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On the theory of light scattering by a dilute gas Bose–Einstein condensate

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Abstract. The semiclassical theory of light scattering by a dilute gas Bose–Einstein condensate is examined, with special attention paid to the key role of the momentum recoil imparted to atoms by photons, a phenomenon that disturbs the interferential quenching of superradiant scattering.

Keywords: Rayleigh scattering, superradiant scattering, Bose–Einstein condensate, superradiant reflection

1. Introduction

One of the fundamental problems in optics centers around understanding the scattering of light from an atomic medium. Early work on this subject included the classical electrodynamics treatment by J Rayleigh [1] and similar subsequent analyses, with some improvements, by L I Mandelstam [2] and M Smoluchowski [3]. The discovery of Raman scattering stimulated the development of the quantum-mechanical description of the light scattering process [4]. It was only quantum electrodynamics [5] that provided an adequate explanation of the phenomenon.

Interest in the optical properties of the Bose–Einstein condensate (BEC) was intensified by the discovery that the condensate can be prepared by laser evaporative cooling in a dilute atomic Bose gas, an achievement which directly confirmed the prediction by Einstein [6] and has brought its authors, E Cornell, W Ketterle, and C Wieman, the Nobel Prize in Physics 2001 (see Nobel Lectures [7, 8] and reviews [9–14]). In an important development shortly afterwards, the Ketterle team studied light scattering off a BEC [15] and found it to be different from ordinary Rayleigh scattering, possibly due to a weakness of relaxation processes and to the condensate retaining its quantum phase memory. This type of scattering is akin to Dicke's superradiance effect [16] and has therefore come to be known as 'superradiant' scattering.

It should be noted that light scattering from a BEC is one of the few examples of how superradiance can be implemented experimentally. A major factor in superradiant scattering is that the atoms are acted upon by scattered (secondary) radiation, leading to a considerable enhancement of the effect. Because of the inherently cooperative nature of superradiant scattering (the coherence of the atomic states is envisaged), its intensity turns out to be proportional to the square of the total number of particles in the condensate. An essential feature of superradiant scattering from a BEC is the formation of coherent atomic waves, a phenomenon that was tentatively interpreted as the realization of a 'one-atom laser' [17].

There were afterwards other experiments that observed superradiant scattering off a dilute BEC [18–24]. In Ref. [15], the dynamics of coherent atomic waves (atom 'clouds') propagating along an irradiating light wave were studied. Experiment [18] demonstrated that decreasing the mismatch between the pumping and resonance frequencies of the atomic transition also leads to the appearance of atom clouds propagating opposite to the direction of the incident radiation (with an asymmetry in density distribution between the forward and backward propagating clouds). Other experimental developments include the precise measurement of the recoil momentum acquired by a BEC atom upon photon scattering [19], the study of light scattering off a BEC in the two-frequency pumping regime [20], and the investigation of how the sign of the mismatch affects the dynamics of the process [21–24].

There has been quite extensive recent literature [25–54] (including our work [29–37, 40–44, 50–54]) on the theoretical interpretation of the above-mentioned experiments based on

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previous work on the theory of superradiance in normal media ([55–57]; see also monograph [58]).

In this paper, we summarize previous results, provide an improved version of the semiclassical theory of light scattering from a BEC, and draw attention to the fact that BEC atoms experience a momentum recoil upon photon scattering [54] — the key effect that determining the superradiant nature of scattering.

The paper outline is as follows. We begin in Section 2 by reminding the reader of the general quantum-electrodynamic formulation of the problem of Raman scattering from a BEC.

The semiclassical approach to the theory of superradiant scattering off a BEC is presented in Section 3, whose main point is the fundamental role of momentum recoil processes that determine the scattering intensity and the lower limit of the superradiant pulse duration.

Section 4 discusses a one-dimensional model of the superradiant reflection from a BEC. The results of the solution to the Maxwell–Schrödinger equations for this case illustrate the properties of superradiant scattering described above.

We conclude by analyzing and discussing the results obtained in Section 5.

2. Linear Rayleigh scattering

Let us first refresh the standard quantum-electrodynamical approach to explaining Rayleigh light scattering based on the Wigner–Weisskopf theory.

We begin by considering the scattering of light by a single atom. The excitation of the atom is caused by the incident classical field, and the subsequent transition of the atom to the ground state and the appearance of scattered radiation are accounted for by the influence of the quantized electromagnetic field. We will assume the incident field to be weak and will calculate the scattering probability in the second-order perturbation theory (first order in the interaction with the incident field, and first order in the interaction with the vacuum of the quantized electromagnetic field).

We model the atom as a two-level electron–nucleus system whose ground (excited) state has the wave function φ_a (φ_b) and eigenvalue E_a (E_b); the excited energy level is assumed to have a radiative width $\Gamma/2$. We also introduce the wave functions of the translational motion of the atom (de Broglie waves):

$$\psi_{\mathbf{p}}(\mathbf{r}, t) = \frac{1}{\sqrt{V}} \exp\left[\mathrm{i}(\mathbf{pr} - \varepsilon_{\mathbf{p}} t)\right], \tag{1}$$

where $\mathbf{r}$ is the atomic center-of-mass radius vector, $\mathbf{p}$ is the atomic wave vector, $\varepsilon_{\mathbf{p}} = \hbar p^2/(2M)$ is the kinetic energy of the atom (in frequency units, M is the atomic mass), and V is the volume of the system. Thus, the basis states of the single atom are as follows:

$$\psi_{a,\mathbf{p}} = \varphi_a \psi_{\mathbf{p}}, \qquad \psi_{b,\mathbf{p}'} = \varphi_b \psi_{\mathbf{p}'}. \tag{2}$$

