{S}} + \frac{4\mathrm{i}a}{\delta^2 K_1^3 r_\mathrm{S}^3 U^\varphi}\,\mathbf{W} = 0\,, \tag{2.88}$$

where the asterisk denotes complex conjugation and

$$f(\alpha, r_\mathrm{S}) = (1 + \alpha^2 - 3\mathrm{i}\alpha K_1^2)\,\frac{r_\mathrm{S}(\mathrm{i} - \alpha)}{\alpha r_\mathrm{S}(\alpha + 2\mathrm{i}) - 6} + \alpha\,. \tag{2.89}$$

We note that Eqn (2.88) was written after passing to the Schwarzschild radial coordinate r_S. In what follows, we wish to consider only the case $a > 0$, i.e., a prograde disk. It can be seen that the problem has two free parameters. First of all, this is the combination $\tilde{\delta} \equiv \delta_*/\sqrt{|a|}$. Clearly, $\tilde{\delta}$ ranges from 0 to ∞ and characterizes the relative role of the hydrodynamic and gravitomagnetic forces acting on the disk rings. Second, (2.88) contains the disk viscosity parameter $0 < \alpha < 1$. Equation (2.88) in the rigorous Newtonian limit with nonzero viscosity reproduces the corresponding equation (2.10) from [37] and, additionally, with post-Newtonian corrections, reproduces Eqn (33) from [25], which was verified in [5] (see paragraph 4.1 therein).

The coefficients of Eqn (2.88) have a singular point at the inner edge of the disk at $r_S = \bar{r}_S \equiv 6$, where L vanishes. The regularity of the solution at $\bar{r}_S$ must yield a condition for the function $\mathbf{W}$. Using this condition as the initial one, we can integrate (2.88) from $\bar{r}_S$ to infinity and obtain the shape of the stationary twisted disk. We expand Eqn (2.88) in a series in the small parameter $x_0 = r_S - \bar{r}_S \ll 1$. In practice, to do this, all quantities that take nonzero values at $\bar{r}_S$ must be set exactly equal to these values and the function L must be expanded to the leading order in x_0. From (2.72), we also find

$$L \approx \frac{x_0^2}{72}\,, \tag{2.90}$$

whence we see that another quantity in (2.88) that vanishes at the inner disk edge, δ, can be written as

$$\delta = \delta_{\mathrm{ms}} x_0^{2\epsilon}\,,$$

where ϵ is the power-law exponent L in Eqn (2.80). Accordingly, δ_{ms} is also given by Eqn (2.80), which is taken at $\bar{r}_S$ and into which we now substitute the coefficient 72^{-1} from (2.90) instead of L.

After that, it is easy to obtain the equation valid for $x_0 \ll 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x_0}\left(x_0^2\,\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}x_0}\right) + C_1 x_0^{2-4\epsilon}\,\mathbf{W} + C_2\,\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}x_0} = 0\,, \tag{2.91}$$

where

$$C_1 = -\frac{2\mathrm{i}}{f(\alpha, r_S)}\,\frac{U^\tau}{U^\varphi}\,\frac{\Omega_{\mathrm{LT}}}{K_1^3 r_S \delta_{\mathrm{ms}}^2}$$

and

$$C_2 = -\frac{216\alpha}{f(\alpha, r_S)}\,\frac{(U^\tau)^2}{r_S}$$

are taken at $\bar{r}_S$. We see that for any finite viscosity, the last term in (2.91) becomes dominant sufficiently close to the disk edge; therefore, the boundary condition can be straightforwardly written as

$$\left.\frac{\mathrm{d}\mathbf{W}}{\mathrm{d}x_0}\right|_{\bar{r}_S} = 0\,. \tag{2.92}$$

On the other hand, from (2.91) with $\alpha = 0$, we obtain a simpler equation whose solution is a Bessel function:

$$\mathbf{W} = C x_0^{-1/2} J_{1/(2-4\epsilon)}(z)\,, \tag{2.93}$$

where

$$z = \sqrt{C_1}\,\frac{x_0^{1-2\epsilon}}{1-2\epsilon}\,. \tag{2.94}$$

As $x_0 \to 0$, (2.93) tends to a nonzero constant but with a zero derivative with respect to x_0. Therefore, in this case, we return to condition (2.92).

