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Stochastic instability and turbulent transport. Characteristic scales, increments, and diffusion coefficients

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Contents

Abstract. **Stochastic instability and its associated turbulent diffusion models are reviewed, with particular attention given to the problem of obtaining estimates and scaling laws that characterize correlation effects and increments. Specific models considered include the quasilinear Kazantsev approximation, stochasticity in a system of convective cells, the Kadomtsev–Pogutse scaling, percolation models, and the Rochester–Rosenbluth balance. The primary goal is to highlight the importance of determining the functional dependence of the stochastic instability increments and transport coefficients on turbulent pulsation amplitudes and other key parameters (characteristic pulsation frequencies, drift velocities, spectral energy flow, etc.) describing the systems under discussion.**

Keywords: stochastic instability, turbulent transport, diffusion coefficients, dynamical chaos, stochastic magnetic field, plasma

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1. Introduction

Stochastic instability is a physical mechanism of utmost importance, which is tightly linked to such phenomena as dynamical chaos, decay of correlations, and mixing [1–6]. Boltzmann, seeking a mechanistic substantiation for irreversibility, already used descriptive representations featuring a spreading 'drop' of phase fluid, the increase in whose 'coarse-grained' phase volume should have served as an apparent analogy to the increase in the entropy of the system as it evolves to equilibrium [7]. Regrettably, these ideas of Boltzmann's remained 'in the shadows' for a long time, while the attention of physicists was attracted by computational possibilities opened up in connection with the method of nonequilibrium system analysis proposed by Gibbs [8].

Substantial progress in studying fundamental processes of nonequilibrium system dynamics was achieved only after the appearance of Bogoliubov's work [9]. We note, however, that this contribution was preceded by the work of Leontovich [10], Hopf [11], Khinchin [12, 13], and Krylov [14]. Krylov's work [1, 14] for the first time offered an intuitive interpretation of the phenomenon of stochastic instability. The first paper by Bogoliubov devoted to questions of statistical physics also appeared in co-authorship with Krylov [15]. We note that the effect of mixing

associated with stochastic instability was well known. For example, Borel's lectures given in the early 1920s [16] proposed an illustrative scheme for the evolution of a phase volume element modeling the motion of an ensemble of free particles in a region confined by rigid walls. In accordance with Boltzmann's ideas, this simple example demonstrated strong distortions of the phase volume element. Nevertheless, the link between the phenomenon of mixing and the effects of instability of phase trajectories evaded attention for the following decade and a half.

The work by Krylov and Bogoliubov launched a broad discussion on the fundamental aspects of irreversibility [17–20]. Here, it is important to mention the paper by Davydov [17], whose other work on physical kinetics [21, 22] won recognition already in the late 1930s–early 1940s. These debates were particularly heated due to the need to construct fundamentally novel kinetic models of high-temperature plasmas [23–26]. At that time, in the framework of the kinetic approach, the processes of mixing and related correlation effects were taken into account indirectly, through the use of a diffusive term in velocity space [21] or by a qualitative consideration of the effects of particle confinement by plasma waves [27, 28].

A different situation evolved in the theory of turbulence. Here, the correlation method became one of the main tools and mixing processes were studied in relation to atmospheric and oceanographic research. For the purposes of this review, it is important to mention that, in fact, simultaneously with the publication of Krylov's work, research was launched on stochastic instability, concerned with anomalous diffusion in turbulent flows [31–34]. In the work by Batchelor [31] published in 1952, we find a clear understanding of the importance of exponential stretching of fluid elements in a turbulent flow. It is interesting to note that in the domestic scientific literature, the name Batchelor is commonly associated with his work on the Kolmogorov spectra in turbulence and, certainly, his book [35]. But without a doubt, Batchelor's recognition of the significance of stochastic instability as a key physical mechanism underpinning the evolution of Lagrange particle trajectories in a turbulent flow contributed quite substantially to turbulence theory.

The effects of stochastic instability and mixing were put on a firm mathematical basis in studies by Kolmogorov [36, 37] who, using ideas from probability theory and methods of information theory, introduced the notion of entropy based on the characteristic properties of dynamic system trajectories. The formalism created by Kolmogorov for systems that preserve a measure (Hamiltonian systems) was elaborated further in the work by Sinai [38–40] and later in papers by D V Anosov, V A Rokhlin, L A Bunimovich, R L Dobrushin, and others. The results of these studies played an important role in substantiating the fundamental principles of nonequilibrium statistical mechanics and establishing stochastic dynamics as a bridge between deterministic and statistical theories [41–45].

Sinai's mathematically rigorous result, which is important for the purposes of this review, can be schematically represented as an estimate of the rate of entropy growth based on the exponential instability effect. A characteristic feature of chaotic motion is its high sensitivity to small variations in initial conditions. In chaotic systems, two initially close phase trajectories on average diverge from each other according to an exponential law, whereas they diverge linearly for regular motions. Formally, we can write

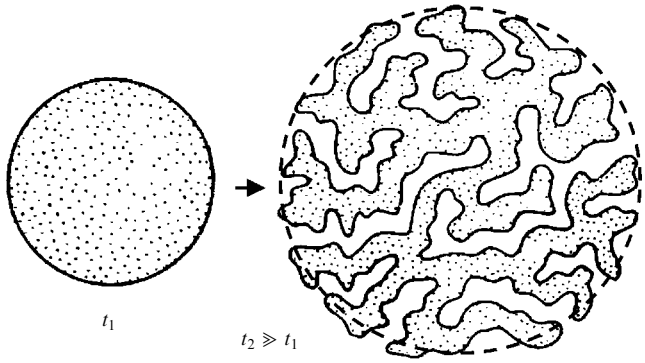

Figure 1. The evolution of a phase droplet. The dashed line shows a coarse-grained phase volume.

the exponential dependence as

$$l(t) = l_0 \exp\left(\frac{t}{\tau_{\mathrm{K}}}\right), \tag{1}$$

where l_0 is the initial distance between points in the phase space and t is time. The quantity $h_{\mathrm{K}} = 1/\tau_{\mathrm{K}}$ was called the Kolmogorov–Sinai entropy [4, 5]:

$$h_{\mathrm{K}} = \lim_{l_0 \to 0,\, t \to \infty} \left(\frac{1}{t} \ln \frac{l(t)}{l_0}\right). \tag{2}$$

It is well known that in the framework of classical mechanics, the phase droplet evolves in accordance with the Liouville theorem such that its volume is preserved: $\Delta\Gamma_0 = \Delta\Gamma(t)$ [2–7]. However, the topological structure of a droplet undergoes substantial changes (Fig. 1). In fact, the 'indentation' of the hypersurface bounding the phase volume constantly increases, leading to an increase in the coarse-grained phase volume $\langle \Delta\Gamma(t) \rangle$ because of stochastic instability. The Liouville theorem is not applicable to the coarse-grained phase volume, and just $\langle \Delta\Gamma(t) \rangle$ is used in the Boltzmann definition of entropy. The model of evolution for the coarse-grained phase volume also has an exponential character,

$$\langle \Delta\Gamma(t) \rangle = \Delta\Gamma_0 \exp\left(\frac{t}{\tau_{\mathrm{K}}}\right), \tag{3}$$

and the Boltzmann expression for the entropy growth allows a linear form $S(t) \propto \ln[\langle \Delta\Gamma(t) \rangle] \propto h_{\mathrm{K}} t$. The divergence of neighboring trajectories (the positivity of the Lyapunov exponents) leads to a substantial distortion of phase space elements, which is the mechanism of mixing. Stochastic instability is a necessary condition for the realization of a scenario describing the evolution of systems with mixing [4–6, 56, 57].

The importance of mixing effects was already recognized during the time of broad discussions on problems of ergodicity [58–60]. The mechanism of mixing, as a key one for the emergence of irreversibility, was already invoked repeatedly by Ehrenfest for a qualitative analysis of the Boltzmann phase drop spreading. As concerns the description of irreversibility effects, the condition of mixing is stronger than the condition of ergodicity, and insures the equality of time and phase means. Extensive literature is available today on general questions of the statistical description of dynamical systems [2, 4–6, 56–58, 61, 62], and

we therefore touch these questions here only briefly. We show in what follows that despite the seemingly special character of the problems related to the description of turbulent diffusion, the results from this area are needed to explain fundamental questions from transport theory [63–65].

Ideas about the influence of stochastic instability on transport processes evolved in the context of different physical problems. In 1959, Chirikov [66] proposed an original method to analyze stochastization in Hamiltonian systems containing separatrices. This approach was applied in a nontrivial way to the description of the effects of mixing in the velocity space in studies of weak plasma turbulence [67, 68]. In this case, a package of waves and their interaction with particles had already been considered. A trigger mechanism for the build-up of a stochastic layer is furnished here by the Landau damping, which supplies the required effect of 'modulation'. We note that it took nearly 20 years to move from problems of quasilinear diffusion [67] to models of transport under conditions of structured turbulence [69]. Indeed, only in 1979 was it proposed to use the splitting of separatrices in a system of convective cells as a mechanism capable of forming mesoscale (percolation) vortex structures in two-dimensional turbulence. Moreover, a rigorous analysis of transport in a system of regular vortex cells was carried out only in 1986 in Ref. [70], while the percolation models mentioned above started to be actively used only in the early 1990s [71–74].

Currently, the exponential divergence of streamlines (phase trajectories) has become one of most important elements of stochastic dynamics [3–5], while accounting for increments of stochastic instability was instrumental in explaining the diversity of transport regimes of charged particles in a stochastic magnetic field [69, 75]. The phenomena of anomalous transport in two-dimensional flows containing separatrices [76–84] are also a subject of active research.

