

LETTERS TO THE EDITORS

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Relation between energy conservation and equations of motion
(a reply to the comment by E A Arinshtein
[*Phys. Usp.* **58** 309 (2015); *Usp. Fiz. Nauk* **185** 333 (2015)]
on "Analytical mechanics and field theory:
derivation of equations from energy conservation"
[*Phys. Usp.* **57** 593 (2014); *Usp. Fiz. Nauk* **184** 641 (2014)])

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Abstract. **Equations of motion that preserve a given function (called energy) of generalized coordinates and velocities are derived. These equations differ from Lagrange equations by the presence of additional terms describing generalized gyroscopic forces. The relation between energy conservation and the d'Alembert principle is noted.**

Keywords: energy conservation, Lagrange equations

The letter by Arinshtein [1] formulates interesting arguments on the link between the energy conservation law and equations of motion.

The first formula in Ref. [1] (with some amendment),

$$0 = \mathrm{d}E(\mathbf{q}, \mathbf{v}) = Q_i \, \mathrm{d}q^i \,, \tag{1}$$

represents the d'Alembert principle (see, e.g., Ref. [2]), where the full force Q_i also includes inertia forces. The energy conservation law

$$0 = \mathrm{d}E(\mathbf{q}, \mathbf{v}) = \frac{\partial E}{\partial q^i} \, \mathrm{d}q^i + \frac{\partial E}{\partial v^i} \, \mathrm{d}v^i \tag{2}$$

can be reduced to form (1) if the energy E can be represented as

$$E(\mathbf{q}, \mathbf{v}) = \frac{\partial L(\mathbf{q}, \mathbf{v})}{\partial v^i} \, v^i - L(\mathbf{q}, \mathbf{v}) \,. \tag{3}$$

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Indeed, substituting (3) in Eqn (2), we obtain

$$0 = \mathrm{d}E = \frac{\partial^2 L}{\partial v^i \partial q^k} \, v^i \, \mathrm{d}q^k + \frac{\partial^2 L}{\partial v^i \partial v^k} \, v^i \, \mathrm{d}v^k - \frac{\partial L}{\partial q^k} \, \mathrm{d}q^k$$

$$= \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial v^k} - \frac{\partial L}{\partial q^k} \right) \mathrm{d}q^k \,. \tag{4}$$

Hence, if, solving partial differential equation (3), we can find the Lagrangian L, then d'Alembert principle (1) follows from the energy conservation law. From the d'Alembert principle in form (4), generalized Lagrange equations follow directly (formula (11a) in [3]),

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L_1}{\partial v^i} - \frac{\partial L_1}{\partial q^i} = G_{ij}(\mathbf{q}, \mathbf{v}, \dot{\mathbf{v}}, t) v^j \,, \tag{5}$$

where $G_{ij} = -G_{ji}$ is an antisymmetric tensor describing generalized gyroscopic forces.

If the form of gyroscopic forces G_{ij} is exact, it can be represented as $\partial a_i(\mathbf{q})/\partial q^j - \partial a_j(\mathbf{q})/\partial q^i$ and generalized Lagrange equations (5) reduce to the standard form (i.e., without the right-hand side) with the Lagrangian function $L + a_j v^j$. In general, generalized gyroscopic forces cannot be described by standard Lagrange equations. As a consequence, the energy conservation law allows 'non-Lagrangian' interactions (which cannot be described by terms in a Lagrangian function).

In Ref. [3], generalized Lagrange equations (5) are given without derivation, with the only goal to show that standard Lagrange equations cannot be the most general form of an equation that preserves energy. The pertinent questions and remarks by Arinshtein are most likely partly related to this 'vagueness'. As an excuse, I can say that it is due to the publishing rules of *Physics–Uspekhi*: this journal does not publish original results. Unfortunately, my paper devoted to generalized Lagrange equations was declined without reviewing by several leading physical journals, including *JETP*.

I fully agree with Arinshtein that concrete problems can be solved using solely the equations of motions, and that the energy conservation law allows the presence of gyroscopic forces, which require separate treatment. On the other hand, in many branches of modern theoretical physics, it is proposed to modify the motion equations. In that case, it is necessary to understand the limitations imposed on possible modifications. One of the most important limitations is the energy conservation law. For example, one may consider 'gyroscopic fields' that are not described by a Lagrangian function, but nevertheless make a contribution to the right-hand side of the generalized Lagrange equations.

References

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