Due to the linearity of the problem, it suffices to take an arbitrary nonzero value of $\mathbf{W}$ in $\bar{r}_S$, to set the first derivative of $\mathbf{W}$ in $\bar{r}_S$ equal to zero, and with these boundary conditions to integrate (2.88) up to infinity. The modulus and phase of $\mathbf{W}$ give the profiles $\beta(r_S)$ and $\gamma(r_S)$ for a stationary twisted disk. In what follows, we implicitly normalize the profile β at infinity to unity.

2.4.2 Disk with a marginally small viscosity. We discuss the disk with a very low viscosity separately. Clearly, it is possible to analytically treat the accretion disk by formally setting $\alpha \to 0$, if simultaneously $\dot{M} \to 0$. In such a disk, $U^r \to 0$; however, it then has real profiles of Σ and h.

In addition, to obtain an analytic solution, we consider the case $\tilde{\delta} \ll 1$; in other words, we assume a sufficiently thin disk around a rapidly rotating black hole.

Setting $\alpha = 0$ in (2.88) yields

$$\frac{\mathrm{d}}{\mathrm{d}r_S}\left(b\,\frac{\mathrm{d}}{\mathrm{d}r_S}\,\mathbf{W}\right) + \lambda\mathbf{W} = 0\,, \tag{2.95}$$

where

$$b = \frac{r_S^{5/2} L}{K_1 U^\tau}\,, \quad \lambda = \frac{24aL}{\delta^2 K_1^4 U^\varphi r_S^{5/2}}\,. \tag{2.96}$$

The coefficients in (2.95) take real values; therefore, there are real solutions of this equation. This means that in the absence of viscosity in the stationary twisted disk, $\gamma = \mathrm{const}$, which can be set equal to zero by the corresponding choice of the reference frame. Therefore, the variable $\mathbf{W}$ is identical to the angle β in this section.

The form of the disk near its inner edge. In the foregoing, we have already presented the solution near the inner edge of the inviscid disk [see Eqn (2.93)]. The constant C_1 in this case has the explicit form

$$C_1 = \frac{24aU^\tau}{r_S^5 K_1^3 U^\varphi \delta_{\mathrm{ms}}^2}\,, \tag{2.97}$$

and is taken at $r_S = \bar{r}_S$.

Using the known approximation for the Bessel function of a small argument, we obtain a relation between the constant C in (2.93) and the value of $\mathbf{W}$ at $\bar{r}_S$, $\mathbf{W}(\bar{r}_S) \equiv \mathbf{W}_0$:

$$C = \Gamma\left(\frac{3-4\epsilon}{2(1-2\epsilon)}\right)\left(\frac{\sqrt{C_1}}{2(1-2\epsilon)}\right)^{-1/2(1-2\epsilon)}\mathbf{W}_0\,, \tag{2.98}$$

where $\Gamma(x)$ is the gamma function.

In addition, we need the asymptotic form of (2.93) for $z \gg 1$. Clearly, z can be large even for $x \ll 1$ because $\sqrt{C_1} \sim \tilde{\delta}^{-1} \gg 1$. Hence, for $z \gg 1$, we obtain

$$\mathbf{W} \approx C\sqrt{\frac{2}{\pi x z}}\cos\left(z - \frac{\pi}{2}\,\frac{1-\epsilon}{1-2\epsilon}\right). \tag{2.99}$$

Shape of the disk at long distances. We consider Eqn (2.88) for $r_S \gg 1$ and $\alpha \to 0$. Importantly, we cannot set all variables in (2.88) to their Newtonian values and have the viscosity simultaneously vanish. This already follows from the fact that then $f(\alpha, R) \to 1/(2\alpha) \to \infty$. Physically, this reflects the fact that, as mentioned above, in the absence of viscosity in the strictly Newtonian potential, a Keplerian resonance occurs when the circular and epicyclic frequencies coincide, and perturbations in the twisted disk grow infinitely due to the action of the radial projection of the vertical pressure gradient. Therefore, a stationary twist is impossible in this case. Taking the next-order term in the expansion of $f(\alpha, r_{\rm S})$ in small $r_{\rm S}^{-1}$ into account, we obtain

$$f(\alpha, r_{\rm S}) \approx \frac{1}{2\alpha\big(1 + 3{\rm i}/(\alpha r_{\rm S})\big)}\,. \tag{2.100}$$