On the other hand, exploration of relative diffusion of tracers under conditions of oceanic and atmospheric quasi-two-dimensional turbulence convincingly prove the correctness of Batchelor's ideas on the importance of the hierarchy of spatial and temporal scales in describing different stages of turbulent transport [34, 85–91]. In the framework of the already classical Bogoliubov hierarchy of time scales, the characteristic relaxation time in the velocity space is much shorter than the system relaxation time in the configurational space. This can easily be explained if we take into account that macroscopic quantities do not change in particle collisions, and collisions only redistribute these quantities between the particles. When considering questions of turbulent diffusion, we also need to distinguish the characteristic time scales pertaining to stochastic instability, processes of reconnection of field or stream lines, and diffusive decorrelation of particles located in the region of turbulence [92–95].

In this review, we pay special attention to the methods that are directly related to computations of turbulent transport coefficients in both hydrodynamical and plasma flows. This allows us to focus on questions of obtaining scalings that are widely used in the analysis of realistic flows and numerical experiments [96–99]. It is particularly important in this respect to determine the character of the dependence of transport coefficients on the amplitude of turbulent pulsations. In problems of fluid dynamics, this is the dependence on velocity pulsations, whereas in problems dealing with the description of transport in stochastic electromagnetic fields, this is the influence of perturbations in magnetic or electric fields. These questions are reflected insufficiently well in the existing review literature dealing with research on dynamical stochasticity.

2. Mixing effects and the characteristic Kubo and Peclet numbers

On the physical level of rigor, stochastic instability was first explored by Krylov in his thesis "Mixing processes in phase space" [14] defended in Leningrad in July 1941. The expression for the instability increment was obtained by Krylov by considering a gas of elastic spheres [1, 14, 56, 100, 101]. We very briefly discuss the estimates that allow exposing those aspects of the 'mixing' process which are common to physical kinetics and turbulence.

We assume that the thermal velocity of a molecule V_T acquires a perturbation V' such that its trajectory follows the dashed line in Fig. 2, deflecting by a small angle $\varphi_0 = V'/V_T$ from the original, unperturbed trajectory. It is easy to see that upon the next collision, this angle increases by the factor l_c/r_* and, after the Nth collision, attains a magnitude of the order of

$$\varphi(N) \propto \varphi_0 \left(\frac{l_c}{r_*}\right)^N , \tag{4}$$

where l_c is the mean free path between the particle collisions, $l_c \propto 1/(nr_*^2)$, n is the concentration of particles, and r_* is the radius of scattering centers. In view of the obvious relation $V_T t \approx N l_c$, we obtain that the scattering angle increases exponentially,

$$\varphi(t) = \varphi_0 \exp\left(N \ln \frac{l_c}{r_*}\right) = \varphi_0 \exp\left(\frac{t}{\tau_s}\right) , \tag{5}$$

where the instability increment γ_s and the corresponding characteristic time τ_s are

$$\gamma_s = \frac{1}{\tau_s} = \frac{V_T}{l_c} \ln \frac{l_c}{r_*} , \tag{6}$$

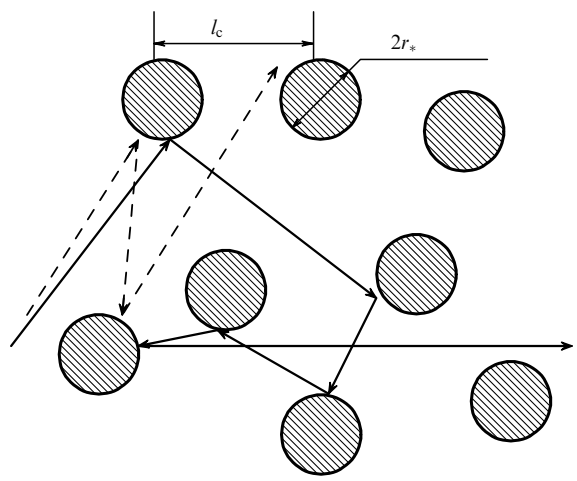

Figure 2. Exponential instability in a system of scattering disks: r_* is the characteristic disk radius and l_c is the mean distance between the scattering centers.

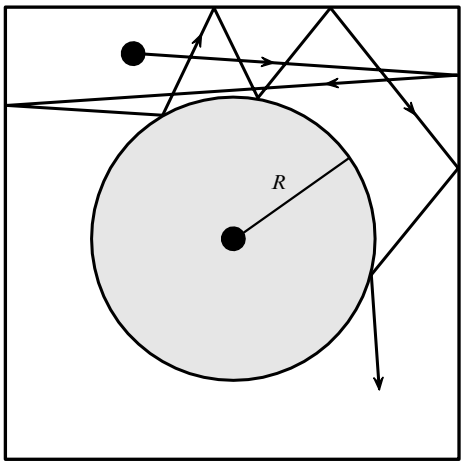

Figure 3. Sinai billiard. The trajectory of a particle reflected by the disk wall. R is the radius of the scattering disk.

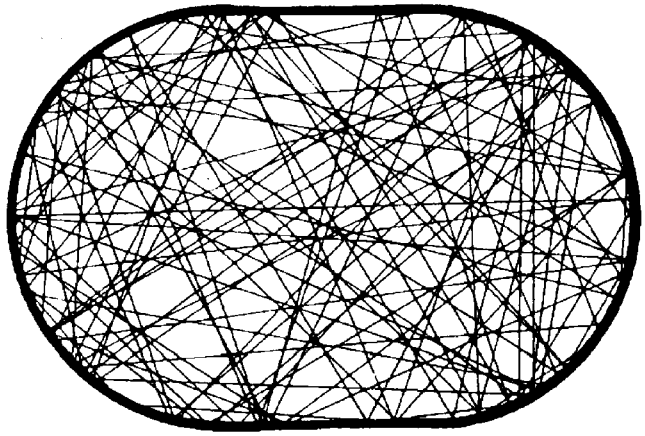

Figure 4. Bunimovich billiard. An ensemble of trajectories followed by a system of particles undergoing scattering.

where γ_s is simultaneously the Kolmogorov–Sinai entropy [4, 5]. Here, $1/\tau_0 = V_T/l_c$ is the characteristic inverse time scale. It follows from these estimates that even for a very small initial perturbation $\varphi_0 \ll 1$, φ rapidly reaches values of the order of unity. In this manner, the motion of a molecule in a gas turns out to be extremely sensitive to initial perturbations.

The model proposed by Krylov was only the first step in the analytic investigation of scattering processes based on ideas of stochastic instability. The next important step was research on billiards performed by Sinai [41, 42]. In particular, the simplest planar Sinai billiard is a square with a circular inner wall (Fig. 3). The proof of the mixing property and the existence of the Kolmogorov–Sinai entropy can be found in numerous mathematical publications [41–55]. For the sake of illustration, we present only schematics of billiard models of other types (Figs 4 and 5). A special role in billiard systems is delegated to determining the mean free path and distribution functions. Because these problems mostly pertain to models of physical kinetics and not to questions related to turbulent transport, we direct the reader to the vast literature dealing with models of physical kinetics [43–55].

Numerous studies have proved the universality of formula (6) for τ_s, with $K_m = l_c/r_*$ being the parameter of the multiplicative map (mapping parameter) considered above [102]. Indeed, in the framework of a simple discrete model, we can consider the evolution of a phase element δx_0 in the form

$$\delta x_N(t) \approx K_m^N \, \delta x_0 \approx \delta x_0 \exp\left(N(t) \ln K_m\right), \tag{7}$$

where N is the number of iterations, δx_0 is the initial size of the phase element, and δx_N is the size of the phase element after N iterations. Accordingly, for the increment of stochastic instability, we obtain

$$\gamma_s = \frac{N(t)}{t} \ln K_m = \frac{1}{\tau_0} \ln K_m . \tag{8}$$

It is straightforward to propose a hydrodynamical interpretation of this map and of the parameter K_m by considering the deformation of a fluid element in a chaotic velocity field with the scale V_0. Let L_* be the characteristic size of the fluid element and ω be the characteristic frequency of velocity variations. These parameters allow considering the

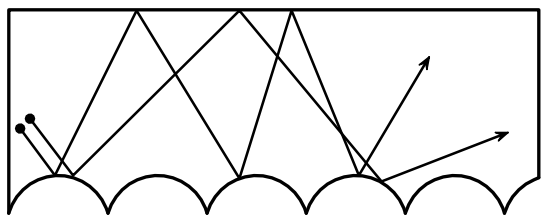

Figure 5. 'Caterpillar' billiard. The exponential divergence of scattered particles.

deformation of the fluid element in the characteristic time τ_0 (Fig. 6),

$$L_1 \approx L_* \frac{\delta L}{\lambda} \approx L_* \frac{V_0}{\omega\lambda} , \tag{9}$$

where λ is the characteristic spatial scale of the velocity field. Then the evolution of $L(t)$ at times larger than τ_s is described by the formula

$$L(t) \approx L_* \left(\frac{V_0}{\omega\lambda}\right)^{t/\tau_0} = L_* \exp\left(\frac{t}{\tau_0} \ln \mathrm{Ku}\right) \approx L_* \exp\left(\frac{t}{\tau_s}\right). \tag{10}$$

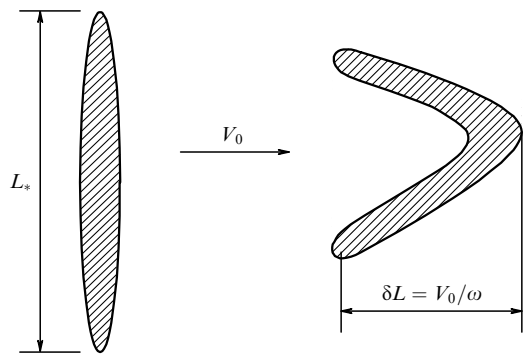

Figure 6. Deformation of a fluid element in a random flow with a characteristic initial velocity scale V_0. L_* is the characteristic initial scale, δL is the scale of transverse deformation, and ω is the characteristic frequency.

