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The Hall effect and its analogs

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Contents

Abstract. We draw attention to a similarity between mutually related kinetic material phenomena that are odd in the magnetic field and produce an electric current or heat flow perpendicular (1) to the magnetic field, (2) to the electric field strength or to the temperature gradient. These phenomena include the Hall effect, the Righi–Leduc effect in nonmagnetic metals, the anomalous Hall effect in magnets, the odd Senftleben–Beenakker effect in molecular gases, and the phonon Hall effect in dielectrics. While these phenomena have much in common in terms of geometry, their formation mechanisms — dynamic and dissipative — are different. However, in all cases, the flow perpendicular to the magnetic field arises from the spin–orbit interaction of carriers with magnetic moments.

Keywords: Hall effect, anomalous Hall effect, spin Hall effect, phonon Hall effect, Righi – Leduc effect, Senftleben – Beenakker effect

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1. Introduction

The Hall effect — the occurrence of a transverse potential difference (also known as the Hall voltage) in a direct current-carrying conductor placed in a magnetic field — was discovered by Edwin Hall in 1879 in thin gold films [1]. Because in those days the publication of a paper even in a high-status journal did not prevent it from being republished elsewhere, it comes as no surprise that Hall's pioneering work "On a New Action of the Magnet on Electric Current" [1] reappeared as Refs [2–4]. Apart from Ref. [1], the early bibliography on the Hall effect includes Edwin Hall's subsequent work on the subject [5–14] and responses to it [15–17].

The Hall effect in metals is due to the electron drift in crossed electric and magnetic fields **E** and **B**. In magnets, it results from the spin–orbit interaction between the conduction electrons and magnetic moments. In dielectrics, the magnetic field changes the phonon polarization and affects the rate of phonon collisions with the magnetic subsystem. In gases, the precession of magnetic moments in a magnetic field changes the collision rate of nonspherical molecules. This review is limited to studies in weak magnetic fields and does not discuss the quantum Hall effect.

According to the Onsager–Casimir theory of irreversible processes, the generalized flows at small deviations from equilibrium are proportional to the generalized forces [18],

$$J_i = \sigma_{ik} X_k \,, \tag{1}$$

where σ_{ik} is the electrical conductivity tensor, or equivalently,

$$X_i = \rho_{ik} J_k \,, \tag{2}$$

where $\rho_{ik} = \sigma_{ik}^{-1}$ is the generalized resistivity tensor. In accordance with the Onsager principle,

$$\rho_{ik}(\mathbf{B}) = \rho_{ki}(-\mathbf{B}) \,. \tag{3}$$

We note that all phenomena to be considered below have already been discussed one by one, in particular, in Refs [19–

23]. The primary objective of this review is to identify the features common to all these effects. All kinetic effects odd in a magnetic field produce flows in the transverse direction,

$$\mathbf{J} \sim \mathbf{B} \times \mathbf{x} .$$

Interestingly, however, different mechanisms produce transverse flows in different materials. In nonmagnetic materials, this is the classical electron drift in crossed fields **E** and **B**. In magnets, there are two mechanisms that lead to the anomalous Hall effect [19–21, 24].

First, we have a *dynamical* mechanism (the left-hand side of the kinetic equation), according to which trajectories of conduction electrons are bent due to their spin–orbit (SO) coupling to magnetic moments. Second, there is a *dissipative* mechanism (the right-hand side), in which the dominant role is played not by the smooth bending of the trajectory in the Weiss field but by electron collisions with magnetic atoms.

In dielectrics, similarly, there are also two formation mechanisms for the transverse heat flow:

(1) a change in the phonon polarization due to the magnetic field;

(2) the field dependence of the rate of phonon collisions with magnetic impurities.

Finally, two mechanisms also operate in molecular gases in which phonons are produced by nonspherical molecules: the precession of rotational moments in a magnetic field (*dynamical* mechanism) and the change in the collision rate (*dissipative* mechanism).

In this review, we highlight the relation between the Hall constant and the Fermi surface curvature and provide an elementary derivation of the anomalous Hall effect for various systems in the case where the SO interaction produces the effect.

The outline of the review is as follows. In Section 2, we examine the Hall effect in metals and discuss the features of the effect in doped magnets with a strong interaction between charge carriers and the magnetic background. In Section 3, we discuss the anomalous Hall effect in magnets and briefly mention the spin Hall effect. The Righi–Leduc effect (transverse thermal conductivity in metals), the Senftleben–Beenakker effect (an analog of the Hall effect in molecular gases), and the phonon Hall effect in dielectrics are discussed in Sections 4, 5, and 6.

2. Hall effect in metals

When applied to a metal, Eqn (2) takes the form of Ohm's law,

$$E_i = \rho_{ik} J_k \,, \qquad \rho_{ik} = \rho \delta_{ik} + R e_{ikl} B_l \,, \tag{4}$$

where E_i is the electric field strength, J_i is the vector of the electric current density, ρ is the resistivity, R is the Hall constant, δ_{ik} is the Kronecker symbol, and e_{ikl} is the totally antisymmetric unit tensor. As usual, it is assumed that $J_i \parallel x$, $B_i \parallel z$. The magnetic field dependence of ρ_{ik} is found by solving the stationary Boltzmann kinetic equation,

$$V_i \frac{\partial f}{\partial r_i} + \left(eE_i + \frac{e}{c} [\mathbf{V} \times \mathbf{B}]_i \right) \frac{\partial f}{\partial p_i} + \mathrm{St} f = 0 \,, \tag{5}$$

(where **V** is the velocity and St is the collision operator) by introducing a small deviation from the equilibrium distribution function, $f^{(1)} = f - f^{(0)}$. In the linear order in E, Eqn (5) becomes

$$eE_i \frac{\partial f^{(0)}}{\partial p_i} + \frac{e}{c} [\mathbf{V} \times \mathbf{B}]_i \frac{\partial f^{(1)}}{\partial p_i} + \mathrm{St} f^{(1)} = 0 \,. \tag{6}$$

For the Fermi distribution, the first term is

$$-eE_i V_i \left| \frac{\partial f^{(0)}}{\partial \varepsilon} \right| , \quad \left| \frac{\partial f^{(0)}}{\partial \varepsilon} \right| = \frac{1}{T} f^{(0)} (1 - f^{(0)}) \,.$$