Taking the atom to be initially in the ground state $\psi_{a,0}$, the second-order time-dependent perturbation theory yields the following result for the probability per unit time (or more precisely, for the spectral probability density) that the scattering of the incident field of frequency ω_0 and wave vector $\mathbf{k}_0$ will cause the emission of a photon of frequency ω and wave vector $\mathbf{k}$ [5]:

$$W(\mathbf{k}_0, \mathbf{k}) = \frac{|v|^2 |u|^2 \hbar^{-4}}{(\omega_0 - \omega_{ba})^2 + \Gamma^2/4} \pi\delta(\omega_0 - \omega - \varepsilon_{\mathbf{k}_0 - \mathbf{k}}), \tag{3}$$

where $\omega_{ba} = (E_b - E_a)/\hbar$ is the resonant frequency of the atomic transition, and v and u are the interaction matrix elements between the atom and the incident and quantized electromagnetic fields:

$$v = \left\langle b, \mathbf{k}_0 | \hat{\mathbf{d}}\, \mathbf{E}_0 | a, 0 \right\rangle, \qquad u = \left\langle a, \mathbf{k}_0 - \mathbf{k}, \{\mathbf{k}\} | \hat{\mathbf{d}}\, \hat{\mathbf{E}}_{\mathbf{k}} | b, \mathbf{k}_0, \{0\} \right\rangle. \tag{4}$$

Here, $\hat{\mathbf{d}}$ is the dipole moment operator of the atom, $\mathbf{E}_0$ is the electric vector of the incident plane wave of wave vector $\mathbf{k}_0$, and $\hat{\mathbf{E}}_{\mathbf{k}}$ is the electric field strength operator corresponding to a radiation oscillator of wave vector $\mathbf{k}$. The curly brackets denote the states of the quantized electromagnetic field: $\{0\}$ is the vacuum state, $\{\mathbf{k}\}$ is a state with a single photon of wave vector $\mathbf{k}$. Further, we have

$$\left\langle a, \mathbf{k}_0 - \mathbf{k}, \{\mathbf{k}\} | \hat{\mathbf{d}}\, \hat{\mathbf{E}}_{\mathbf{k}} | b, \mathbf{k}_0, \{0\} \right\rangle = \mathrm{i} d_{ba} \sqrt{\frac{2\pi\hbar c k}{L^3}} \sin\vartheta, \tag{5}$$

where d_{ba} is the magnitude of the matrix element of the transition dipole moment $\mathbf{d}_{ba}$, ϑ is the angle between this vector and the wave vector $\mathbf{k}$, L^3 is the quantization volume, and the δ function on the right-hand side of Eqn (3) expresses the conservation of energy: the frequency of the photon scattered in the direction of the vector $\mathbf{k}$ is defined as $\omega = \omega_0 - \varepsilon_{\mathbf{k}-\mathbf{k}_0}$.

The scattering intensity $\mathrm{d}I$ into a solid angle $\mathrm{d}\Omega$ in a given direction is obtained by additionally multiplying probability (3) by the magnitude of the quantum $\hbar\omega$ and by the 'number of final states' of the field, namely

$$\frac{k^2\, \mathrm{d}k\, \mathrm{d}\Omega}{(2\pi/L)^3} = \frac{L^3 \omega^2\, \mathrm{d}\omega\, \mathrm{d}\Omega}{(2\pi c)^3}, \tag{6}$$

and then integrating over frequency to give

$$\mathrm{d}I = \frac{(1/4\pi)|\mathbf{d}_{ba}\mathbf{E}_0|^2 |d_{ba}|^2 \hbar^{-2} \omega_0^4 c^{-3} \sin^2\vartheta\, \mathrm{d}\Omega}{(\omega_0 - \omega_{ba})^2 + \Gamma^2/4}. \tag{7}$$

Using an expression for the radiation constant, $\Gamma = (4\omega_{ba}^3 |d_{ba}|^2)/3\hbar c^3$, Eqn (7) can be rewritten as

$$\mathrm{d}I = \frac{3}{8} |\mathbf{d}_{ba}\mathbf{E}_0|^2 \hbar^{-1} \omega_0 \sin^2\vartheta \frac{\Gamma/2\pi}{(\omega_0 - \omega_{ba})^2 + \Gamma^2/4}\, \mathrm{d}\Omega. \tag{8}$$

The extension of this relation to the case of a BEC of an ideal gas of N bosonic atoms is obtained by replacing one-particle matrix elements by multiparticle ones in expression (3). The term 'bosonic atom' refers to an atom with an integer total (electron–nucleus) spin. Because of the indistinguishable nature of particles (atoms) and due to the Bose–Einstein statistics, this reduces to multiplying the matrix element v by $\sqrt{N}$. The matrix element u remains unchanged in value. As a result, the scattering probability from a BEC differs by a factor of N from the single-particle case. The scattering intensity so obtained is in agreement with classical theory: *the light scattering intensity from a system of atoms is equal to*

the sum of scattering intensities from all the atoms. In the classical case, this is explained by the presence of density fluctuations. We see that we obtain the same results when treating an ideal gas BEC in a consistent quantum-mechanical framework without explicitly invoking the concepts of high-precision atomic localization and of density fluctuations.

3. Semiclassical theory of superradiant scattering

The only processes allowed in our quantum-mechanical treatment of Rayleigh scattering are the single excitation of the atom by an incident field and its subsequent transition to the ground electronic state. In this picture, the atom acquires a translational momentum, i.e., its state of translational motion becomes different from the original one. When in this new state, the atom can again be excited by the external field, a fact that has been neglected above in the discussion in Section 2 (as was the influence exerted on the atom by the scattered light). These effects can, in principle, be considered in the quantum-electrodynamic approach — but at the cost of considerable computational effort. A simpler alternative is the semiclassical approach which has been widely used in the theory of superradiance. In our case, the semiclassical approach (which could as well be called semi-quantum) simply means that the radiation field and the scattering field are treated as classical, whereas the evolution of the atomic state is described quantum mechanically. The reason for the appearance of a scattered wave is then a quantum fluctuation of the atomic polarization [58, 59].