In our case, the role of the mapping parameter is taken by the Kubo number Ku characterizing the intensity of turbulence,

$$\mathrm{Ku}\,(V_0) = \frac{V_0}{\omega\lambda}\,. \tag{11}$$

From the standpoint of the description of passive tracer transport by a vortex flow, the parameter $\mathrm{Ku}\,(V_0)$ allows comparing the particle path length with the size of eddies. The characteristic time of stochastic instability in the case of streamline chaos is given by

$$\tau_\mathrm{s} \approx \frac{1}{\gamma_\mathrm{s}} \approx \frac{\tau_0}{\ln \mathrm{Ku}} \approx \frac{\tau_0}{\ln\left[V_0/(\omega\lambda)\right]}\,. \tag{12}$$

The possibility of considering the increment of stochastic instability in terms of the Kubo number turns out to be of immense importance for describing turbulent transport. In fact, the task of a researcher reduces to finding the functional dependence of the characteristic time τ_0 on the model parameters. In this simplest model, we assume that the sizes of all eddy features of the turbulent flow are integer multiples of λ. In such a monoscale approximation, a natural estimate is $\tau_0 \propto \lambda/V_0$.

It is necessary to consider the impact of effects caused by molecular diffusion on the development of stochastic instability. In the simplest formulation, the problem reduces to describing the fluid element evolution under the action of a chaotic velocity field with a characteristic spatial scale L_0 and a velocity scale V_0. In the framework of the monoscale model used here, the velocity gradient and the increment of stochastic instability are given by V_0/L_0. Under the action of chaotic advection, a patch made of scalar particles elongates and deforms as shown in Fig. 7. For a stationary incompressible flow, the elongation of the patch accompanied by the reduction in the width of its 'tentacles' proceeds exponentially. The evolution of the small characteristic scale of the 'tentacles' can be estimated with the help of the formula

$$\Delta(t) = L_0 \exp\left(-\gamma_\mathrm{s} t\right). \tag{13}$$

In the presence of molecular (seed) diffusion, the scalar density gradients smooth out with time. The characteristic diffusive time τ_d is determined by the expression

$$\gamma_\mathrm{d} = \tau_\mathrm{d}^{-1} \approx \frac{D_0}{\Delta_\mathrm{mix}^2}\,. \tag{14}$$

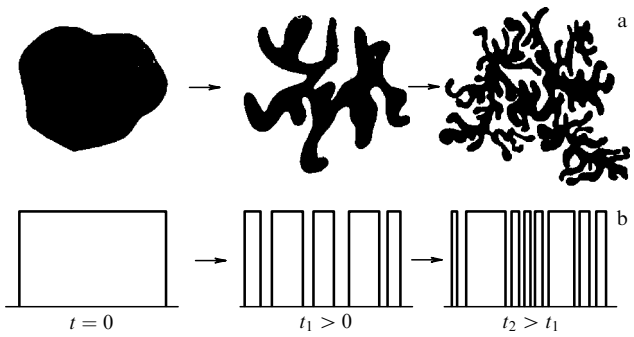

Figure 7. (a) Change in the shape of a volume occupied by a tracer as a result of turbulent diffusion. (b) A schematic shape of the distribution of tracer concentration along a straight line intersecting this volume.

Here, D_0 is the coefficient of seed (molecular) diffusion. It is straightforward to estimate the minimum spatial scale of tentacles Δ_mix evolving in the process of mixing by resorting to the balance of characteristic time scales. Equating the diffusive increment to that of stochastic instability, $\gamma_\mathrm{s} \approx \tau_\mathrm{d}^{-1}$, we obtain

$$\frac{D_0}{\Delta_\mathrm{mix}^2(t_\mathrm{mix})} \approx \frac{V_0}{L_0}\,. \tag{15}$$

It is now easy to obtain the scaling law for the dissipative scale in terms of the Peclet number $\mathrm{Pe} = V_0\lambda/D_0$, which is an analogue of the Reynolds number for problems of turbulent diffusion,

$$\Delta_\mathrm{mix}(\mathrm{Pe}) \propto \sqrt{\frac{L_0 D_0}{V_0}} = \frac{L_0}{\sqrt{\mathrm{Pe}}}\,. \tag{16}$$

Here, it is assumed that the Peclet number is large, $\mathrm{Pe} \gg 1$ (well-developed turbulence). The characteristic mixing time in the monoscale case is now written as

$$\frac{L_0}{\sqrt{\mathrm{Pe}}} = L_0 \exp\left(-\gamma_\mathrm{s} t_\mathrm{mix}\right). \tag{17}$$

By elementary manipulations, we arrive at the scaling

$$t_\mathrm{mix}(\mathrm{Pe}) \propto \frac{\ln \mathrm{Pe}}{\gamma_\mathrm{s}(V_0)} = \frac{L_0}{V_0} \ln \mathrm{Pe}\,. \tag{18}$$

We see that the dependence on the coefficient of molecular diffusion is rather weak, but in strong turbulence, $\mathrm{Pe} > 100$, the characteristic time scales of the problem can differ by more than an order of magnitude, $t_\mathrm{mix} \gg \tau_\mathrm{s} = 1/\gamma_\mathrm{s}$. In this simplest model, we discarded the effects related to the Kolmogorov cascade, rearrangement of the flow topology, and others. Nevertheless, even an estimate that simple turns out to be adequate, for example, when dealing with the currently relevant tasks of reagent mixing in micro-channels [103, 104].

We note that in geophysical fluid dynamics, the effects of mixing of Lagrangian trajectories and processes of turbulent transport related to these phenomena were already attracting attention in the 1940s. Such effects were discovered by Ertel [29], Eckart [30], and Welander [105], yet a clear understanding of the nature of these phenomena was lacking. In this review, we address the methods of deriving scaling laws for characteristic time scales and the coefficient of turbulent diffusion in problems where the main decorrelation mechanism is associated with stochastic instability.

3. Stochastic instability and relative diffusion

The paper by Batchelor [31] mentioned in the Introduction is devoted to the effect of homogeneous isotropic turbulence on the stretching of material lines and surfaces. The result derived in Ref. [31] for the elongation of a material element $\delta L(t_0)$ in a turbulent field,

$$\delta L(t) \approx \delta L(t_0) \exp\left(\frac{t - t_0}{\tau_\mathrm{s}}\right), \tag{19}$$

has allowed improving the already established views on turbulent diffusion. Here, τ_s is the characteristic time of

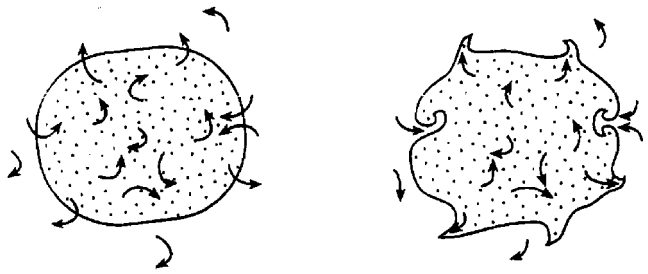

Figure 8. Evolution of a scalar cloud in the field of turbulence. The arrows show pulsations of velocity that mix the scalar (tracer).

instability development. However, the idea of the exponential divergence of two initially close scalar particles (passive tracer) in a turbulent flow was explicitly formulated only in review [106], which summarized the first results of turbulent transport studies relying on the Kolmogorov model of homogeneous isotropic turbulence. One of the main problems here is a detailed description of relative diffusion [34, 85, 87]. The actual point is a qualitatively new type of diffusion discovered by Richardson [107] in 1926. Instead of exploring diffusion from a fixed source, the question about a spreading particle cloud (Fig. 8) was formulated. An analysis of experimental results led to the following expression for the coefficient of relative diffusion for two selected particles in the field of atmospheric turbulence:

$$D_{\mathrm{R}} \propto \frac{1}{6}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle l_{\mathrm{R}}^2(t)\right\rangle \approx C_{\mathrm{R}}\left\langle l_{\mathrm{R}}^2(t)\right\rangle^{2/3}. \tag{20}$$

Here, l_{R} is the distance between the two particles located at points $x_1(t)$ and $x_2(t)$, $l_{\mathrm{R}}(t) = x_2(t) - x_1(t)$, and the constant C_{R}, introduced by Richardson, is estimated as $C_{\mathrm{R}} \approx 0.2\ \mathrm{cm}^{2/3}\ \mathrm{s}^{-1}$ [34]. Dependence (20) expresses the accelerated character of the particle relative motion. Expression (20) can be written in a simplified way as a superdiffusive scaling $l_{\mathrm{R}}^2(t) \propto t^3$. This result is nontrivial because it strongly differs not only from the diffusive scaling

$$\left\langle x^2(t)\right\rangle = R^2(t) \propto t, \tag{21}$$

but also from the ballistic one, $R \propto t$. The lack of proper understanding of physical mechanisms behind turbulent transport called for a substantial modification of views on correlation effects. At first glance, the decay of correlations can only double the turbulent diffusion coefficient D_T:

$$\left\langle l_{\mathrm{R}}^2(t)\right\rangle \approx \left\langle x_1^2(t)\right\rangle + \left\langle x_2^2(t)\right\rangle \approx 2\left\langle x_1^2(t)\right\rangle \approx 2(2D_T)t. \tag{22}$$

The scaling proposed by Richardson agrees well with observational data in a very broad range of parameters ($10^2 - 10^6$ cm) [34]. We note that the magnitude of the relative diffusion coefficient is substantially larger than the molecular diffusivity. For example, in the atmosphere, $D_{\mathrm{R}} \approx 10^4 - 10^5\ \mathrm{m}^2\ \mathrm{s}^{-1}$, whereas the molecular diffusion coefficient is $D_0 \approx 1\ \mathrm{m}^2\ \mathrm{s}^{-1}$. In fact, expression (22) reflects the nonlocal character of transport effects under conditions of atmospheric turbulence, when the distance between diffusing particles evolves essentially under the action of eddies with sizes comparable to the distance separating these particles.