Introducing χ_i, $\hat{\Omega}$, and $\hat{\Lambda}$ as

$$\begin{aligned} & f^{(1)} = eE_i \chi_i \left| \frac{\partial f^{(0)}}{\partial \varepsilon} \right| , \qquad \mathrm{St} f^{(1)} = \left| \frac{\partial f^{(0)}}{\partial \varepsilon} \right| eE_i \hat{\Omega} \chi_i \,, \\ & \hat{\Lambda} = \frac{e}{c} [\mathbf{V} \times \mathbf{B}]_l \frac{\partial}{\partial p_l} \,, \end{aligned} \tag{7}$$

and canceling the factor $eE_i |\partial f^{(0)} / \partial \varepsilon|$, we rewrite Eqn (6) in a more compact form:

$$V_i = (\hat{\Lambda} + \hat{\Omega}) \chi_i \,. \tag{8}$$

In the relaxation time approximation, the self-adjoint collision operator $\hat{\Omega}$ ($\Omega_{ik} = \Omega_{ki}$, $\det \hat{\Omega} > 0$) is replaced with $1/\tau$. The electric current density is given by

$$J_i = \sum_{p\sigma} eV_i f = e^2 E_k \sum_{p\sigma} V_i \chi_k \left| \frac{\partial f^{(0)}}{\partial \varepsilon} \right| = e^2 \langle V_i \chi_k \rangle E_k \,,$$

where we introduce averaging near the Fermi surface,

$$\langle \ldots \rangle = \sum_{p\sigma} \left| \frac{\partial f^{(0)}}{\partial \varepsilon} \right| (\ldots) \,.$$

(For a spherical Fermi surface, $\langle 1 \rangle = p_{\mathrm{F}}^3 / (2\pi^2 \varepsilon_{\mathrm{F}})$, $N = p_{\mathrm{F}}^3 / (3\pi^2)$, $\langle 1 \rangle = 3N / (2\varepsilon_{\mathrm{F}})$, where p_{F} and ε_{F} are the Fermi momentum and energy.) As a result, the electrical conductivity tensor takes the form

$$\sigma_{ik} = e^2 \langle \chi_k V_i \rangle \,. \tag{9}$$

Next, we rewrite Eqn (9) by replacing the dynamical terms with the right-hand side of Eqn (8) and thereby making the symmetry of this tensor manifest:

$$\sigma_{ik} = e^2 \left\langle \chi_k (\hat{\Omega} + \hat{\Lambda}) \chi_i \right\rangle = \sigma_{ik}^{(+)} + \sigma_{ik}^{(-)} \,. \tag{10}$$

The first term in the right-hand side of Eqn (10), $\sigma_{ik}^{(+)} = e^2 \langle \chi_k \hat{\Omega} \chi_i \rangle$, is a symmetric tensor with no components odd in a magnetic field.

To calculate the part of σ_{ik} that is linear in the magnetic field, $\sigma_{ik}^{(-)} = e^2 \langle \chi_k \hat{\Lambda} \chi_i \rangle$,

$$\sigma_{ik}^{(-)} = \frac{e^3}{c} \left\langle \chi_k [\mathbf{V} \times \mathbf{B}]_l \frac{\partial \chi_i}{\partial p_l} \right\rangle , \tag{11}$$

it suffices to take the zeroth order of the vector $\boldsymbol{\chi}$ in the field (here acquiring the meaning of the mean free path),

$$\chi_i^{(0)} = L_i = \Omega_{ik}^{-1} V_k \,. \tag{12}$$

The Hall conductivity in a weak field $\mathbf{B} \parallel z$ is given by the antisymmetric half-difference

$$\sigma_{xy}^{(-)} = \frac{1}{2}\left(\sigma_{xy}^{(-)} - \sigma_{yx}^{(-)}\right) = \frac{e^3 B}{2c}\left(\left\langle L_y V_y \frac{\partial L_x}{\partial p_x}\right\rangle - \left\langle L_y V_x \frac{\partial L_x}{\partial p_y}\right\rangle + \left\langle L_x V_x \frac{\partial L_y}{\partial p_y}\right\rangle - \left\langle L_x V_y \frac{\partial L_y}{\partial p_x}\right\rangle\right).$$

In the τ approximation $L_i = \tau V_i$, we obtain

$$\sigma_{xy}^{(-)} = \frac{e^3 \tau^2 B}{2c}\left(\left\langle V_x^2 \frac{\partial V_y}{\partial p_y}\right\rangle + \left\langle V_y^2 \frac{\partial V_x}{\partial p_x}\right\rangle - 2\left\langle V_x V_y \frac{\partial V_x}{\partial p_y}\right\rangle\right). \tag{13}$$

The section of a Fermi surface by a plane perpendicular to the z axis has the curvature

$$K_z(\mathbf{p}) = \frac{2\varepsilon_x \varepsilon_y \varepsilon_{xy} - \varepsilon_x^2 \varepsilon_{yy} - \varepsilon_y^2 \varepsilon_{xx}}{(\varepsilon_x^2 + \varepsilon_y^2)^{3/2}}\,, \tag{14}$$

$$V_i = \varepsilon_i = \frac{\partial \varepsilon}{\partial p_i}\,, \qquad \varepsilon_{ik} = \frac{\partial V_k}{\partial p_i}\,, \qquad \left\langle V_\parallel^3 K_z(\mathbf{p})\right\rangle = \frac{N}{m^2}\,,$$

$$K_z(\mathbf{p}) = \frac{2 V_x V_y \varepsilon_{xy} - V_x^2 \varepsilon_{yy} - V_y^2 \varepsilon_{xx}}{V_\parallel^3}$$

at a point $\mathbf{p}$ (see, e.g., Ref. [25]), and therefore the transverse conductivity, Eqn (13), can be represented as a weighted average of curvature (14) [26], which gives

$$\sigma_{xy}^{(-)} = -\frac{\tau^2 e^3 B}{2c}\left\langle V_\parallel^3 K_z\right\rangle, \qquad V_\parallel^2 = V_x^2 + V_y^2\,. \tag{15}$$

Hence, the Hall constant is found to be

$$\rho_{yx} = RB = \frac{\sigma_{xy}^{(-)}}{\sigma_{xx}^{(+)} \sigma_{yy}^{(+)}}\,. \tag{16}$$

We note that for a cubic crystal, Eqn (15) can be replaced with the more symmetric formula

$$\sigma_{xy}^{(-)} = -\frac{\tau^2 e^3 B}{6c}\left\langle V^3 k_3\right\rangle, \tag{17}$$

where k_3 is the sum of the principal curvatures of the three-dimensional surface,

$$k_3 = (\varepsilon_\gamma^2)^{-3/2}(\varepsilon_\alpha \varepsilon_\beta \varepsilon_{\alpha\beta} - \varepsilon_\alpha \varepsilon_\alpha \varepsilon_{\beta\beta})\,. \tag{18}$$

To the best of our knowledge, the relation given by Eqns (15) and (16) between the Hall constant and the Fermi surface curvature has never been published before (except for Ref. [26], which is not easily accessible).

We also note that in the two-dimensional case, an alternative arises [27] to represent the Hall conductivity $\sigma_{xy}^{(-)}$ geometrically as the area $A = \int d\mathbf{L} \times \mathbf{L}$ swept by the vector $\mathbf{L} = \mathbf{V}\tau$ as it moves along the Fermi surface.