For the case of an ideal gas BEC, the discussion below will consider the influences of the field on different atoms as independent of each other and will construct the BEC wave function as the product of the identical wave functions of the individual atoms. Because the scattering field produced also exerts an influence on the atoms, it can be argued that the atoms interact with each other through the mediation of the radiation field. However, if the number of atoms is sufficiently large, the scattering field can, as far as its influence on an individual atom is concerned, be considered as an 'external' field, allowing the use of the model of a *self-consistent radiation* field. Importantly, we will assume that all the atoms reside in the same state and will again retain the wave function as the product of the individual atomic wave functions.

Suppose a dilute gas BEC is exposed to the external field

$$E_0 \exp\left(-\mathrm{i}\omega_0 t + \mathrm{i}\mathbf{k}_0\mathbf{r}\right) + \text{c.c.} \tag{9}$$

with the wave vector $\mathbf{k}_0$, $k_0 = \omega_0/c$, detuned from the resonance by $\Delta = \omega_0 - \omega_{ba}$.

Similar to the discussion above, each atom will be considered as a two-level electronic system possessing translational degrees of freedom.

Consider a scattered wave propagating along the wave vector $\mathbf{k}$, $k = k_0$:

$$E_{\mathbf{k}}(x,t) \exp\left(-\mathrm{i}\omega_0 t + \mathrm{i}\mathbf{k}\mathbf{r}\right) + \text{c.c.} \tag{10}$$

For simplicity, both waves are assumed to be polarized perpendicular to the plane of the vectors $\mathbf{k}_0$, $\mathbf{k}$.

It is further assumed that the atomic system (a dilute gas BEC in our case) is much larger in size than the radiation wavelength and that the scattered amplitude varies slowly in time and with the spatial coordinate x along the scattering direction (the approximation of a slowly varying amplitude).

If the atom resides in its ground state, the incident field will promote it to the excited electronic state with momentum $\hbar\mathbf{k}_0$ and then return it. The scattering field transfers the atom from this excited state to the ground state of momentum $\hbar\mathbf{k}_0 - \hbar\mathbf{k}$. Then, the incident field can cause the atom to make a transition from this state to an excited electronic state with momentum $2\hbar\mathbf{k}_0 - \hbar\mathbf{k}$, etc. Our discussion will also include the excitation of atoms by the scattered field and the emission stimulated by the incident field. Thus, the basis set of atomic states for the problem of interest has the form

$$\phi_{\sigma,n,n'} = \frac{1}{\sqrt{V}} \exp\left[\mathrm{i}(\mathbf{k}_0 n - \mathbf{k}n')\,\mathbf{r}\right] \varphi_\sigma\,, \tag{11}$$

where the indices n and n' can be either positive or negative, $n' = n,\ n-1$ (notice that for $n' = n$, $\sigma = a$, and for $n' = n-1$, $\sigma = b$).

The solution for the atomic wave function is sought in the form

$$\psi = \sum_{n=0,\,\pm2,\,\ldots} a_n \phi_{a,n,n} + \exp\left(-\mathrm{i}\omega_0 t\right) b_{n+1}\phi_{b,n+1,n}\,, \tag{12}$$

where $a_n(x,t)$ and $b_{n+1}(x,t)$ $(n = 0, \pm2, \pm4, \ldots)$ vary with time and with the spatial coordinate along the scattering direction. Notice that the exponential separated out in Eqn (12) contains the incident, not the resonant, frequency of the electronic transition.

The scattered field is created by a system ability to be polarized, which we will calculate as the expectation value of the polarization operator

$$\hat{P}(\mathbf{r}) = \sum_{i=1}^{N} \hat{d}_i\, \delta(\mathbf{r} - \mathbf{r}_i)\,. \tag{13}$$

Here, $\hat{d}_i$ is the dipole moment operator of the i-th atom. The amplitude of the polarization wave with wave vector $\mathbf{k}$ can then be written out in the form

$$P_{\mathbf{k}} = N_0 d \sum_{n=0,\,\pm2,\,\ldots} \bar{a}_n b_{n-1}\,, \tag{14}$$

where d (assumed, for simplicity, to be real) is the matrix element of the dipole moment transition between the ground and excited electronic states, and N_0 is the concentration of atoms.

From Maxwell's equations, or, more precisely, from their consequence, the inhomogeneous wave equation

$$\nabla^2 E - \frac{1}{c^2}\,\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\,\frac{\partial^2 P}{\partial t^2}\,, \tag{15}$$

it follows in the slowly-varying-amplitude approximation that

$$E_{\mathbf{k}}(x,t) = \mathrm{i}2\pi k_0 \int_0^x P_{\mathbf{k}}(x',t')\,\mathrm{d}x'\,, \tag{16}$$

where $t' = t - (x - x')/c$ is the retarded time.