Richardson realized very clearly that the process of relative diffusion of two particles in a turbulent flow that he considered fundamentally differs from classical particle diffusion. Moreover, Richardson stressed the significance of this new phenomenon by considering the law discovered by him not as an empirical dependence, where the power-law exponent is known only approximately, but as a fundamental law. Just for that reason, the fractional exponent appearing in his scaling $D_{\mathrm{R}} \propto \langle l_{\mathrm{R}}^2\rangle^{2/3}$ was not replaced by an approximate estimate [107]. Further theoretical development confirmed the strength of Richardson's vision, for this scaling finds its explanation in the framework of the theory of homogeneous isotropic turbulence [34, 35, 85, 108, 109]. Kolmogorov and Obukhov have shown in [108, 109] that if the energy dissipation rate ε_{K} is the only dimensional characteristic of the field of homogeneous turbulence in a broad range of scales l, then a possibility emerges of building a phenomenological model of turbulence by considering a cascade process of energy transfer from large to small eddies. In the framework of this approach, we can write the scaling for the diffusion coefficient relying on the dimensional character of the quantity $\varepsilon_{\mathrm{K}} = [L^2/T^3]$ and using the local spatial eddy scale $l = l(k)$: $k \approx 1/l(k) = [1/L]$. Then, performing elementary computations, we can derive a dimensional scaling for the Richardson relative diffusion coefficient:

$$D_{\mathrm{R}}(l) = \left[\frac{L^2}{T}\right] \approx \varepsilon_{\mathrm{K}}^{1/3}\frac{k^{2/3}}{k^2} \approx \varepsilon_{\mathrm{K}}^{1/3}\frac{1}{k^{4/3}} \approx \varepsilon_{\mathrm{K}}^{1/3} l^{4/3}. \tag{23}$$

Owing to the relative simplicity of experiments, data on the relative diffusion allow obtaining numerous estimates. Obukhov used the value $\varepsilon_{\mathrm{K}} \approx 5\ \mathrm{cm}^2\ \mathrm{s}^{-3}$ for atmospheric turbulence, assuming, according to Brunt [110], that only 2% of solar energy is transformed into the kinetic energy of air masses. This enabled Obukhov to derive the estimate for the coefficient in the Richardson formula [11, 112]. Additionally, the explanation of the Richardson relative diffusion law is closely connected to the turbulent energy spectrum $E(k)$, because it relies on the turbulent cascade phenomenology. Indeed, dimensional estimates allow directly linking the relative displacement of two particles in a turbulent flow to the expression for the energy spectrum,

$$D_{\mathrm{R}}(l) \propto V(l)l \propto l\sqrt{E(k)k}\,\Big|_{k\propto 1/l}, \tag{24}$$

which in the case of the classical Kolmogorov spectrum $E(k) \propto k^{-5/3}$ immediately leads to the Richardson scaling $D_{\mathrm{R}} \propto l^{4/3}$ and the formula relating the velocity scale to the size of a 'carrier' eddy, $V(l) \propto (\varepsilon_{\mathrm{K}} l)^{1/3}$.

For the purposes of this review, it is important that in the region of viscous scales (small l), we can use a linear dependence that arises as the asymptotic form of the Kolmogorov scaling $V(l) \propto l^{1/3}|_{l\to 0} \propto l$. The modified law for the relative diffusion takes the form $D_{\mathrm{R}}(l) \propto V(l)l \propto \mathrm{const}\, l^2$ predicted by Batchelor, because we now have the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle l^2(t)\right\rangle = \mathrm{const}\left\langle l^2(t)\right\rangle, \tag{25}$$

whose exponential solution describes the relative diffusion in the range of scales where viscosity effects are essential,

$$l^2(t) = l_0^2 \exp\left(\frac{t}{\tau_0}\right). \tag{26}$$

Here, l_0 is the initial distance between particles in the dissipation range.

In the framework of the Kolmogorov phenomenological approach, in the range of small spatial scales, the main flow parameters should depend only on the viscosity $\nu_f = [L^2/T]$ and the spectral energy flux $\varepsilon_K = [L^2/T^3]$. Combinations with dimensions of length and time can be composed from these quantities in a unique way. Then the spatial scale limiting the exponential 'divergence' of particles corresponds to the Kolmogorov dissipation scale

$$l_\nu = \left(\frac{\nu_f^3}{\varepsilon_K}\right)^{1/4} \gg l_0 \,. \tag{27}$$

The characteristic time scale related to the process of dissipation is $\tau_\nu = (\nu_f/\varepsilon_K)^{1/2}$. In the range of scales $l \gg l_\nu$, the exponential regime is replaced by the Richardson law. However, between them (in an intermediate range of scales), a transitional quasi-ballistic regime is realized [34],

$$D_R(t) \propto (l_* \varepsilon_K)^{2/3} t \,. \tag{28}$$

The exponential character of particle divergence in a turbulent flow is manifested not only in considering averaged flow characteristics. Observations of the evolution of scalar particle patches in eddy flows allow regarding stochastic instability as a fundamental mechanism of mixing, which, in turn, plays an important role in problems of the substantiation of statistical mechanics. In Sections 4–15, we concentrate on questions pertaining to the derivation of concrete scalings for the increment of stochastic instability and its influence on turbulent transport processes.

4. Quasilinear estimates and the Kazantsev increment

The formulas derived by Batchelor for the exponential stretching of a fluid element rely explicitly on the corresponding instability increment γ_s. Naturally, an important tasks is then to determine γ_s and learn about its dependence on physical parameters characterizing the turbulent flow. In this respect, we mention Ref. [113], where a formula is derived that allows estimating γ_s based on the characteristics of the mean squared vorticity of the flow

$$\gamma_s^2 \approx \left\langle (\mathrm{rot}\,\mathbf{V})^2 \right\rangle . \tag{29}$$

Result (29) is thoroughly discussed in the monograph by Monin and Yaglom [34, Vol. 2]. Recognizing the 'loose' character of this definition for well-developed turbulent flows, the authors of Ref. [34] propose a purely dimensional estimate of the increment, relying on the key parameters of the Kolmogorov model, the spectral energy flux ε_K and the viscosity ν_f:

$$\gamma_s \approx \sqrt{\frac{\varepsilon_K}{\nu_f}} = \left[\sqrt{\frac{\mathrm{m}^2}{\mathrm{s}^3}\,\frac{\mathrm{s}}{\mathrm{m}^2}}\right] . \tag{30}$$

Moreover, it is rigorously proved in [114, 115] that asymptotically

$$\gamma_s^2(t)\Big|_{t\to\infty} \to 0 \,. \tag{31}$$

We mention that simultaneously with the book by Monin and Yaglom, an important work by Kazantsev [116] appeared dealing with amplification of the magnetic field in a conducting turbulent medium. We note that the approach proposed in Ref. [116] was repeatedly discussed by Zel'dovich in the context of the turbulent dynamo problem [117–119]. For us, however, it is important that the derivation of an expression for the increment of exponential divergence was implicitly proposed in Ref. [116] based on the idea of the specific correlation (delta-correlation) of velocities in a model turbulent flow [34, 85]. We briefly discuss the derivation of this formula, because, in hindsight, it was the first time that a correct scaling was obtained for the stochastic instability increment in a developed turbulent flow in the absence of large-scale coherent structures.

Following the ideas of Richardson and Taylor, Kazantsev considered the relative distance $\boldsymbol{\rho}(t) = \mathbf{r}_1 - \mathbf{r}_2$ between two particles in a turbulent flow using a simple estimate for the turbulent diffusion [120]:

$$D_T = \int_0^\infty \left\langle V(t)V(0)\right\rangle \mathrm{d}t \propto V_0^2 \tau \,, \tag{32}$$

where $V(t)$ is the velocity of a Lagrangian particle, V_0 is the amplitude of turbulent pulsations, and τ is the characteristic correlation time. Then, from the formal expression for the evolution of ρ^2, it follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_1 - \mathbf{r}_2)^2 &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_1 + \mathbf{r}_2)^2 - 2\,\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_1\mathbf{r}_2) \\ &= \frac{4}{3}\,V_0^2\tau - 2\,\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}_1\mathbf{r}_2) \,. \end{aligned} \tag{33}$$

It is well known that the mean square value $\langle \exp A\rangle$ for Gaussian distributions is given by

$$\langle \exp A\rangle = \exp\frac{A^2}{2} \,. \tag{34}$$

If $\mathbf{r}$ is a random Gaussian quantity, then, considering the expression for the evolution of the mean square relative distance $\boldsymbol{\rho}$ between the particles, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\rho^2\rangle = \frac{8}{3}\,\tau \int \left[1 - \exp\left(-\frac{k^2\langle\rho^2\rangle}{2}\right)\right] E(k)\,\mathrm{d}k \,, \tag{35}$$

where we use the definition of the turbulent energy spectrum in the stationary form

$$\frac{V^2}{2} = \int E(k)\,\mathrm{d}k \tag{36}$$

and the condition that the velocity field can be represented as $\tilde{V}(k,t) = U(k)\delta(t)$. Considering the initial stage of the 'divergence' evolution of initially close particles, in the limit $\rho^2 k^2 \ll 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\rho^2\rangle = \frac{4}{3}\langle\rho^2\rangle \int E(k)k^2\,\mathrm{d}k \,. \tag{37}$$

In fact, this is an equation for exponential divergence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\rho^2\rangle = \gamma_s\langle\rho^2\rangle \,, \tag{38}$$

where the increment of stochastic instability is given by the quasilinear scaling (quadratic in velocity)

$$\gamma_s \approx V_0^2 k^2 \tau \approx \frac{V_0^2}{\lambda^2}\,\tau \propto \frac{1}{\tau}\,\mathrm{Ku}^2 \,. \tag{39}$$

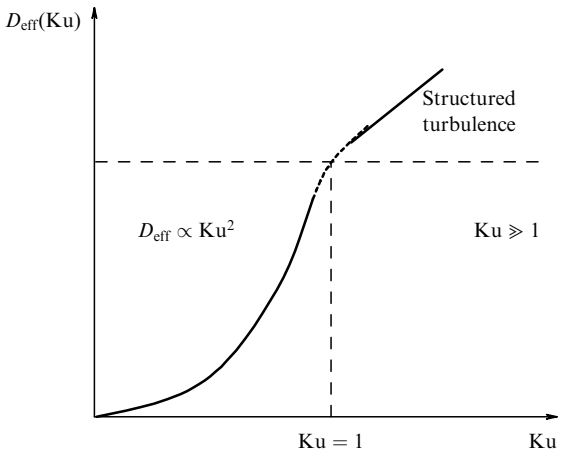

Figure 9. Dependence of effective diffusion coefficient on the Kubo number. $D_{\rm eff}$ is the effective turbulent diffusion coefficient and Ku is the Kubo number characterizing the intensity of turbulent pulsations.