For an isotropic metal with the spectrum $\varepsilon = p^2/(2m)$, Eqn (16) produces the usual expression

$$R = \frac{1}{Nec}\,. \tag{19}$$

For a metal with a two-dimensional narrow band with nesting,

$$\varepsilon = -t\gamma\,, \qquad \gamma = \cos(p_x a) + \cos(p_y a)\,,$$

the curvature and hence the Hall constant are negative (positive) in the lower (higher) half of the band, and near the half-filling $N_{0.5}$, we have

$$R \sim (N - N_{0.5})\,. \tag{20}$$

If the crystal section perpendicular to the magnetic field has a rectangular symmetry and if the electron spectrum is given by

$$\varepsilon = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}\,,$$

then the conductivity tensor component $\sigma_{yx}^{(-)}$ can be written as

$$\sigma_{yx}^{(-)} \sim \left\langle L_x \left[\mathbf{V} \frac{\partial}{\partial \mathbf{p}}\right]_z L_y \right\rangle \sim \left(\frac{1}{m_1}\right)^2 \frac{1}{m_2}(2m_1)^{3/2}\sqrt{2m_2}\,. \tag{21}$$

We therefore conclude that Eqn (19), commonly used to determine the sign and density of charge carriers, is valid only in the simplistic undergraduate-level case of free electrons with the spectrum $\varepsilon = p^2/(2m)$.

The calculation of the Hall effect in a real metal requires that

(1) the features of the Fermi surface be taken into account;

(2) the carrier velocity and the Fermi surface curvature be allowed to be direction-dependent; and

(3) the τ approximation be discarded and the conductivity tensor calculated taking collisional anisotropy into account in the multimoment approximation (see, e.g., Refs [26, 28]).

We do not elaborate here on the last item for brevity. The most important point — the dependence of the Hall effect on the Fermi surface curvature — is illustrated by Eqn (15).

This makes it all the more dubious to apply the τ approximation to systems with strong electron correlations, including doped antiferromagnets and high-temperature superconductors (HTSCs).

The strong coupling of charge carriers to the magnetic spin background results in the bare carrier (which we treat as a hole, similarly to the cases of lanthanum and yttrium HTSCs) being 'dressed' by spin excitations, thus markedly changing the bare spectrum. The hole is scattered on the spin subsystem, exciting a spin wave

$$S_{\mathbf{q}}^\alpha = \sum_{\mathbf{R}} \exp(-\mathrm{i}\mathbf{q}\mathbf{R}) S_{\mathbf{R}}^\alpha\,,$$

with $\alpha = x, y, z$ being the polarization direction. The scattering is described by the interaction

$$\hat{H}_{\text{int}} = J \sum_{\mathbf{k},\mathbf{q}} a_{\mathbf{k}+\mathbf{q},\gamma_1}^\dagger S_{\mathbf{q}}^\alpha \hat{\sigma}_{\gamma_1\gamma_2}^\alpha a_{\mathbf{k}\gamma_2}\,,$$

where J is the spin–hole coupling strength and $\hat{\sigma}^\alpha$ are the Pauli matrices. The 'dressed' hole $\tilde{a}_{\mathbf{k}\sigma}$ is a quasiparticle that can be viewed as a superposition of the states of a bare hole with spin excitations,

$$\tilde{a}_{\mathbf{k}\sigma} = u_0(\mathbf{k}) a_{\mathbf{k}\sigma} + \sum_{\mathbf{q}} u_1(\mathbf{k},\mathbf{q}) S^{\alpha}_{\mathbf{q}} \sigma^{\alpha}_{\sigma\sigma'} a_{\mathbf{k}-\mathbf{q},\sigma'} + \sum_{\mathbf{q},\mathbf{q}'} u_2(\mathbf{k},\mathbf{q},\mathbf{q}') S^{\alpha}_{\mathbf{q}} S^{\beta}_{\mathbf{q}'} (\sigma^{\alpha}\sigma^{\beta})_{\sigma\sigma'} a_{\mathbf{k}-\mathbf{q}-\mathbf{q}',\sigma'} + \ldots, \quad (22)$$

with certain coefficients $u_j(\mathbf{k},\mathbf{q},\ldots)$ determined by the structure of the magnetic background. In this case, solving the Boltzmann equation requires the use of the multimoment approximation. Notably, the choice of the moments is not unique; to give an example, polynomials in the velocity components and their derivatives can be used. When treated in this way, the Hall effect acquires a temperature dependence. An implementation of this approach is described in detail in Refs [29–32].

3. Anomalous Hall effect in ferromagnets

We next consider the transverse electrical conductivity in ferromagnets — the so-called anomalous Hall effect (AHE) first observed by Kikoin [33, 34]. The AHE is several orders of magnitude stronger than the classical Hall effect. In ferromagnets, the strong SO coupling $H^{(SL)} \sim \mathbf{SL}$ operates in addition to the Lorentz force (see, e.g., Ref. [35]).

A point to note is that the orbital momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ of conduction electrons does not reduce to the orbital momentum of the atoms. The simplest possible AHE mechanism is due to the coupling of conduction electrons to the coherent subsystem of spontaneous magnetic moments; it manifests itself as an effective (molecular) Weiss field, which is proportional to the spontaneous magnetization of the body, $\mathbf{B}^M = \gamma\mathbf{M}$, and is much stronger than $\mathbf{B}$. The description of electron motion is then modified by replacing the Lorentz force term in Eqn (8) with the following expression containing the Weiss field:

$$\hat{\Lambda}^M = \frac{e}{c}\,[\mathbf{V} \times \gamma\mathbf{M}]_a \frac{\partial}{\partial p_a}\,, \quad (23)$$

and the transverse component of the electrical resistivity becomes

$$\rho^M_{yx} = \frac{\gamma M}{Nec}\,. \quad (24)$$

This *dynamical* mechanism results from the effect of the average magnetization on the trajectory of a conduction electron.

More interesting is the AHE mechanism associated with the SO anisotropy of electron scattering by the magnetic moments $\mathbf{M}$ of a ferromagnetic lattice. This *dissipative* mechanism was treated theoretically in Refs [19–21, 24] (see also review [36]) and arises due to the SO scattering of conduction electrons on fluctuations of the magnetic moments of inner-shell electrons. The magnitude of the effect is estimated by going beyond the Born approximation in considering scattering (i.e., by calculating through the second order in the potential interaction with impurities, $H^{(\mathrm{imp})}$, and through the linear order in $H^{(SL)}$).