In the representation used, the Schrödinger equation for a single atom can be written out as

$$\begin{aligned} \mathrm{i}\,\frac{\partial a_n}{\partial t} &= -\bar{F}_0 b_{n+1} - \bar{F} b_{n-1} + \varepsilon_{n,n} a_n\,, \\ \mathrm{i}\,\frac{\partial b_{n+1}}{\partial t} &= -F_0 a_n - F a_{n+2} + \left(\varepsilon_{n+1,n} - \Delta - \mathrm{i}\,\frac{\Gamma}{2}\right) b_{n+1}\,, \end{aligned} \tag{17}$$

where $F_0 = \mathrm{d}E_0/\hbar$, $F = \mathrm{d}E_{\mathbf{k}}/\hbar$, Γ is the radiative rate constant of the excited state, and $\varepsilon_{n,n'}$ are the frequency shifts associated with the atom translational kinetic energy:

$$\varepsilon_{n,n'} = \frac{\hbar(n\mathbf{k}_0 - n'\mathbf{k})^2}{2M}\,. \tag{18}$$

The term with the radiative rate constant in the second of equations (17) essentially describes spontaneous radiation in arbitrary directions.

If it is assumed that $|\Delta|$ is sufficiently large to ensure that $|\Delta\, b_{n+1}| \gg |\partial b_{n+1}/\partial t|$, then, neglecting fast oscillations at frequency $|\Delta|$, the amplitudes b_{n+1} corresponding to the atomic excited electronic states can be expressed in the adiabatic approximation in terms of the amplitudes of the ground electronic states. Then, the system of equations (17) becomes

$$\begin{aligned} \mathrm{i}\,\frac{\partial a_n}{\partial t} &= -\bar{F}_0 b_{n+1} - \bar{F} b_{n-1} + \varepsilon_{n,n} a_n\,, \\ b_{n+1} &= \frac{1}{\varepsilon_{n+1,n} - \Delta - \mathrm{i}\Gamma/2}\,(F_0 a_n + F a_{n+2})\,. \end{aligned} \tag{19}$$

Assuming further that $|\Delta| \gg \varepsilon_{n+1,n}, \Gamma$, we obtain

$$\mathrm{i}\,\frac{\partial a_n}{\partial t} = \frac{1}{\Delta}\left\{(\bar{F}_0 F_0 + \bar{F}F)\, a_n + \bar{F}_0 F a_{n+2} + \bar{F} F_0 a_{n-2}\right\} + \varepsilon_{n,n} a_n\,. \tag{20}$$

Now, the matrix of the 'reduced' Hamiltonian in Schrödinger equation (20) can be written down as

$$\begin{aligned} H_{n,m} = \frac{1}{\Delta}\left\{(\bar{F}_0 F_0 + \bar{F}F)\,\delta_{m,n} + \bar{F}_0 F \delta_{m,n+2} + \bar{F} F_0 \delta_{m,n-2}\right\} \\ + \varepsilon_{n,n}\delta_{n,m}\,. \end{aligned} \tag{21}$$

It should be noted that the hermiticity of this matrix ensures that the population normalization condition holds for the states corresponding to the ground electronic state with different translational momenta.

In the approximation used here (with the excited electronic states being adiabatically eliminated), polarization amplitude (14) is written out as

$$P_{\mathbf{k}} = \frac{-N_0 d}{\Delta} \sum_{n=0\,\pm 2,\,\ldots} (F_0 \bar{a}_n a_{n-2} + F \bar{a}_n a_n)\,, \tag{22}$$

so that the polarization operator matrix takes the form

$$P_{n,m} = \frac{-N_0 d}{\Delta}\,(F_0 \delta_{m,n-2} + F \delta_{n,m})\,. \tag{23}$$

For the initial condition $a_0(x,0) = 1$, using Eqn (19) for $|\Delta| \gg \varepsilon_{n+1,n}, \Gamma$, we find that $b_1 = -F_0/\Delta$, i.e., we have the effective population of the excited state '1,0'. The scattering is initiated by the transition induced by the quantized electromagnetic field from this state to the state '1,1'. When treated semiclassically, this transition is described as a fluctuation of the dipole moment of an atom resided in an excited state (see Refs [58, 59]). Assuming the individual atomic dipole moments to be statistically independent, the polarization dispersion is the sum of the dispersions of all atomic dipole moments. Then, the dimensionless amplitude of the fluctuation polarization per one atom is estimated as

$$\bar{a}_2 b_1 \sim \frac{1}{\sqrt{N}}\left(-\frac{F_0}{\Delta}\right), \tag{24}$$

implying a nonzero initial state imposed in the form

$$a_2(x,0) \sim \frac{1}{\sqrt{N}}\,, \tag{25}$$

with N being the total number of atoms in the system.

We now proceed to show that, neglecting the recoil energy and radiation relaxation in the Schrödinger equation, the polarization amplitude of the scattered wave does not vary with time and retains its initial value (24), i.e., one has

$$\frac{\partial}{\partial t} \sum_{n=0,\,\pm 2,\,\ldots} \bar{a}_n b_{n-1} = 0\,. \tag{26}$$

This follows from the fact that, setting $\varepsilon_{n,n'} = 0$ in this case, the matrix of Hamiltonian (21) commutes with the matrix of the polarization amplitude operator (23). This, of course, can also be seen by directly calculating the derivative written above.

Thus, neglecting recoil energies results in destructive interference developing between the polarization amplitudes, and superradiance-enhanced scattering does not materialize.

Including the recoil energy in the Schrödinger equation will lead to a phase shift in the polarization amplitudes and to the appearance of superradiant scattering with an intensity proportional to the square of the number of BEC atoms. Clearly, the time scale to observe a noticeable phase shift due to recoil energies is on the order of $\varepsilon_{n,n'}^{-1}$, setting a lower threshold for the incident pulse duration at which the superradiant scattering can be observed. Notice that this estimate is valid only for a sufficiently large detuning, when eliminating excited states [see Eqn (19)] is a justifiable approximation.