As $\alpha \to 0$, $f(\alpha, r_{\rm S})$ now remains finite at any finite $r_{\rm S}$. Nevertheless, it makes the leading contribution due to relativistic effects, and all other variables in (2.88) can now be set equal to their Newtonian values $U^\tau = 1$, $U^\varphi = r_{\rm S}^{-1/2}$, $L = 1$, and $K_1 = 1$. Moreover, we neglect the weak dependence of δ on $r_{\rm S}$ far from the black hole and set $\delta = \delta_*$.

After that, by introducing the new independent variable $x_1 \equiv r_{\rm S}^{-1/2} \ll 1$, we obtain the equation

$$x_1 \frac{{\rm d}^2}{{\rm d}x_1^2}\,\mathbf{W} - 2\frac{\rm d}{{\rm d}x_1}\,\mathbf{W} + 96\tilde{\delta}^{-2}x_1^4\mathbf{W} = 0\,. \tag{2.101}$$

The solution of (2.101) can again be expressed in terms of a Bessel function:

$$\mathbf{W} = x_1^{3/2}\big(A_1 J_{-3/5}(z_1) + A_2 J_{3/5}(z_1)\big)\,, \tag{2.102}$$

where

$$z_1 = \frac{8}{5}\sqrt{6}\,\tilde{\delta}^{-1}x_1^{5/2}\,, \tag{2.103}$$

and A_1 and A_2 are constants.

When $r_{\rm S}$ is so large that $z_1 \ll 1$, the first and second terms in (2.102), multiplied by $x_1^{3/2}$, respectively tend to a nonzero constant and to zero. This allows expressing the constant A_1 in terms the value of $\mathbf{W}$ at infinity, $\mathbf{W}_\infty$:

$$\mathbf{W}_\infty = \left(\frac{5}{4\sqrt{6}}\right)^{3/5} \frac{\tilde{\delta}^{3/5}}{\Gamma(2/5)}\,A_1\,. \tag{2.104}$$

In the opposite case $z_1 \gg 1$, i.e., closer to the black hole, we obtain another asymptotic form:

$$\mathbf{W} \approx \sqrt{\frac{5\tilde{\delta}}{2\pi\sqrt{24}}}\,r_{\rm S}^{-1/8}\left[A_1\cos\left(z_1 + \frac{\pi}{20}\right) + A_2\sin\left(z_1 - \frac{\pi}{20}\right)\right]. \tag{2.105}$$

WKB solution for the shape of the disk. Everywhere in the disk, asymptotic solutions (2.99) and (2.105) can be matched with a WKB solution of Eqn (2.95). Indeed, because we are considering the case $\tilde{\delta} \ll 1$, the ratio of λ and b in (2.95), $\tilde{\lambda} = \lambda/b$, is large at all $r_{\rm S}$ such that z and z_1 are large.

The WKB solution has the form

$$\mathbf{W} \approx \frac{C_3}{(\lambda b)^{1/4}}\cos\left(\int_{\bar{r}_{\rm S}}^{r_{\rm S}} \sqrt{\tilde{\lambda}}\,{\rm d}r_{\rm S} + \phi_{\rm WKBJ}\right), \tag{2.106}$$

where the constants C_3 and $\phi_{\rm WKBJ}$ should be chosen such that (2.106) is smoothly matched with formula (2.99) in the corresponding region. It can be verified that this yields

$$\phi_{\rm WKBJ} = -\frac{\pi}{2}\,\frac{1-\epsilon}{1-2\epsilon} \tag{2.107}$$

and

$$C_3 = 6^{1/4}\sqrt{\frac{1-2\epsilon}{\pi K_1 U^\tau}}\,C\,, \tag{2.108}$$

where we assume that K_1 and U^τ are taken at $r_{\rm S} = \bar{r}_{\rm S} = 6$, and $L \approx x^2/72$ near $\bar{r}_{\rm S}$.