Next, the dissipative part of the Hall current is obtained by means of a calculation that neglects dynamical mechanism (23) in Eqn (8) but in which the collision integral includes not only the potential scattering on impurities $\hat{\Omega}^{(0)}$ but also the $\mathbf{M}$-linear contribution from the SO scattering $\hat{\Omega}^{(SL)}$, giving

$$V_i = (\hat{\Omega}^{(0)} + \hat{\Omega}^{(SL)})\chi_i\,. \quad (25)$$

Conductivity tensor (9) can be written in the form

$$\sigma_{ik} = e^2\langle V_i \chi_k\rangle = e^2\big\langle \chi_k (\hat{\Omega}^{(0)} + \hat{\Omega}^{(SL)})\chi_i\big\rangle = \sigma^{(0)}_{ik} + \sigma^{(SL)}_{ik}\,, \quad (26)$$

where $\sigma^{(0)}_{ik} = (e^2/\tau)\langle\chi_k\chi_i\rangle$ is a symmetric $\mathbf{M}$-even tensor. The tensor

$$\sigma^{(SL)}_{ik} = e^2\langle\chi_k \hat{\Omega}^{(SL)}\chi_i\rangle \quad (27)$$

transforms under coordinate rotations as the product of two polar vectors ($\boldsymbol{\chi}$) and is linear in $\mathbf{M}$. Therefore, neglecting the anisotropy of the medium, $\mathbf{M}$ is the dual vector to the tensor $\langle\chi_k \hat{\Omega}^{(SL)}\chi_i\rangle$, and the antisymmetric part of the conductivity tensor can be written as [18]

$$\sigma^{(SL)}_{ik} = \beta e_{ikl} M_l\,, \qquad \sigma^{(SL)}_{yx} = -\beta M\,. \quad (28)$$

Below the Curie temperature, the vector $\mathbf{M}$ is the spontaneous magnetic moment; in the paramagnetic domain, $\mathbf{M} = \chi\mathbf{H}$.

To determine which of the mechanisms considered above is more important, the coefficients γ in Eqn (24) and β in Eqn (28) should be calculated for a specific SO interaction model. This, however, is beyond the scope of the present review (see Refs [19–21, 24]).

We mention the so-called *spin Hall effect* predicted theoretically by Dyakonov and Perel [37, 38] as far back as 1971 and thus named by Hirsch [39] in 1999. Similar to the AHE, the spin Hall effect does not require the presence of an external magnetic field. There are two varieties of this effect, *external* and *internal*. The former arises due to the anisotropy of the electron scattering by Coulomb centers, the anisotropy originating from the SO coupling. As the current flows through the material, electrons with spin up relative to the plane scatter predominantly to the right and those with spin down scatter to the left, as in the anomalous Hall effect. As a result, one side edge acquires an excess of spin-up electrons, and the other of spin-down electrons, similar to the excess charge in the ordinary Hall effect. For the internal spin effect, it is the SO coupling which pushes the opposite-spin carriers apart.

The existence of the spin Hall effect has recently been demonstrated experimentally not only in semiconductors [40–42] but also in metals [43–45].

4. Righi–Leduc effect

We now consider the transverse thermal conductivity in metals — the anomalous Righi–Leduc effect, which was discovered simultaneously by the Italian physicist Augusto Righi and the French physicist Sylvester Leduc. Their two 1887 papers, "Sulla conducibilitá calorifica del bismuto posto in un campo magnetico" ("Thermal conductivity of bismuth in a magnetic field") by Righi [46] and "Sur la conductibilité calorifique du bismuth dans un champ magnétique et la déviation des lignes isothermes" ("On the calorific conductibility of bismuth in a magnetic field, and on the deviation of the isothermal lines") by Leduc [47], are commonly referred to as pioneering, although in actual fact both physicists started studying and reporting on the effect in 1883–1884 [48–54] (here, references are given only for the first publications, and reprints are excluded).

Like Edwin Hall, Righi and Leduc did quite a considerable amount of original work, enough for us to avoid citing it in full. An exhaustive bibliography drawing together the early literature on galvanomagnetic and thermomagnetic phenomena can be found in book [55].

In considering the transverse thermal conductivity in metals, the logarithm of the temperature gradient and the heat flow play the respective roles of a generalized force and a generalized flow. Linearized Boltzmann equation (6) should be modified by replacing the first term according to the equation

$$(\mathbf{V}\boldsymbol{\nabla})f^{(0)} = (\varepsilon-\mu)(\mathbf{V}\boldsymbol{\nabla}\ln T)\left|\frac{\partial f^{(0)}}{\partial\varepsilon}\right|,$$

which gives

$$(\varepsilon-\mu)(\mathbf{V}\boldsymbol{\nabla}\ln T)\left|\frac{\partial f^{(0)}}{\partial\varepsilon}\right| + \frac{e}{c}\,[\mathbf{V}\times\mathbf{B}]_i\,\frac{\partial f^{(1)}}{\partial p_i} + \mathrm{St}\,f^{(1)} = 0\,. \tag{29}$$

Defining

$$\tilde{V}_i = (\varepsilon-\mu)V_i\,, \qquad f^{(1)} = (\nabla\ln T)_i\chi_i\left|\frac{\partial f^{(0)}}{\partial\varepsilon}\right|$$

reduces this problem to that considered in Section 2, but this time, instead of the current density, we have to find the heat flux density carried by electrons (see Ref. [56]):

$$q_i = \sum_{p\sigma}(\varepsilon-\mu)V_i f = \langle\tilde{V}_i\chi_k\rangle(\nabla\ln T)_k\,. \tag{30}$$

Equation (8) is replaced by

$$\tilde{V}_i = \hat{\Lambda}\chi_i + \hat{\Omega}\chi_i\,, \tag{31}$$

and Eqn (11) by the expression for the thermal conductivity coefficient $\varkappa$,

$$\varkappa_{ik}^{(-)} = -\frac{1}{T}\,\langle\chi_k^{(0)}\hat{\Lambda}\chi_i^{(0)}\rangle\,, \tag{32}$$

where the function $\chi_i^{(0)}$ is a solution of the Boltzmann equation in the absence of a magnetic field,

$$\tilde{V}_i = \hat{\Omega}\chi_i^{(0)}\,.$$

The calculations are similar to those in Section 2 and are not repeated here. In Section 5, we discuss the Senftleben–Beenakker effect.

5. Senftleben–Beenakker effect

We consider the Senftleben–Beenakker effect, a molecular gas analog of the Hall effect [57–61]. In metals, the magnetic field (the Lorentz force) acts directly on the orbital motion of particles (electrons). In gases, the effect of the field on molecular motion is indirect, via the subsystem of the molecular rotational moments. The field causes these to precess, and this precession changes the collision rate of nonspherical molecules and causes the molecular motion to relax in a field-dependent manner.

We find the thermal conductivity tensor by solving the thermal conduction problem for a gas with rotational degrees of freedom in a magnetic field,

$$(\mathbf{V}\boldsymbol{\nabla})f^{(0)} + \gamma[\mathbf{M}\times\mathbf{B}]\,\frac{\partial f}{\partial\mathbf{M}} + \mathrm{St}\,f = 0\,. \tag{33}$$

Unlike Eqn (6), the Boltzmann equation for metals in which the magnetic Lorentz force acts on the electron momentum, Eqn (33) involves the moment of force of the molecular rotation moment $\gamma[\mathbf{M}\times\mathbf{B}]$. The local equilibrium distribution is then the Maxwell–Boltzmann function at constant pressure,

$$f^{(0)} = \mathrm{const}\,\frac{p}{T^{c_p}}\exp\left[-\frac{1}{T}\left(\frac{mV^2}{2}+\frac{M^2}{2I}\right)\right], \tag{34}$$

and the temperature gradient plays the role of the electric field strength. The distribution is sought in the form $f = f^{(0)}(1+(\nabla\ln T)\boldsymbol{\chi})$, and the 'field' term in Eqn (33) is written as

$$(\mathbf{V}\boldsymbol{\nabla})f^{(0)} = f^{(0)}(\nabla\ln T)\mathbf{Q}\,, \quad \mathbf{Q} = \frac{\mathbf{V}}{T}\left(\frac{mV^2}{2}+\frac{M^2}{2I}-\frac{7}{2}\,T\right). \tag{35}$$