Let us now estimate the modulation frequency of the individual terms in the polarization expression (14), i.e., the $\bar{a}_n b_{n-1}$ amplitudes. In the approximation of eliminated excited state we have

$$\bar{a}_n b_{n-1} = -\frac{1}{\Delta}\,(F_0 \bar{a}_n a_{n-2} + F \bar{a}_n a_n)\,. \tag{27}$$

It should be noted that the matrix of the corresponding operator has only two nonzero elements. Further, as before, we will neglect recoil energies and take F_0 and F to be time-independent. Then, calculating the second derivative of this amplitude using the double commutator with the Hamiltonian yields (see Appendix)

$$\frac{\partial^2}{\partial t^2}\,\bar{a}_n b_{n-1} \approx -\left(\frac{2|F_0 F|}{\Delta}\right)^2 \bar{a}_n b_{n-1}\,. \tag{28}$$

The approximate nature of this equation is due to neglecting the contribution from the adjacent partial polarization amplitudes on the right-hand side.

From equation (28), the partial polarization amplitudes are estimated to have the oscillation frequency

$$\Omega_{\mathrm{mod}} = 2\left|\frac{F_0 F}{\Delta}\right|. \tag{29}$$

Oscillations at frequency Ω_{mod} can be interpreted as Rabi oscillations of the scattering field for the atom 'dressed' by the incident field. We have shown above that, neglecting the recoil energies, these oscillations of different partial polarization amplitudes mutually cancel out. But they should appear if the recoil energies are considered.

4. Superradiant reflection (one-dimensional model)

Let us now consider the problem of superradiant light scattering off a BEC in one dimension. Let the incident wave propagate along an oblong sample of length L. We will consider only forward and backward scattering; sideward light scattering will be ignored. All field and atomic characteristics will formally be taken as uniform over any cross section perpendicular to the incident wave direction. Backward scattering (or backscattering) can be interpreted as 'bulk superradiant reflection'.

Now, the approach outlined above can be applied by considering the forward scattering and backscattering with respective amplitudes F^+ and F^-. In accordance with equation (16), we have (neglecting retardation)

$$\begin{aligned} F^+(x,t) &= \frac{d}{\hbar}\left(E_0 + \mathrm{i}2\pi k_0 N_0 d \sum_{n=0,\,\pm 2,\,\ldots} \int_0^x \mathrm{d}x'\, \bar{a}_n(x',t) \right. \\ &\quad \left. \times\, b_{n+1}(x',t)\right), \\ F^-(x,t) &= \frac{d}{\hbar}\left(\mathrm{i}2\pi k_0 N_0 d \sum_{n=0,\,\pm 2,\,\ldots} \int_x^L \mathrm{d}x'\, \bar{a}_n(x',t) \right. \\ &\quad \left. \times\, b_{n-1}(x',t)\right). \end{aligned} \tag{30}$$

Choosing the 'superradiance time' τ_{R} as the time unit, i.e.

$$\tau_{\mathrm{R}}^{-1} = 2\pi \frac{d^2}{\hbar} k_0 N_0 L = \frac{3}{8\pi} \Gamma N_0 \lambda^2 L, \tag{31}$$

and the sample length L as the length unit, field (30) amplitude becomes

$$\begin{aligned} F^+(x,t) &= F^0 + \mathrm{i}\int_0^x \sum_{n=0,\,\pm 2,\,\ldots} \bar{a}_n(x',t)\, b_{n+1}(x',t)\,\mathrm{d}x', \\ F^-(x,t) &= \mathrm{i}\int_x^1 \sum_{n=0,\,\pm 2,\,\ldots} \bar{a}_n(x',t)\, b_{n-1}(x',t)\,\mathrm{d}x', \end{aligned} \tag{32}$$

where F_0 is the field amplitude of the incident wave E_0, also in units of $\tau_{\mathrm{R}}^{-1}\hbar/d$. Schrödinger equation (17) then takes the form

$$\mathrm{i}\frac{\partial a_n}{\partial t} = -\bar{F}^+ b_{n+1} - \bar{F}^- b_{n-1} + \varepsilon_n a_n, \tag{33a}$$

$$\mathrm{i}\frac{\partial b_{n+1}}{\partial t} = -F^+ a_n - F^- a_{n+2} + \left(\varepsilon_{n+1} - \Delta - \mathrm{i}\frac{\Gamma}{2}\right) b_{n+1}. \tag{33b}$$

Here, ε_n is the kinetic energy of an atom with momentum $n\hbar k_0$ (as expressed in terms of our frequency unit τ_{R}^{-1}: $\varepsilon_n = n^2\hbar k_0^2 \tau_{\mathrm{R}}/(2M)$); the same unit is invoked for writing down the detuning Δ and radiative rate constant Γ.

The excited electronic states are eliminated through the substitutions

$$\begin{aligned} b_{n+1} &= \frac{1}{\varepsilon_{n+1} - \Delta - \mathrm{i}\Gamma/2}\,(F^+ a_n + F^- a_{n+2}), \\ b_{n-1} &= \frac{1}{\varepsilon_{n-1} - \Delta - \mathrm{i}\Gamma/2}\,(F^+ a_{n-2} + F^- a_n). \end{aligned} \tag{34}$$

When numerically solving the system of Schrödinger–Maxwell equations, the key parameters involved were taken to be of the same order as in experiment [18], in which the BEC sample studied consisted of about two million rubidium atoms and measured about 200 μm in length and 15 μm across. Other parameters used were: the intensity of the pumping laser beam, about 63 mW cm^{-2}; the dipole moment of the optical transition, $d \sim 2 \times 10^{-29}$ K m; the corresponding wavelength, $\lambda \sim 780$ nm; the frequency shift $\varepsilon_1 \sim 4.7 \times 10^4$ s^{-1}; the duration of the longer pulse, $T_{\mathrm{p}} \sim 800$ μs, and the detuning, $\Delta = -4400$ MHz. For such experimental conditions, our time scale is $\tau_{\mathrm{R}} \sim 10^{-10}$ s and, correspondingly, the key parameters involved were taken to be (in our units): $F^0 = 6 \times 10^{-3}$, $T_{\mathrm{p}} = 1.8 \times 10^7$, $\Delta = -1.5$, $\varepsilon_1 = 1.25 \times 10^{-6}$, and $\Gamma = 4 \times 10^{-3}$. We utilized the relations $a_0(0) = [1 - a_2^2(0)]^{1/2}$, $a_2(0) = 10^{-3}$ as nonzero initial conditions to impose on the atomic states to effectively account for the quantum polarization fluctuations for a given number of atoms (10^6) in the condensate.