It is assumed here that the characteristic size of eddies can be estimated as $\lambda \propto 1/k$.

In considering transport coefficients, the increment of stochastic instability can be regarded as one possible approximation of the effective correlation time. Present-day experiments point out that the form of the dependence of turbulent diffusion differs essentially from that predicted in quasilinear approximations under conditions of strong turbulence [121–128],

$$\mathrm{Ku} = \frac{V_0}{\lambda\omega} \geqslant 1\,, \qquad \mathrm{Pe} = \frac{V_0\lambda}{D_0} \geqslant 1\,. \tag{40}$$

Scaling estimates demonstrate similarity in the character of the dependence of the diffusion coefficient and the corresponding correlation time on the amplitude of turbulent velocity pulsations. In strongly turbulent regimes, this dependence becomes substantially softer (Fig. 9). This provides a solid argument to believe that the dependence of the stochastic instability increment also deviates from the quasilinear dependence obtained by Kazantsev. We consider this question in detail in Sections 7–11 devoted to the transport of scalar particles and the transport of electrons in a stochastic magnetic field.

5. Criticism of the quasilinear approach

Contemporary research of the transport coefficients in strongly turbulent hydrodynamical flows and analyses of anomalous regimes of diffusion in strongly magnetized plasmas clearly point out the necessity of accounting for the effect of coherent vortex structures [122–128]. An important criterion here is the Kubo number introduced in Section 2. For Ku $\geqslant$ 1, we are dealing with more complex decorrelation effects than in the classical Taylor approach [120], where the turbulent velocity pulsations are represented by equations of motion in the Lagrangian form

$$\mathbf{V}(t) = \frac{\mathrm{d}\mathbf{r}(\mathbf{r}_0, t)}{\mathrm{d}t}\,, \tag{41}$$

and the diffusion coefficient is directly related to the Lagrangian velocity autocorrelation function

$$D_T = \int_0^\infty \big\langle V(0)V(\tau)\big\rangle\,\mathrm{d}\tau\,. \tag{42}$$

This approach facilitated effective solutions of weak turbulence problems (quasilinear theory) in which the estimate of the turbulent diffusion coefficient took the form of scaling, and the scale of velocity fluctuations V_0 and characteristic correlation time τ were introduced: $D_T \propto V_0^2\tau$. Such an approach assumes that the velocity fluctuations are short-correlated. This is obviously a serious drawback in the case where the velocity field contains large-scale vortex structures or zonal currents. However, it offered a way to incorporate the amplitudes of turbulent pulsations.

Molecular (seed) diffusion affects the correlation effects substantially. Corrsin [32] was among the first to propose a concrete model including the effects of molecular diffusion in the correlation analysis. He recognized that the definition of the correlation function proposed by Taylor [120] is based on the Lagrangian velocities $V(x_0, t)$, but their measurement encounters serious difficulties. For this reason, the Eulerian representation of the correlation function is frequently used, involving the correlations of velocity at points separated by a distance λ:

$$C_{\rm E}(\lambda, t) = \big\langle u(x_0, T)\, u(x_0 + \lambda, T + t)\big\rangle\,.$$

In this representation, the formula for the correlation function is more convenient for carrying out experiments. We can also express the Lagrangian correlation function in terms of the Eulerian velocity,

$$C(t) = \big\langle u(x_0; T)\, u\big(x(x_0, T + t); T + t\big)\big\rangle\,,$$

where $u(x_0, T)$ is the Eulerian velocity at the point x_0 and T is time. However, a unique relation between the Lagrangian and Eulerian correlation functions is absent. Indeed, in the expression for the Eulerian correlation function, the Lagrangian connection between points x_0 and $x_0 + \lambda$ is absent. Here, λ is merely 'some' arbitrary displacement.

Corrsin proposed a nontrivial approximating formula by expressing the Lagrangian correlation function by randomizing the Eulerian correlation function with some probability distribution $\rho(x, t)$:

$$C(t) = \int_{-\infty}^{\infty} \rho(\lambda, t)\, C_{\rm E}(\lambda, t)\,\mathrm{d}\lambda\,.$$

An even more important contribution was the idea of the diffusive nature of the shift λ, because Corrsin took $\rho(\lambda, t)$ to be a classical solution of the diffusion equation in three-dimensional space (the Gaussian distribution),

$$\rho(\lambda, t) = \frac{1}{(4\pi D_0 t)^{3/2}} \exp\left(-\frac{\lambda^2}{4D_0 t}\right).$$

This formula includes the coefficient of molecular diffusion D_0. As a consequence, it becomes possible to consider both turbulent transport and molecular diffusion. Eventually, the integral expression becomes

$$C(t) = \int_{-\infty}^{\infty} \frac{C_{\rm E}(\lambda, t)}{(4\pi D_0 t)^{3/2}} \exp\left(-\frac{\lambda^2}{4D_0 t}\right)\mathrm{d}\lambda\,.$$

From this new standpoint, λ is simultaneously a spatial scale and a diffusive displacement. In fact, instead of the formal averaging of the form

$$\big\langle V\big(x(0)\big)\, V\big(x(t)\big)\big\rangle = \int_{-\infty}^{\infty} \big\langle V(0)\, V(y)\, \delta\big(y - x(t)\big)\big\rangle\,\mathrm{d}y\,,$$

the factorization ('the independence hypothesis')

$$\langle V(0)\, V(y)\, \delta(y - x(t))\rangle = \langle V(0)\, V(x)\rangle \langle \delta(y - x(t))\rangle$$

has been used. Moreover, to describe the correlation of trajectories, Corrsin used the Gaussian distribution

$$\langle \delta(y - x(t))\rangle \approx \rho(y, t)\,.$$

In plasma problems, where diffusion is considered in the velocity space, we arrive at an expression analogous to the Taylor formula, but in terms of the electric field fluctuations δE [121–124],

$$D_V \propto \left(\frac{e}{m_{\mathrm{e}}}\right)^2 (\delta E)^2 \tau\,, \tag{43}$$

where e and m_{e} are the charge and mass of the electron. The specifics of plasma problems formulated in the early 1960s was related to the description of interaction between waves and particles based on the view of Landau damping as the mechanism providing stochastic behavior. In the time that followed, the use of the short-range correlation approximation met with difficulties in describing strongly turbulent plasma states [129, 130].

In reality, it had already become clear at the end of the 1960s that quasilinear estimates fail to reflect the real character of transport in conditions of strong plasma or hydrodynamic turbulence [122–128]. Among numerous causes of this disparity, the failure to take the effects connected with exponential divergence into account was noted. In particular, in a short article [131], van Kampen criticized the Taylor formula for the turbulent diffusion coefficient and questioned the validity of the results of the linear response theory by Kubo [61, 68].

Indeed, the classical Kubo–Green theory gives a rigorous relation between the transport coefficients and correlation functions in the integral form

$$D = \int_0^\infty C(\tau)\, \mathrm{d}\tau\,. \tag{44}$$

Such an approach can be directly linked to the Langevin representation of fluctuations, and the classical results by Einstein can be interpreted in terms of fluctuation–dissipation relations. A simple example of such a relation is offered by the formula $k_{\mathrm{B}} T/(m\beta_t) = D$, which connects the Langevin 'friction' coefficient β_t with the system temperature and diffusion coefficient.

Considering transport problems on the basis of linear response theory allows using a well-developed and rather general approach of nonequilibrium thermodynamics. In the presence of small perturbations of the field $\alpha(\mathbf{r}, t)$ in time and space, the system evolution is described by the system of Onsager equations [68]

$$\frac{\partial \alpha(\mathbf{r}, t)}{\partial t} = -\nabla \mathbf{J}(\mathbf{r}, t)\,, \tag{45}$$

$$\mathbf{J}(\mathbf{r}, t) = L_J\, \boldsymbol{\chi}(\mathbf{r}, t)\,, \tag{46}$$

$$\boldsymbol{\chi}(\mathbf{r}, t) = -\nabla \alpha(\mathbf{r}, t)\,, \tag{47}$$

where $\mathbf{J}(\mathbf{r}, t)$ is the flux, $\boldsymbol{\chi}(\mathbf{r}, t)$ is the thermodynamic 'force', and L_J is the corresponding transport coefficient, which is represented by the Kubo–Green correlation expression that is traditional for thermodynamics,

$$L_J = \mathrm{const} \int_0^\infty \langle J(0)\, J(\tau)\rangle\, \mathrm{d}\tau\,. \tag{48}$$

As we have seen, in the case of turbulent diffusion of scalar particles, the relevant expression is written in terms of the velocity autocorrelation function.