Next, in the limit $r_{\rm S} \to \infty$, we can set λ and b before the cosine in (2.106) equal to their Newtonian values. In addition, the integral in (2.106) can be represented as

$$I(r_{\rm S}) \equiv \int_{\bar{r}_{\rm S}}^{r_{\rm S}} \sqrt{\tilde{\lambda}}\,{\rm d}r_{\rm S} = I - \int_{r_{\rm S}}^{\infty} \sqrt{\tilde{\lambda}}\,{\rm d}r_{\rm S}\,,$$

where $I = \int_{\bar{r}_{\rm S}}^{\infty}\sqrt{\tilde{\lambda}}\,{\rm d}r_{\rm S}$. With the Newtonian value $\tilde{\lambda} = 24\tilde{\delta}^{-2}R^{-9/4}$, we have

$$\int_{r_{\rm S}}^{\infty} \sqrt{\tilde{\lambda}}\,{\rm d}r_S \approx \frac{8\sqrt{6}}{5}\,\tilde{\delta}^{-1}r_{\rm S}^{-5/4}\,,$$

and therefore

$$\mathbf{W} \approx C_3\,\frac{\tilde{\delta}^{1/2}}{24^{1/4}}\cos\left(\frac{8\sqrt{6}}{5}\,\tilde{\delta}^{-1}r_{\rm S}^{-5/4} - I - \phi_{\rm WKBJ}\right). \tag{2.109}$$

Solution (2.109) must be smoothly matched with expression (2.105) in the corresponding region, which yields the constants A_1 and A_3. It can be verified that they are

$$A_1 = \sqrt{\frac{2\pi}{5}}\,C_3\cos\left(I + \phi_{\rm WKBJ} - \frac{\pi}{20}\right)\cos^{-1}\frac{\pi}{10}\,,$$
$$A_2 = \sqrt{\frac{2\pi}{5}}\,C_3\sin\left(I + \phi_{\rm WKBJ} + \frac{\pi}{20}\right)\cos^{-1}\frac{\pi}{10}\,. \tag{2.110}$$

Thus, Eqns (2.93), (2.106), and (2.102), jointly with coefficients (2.108), (2.110) and phase (2.107), determine the shape of an inviscid stationary relativistic twisted disk at all distances in the range from $r_{\rm S} = \bar{r}_{\rm S}$ to $r_{\rm S} = \infty$.

Resonance solutions in the low-viscosity disk. We note that Eqns (2.98), (2.108), (2.110), and (2.104) yield a relation between $\mathbf{W}_0$ and $\mathbf{W}_\infty$,

$$\mathbf{W}_\infty = C_{\rm tot}(\tilde{\delta})\,\mathbf{W}_0\,, \tag{2.111}$$

where the explicit form of $C_{\rm tot}(\tilde{\delta})$ follows from the formulas. In particular, as follows from (2.104) and (2.110), $C_{\rm tot}(\tilde{\delta}) \propto \cos(I + \phi_{\rm WKBJ} - \pi/20)$.

We hence conclude that for some discrete set of $\tilde{\delta}$ for which $\cos(I + \phi_{\rm WKBJ} - \pi/20) = 0$, $\mathbf{W}_\infty = 0$, but $\mathbf{W}_0 \neq 0$.

From Eqns (2.96), it is possible to obtain the integral I in the form $I = \tilde{\delta}^{-1}\tilde{I}$, where $\tilde{I}$ does not depend on $\tilde{\delta}$, allowing us to write the explicit form of the singular values of $\tilde{\delta}$:

$$\tilde{\delta}_k = \frac{\tilde{I}}{\pi/2\big(11/10 + (1-\epsilon)/(1-2\epsilon) + 2k\big)}\,, \tag{2.112}$$

where k is an integer number.