Let us first estimate the accuracy of eliminating excited electronic states. For this purpose, we obtain the solutions of the system of Maxwell–Schrödinger equations with and without this approximation. As can be seen from Fig. 1, which presents the results for systems (32), (33) and (32), (33a), and (34) for the initial stage of scattering, the inclusion of excited states leads to the appearance of fast field-amplitude oscillations at a frequency of $\sim |\Delta|$. As time goes on, however, these oscillations damp out relatively quickly, implying justification for eliminating excited states. It is this approximation which was applied to calculate the results presented in the remainder of this paper.

Now, let us illustrate how the momentum recoil effects influence the intensity of the superradiant scattering.

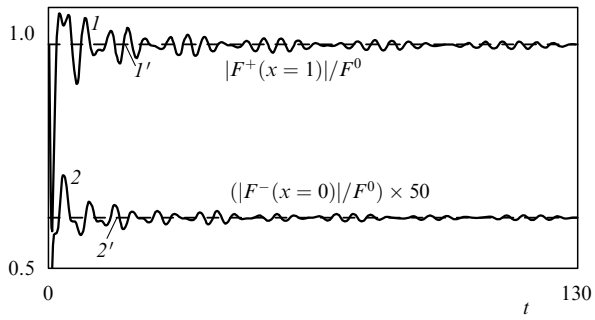

Figure 1. Amplitude $F^{\pm}$ dynamics of the fields as obtained for the initial pulse stage when eliminating (dashed lines *1′*, *2′*) and without eliminating (solid lines *1*, *2*) excited states for the following values of the dimensionless parameters involved: $F^0 = 6 \times 10^{-3}$, $\Delta = -1.5$, $\varepsilon_1 = 1.25 \times 10^{-6}$, $\Gamma = 4 \times 10^{-3}$, and $a_2(x, t=0) = 10^{-3}$.

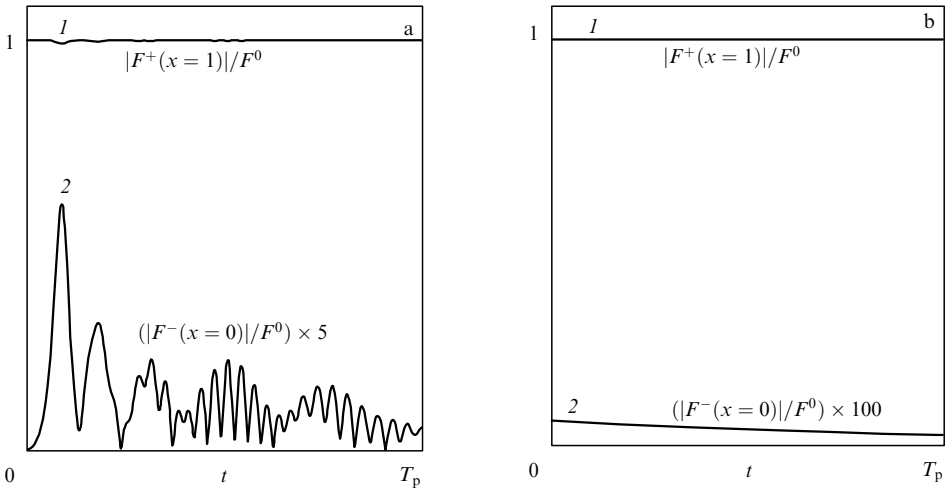

Figure 2. Amplitude $F^{\pm}$ dynamics of the fields as obtained including (a) and excluding (b) frequency shifts ε_n. Pumping pulse duration is $T_p = 1.8 \times 10^7$; the remaining parameters are the same as in the caption to Fig. 1.

Figure 2 demonstrates the calculated evolution of the reflected and transmitted fields. It is seen that the reflected field amplitudes $|F^-(x=0)|$ at the output are relatively large when calculated including frequency shifts (Fig. 2a, line *2*) and that neglecting these shifts results in a considerable suppression of reflection (Fig. 2b, line *2*; note the scale factor of 100). This fact is in complete agreement with the interference quenching effect described in Section 3 and, incidentally, initially discovered by the present authors when numerically solving the system of Maxwell–Schrödinger equations.

Neglecting recoil energies, the time variation of the reflected amplitude is slow compared with the initial value $F^0/(\Delta\sqrt{N})$ equal to the amplitude of ordinary Rayleigh scattering (more precisely, as seen from a comparison with Eqn (8), to the Rayleigh scattering amplitude divided by the Fresnel number $S/(\lambda L)$, where S is the cross section area; for the experimental data considered, the Fresnel number is on the order of unity). The reason for the monotonic decrease in scattering intensity is spontaneous relaxation ($\Gamma \neq 0$).