The collision integral in the linear order in ∇T is presented in the form

$$\mathrm{St}\,f = (\boldsymbol{\nabla}\ln T)f^{(0)}(\hat{\Omega}\boldsymbol{\chi})\,, \qquad \Omega_{ik} = \Omega_{ki}\,.$$

Introducing the precession operator $\hat{\Lambda} = \gamma[\mathbf{M}\times\mathbf{B}]\,\partial/\partial\mathbf{M}$ for brevity, we rewrite Eqn (33) as

$$f^{(0)}(\nabla\ln T)_i Q_i + f^{(0)}(\nabla\ln T)_i\hat{\Lambda}\chi_i = -f^{(0)}\nabla_i\ln T(\hat{\Omega}\chi)_i\,,$$
$$\Omega_{ik} = \Omega_{ki}\,.$$

Canceling by $f^{(0)}(\nabla\ln T)$, we obtain [cf. Eqn (8)]

$$Q_i = -(\hat{\Omega}+\hat{\Lambda})\chi_i\,. \tag{36}$$

The heat flow in the rest frame of the gas as a whole is

$$q_i = \int \mathrm{d}\Gamma\,V_i\left(\frac{mV^2}{2}+\frac{M^2}{2I}-\frac{7}{2}\,T\right)f$$
$$= \int \mathrm{d}\Gamma\,TQ_i\big(1+(\nabla\ln T)_k\chi_k\big)f^{(0)} = -\varkappa_{ik}(\nabla\ln T)_k\,,$$
$$\langle\ldots\rangle = \int \mathrm{d}\Gamma\,Tf^{(0)}\,, \qquad \mathrm{d}\Gamma = \frac{1}{M^n}\,\mathrm{d}^3M\,\mathrm{d}^3p\,, \qquad \langle 1\rangle = TN\,.$$

In the element of integration over the rotation moments, $n=1$ for two-atomic molecules and $n=0$ for polyatomic ones.

In using the method of moments to find the magnetic dependence of the tensor

$$\varkappa_{ik} = -T\langle Q_i\chi_k\rangle\,, \tag{37}$$

it should be taken into account that the vector function χ_k depends on the rotation moment, which requires using several moments, making the calculation is fairly tedious. An easier way to obtain the answer is to transform Eqn (37) into a more symmetric form [as in Eqn (10)] by simply replacing in (37) the left-hand side of Eqn (36) with the right-hand side, with the result

$$\varkappa_{ik} = -\langle\chi_k Q_i\rangle = \big\langle\chi_k(\hat{\Omega}+\hat{\Lambda})\chi_i\big\rangle\,. \tag{38}$$

The tensor $\varkappa_{ik}^{(0)} = \langle \chi_k \hat{\Omega} \chi_i \rangle$ is manifestly symmetric and contributes nothing to the **B**-linear part of the heat flow. An effect odd in the field can only result from

$$\varkappa_{ik}^{(1)} = \langle \chi_k \hat{\Lambda} \chi_i \rangle \,. \tag{39}$$

Here, we can use the solution of Eqn (36) of the zeroth order in the field ($\boldsymbol{\chi}^{(0)} = \mathbf{F}$),

$$\hat{\Omega}\,\mathbf{F} = -\mathbf{Q} \,. \tag{40}$$

The tensor $\varkappa_{ik}^{(1)}$ is nonzero only if $\hat{\Lambda}\,\mathbf{F} \neq 0$, which requires that the vector **F** be **M**-dependent.

The quantity **Q** is a polar vector [see Eqn (35)] and is independent of the direction of **M**. However, the solution **F** of Eqn (40) depends on the direction of **M** if the collision operator $\hat{\Omega}$ involves nonspherical collisions. This means that in addition to the contribution from the τ approximation, $\mathbf{F}^{(0)} = -\tau \mathbf{Q}$, the solution of Eqn (40) has a part $\mathbf{F}^{(m)}$ that is a polar vector odd in **V** and even in **M**. The simplest possible **M**-dependent vector has the form

$$\mathbf{m} = \xi(\mathbf{VM})\mathbf{M} \,. \tag{41}$$

This vector is known in the literature as the Kagan vector.

It should be expected that a deformation of the distribution of this form is sufficiently close to the exact solution of Eqn (40). The reason is that the operator $\hat{\Omega}$ involves integration over the directions of **V** and **M**, resulting in all higher harmonics being smoothed and reduced to zero. There is no reason why the dependence we are discussing should have a more complex structure than Eqn (41) (see Ref. [56]).

To find the parameter ξ, we should — as in the discussion of the phonon Hall effect in Section 6 — go beyond the Born approximation in calculating the collision rate entering the collision operator $\hat{\Omega}$. To avoid repetition, we omit this procedure here.

In this model, tensor (39) has the form

$$\begin{aligned} \varkappa_{ik}^{(1)} &= \left\langle F_k^{(m)} \gamma e_{abz} M_b B_z \frac{\partial}{\partial M_a} F_i^{(m)} \right\rangle \\ &= \xi^2 \gamma e_{abz} B_z \left\langle \frac{M_k M_b}{M^4} (\mathbf{MV}) \big((\mathbf{MV}) \delta_{ai} + V_a M_i \big) \right\rangle , \end{aligned} \tag{42}$$

which, when averaged over the velocity directions, yields

$$\begin{aligned} \varkappa_{ik}^{(1)} &= -\xi^2 \gamma e_{abz} B_z \left\langle \frac{M_k M_b}{3M^4} V^2 (M^2 \delta_{ai} + M_a M_i) \right\rangle \\ &= -\xi^2 \gamma e_{abz} B_z \left\langle V^2 \frac{M_k M_b}{3M^2} \delta_{ai} + \frac{V^2}{3M^4} M_k M_b M_a M_i \right\rangle . \end{aligned}$$

Multiplying e_{abz} by the product of four M immediately yields zero. The first term is easily averaged over the directions of the rotation moment to give the sought effect:

$$\varkappa_{ik}^{(1)} = \frac{1}{9}\, \xi^2 \gamma e_{ikz} B_z \langle V^2 \rangle = \frac{1}{9} \frac{TN}{m}\, \xi^2 \gamma e_{ikz} B_z \,.$$

Explicitly, we have

$$\mathbf{q} = \frac{1}{9} \frac{TN}{m}\, \xi^2 \gamma [\nabla T \times \mathbf{B}] \,.$$

An oxygen molecule has the spin $S = 1$ and three projections $S_z = 0, \pm 1$ on the total moment J, which is close to the rotational moment. The components with $S_z = \pm 1$ have gyromagnetic ratios of opposite signs and do not produce an odd effect. An effect linear in the field comes from the component $S_z = 0$.