The values $\tilde{\delta}_k$ correspond to such a balance between the external gravitomagnetic force and the internal pressure gradient in the disk that leads to the disk twisting even if the matter flowing into the disk at infinity moves in the equatorial plane of the black hole. We note that the solution

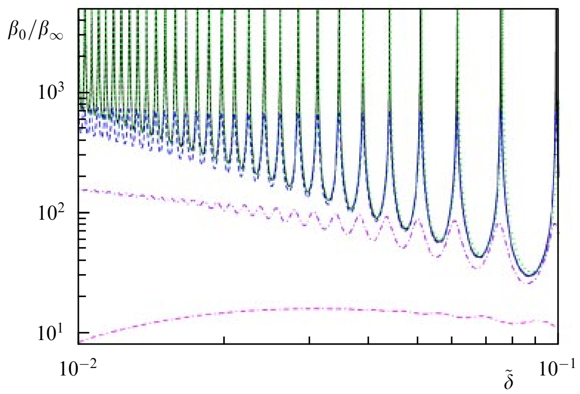

Figure 1. Ratio of the tilt angle of the inner disk edge to the tilt at infinity, β_0/β_∞, as a function of the parameter $\tilde{\delta}$. The solid curve shows the numerical solution of Eqn (2.88) with $\alpha = 0$, the dotted curve represents the analytic dependence $C_{\text{tot}}^{-1}(\tilde{\delta})$, where C_{tot} is given by Eqn (2.111). The dashed, dashed-dotted, and dashed-dotted-dashed curves are respectively obtained by numerical integration of Eqn (2.88) with $\alpha = 10^{-4}$, 10^{-3}, and 10^{-2}.

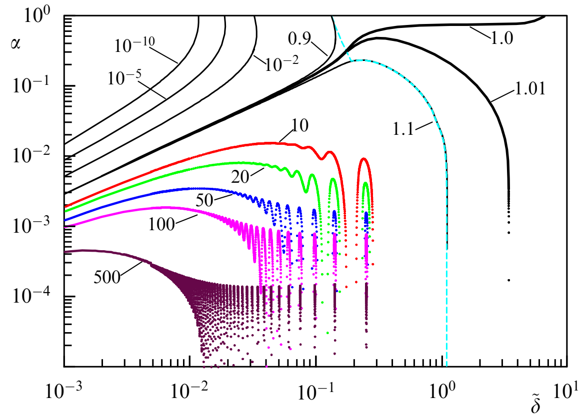

Figure 2. Contours of constant ratio β_0/β_∞ on the parameter plane $(\tilde{\delta}, \alpha)$. The numbers show the value of β_0/β_∞ for each curve. The dashed curve in the right part of the figure separates the region where the change in β with r_{S} is more than 10% of β_∞ (to the left) from the region where the disk twist is insignificant and β deviates by less than 10% from β_∞ (to the right).

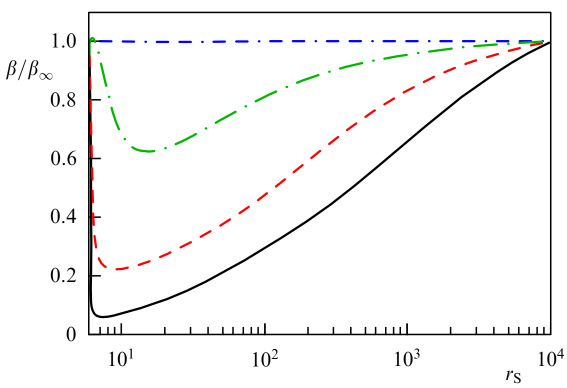

Figure 3. The dependence of β on r_{S} along the curve in Fig. 2 for which $\beta_0/\beta_\infty = 1$. $\tilde{\delta}$ takes the respective values 10^{-3}, 10^{-2}, 10^{-1}, and 1 for the solid, dashed, dashed-dotted, and dashed-doted-dashed curves.

in the form of a flat disk lying entirely in the black hole equatorial plane, of course, also exists for these $\tilde{\delta}_k$. This nonuniqueness of the solution disappears for any small viscosity in the disk, for which $\mathbf{W}_\infty = 0$ always implies $\mathbf{W}_0 = 0$. For small $\alpha \ll 1$, the disk 'feels' these 'resonance' solutions, and its inner parts deviate significantly from the equatorial plane of the black hole, even when the outer parts of the disk lie almost in the equatorial plane. Figure 1 shows the curve corresponding to analytic solution (2.111), as well as several curves for a viscous twisted disk obtained by integrating the original equation (2.88). We see that already for $\alpha = 10^{-3}$, the discussed resonances are almost entirely suppressed.