Thus, we conclude that the superradiant enhancement of reflection occurs because of a phase mismatch in scattering events, which is *due to recoil processes*. Notice also that over times an order of magnitude shorter than the inverse of the recoil frequency shift — a situation in which phase mismatch is insignificant — no superradiant enhancement of scattering (reflection) appears. This imposes a duration restriction on the excitation pulse at which superradiant scattering can appear: *the excitation duration should not be less, in order of magnitude, than the inverse of the recoil frequency shift*.

Figure 3 illustrates the time variation of the populations of atomic clouds that move parallel and antiparallel to the incident field direction, the populations being defined as

$$S_n \equiv \int_0^1 \mathrm{d}x \left|a_n(x)\right|^2 . \tag{35}$$

Given the assumed incident field intensities, the dominant process in the system is the formation of an atomic cloud with $n = 2$, and the phase shift ε_2 actually occurs. Then, the destruction time of interference quenching is determined by ε_2^{-1}. For example, this value is on the order of 10^{-4} s for sodium, in accordance with Eqn (18).

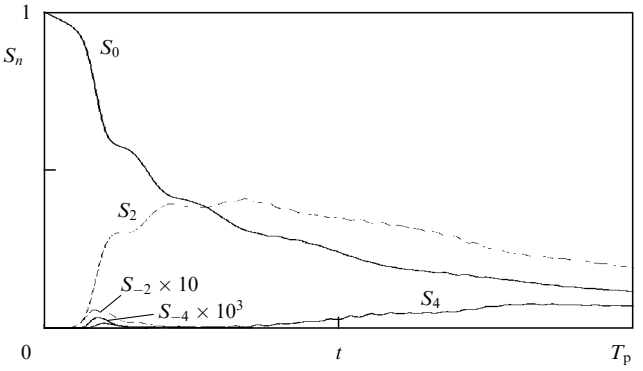

Figure 3. Population S_n dynamics of atomic states with translational momenta $\hbar k_0 n$, $n = 0, \pm 2, \pm 4$. Line S_0 illustrates the depletion of the BEC ground state.

It is seen that the atomic states with negative momenta ($n = -2, -4$) exhibit negligible populations (as have the states — not shown in the figure — with progressively more negative n). Advanced development occurs for the atomic state with $n = 2$, but as the momentum develops, the population of the $n = 4$ state also becomes noticeable.

The oscillatory nature of the reflected field must also manifest itself in its spectrum, which can be defined as

$$\Phi^-(\omega) = \left| \int_0^{T_p} \mathrm{d}t \exp(\mathrm{i}\omega t)\, F^-(x=0,t) \right|^2 . \tag{36}$$

Figure 4 shows the spectrum obtained from the reflected field dynamics. The spectrum is shifted towards the red relative to the incident field spectrum, as it should in accordance with the energy conservation law. The shift by $-4\varepsilon_1$ corresponds to the Raman transition $n = 0 \to n = 2$, in which an atom acquires a kinetic energy $4\varepsilon_1$ as a result of recoil. Corresponding to the Raman transition $n = 2 \to n = 4$ is the spectral shift by $-12\varepsilon_1$, and the splitting of this spectral component into a doublet is possibly due to the modulation of the polarization amplitude at a frequency given by Eqn (29). At the same time, the spectrum in the region near the frequency $\omega = -4\varepsilon_1$ has a complex structure, possibly due to the considerable inter-

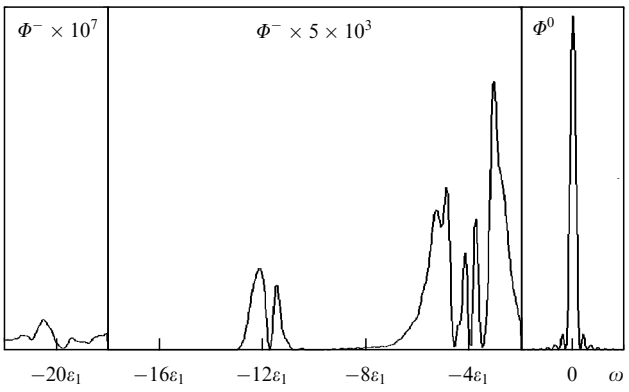

Figure 4. Comparison of the reflected ($\Phi^-(\omega)$) versus incident ($\Phi^0(\omega)$) field spectra.

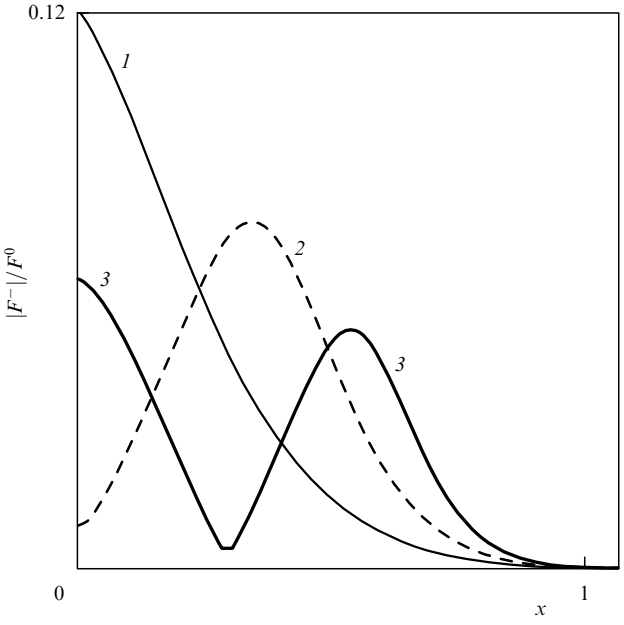

Figure 5. How the amplitudes $|F^-(x)|$ of the reflected field vary with coordinate at the first maximum, the first minimum, and the second maximum $|F^-(x=0)|$ (Fig. 2. line *2*) (lines *1*, *2*, and *3*, respectively).

ference $\bar{a}_n b_{n-1}$ between individual partial polarization amplitudes.