It is nevertheless important to remember that the linear response theory formulated in the statistical mechanics framework does not assume substantial deviations from the equilibrium. This leads to serious problems in describing turbulence. The aspect pertaining to stochastic instability was clearly formulated by van Kampen [31]. In considering the evolution of dynamical system perturbations $\delta x_i(t)$, the Kubo–Green theory resorts to the Taylor series expansion, which allows connecting perturbed and unperturbed quantities,

$$\delta x_i(t) = \sum_j \frac{\partial x_i(t)}{\partial x_j(0)}\, \delta x_j(0) + O\big(\delta \mathbf{r}^2(0)\big)\,, \tag{49}$$

where $\delta \mathbf{r} = \{\delta x_i\}$. But in turbulent flows, the factor $\partial x_i(t)/\partial x_j(0)$ initially grows exponentially with time because of stochastic instability [114, 115]. In the case of turbulent particle transport, this is vividly demonstrated by even simpler single-vortex models. For example, Novikov [132] considered 'spiraling' of the tracer spot by a vortex and demonstrated the importance of exponential stretching effects for describing the evolution of a scalar (Fig. 10).

It follows from these remarks that the applicability region for expressions based on linear estimates should be quite limited. Nevertheless, the applications of linear theory and related Kubo–Green formulas for the transport coefficient are rather successful in a wide range of parameters [68]. The main reason for this success stems from considering transport phenomena using the concept of self-averaging, because the values measured by us are the results of averaging over ensembles of trajectories. From this perspective, stochastic instability represents an important decorrelation mechanism providing the 'mixing' needed for self-averaging. Indeed, observing a single particle cannot provide information on the ensemble behavior.

Van Kampen's criticism of the linear response theory played an important role by simulating the treatment of stochastic instability effects on the basis of approaches differing essentially from the quasilinear one. These ideas were efficiently realized in the analysis of various turbulent transport models involving stochastic layers, large-scale vortex structures, or reconnection processes as their important ingredients.

6. Lagrangian turbulence and Arnold–Beltrami–Childers flows

An intuitive perception of turbulence identifies a turbulent flow with chaos in time or space. Nevertheless, if a Eulerian field is regular and has a simple analytic form, this in no way excludes the presence of stochasticity for the Lagrangian trajectories of fluid particles, and, as a consequence, the Lagrangian velocities can be stochastic as well. Surprisingly enough, this fact was perceived only in the mid-1960s by Arnold [133, 134] who, working on problems of

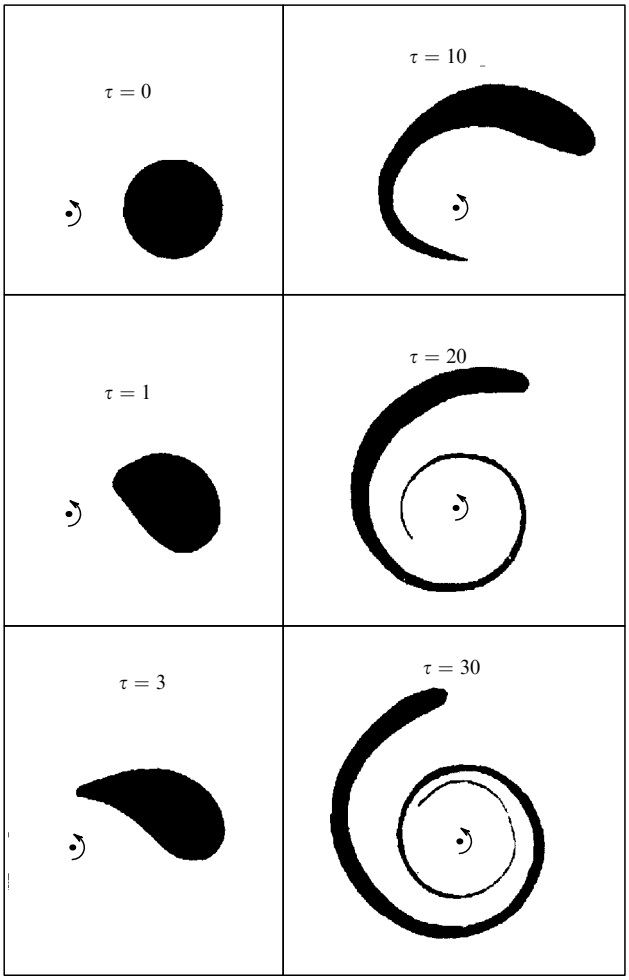

Figure 10. Stretching a tracer patch by a vortex. The arrow indicates the vortex rotation sense.

dynamical system theory, paid attention to the three-dimensional stationary velocity field expressible in elementary functions:

$$V_x = A\sin z + C\cos y\,, \tag{50}$$

$$V_y = B\sin x + A\cos z\,, \tag{51}$$

$$V_z = C\sin y + B\cos x\,. \tag{52}$$

Numerical modeling carried out by Hénon [135] demonstrated that Lagrangian trajectories of fluid particles in the field of the ABC flow (thus dubbed after Arnold, Beltrami, and Childers) can undoubtedly be considered stochastic [76, 78, 80, 85]. We note that for incompressible flows similar to the ABC model, in which $\operatorname{div}\mathbf{V} = 0$, the system dynamics is conservative (Hamiltonian), whereas, formally, the ABC flows satisfy the Beltrami condition $\operatorname{rot}\mathbf{V} = \alpha_V\mathbf{V}$, where α_V is a constant equal to unity for the ABC flows.

Indeed, the ABC system of three equations can be written in a form close to the Hamiltonian one,

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{1}{K}\frac{\partial H(x,y,z)}{\partial y}\,, \tag{53}$$

$$\frac{\mathrm{d}y}{\mathrm{d}z} = -\frac{1}{K}\frac{\partial H(x,y,z)}{\partial x}\,, \tag{54}$$

where the potentials $K(x,y)$ and $H(x,y,z)$ are

$$K(x,y) = C\sin y + B\cos x\,, \tag{55}$$

$$H(x,y,z) = K(x,y) + A(y\sin z - x\cos z)\,. \tag{56}$$

Introducing a new variable p and using the nonlinear transformation

$$y = y(p,x)\,,\qquad p = \int_0^y K(x,y')\,\mathrm{d}y' = By\cos x + C(1-\cos y)\,, \tag{57}$$

we arrive at Hamilton equations for the ABC flows of the canonical form

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{\partial H(p,x,z)}{\partial p}\,, \tag{58}$$

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial H(p,x,z)}{\partial x}\,, \tag{59}$$

where the Hamiltonian function is

$$H(p,x,z) = C\sin y + B\cos x + A(y\sin z - x\cos z)\,. \tag{60}$$

The investigation of exponential instability started when Hénon [135], prompted by Arnold, demonstrated exponential divergence of close trajectories, also in the case of equal coefficients $A = B = C$:

$$l(z) \propto l_0\exp(\gamma_z z)\,. \tag{61}$$

Here, l_0 is the initial distance and γ_z is the exponential instability increment.

At first glance, the emergence of chaos in Lagrangian trajectories (chaotic advection) in a flow whose Eulerian equations describe a regular velocity distribution seems surprising. But we note that it is well known in the theory of dynamical systems that for a system of three ordinary differential equations similar to the Lagrangian equations above, the cases where it is nonintegrable, leading to chaotic dynamics, are not an exception [2–6]. Chaotic behavior arises even for fairly simple functional dependences in the right-hand sides of these equations [136]. In agreement with that, the Lagrangian trajectories in a three-dimensional space may demonstrate chaotic flows, albeit independent of time.

Arnold [133] was the first to clearly point out a special feature of Beltrami flows in which the vorticity and velocity vectors are always parallel: the vector product of velocity and vorticity, being the gradient of the Bernoulli integral in an ideal (inviscid) flow, is everywhere zero,

$$\mathbf{L} = \mathbf{V}\times\operatorname{rot}\mathbf{V} = 0\,. \tag{62}$$

Condition (62) lifts the constraints on Lagrangian trajectories, which, in this case, are 'allowed' to move not only along two-dimensional surfaces (invariant tori). The free walk of particles in some flow domain is here a manifestation of the nonintegrability of the advection equations.

Indeed, in accordance with the Kolmogorov–Arnold–Moser theory (KAM), we are dealing with a nonintegrable

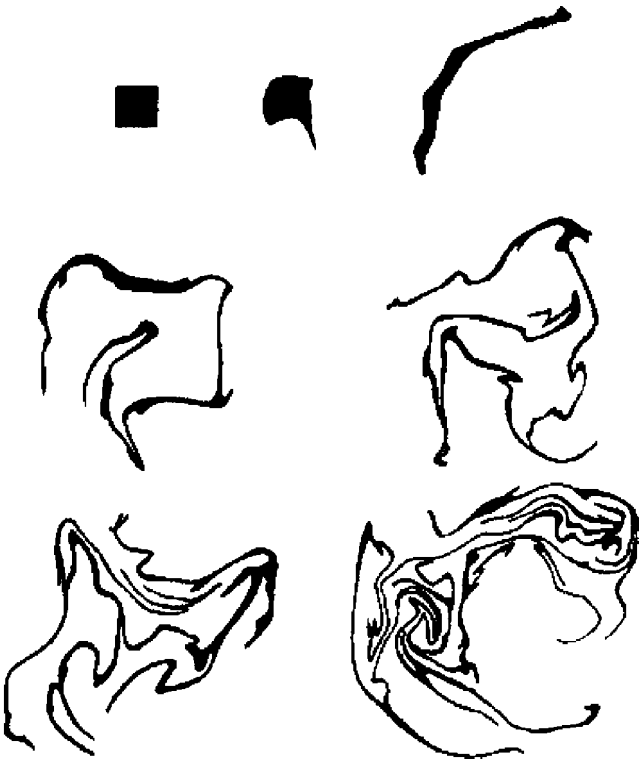

Figure 11. Tracer evolution in the experiment by Welander as an example of chaotic advection.

system [2–4]. But it can be reduced to an integrable two-dimensional one, for example, by setting $A = 0$. This case is characterized by the presence of the energy integral

$$H_0 = B\cos x + C\sin y = \text{const}\,; \tag{63}$$

hence, we have an ensemble of cylindrical surfaces that correspond to different values of the constant H_0. Small perturbations of this system of surfaces, which correspond to small values of the problem parameter A, lead to only small deformations for most tubes. In the vicinity of separatrix surfaces, the domains of stochasticity form (where regular surfaces are destroyed), entailing chaos with the streamlines. For $A \approx 1$, this chaos extends to large spatial domains.