In a gas of diamagnetic molecules, in the case $S = 0$, the magnetic moment of a molecule is due to the rotation of nuclei, and $\gamma \simeq e\hbar/(2Mc)$. Although this is a very small moment, the high accuracy of measuring the heat conductivity allows observing the transverse heat flow in a magnetic field even in a gas of diamagnetic molecules.

We also note that the effect considered in Section 4 — the appearance in molecular gases of a heat flow perpendicular to both the temperature gradient and the magnetic field — can be extended to include solids in which molecules vibrating around lattice sites are at the same time free to rotate at temperatures above the freezing temperature of the rotational degrees of freedom. Clearly, heat is transmitted not by molecules tied to the sites but by phonons. It was shown in [62] that in molecular crystals in which quasifree rotation of molecules is possible in a wide temperature range, a heat flow perpendicular to the applied magnetic field and temperature gradient should also be observed.

6. Phonon Hall effect

Finally, there is one more effect to consider, which was discovered in the late 20th century and which consists in the appearance of a transverse heat flow in a dielectric due to the phonon flow on the background of the spin subsystem.

This analog of the Hall effect was recently discovered [63–65] in the dielectric compound $Tb_3Ga_5O_{12}$. The authors of Refs [63–65] observed that the heat flow has a component perpendicular to both the temperature gradient and the field, $q_\perp \sim \nabla T \times \mathbf{B}$, and named this phenomenon the phonon Hall effect (PHE). Dielectrics have no free charge carriers, and therefore the physical nature of this phenomenon is significantly different from that of the Hall effect mechanism in metals. Nor are there present rotational degrees of freedom that could lead to a process similar to the Beenakker effect in gases (see Section 5) [57–60]. The authors of Ref. [63] believed that the PHE is due to the SO coupling of phonons and the magnetized spins of the paramagnetic ions. In this sense, the PHE is akin to the AHE in ferromagnets.

In Refs [33, 34, 66–68], the AHE is related to the uniform part of the SO interaction, a part that leads to the renormalization of the electron group velocities (the Berry phase). This idea was used in [69] to describe the PHE. However, the renormalization of ion group velocities cannot lead to the PHE, because ions, unlike quasifree electrons, vibrate near the lattice sites and averaging the SO renormalization of ion velocities always yields zero. In the linear order in the SO coupling, the phonon velocity is not renormalized (see below), and the effect of the SO coupling on the heat conductance tensor can manifest itself through the elliptic renormalization of the phonon polarization [22, 70, 71] (magnetoacoustic effect). This mechanism is very different from those discussed above and is to be considered last. Closer to the previous mechanism is that of the PHE, in which this process is viewed, by analogy with the AHE, as a kinetic phenomenon with an anisotropic SO scattering (see Section 3).

We consider a model where a scattered phonon retains its mode number. In this case, the overall heat conductivity of the dielectric can be represented as the sum of the heat conductivities of each individual mode.

As already stressed, the magnetic field does not exert a direct effect on the motion of phonons, and the Boltzmann equation has the form

$$\omega(\mathbf{V}\nabla\ln T)\left|\frac{\partial f^{(0)}}{\partial\varepsilon}\right| + \mathrm{St}\, f^{(1)} = 0\,, \tag{43}$$

where $f^{(0)}$ is the Bose–Einstein distribution,

$$\left|\frac{\partial f^{(0)}}{\partial\varepsilon}\right| = \frac{1}{T}f^{(0)}(1+f^{(0)})\,.$$

As before, we introduce

$$f^{(1)} = \omega(\boldsymbol{\chi}\nabla\ln T)\left|\frac{\partial f^{(0)}}{\partial\varepsilon}\right|,$$
$$\mathrm{St}\, f^{(1)} = \left|\frac{\partial f^{(0)}}{\partial\varepsilon}\right|\omega(\nabla\ln T\hat{\Omega}\boldsymbol{\chi})\,, \quad \tilde{\mathbf{V}} = \omega\mathbf{V}\,, \tag{44}$$

and arrive at the equation of form (25),

$$\tilde{V}_i = (\hat{\Omega}^{(0)} + \hat{\Omega}^{(SL)})\chi_i\,, \tag{45}$$

where $\hat{\Omega}^{(0)}$ is the collision integral for the scattering of phonons on impurities without the participation of magnetic moments (scattering by phonons at low temperatures is negligibly small), and $\hat{\Omega}^{(SL)}$ is the contribution from the spin–phonon scattering. It can be shown that $\hat{\Omega}^{(0)}$ appears already in the Born approximation, and calculating $\hat{\Omega}^{(SL)}$ requires going beyond the Born approximation linearized with respect to the SO coupling.

Similarly to ferromagnets [see Eqns (27), (28)], the transverse part of the conductivity tensor is again found to be

$$\varkappa_{ik}^{(\mathrm{SO})} = -\frac{1}{T}\langle\chi_k\hat{\Omega}^{(SL)}\chi_i\rangle \sim \xi e_{ikl}M_l\,. \tag{46}$$

We now treat the PHE as resulting from the influence of the magnetic field on the phonon polarization (magneto-acoustic effect). We find the renormalization of the acoustic phonons due to the SO coupling by taking the self-averaging of the magnetization ($\mathbf{M}_m \to \langle\mathbf{M}_m\rangle = \mathbf{M}$) into account on the scale of the acoustic phonon wavelength.

The Hamiltonian of the spin–phonon interaction has the form

$$H_{\mathrm{SO}} = -\sum_n\big([\mathbf{u}_n\times\mathbf{p}_n],\mathbf{S}\big)\,, \quad \mathbf{S}\sim\mathbf{M}\,. \tag{47}$$

We add this to the Hamiltonian of lattice vibrations taken in the harmonic approximation ($u_{ij}^a = u_i^a - u_j^a$, $V_{i=j}^{ab} = 0$, $V_{i\neq j}^{ab} \geqslant 0$):

$$H = \sum_i\frac{p_i^2}{2m} + \frac{1}{4}\sum_{ij}V_{ij}^{ab}u_{ij}^a u_{ij}^b - \sum_i\big([\mathbf{u}_i\times\mathbf{p}_i],\mathbf{S}\big)\,.$$

In the harmonic approximation, the classical and quantum equations of motion are identical in form:

$$v_i^a = \dot{u}_i^a = -\mathrm{i}[u_i^a, H] = \frac{p_i^a}{m} + [\mathbf{u}_i\times\mathbf{S}]^a\,, \tag{48}$$

$$\dot{p}_i^a = -\mathrm{i}[p_i^a, H] = -\sum_j V_{ij}^{ab}u_{ij}^b + [\mathbf{p}_i\times\mathbf{S}]^a\,. \tag{49}$$

The equation for lattice vibrations is obtained as

$$\ddot{u}_i^a = -\frac{1}{m}\sum_j V_{ij}^{ab}u_{ij}^b + 2[\dot{\mathbf{u}}_i\times\mathbf{S}]^a\,. \tag{50}$$

Here and hereafter, we use the linear approximation to include the SO coupling.