2.4.3 Disk behavior on the plane of parameters α and $\tilde{\delta}$. In conclusion, we present the full study of regimes of behavior of a stationary twisted relativistic disc near a rotating black hole. It is convenient to show the results of numerical integration of Eqn (2.88) on the plane of free parameters of the problem, $\tilde{\delta}$ and α. The first parameter varies in the range $10^{-3} < \tilde{\delta} < 10$, and the second parameter in the range $0 < \alpha < 1$. As follows from Fig. 2, at small $\tilde{\delta}$, i.e., when the gravitomagnetic force exceeds the internal forces in a twisted disk, it either lies in the equatorial plane of the black hole, i.e., $\beta_0/\beta_\infty \to 0$, or, conversely, the tilt of its rings strongly increases in the inner parts of the disk, with oscillations of $\beta(r_{\text{S}})$ along the radial coordinate. We note that for small viscosity, these oscillations become so strong that the corresponding gradient of the tilt angle, β', gives rise to supersonic perturbations of the velocity components v^r and v^φ at heights of the order of the disk thickness, $\xi \sim h$. This, in turn, must lead to the generation of various hydrodynamic instabilities and sound waves, which cause additional disk heating (and hence also an increase in $\tilde{\delta}$), as well as the growth of α. These processes should partially suppress the oscillations of β discussed above.

An alignment of the disk into the equatorial plane of the black hole occurs at sufficiently high viscosity, when the condition $\alpha > \tilde{\delta}$ is satisfied with a large margin, and is referred to as the Bardeen–Petterson effect [38]. It can be seen from Fig. 2 that this effect occurs only in sufficiently viscous and thin disks. But already for $\tilde{\delta} \sim \alpha$, the ratio β_0/β_∞ becomes of the order of unity, which means the absence of disk alignment. At the same time, the oscillations of β disappear. Figure 3 shows the profiles of $\beta(r_{\text{S}})$ with $\beta_0/\beta_\infty = 1$ for several $\tilde{\delta}$. It is seen that for not very small $\tilde{\delta}$, the twisted disk has a sufficiently smooth shape, which suggests the possibility of the existence of such configurations in nature. We note that β behaves nonmonotonically: it first decreases and then increases with r_{S}. This can have important implications both for the disk structure itself and for its observational manifestations. For example, the hot inner regions of such a disk should illuminate its outer parts much more strongly than in the case of a flat disk. Clearly, this is due to the disk inner parts being tilted with respect to the outer parts.

In the region where $\tilde{\delta}$ is of the order of or greater than unity, the action of the gravitomagnetic force becomes insignificant, and the disk is weakly twisted. In Fig. 2, the area to the right of the dashed line is where $\beta(r_{\text{S}})$ deviates from β_∞ by less than 10%. It is also worth noting that for $\tilde{\delta} > 0.1$, the Bardeen–Petterson effect is completely absent at any α.

3. Conclusion

We have presented a detailed technical derivation of the governing equations for the evolution of the shape of a relativistic twisted disk, as well as for perturbations of the velocity and density inside it. Only three simplifying assumptions have been used: the smallness of the disk aspect ratio $\delta \ll 1$, the slowness of the black hole rotation $a \ll 1$, and the smallness of the disk ring tilt to the equatorial plane of the black hole, $\beta \ll 1$. This allowed us to formulate Eqns (2.58), (2.59), and (2.62) for three variables describing Eulerian perturbations of the azimuthal velocity, v^r and v^φ, and the geometrical shape of the disk Z. In general, the dependence of v^r and v^φ on the twisted coordinates r, ξ, and τ, and the dependence of Z on r and τ should be found. In accordance with Eqn (2.28), all these variables depend harmonically on the azimuthal coordinate. The governing equations contain the profiles of the background solution, representing an accretion disk with a similar radial and vertical structure but lying in the equatorial plane of the black hole. We note once again that not only the twisted disk but also the background itself can be nonstationary, because only one assumption $\delta \ll 1$ was used in deriving the system of equations (2.58), (2.50), and (2.62) for the background. Therefore, the twisted equations also enable the study of the evolution of tilted/twisted gaseous tori/rings near rotating black holes during their expansion in the radial direction, in other words, the evolution of nonstationary accretion due to turbulent viscosity.