It is important to note that Arnold and Hénon had predecessors whose results did not attract the attention they deserved for a long time. In the Introduction, we already mentioned Carl Eckart and Pierre Welander. In 1948, Eckart pointed out [30] the equivalence of advection and mixing in problems of geophysical fluid dynamics. Welander [105] demonstrated in an experiment that by driving a laminar flow in a two-dimensional fluid with streamlines forming a system of enclosed ellipses, dye (passive scalar) patches of intricate shapes can be created (Fig. 11). Welander sought the explanation with the help of the ergodicity concept, resorting to analogies with the coarse-grained phase volume. Moreover, approximately a decade and a half after the work by Arnold [133] and Hénon [135] was published, research by Berry and co-authors appeared [137], dedicated to the morphology of material lines in a periodic incompressible flow. The authors of Ref. [137] regarded two types of fixed points for two-dimensional area-preserving maps. The elliptic points were related to 'bends', while the hyperbolic ones were related to dilations and contractions in the form of 'curls' (Figs 12 and 13).

Figure 12. Oscillations of a perturbed separatrix on approaching a hyperbolic point.

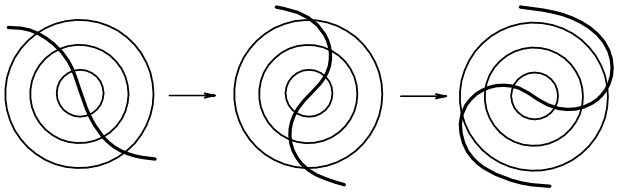

Figure 13. Formation of Berry 'curls' in the vicinity of a center.

We see that the effects of chaos related to advection were well known, but the term proper was coined only in the 1980s, when the relations between the onset of chaos in Hamiltonian systems, problems of integrability, and numerical results demonstrating the chaotic behavior of Lagrangian trajectories were finally established. At present, rich journal literature and several monographs devoted to the topic exist [63, 77–81, 102, 138].

7. Stochastic layers in planar flows

Despite the obvious deep connection between the problems of dynamical chaos in Hamiltonian dynamics, where the ideas by Kolmogorov, Arnold, and Moser [2, 134, 139–141] found wide dissemination, and the issues of stochasticity in hydrodynamical flows, numerous results related to the chaotic advection theory were obtained only very recently [76–81]. Two-dimensional flows are explored most thoroughly. In this case, the equations of motion for a Lagrangian particle have Hamilton's form, the role of the Hamilton function is played by the stream function, and the phase space coincides with the configurational space. In fact, it becomes possible here to visualize the phase space.

In the case of stationary planar incompressible flows, stochasticity is absent, and the Lagrangian trajectories coincide with streamlines determined by the stream func-

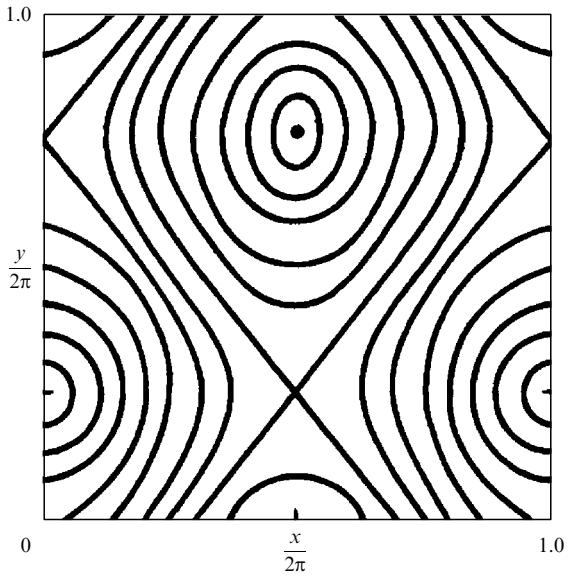

Figure 14. Integrable case of an ABC flow with $A = 0$. The streamlines in the (x, y) plane.

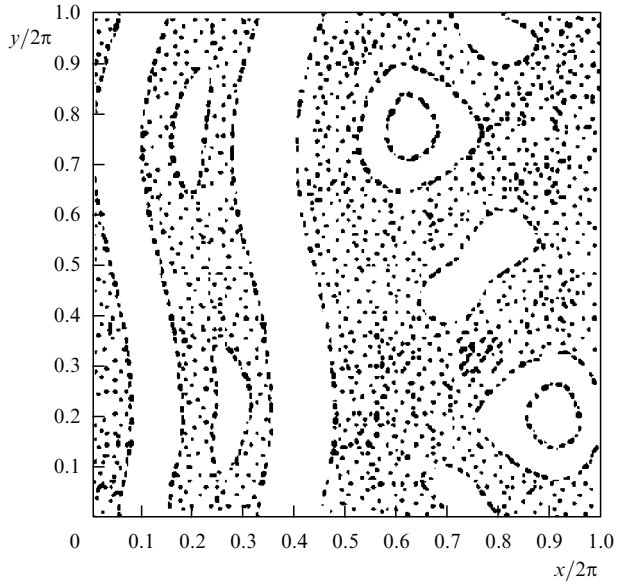

Figure 15. Onset of chaos in an ABC flow in the vicinity of separatrices.

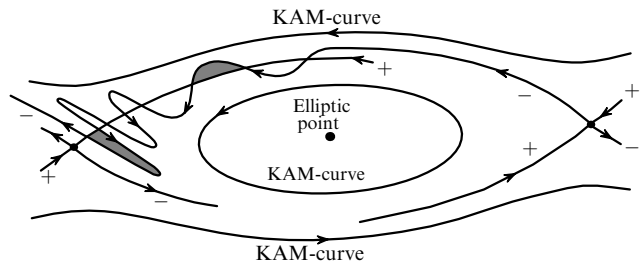

Figure 16. Buildup of a stochastic layer in the vicinity of a separatrix.

tion Ψ:

$$V_x = -\frac{\partial \Psi(x, y)}{\partial y}\,, \tag{64}$$

$$V_y = \frac{\partial \Psi(x, y)}{\partial x}\,. \tag{65}$$

The absence of stochastic behavior in a two-dimensional stationary incompressible flow is directly rooted in the absence of exponential stretching of fluid particles. This fact was first mentioned by Zel'dovich in Ref. [142] in analyzing dynamo effects [143–145]. Indeed, numerical experiments indicate that for most initial conditions, the separation between two initially close particles in a stationary two-dimensional flow of an incompressible fluid grows only linearly in a bounded domain.

A simple integrable case occurs for the ABC flows with $A = 0$ (the BC flow),

$$V_x = \dot{x} = C \cos y\,, \tag{66}$$

$$V_y = \dot{y} = B \sin x\,, \tag{67}$$

which allows studying periodic vortex structures (cells) in terms of the elementary stream function (Fig. 14):

$$\Psi(x, y) = C \sin y + B \cos x\,. \tag{68}$$

This flow has a set of special streamlines (separatrices) passing through saddle points. Numerical simulations of a weakly perturbed BC flow with the Hamiltonian

$$H(x, y, z) = \Psi(x, y) + \varepsilon(y \sin z - x \cos z) \tag{69}$$

point to the appearance of stochastic instability. For the parameter characterizing the amplitude of perturbations $\varepsilon \ll 1$, in accordance with the KAM theory, narrow stochastic layers form in the vicinity of separatrices. Chaos spreads over substantial portions of space when $\varepsilon \equiv A \approx 1$ (Fig. 15).

A similar situation occurs when the stream function of a two-dimensional flow with separatrices acquires time dependence, $\Psi(x, y, t)$. Such two-dimensional models were called systems with $3/2$ degrees of freedom [4, 6, 62, 76, 102]. Here, as well as in many other cases related to statistical descriptions of phenomena, the problem dimension plays an important role. Obviously, domains of stochasticity emerge in the vicinity of separatrices (Fig. 16). Interesting examples can be furnished by nonstationary generalizations of BC flows (vortex cells), offering a possibility of exploring the rearrangement of streamline topology.

In problems of stochastic streamlines in planar flows, we can use qualitative estimates based on the ideas of Chirikov and Zaslavsky on nonlinear resonance [146]. In the first approximation, we can consider stochastic layers around separatrices of complex flows, relying on the estimates for the stochastic layer thickness Δ for the nonlinear resonance perturbed by a periodic force with a characteristic frequency ω_0. In [147], estimates of Δ are given for both high-frequency perturbations,

$$\Delta \propto \lambda \exp\left(-\frac{\omega_0}{\omega_B}\right), \tag{70}$$

and low-frequency modes, interesting for the turbulent transport theory (large Kubo numbers, $\mathrm{Ku} > 1$),

$$\Delta(\varepsilon_\omega) \propto \varepsilon_\omega \lambda \left(\frac{\omega_0}{\omega_B}\right) \propto \varepsilon_\omega\,. \tag{71}$$

Here, ω_B is the characteristic frequency of the principal mode and ε_ω is the amplitude of perturbations. It follows from these estimates that specifically in low-frequency modes, the effect turns out to be substantial.