Solutions of Eqn (50) describe three SO-corrected acoustic modes, whose spectra and polarizations are determined by the dispersion relation

$$\omega_{\mathbf{k}s}^2 e_{\mathbf{k}s}^a = \tilde{D}_{\mathbf{k}}^{ab}e_{\mathbf{k}s}^b\,, \tag{51}$$

where

$$\tilde{D}_{\mathbf{k}}^{ab} = D_{\mathbf{k}}^{ab} + \mathrm{i}D_1^{ab}\,, \quad D_1^{ab} = 2\omega_{\mathbf{k}s}e_{abz}S_z\,, \tag{52}$$

$$D_{\mathbf{k}}^{ab} = \sum_{\mathbf{R}}V_{\mathbf{R}}^{ab}\big[1-\exp(\mathrm{i}\mathbf{kR})\big]\,, \quad \mathbf{e}_{\mathbf{k}s}^*\mathbf{e}_{\mathbf{k}s'} = \delta_{ss'}\,. \tag{53}$$

The $\tilde{D}_{\mathbf{k}}^{ab}$ tensor has the Onsager–Casimir symmetry $\tilde{D}_{\mathbf{k}}^{ab}(S) = (\tilde{D}_{\mathbf{k}}^{ba}(-S))^*$.

We first treat the problem in the zeroth order in the SO coupling. In the long-wavelength approximation [72],

$$D_{\mathbf{k}}^{ab} = \frac{1}{2}\sum_{\mathbf{R}}D_{\mathbf{R}}^{ab}(\mathbf{kR})^2 = \lambda^{acdb}k_c k_d\,. \tag{54}$$

In the case of a sufficiently high crystal symmetry (cubic, tetragonal, or rhombic), the nondiagonal elements of the dynamic matrix satisfy the proportionality relation $D_{\mathbf{k}}^{ab}\sim k_a k_b$ and hence change sign with the sign of k_a and k_b, i.e., on reflection from the corresponding plane in the reciprocal space. We segregate the nondiagonal terms

$$(\omega_{\mathbf{k}s}^2 - D_{\mathbf{k}}^{aa})e_{\mathbf{k}s}^a = \sum_{b\neq a}D_{\mathbf{k}}^{ab}e_{\mathbf{k}s}^b \tag{55}$$

in dispersion relation (51) (with no SO coupling). The tensor $D_{\mathbf{k}}^{ab}$ is real, and the eigenvectors of Eqn (55) can always be considered real, which corresponds to linear polarization.

Although the phonon spectrum has been calculated repeatedly, little or no attention has been paid to phonon polarization [73]. When the nondiagonal elements $D_{\mathbf{k}}^{ab}$ change sign, it follows from Eqn (55) that the relative sign of the polarization vector components also changes sign. This property can be written as

$$e_{\mathbf{k}s}^a = \tilde{e}_s^a(\mathbf{k})\,\mathrm{sign}\,k_a\,, \tag{56}$$

which, when substituted in dispersion relation (56), gives

$$(\omega_{\mathbf{k}s}^2 - D_{\mathbf{k}}^{aa})\,\tilde{e}_s^a(\mathbf{k}) = \sum_{b\neq a}|D_{\mathbf{k}}^{ab}|\tilde{e}_s^b(\mathbf{k})\,. \tag{57}$$

Because the $|D_{\mathbf{k}}^{ab}|$ in the right-hand side are absolute values, the frequency $\omega_{\mathbf{k}s}$ and the unit vector $\tilde{e}_s^a(\mathbf{k})$ are both invariant under reflection.

We note that the polarization vector $\mathbf{e}_{\mathbf{k}}$ is an element of a displacement vector, i.e., a polar vector, similar to the phonon vector $\mathbf{k}$. Therefore, Eqn (56) can be replaced by the relation $\mathbf{e}_k\sim\mathbf{k}$. The commonly accepted relation $\mathbf{e}_{-\mathbf{k}} = \mathbf{e}_{\mathbf{k}}$ [74] seems to be artificial.

In the presence of nondiagonal elements $D_{\mathbf{k}}^{ab}$, the zeroth-order phonon spectrum is almost always nondegenerate.

Outside the degeneration regions, the SO contribution to the spectrum and polarization renormalizations can be treated in the linear order. Because the SO contribution to dispersion equation (51) is given by the imaginary tensor $\mathrm{i}\delta D^{ab}$, the renormalization of the polarization vector should be imaginary, and the complex nature of the renormalized polarization vector $\mathbf{e}_s^a + \mathrm{i}\delta\mathbf{e}_s^a$ indicates that the renormalized wave has an 'elliptic' nature. It is this fact which ultimately leads to the PHE.

The corrections to the spectrum $\delta\omega_s$ and to the polarization vector $\delta\mathbf{e}_s^a$ are found to be

$$(\omega_s^2 + 2\omega_s\delta\omega_s)(e_s^a + \mathrm{i}\delta e_s^a) = D^{ab}(e_s^b + \mathrm{i}\delta e_s^b) + \mathrm{i}(\delta D^{ab})e_s^b \,. \tag{58}$$

The real part of Eqn (58) yields $\delta\omega_s = 0$, which means that the phonon spectrum and the group velocity $c_{\mathbf{k}s} = \partial\omega_{\mathbf{k}s}/\partial\mathbf{k}$ are not renormalized. The imaginary part of Eqn (58) determines the polarization renormalization,

$$(\omega_s^2\delta^{ab} - D^{ab})\delta e_s^b = (\delta D^{ab})e_s^b \,.$$

Hence,

$$\delta\mathbf{e}_s = \frac{2\omega_s[\mathbf{e}_{s+1}\times\mathbf{e}_s]\mathbf{S}}{\omega_s^2 - \omega_{s+1}^2}\,\mathbf{e}_{s+1} + \frac{2\omega_s[\mathbf{e}_{s+2}\times\mathbf{e}_s]\mathbf{S}}{\omega_s^2 - \omega_{s+2}^2}\,\mathbf{e}_{s+2} \,. \tag{59}$$

Essentially, all acoustic modes contribute to the renormalization $\delta\mathbf{e}_s$.

The secondary-quantized phonon Hamiltonian is given by

$$H = \sum_{\mathbf{k}s}\omega_{\mathbf{k}s}a_{\mathbf{k}s}^+a_{\mathbf{k}s} \,,$$

and the displacement vector and its time derivative

$$\begin{aligned} u_i^a &= \frac{1}{\sqrt{N}}\sum_{\mathbf{k}s}\exp(\mathrm{i}\mathbf{k}\mathbf{R}_i)\left(u_{\mathbf{k}s}^a a_{\mathbf{k}s} + u_{-\mathbf{k}s}^{a*}a_{-\mathbf{k}s}^+\right), \\ u_{\mathbf{k}s}^a &= e_{\mathbf{k}s}^a\sqrt{\frac{1}{2M\omega_{\mathbf{k}s}}}\,, \\ v_i^a &= \frac{\partial u_i^a}{\partial t} = \frac{1}{\sqrt{N}}\sum_{\mathbf{k}s}(-\mathrm{i}\omega_{\mathbf{k}s})\exp(\mathrm{i}\mathbf{k}\mathbf{R}_i)\left(u_{\mathbf{k}s}^a a_{\mathbf{k}s} - u_{-\mathbf{k}s}^{a*}a_{-\mathbf{k}s}^+\right), \end{aligned} \tag{60}$$

are expressed in terms of the phonon absorption and phonon creation operators $a_{\mathbf{k}s}$ and $a_{-\mathbf{k}s}^+$.