Plotted in Fig. 5 is the spatial distribution of the reflected field amplitude. The distribution shows the nonuniformity which develops in time in a nontrivial way, indicating an anomaly in the behavior of the reflected wave group velocity in the BEC.

The group velocity may be estimated from the displacement velocity of the maximum in the spatial distribution of the field amplitude over the sample, yielding a value on the order of 1 m s^{-1}.

5. Conclusions

Applying Maxwell's electrodynamics to explaining the physical nature of light scattering reveals that in a homogeneous medium of atoms at rest scattering is impossible due to the destructive interference of the secondary radiation (see, for example, Ref. [60]). Taking into account the motion of atoms and the Doppler effect, Rayleigh proposed to explain the phenomenon of light scattering. If, however, the number of atoms is sufficiently large, then interference quenching will still manifest itself and so, again, no scattering will occur. Mandelstam [2] and Smoluchowski [3], who noticed this point, suggested a way out by introducing interference-destroying density fluctuations, an idea which led to the now well-known result that the scattering intensity by a quasihomogeneous medium is equal to the sum of scattering intensities by each atom independently.

Light scattering off a BEC occurs in a spatially homogeneous medium, and in explaining it quantum-mechanically there is no need to additionally introduce density fluctuations, because they are involved naturally in a consistent quantum-mechanical description of the state of the scattering medium. The virtual excitation of atoms proceeds due to the incident field action. The initiating factor for the 'secondary' radiation is the interaction with a vacuum of the electromagnetic field or, in other words, quantum polarization fluctuations. This is sufficient for the occurrence of conventional Rayleigh scattering. However, the evolution of superradiant scattering strongly depends on the multiple excitations of atoms by both the incident and scattered fields. Under the condition of conserved phase memory (relaxation collisions have little influence), the partial polarizations due to transitions of different multiplicities add up. If the total medium polarization is nonzero, the light scattering will be of a superradiant nature, with intensity proportional to the atom number squared.

Early theoretical work [25, 26] on superradiant scattering focused on the quantum-electrodynamical treatment of the phenomenon observed, but soon the semiclassical approach was recognized as a more efficient method [27–30] which permitted a detailed comparison with experiment using the solutions to the system of Maxwell–Schrödinger equations. Along these lines, we were able to avoid some of the approximations invoked in previous studies (for example, the mean field approximation) and to obtain realistic estimates for the intensity of coherent atomic waves due to light scattering [31, 32, 42–44, 51, 52]. In particular, the above-mentioned asymmetry in the density distribution of atomic clouds was given an explanation.

The applicability of the approximation of eliminated excited states is discussed in Refs [61, 62]. The approach taken in the present paper to justify the use of this approximation consists in comparing the results obtained from the Maxwell–Schrödinger equations with or without it; see the solutions of Eqns (32), (33a), (34) and the solutions of Eqns (32), (33), respectively (Fig. 1).

Discussing the improved version of the semiclassical superradiant scattering theory which uses the solution to the system of Maxwell–Schrödinger equations, we were led to the following conclusion. Neglecting the recoil frequency shift in atomic electronic transitions results in the interferential quenching of partial polarizations (see Section 3), leaving us only with a conventional incoherent Rayleigh scattering, whose intensity is proportional to the number of atoms. This result can be considered as an indirect analogy of the interferential quenching of the Rayleigh light scattering in a homogeneous medium. Because the recoil frequency shift can exert its influence only over times not shorter than the reverse of the shift, it follows that superradiant scattering should also manifest itself over times longer than this value. This last fact determines the phase mismatch time of polarization waves that arise in a BEC when atoms undergo multiple excitation

under the action of the incident light field. It follows then that for the incident pulse to undergo superradiant scattering, its duration cannot be less to an order of magnitude than the inverse of the recoil frequency shift.

Two other noteworthy results of this work are the scattering spectrum for the one-dimensional BEC model (see Section 4) and an estimate of the group velocity of a scattered wave, whose value turned out to be about 1.0 m s^{-1}, in agreement with the experimental value [63] found from the study of light flux enhancement in a BEC.

Interest in the theoretical interpretation of superradiant scattering still persists, as exemplified by the discussion of the adiabatic elimination of excited atomic states [21, 61, 62]. One of the questions still to be resolved is how the sign of the detuning affects the scattering intensity [21–24, 64]. This effect is due to the interatomic interaction and requires for its description to go from the Maxwell–Schrödinger equations to the system of Maxwell–Gross–Pitaevskii equations that opens interesting prospects for the study of the problem.

6. Appendix

It is indeed the case that neglecting the recoil energy, Hamiltonian (21) consists (up to factors) of an identity matrix and two other matrices which are obtained from it by moving the diagonal to the right and to the left, and which we will call, for brevity, a left and a right unit matrices. A matrix which is a multiple of the unit matrix may be neglected, when calculating the commutator.

The following four rules are now to be kept in mind: (a) multiplying a given matrix on the left by the left identity matrix shifts each element of the given matrix by one step downward; (b) multiplying on the right by the left identity matrix shifts each element of the given matrix by one step to the left; (c) multiplying on the right by the right identity matrix shifts each element of the given matrix by one step to the right, and (d) finally, multiplying on the left by the right identity matrix shifts each element of the given matrix by one step upwards. Therefore, calculating the second-order commutator produces a quadrupled initial matrix, in addition to matrices whose elements are shifted by two steps relative to the initial matrix. Neglecting the shifted matrices and taking into account the values of the coefficients of the matrices in Eqn (21), we arrive at Eqn (28).

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