In the particular case of a stationary, vertically isothermal background with the α-parameterization of the viscosity, the twisted equations have been reduced to simpler equations (2.84), (2.85), and (2.86) for the complex amplitudes **A** and **B** describing the velocity perturbations and **W** describing the disk geometry, which depend only on r and τ. Here. the solution for a flat relativistic disk, which was presented in detail in Section 1, was used. The corresponding stationary problem can be described by a second-order linear differential equation for **W** [see Eqn (2.88)]. The analytic integration of this equation for a formally inviscid disk with $\tilde{\delta} \ll 1$ enabled us to find the singular resonance solutions for a discrete set of $\tilde{\delta}_k$, which in fact correspond to an instability of a flat nontilted disk, when it can acquire a twisted shape near the black hole, even with its outer part lying in the equatorial plane of the black hole. This instability, however, rapidly disappears already for $\alpha \sim 10^{-3}$, and for $\alpha > \tilde{\delta}$ with $\tilde{\delta} < 0.1$, numerical calculations show the Bardeen–Petterson effect. At the same time, already for $\alpha \sim \tilde{\delta}$, the alignment of the inner parts of the disk in the equatorial plane of the black hole is absent, and for $\tilde{\delta} \geqslant 0.1$, smooth but nonmonotonic profiles $\beta(r)$ appear (see Fig. 3), which suggests their stability under perturbations and the possibility of their realization in nature. The last effect is confirmed by the first numerical simulations of tilted thin relativistic accretion disks with $\delta \sim \alpha \sim a \sim 0.1$ carried out in recent papers [39, 40]. In these papers, a comparison with the semi-analytic model based on the solution of the system of equations (2.58), (2.59), (2.62) was also done for a slightly tilted vertically barotropic torus.

Observational confirmations of the existence of twisted accretion disks around rotating black holes have just started emerging. Apparently, one of the most direct pieces of evidence of their existence is the observation of maser sources at subparsec scales in the disk around a supermassive black hole in the nucleus of NGC 4258 [41, 42]. The subsequent modeling in [43, 44] showed that the disk twist in this case can be due to the Bardeen–Petterson effect. In recent paper [45], observations of jets in the nucleus of NGC 4298 were used to independently estimate the Kerr black hole parameter $a \approx 0.7$ and, in a similar model, to calculate the radius of the disk aligned into the equatorial plane of the black hole in agreement with observations. Additional but more indirect arguments favoring the presence of twisted disks in galactic nuclei were obtained, for example, in [46, 47], where the observed profiles of the X-ray iron line K_α were calculated for different accretion disk models. It was concluded that in many cases, the observed line profile can be more easily explained in the model of a twisted disk than in the model of a flat disk, but, for example, with specific radial intensity distribution. In [48], a similar modeling of hydrogen Balmer lines was performed, which should arise due to the heating of the outer parts of a twisted disk by hard emission from its inner parts, which have a much greater tilt than in the case of a flat disk. The presence of twisted disks is also suspected in binary stellar systems with black holes. For example, this can be the case with two microquasars, GROJ1655-40 and V4641 Sgr, in which the tilt of jets relative to the orbital plane was discovered.

As mentioned above, Eqns (2.58), (2.59), and (2.62) also describe the nonstationary dynamics of a torus tilted to the equatorial plane of a black hole. If $\delta > \alpha$, the action of the gravitomagnetic force must lead to a solid-body precession of the torus, because in this case the twist waves propagating at almost the speed of sound smear out the dependence of γ on r due to the Lense–Thirring effect. Similar nonstationary models are invoked to explain the variability of Balmer line profiles, as well as the precession of jets in active nuclei (see, e.g., [51]). In many papers, the precessing tori are used to explain low-frequency quasiperiodic oscillations in X-ray binary systems (see, e.g., [52]). Of special interest is the modeling of observational appearances of a tilted accretion disk around the black hole in the center of our Galaxy [53].

The theory of relativistic twisted disks presented here can also be successfully applied both to constructing self-consistent models of individual objects and to making further theoretical predictions of the dynamics of accretion flows around rotating black holes.

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