We note that as the SO coupling is switched on, the normal-mode expansion retains its form for velocity of the atom and not for the momentum, as the authors of [69] believe.

An explicit dependence on the SO interaction appears only in the renormalization of the polarization vector. Because the phonon spectrum $\omega_{\mathbf{k}s}$ and the group velocity $c_{\mathbf{k}s} = \partial\omega_{\mathbf{k}s}/\partial\mathbf{k}$ are not renormalized, it follows that to describe the PHE it is necessary to modify the standard expression for the phonon energy flow $q = \sum_{\mathbf{k}s}\omega_{\mathbf{k}s}c_{\mathbf{k}s}f_{\mathbf{k}s}$ by including the SO coupling. More specifically, it is necessary, as shown by Hardy [75], to derive the flow relation anew from the quantum expression for the vibration energy density of the crystal. The average phonon energy flow density in the coordinate representation is then given by

$$q_{\mathrm{H}}^\gamma = \frac{1}{2V}\sum_{i\neq j}x_{ij}^\gamma D_{ij}^{ab}\left(u_i^a v_j^b\right), \tag{61}$$

where $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$, and $\mathbf{x}_i$ and $\mathbf{x}_j$ are the coordinates of the ith and jth sites.

We note that the quantity v_i^b in Eqn (61) is not the ion momentum divided by its mass, but the velocity of this ion, $v_i^b = \partial h_i/\partial p_i^b = p_i^b/m + (\mathbf{u}_i\times\mathbf{S})^b$. In the presence of the SO coupling, these two quantities are not the same.

Passing to the momentum representation in Eqn (61) amounts to the replacement $x_{ij}^\gamma \to \mathrm{i}\partial/\partial k^\gamma$. The energy flow operator takes the form

$$q_{\mathrm{H}}^\gamma = -\frac{1}{4V}\sum_{\mathbf{k}ss'}\sqrt{\frac{\omega_{\mathbf{k}s'}}{\omega_{\mathbf{k}s}}}\,\nabla_{\mathbf{k}}^\gamma D_{\mathbf{k}}^{ab}(e_{\mathbf{k}s}^a a_{\mathbf{k}s} + e_{-\mathbf{k}s}^{a*}a_{-\mathbf{k}s}^+) \times (e_{-\mathbf{k}s'}^b a_{-\mathbf{k}s'} - e_{\mathbf{k}s'}^{b*}a_{\mathbf{k}s'}^+)\,.$$

After averaging this operator over the states diagonal in the phonon number, the anomalous averages $\langle a_{\mathbf{k}s}a_{-\mathbf{k}s'}\rangle$ and $\langle a_{-\mathbf{k}s'}^+a_{\mathbf{k}s}^+\rangle$ can be dropped. Changing the summation notation, we arrive at the result

$$\langle q_{\mathrm{H}}^\gamma\rangle = \frac{1}{4V}\sum_{\mathbf{k}ss'}\left[\left(\sqrt{\frac{\omega_{\mathbf{k}s}}{\omega_{\mathbf{k}s'}}} + \sqrt{\frac{\omega_{\mathbf{k}s'}}{\omega_{\mathbf{k}s}}}\right)(\nabla_{\mathbf{k}}^\gamma D_{\mathbf{k}}^{\alpha\beta})e_{\mathbf{k}s}^{\alpha*}e_{\mathbf{k}s'}^\beta\right]\langle a_{\mathbf{k}s}^+a_{\mathbf{k}s'}\rangle\,. \tag{62}$$

In the zeroth order in the SO coupling, Eqn (62) takes the usual form for the phonon energy flow.

Because $\delta e_{\mathbf{k}s}^{\alpha*} = -\delta e_{\mathbf{k}s}^\alpha$, $e_{\mathbf{k}s}^{\alpha*} = e_{\mathbf{k}s}^\alpha$, the terms linear in δe in Eqn (62) are given by

$$\langle q_{\mathrm{SO}}^\gamma\rangle = \frac{1}{4V}\sum_{\mathbf{k}ss'}\nabla_{\mathbf{k}}^\gamma D_{\mathbf{k}}^{\alpha\beta}\left(\sqrt{\frac{\omega_{\mathbf{k}s}}{\omega_{\mathbf{k}s'}}} + \sqrt{\frac{\omega_{\mathbf{k}s'}}{\omega_{\mathbf{k}s}}}\right) \times\left[-(\delta e_{\mathbf{k}s}^\alpha)e_{\mathbf{k}s'}^\beta + e_{\mathbf{k}s}^\alpha(\delta e_{\mathbf{k}s'}^\beta)\right]\langle a_{\mathbf{k}s}^+a_{\mathbf{k}s'}\rangle\,.$$

In this integral, all factors except δe are of the zeroth order in the magnetization $\mathbf{S}$.

The last factor is an element of the one-particle density matrix $\langle a_{\mathbf{k}s}^+a_{\mathbf{k}s'}\rangle \sim \nabla T$ nondiagonal in modes, which is calculated by writing an equation similar to the evolution equation for the Green's function, but without the inhomogeneous term. After proper manipulations, we necessarily arrive at the result that we obtained in Sections 3–5, which states that the SO part of the heat flow density is given by

$$\langle q_{\mathrm{SO}}^\gamma\rangle = \xi[\mathbf{S}\times\nabla T]^\gamma\,.$$

7. Conclusion

The search is still on for effects similar to the classical time-honored Hall effect discovered 136 years ago.

The theory of the magnon (and, in principle, of the spinon) Hall effect was considered in [76, 77]. The experimental realization of the magnon Hall effect (or more precisely, the separation of the magnon and phonon contributions) in $Lu_2V_2O_7$ was reported in [78]. Instead of (or in addition to) the Lorentz force, this situation involves spin chirality defined for three sites i, j, k as $\mathbf{S}_i(\mathbf{S}_j\times\mathbf{S}_k)$. Due to the phase shift acquired along the contour i, j, k, the nonzero chirality is equivalent to a magnetic field, thus leading to the Hall deviation. Nonzero chirality may be due to frustration (either of geometric origin, as in triangular, Kagome, and similar lattices, or caused by competition between the nearest-neighbor and next-to-nearest-neighbor interactions), or, alternatively, it may be due to the Dzyaloshinskii–Moriya interaction. A similar line of research is pursued in Refs [79–

86] which, however, avoid using the term 'magnon Hall effect' and speak instead of a chirality-induced or topological Hall effect. We do not discuss this subject in greater detail.

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