If we consider monoscale vortex flows in which the vortex is of a characteristic size λ and V_0 is the characteristic velocity amplitude, then $\omega_B \propto V_0/\lambda$. Accordingly, the estimate for the stochastic layer thickness in flows with vortex cells is expressed as

$$\Delta \propto \varepsilon_\omega \frac{\omega_0 \lambda^2}{V_0} \propto \varepsilon_\omega \frac{\lambda}{\mathrm{Ku}} \ll \lambda\,, \tag{72}$$

where, for low-frequency perturbations, $\mathrm{Ku} \geqslant 1$ because $\omega_B \gg \omega_0$ in the low-frequency limit.

An estimate for diffusion of stochastic streamlines is also of interest:

$$D_\psi(\mathrm{Ku}) \propto \frac{\Delta^2}{\tau(\mathrm{Ku})} \propto \frac{\varepsilon_\omega^2}{\mathrm{Ku}^2} \frac{\lambda^2}{\tau}\,, \tag{73}$$

where $\tau = \tau(\mathrm{Ku})$. Estimating the effective correlation time τ is nontrivial because the estimate should be based on a concrete decorrelation mechanism that accounts for the specifics of the flow topology and the character of instabilities developing in it. However, on a qualitative level, we can consider the processes of decorrelation in the stochastic layer and the flow as a whole being approximately similar, assuming that

$$\tau_\psi \propto \frac{\Delta^2}{D_\psi} \propto \frac{\lambda^2}{D_T}\,, \tag{74}$$

where $D_T \propto V^2/\omega_0$ is the traditional quasilinear coefficient of diffusion. Substituting, we obtain

$$D_\psi \propto \frac{\Delta^2}{\tau_\psi} \propto \frac{\Delta^2}{\lambda^2} D_T \propto \frac{\varepsilon_\omega^2}{\mathrm{Ku}^2} \frac{V_0^2}{\omega_0} \propto \varepsilon_\omega^2 \lambda^2 \omega_0\,. \tag{75}$$

We note that the estimate of τ_ψ is simultaneously also a qualitative estimate for the stochastic instability increment in 'cell' flows,

$$\gamma_\mathrm{s}(V_0) \propto \frac{1}{\tau_\psi} \propto \omega_0\,\mathrm{Ku}^2 \propto V_0^2\,, \tag{76}$$

which agrees within the order of magnitude with the quasilinear estimate by Kazantsev (39) discussed in Section 4 in relation to the Kolmogorov phenomenology of a constant spectral energy flux in conditions of well-developed isotropic turbulence.

8. Effective transport in convective cells

Effects of stochastic instability, clearly manifested in the vicinity of separatrices delineating vortex cells, work as a decorrelation mechanism that maintains the exchange of particles between the cells if the flow is nonstationary. On the other hand, in a system of steady convective cells, the role of decorrelation mechanism is played by molecular (collisional) diffusion. In both cases, the effective transport increases owing to the existence of convective channels, whose width Δ is the key parameter of the problem. In essence, this is one of the simplest models of anomalous transport in the conditions of structured turbulence (containing eddies).

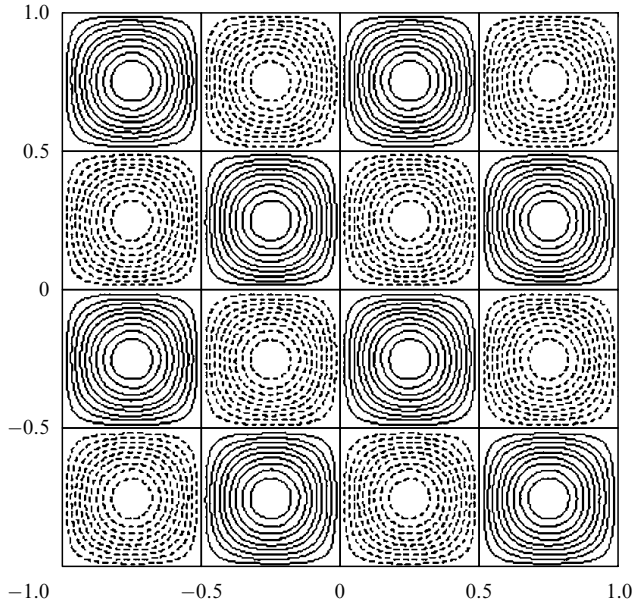

Figure 17. System of regular convective (vortex) cells.

Using straightforward estimates of the particle balance in a stochastic layer, it is possible to show that the width of the channel is directly related to the velocity amplitude of convective flows. Regular vortex cells can be described with the help of the stream function proposed by Taylor (Fig. 17),

$$\Psi(x, y) = \Psi_0 \sin\left(\frac{2\pi}{\lambda} x\right) \sin\left(\frac{2\pi}{\lambda} y\right), \tag{77}$$

because, in this representation, the vortices are located along the coordinate axes, which is more convenient for specifying boundary conditions. Here, $\Psi_0 = \lambda V_0$.

It is assumed that in the limit of large Peclet numbers $\mathrm{Pe} = \lambda V_0/D_0 \gg 1$, scalar particles move along streamlines forming convective channels, and leave them because of the presence of seed diffusion D_0 in the system (molecular diffusion D_m due to collisions or stochastic diffusion D_Ψ).

The term 'passive scalar' is used to refer to tracer particles in the flow that exert no back reaction on the flow. For example, if the temperature of each fluid particle is preserved (it is 'frozen' in the medium), then it can be considered a scalar. A similar situation emerges in problems of transport of a magnetic field, when the magnetic field, being 'frozen in', is transported by the plasma.

In the simplest case where the fluid flow is incompressible, $\operatorname{div} \mathbf{V} = 0$, we can replace the partial derivative over time $\partial n/\partial t$ of the scalar density n in the classical diffusion equation with the Lagrangian derivative. Then the scalar transport equation becomes

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\partial n}{\partial t} + \mathbf{V}\nabla n = D_0 \nabla^2 n\,. \tag{78}$$

In the absence of diffusion (the coefficient of diffusion $D_0 = 0$), we obtain the 'freezing-in' condition $n(\mathbf{r}, t) = n(\mathbf{r}_0, t)$, where $\mathbf{r}_0$ is the particle coordinate at the initial instant, $\mathbf{r}_0 = \mathbf{r}(t = 0)$. The purely Lagrangian behavior of the scalar can lead to arbitrarily large density gradients, but in the presence of diffusion (or heat conduction if the transport problem is considered in terms of temperature), the density inhomogeneities are smoothed out. In that case, it becomes

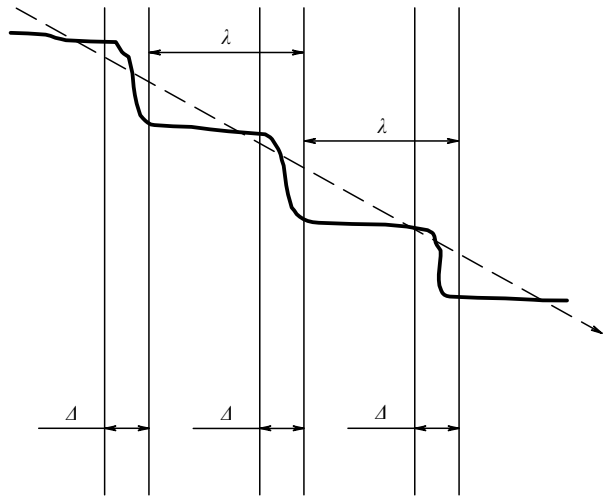

Figure 18. Profile of tracer concentration in a system of convective cells. Δ is the stochastic (diffusive) layer width and λ is the characteristic spatial scale of a cell. The dashed line indicates the direction of concentration decrease.

possible to describe the transport of particles with the help of the effective diffusion coefficient.

In our case, particles leave the convective layer of a width Δ diffusively, $D_0(\delta n/\Delta^2)$, whereas advection along the boundary layer gives $V_0(\delta n/\lambda)$. Comparing these estimates (in essence, they are estimates of characteristic time scales in the diffusive boundary layer), we obtain the width of the stochastic layer in the form

$$\Delta(V_0) = \sqrt{\frac{D_0\lambda}{V_0}} = \lambda\,\frac{1}{\sqrt{\mathrm{Pe}}} \propto \frac{1}{\sqrt{V_0}}\,. \tag{79}$$

If the amplitudes are large (strong turbulence, the Rayleigh number $\mathrm{Ra} \propto \mathrm{Pe} \gg 1$), the stochastic layer is very narrow.

An estimate of the effective transport of a scalar in a system of convective cells (Fig. 18) should include the fraction of space occupied by the stochastic layer, $\lambda\Delta/\lambda^2 \approx \Delta/\lambda$, because we assume that just the convective contribution is decisive:

$$D_{\mathrm{eff}} \approx \lambda V_0 P_\infty \approx \lambda V_0\,\frac{\Delta}{\lambda} \approx V_0\Delta(V_0)\,. \tag{80}$$

Here, P_∞ is the fraction of space responsible for convection. We finally arrive at the following estimate for the turbulent diffusion coefficient [70]:

$$D_{\mathrm{eff}} = \mathrm{const}\sqrt{D_0 V_0 \lambda} \propto V_0^{1/2}\,. \tag{81}$$

This scaling, derived from simple balance arguments, allowed giving a theoretical interpretation of numerical experiments, which attracted great interest, on transport in a system of drift convective cells observed in turbulent magnetized plasmas of tokamaks [148]. In these experiments, particle diffusion in the field of several electrostatic waves was considered. From a formal standpoint, the result should be describable by the classical quasilinear theory; but strong deviations from the quasilinear behavior have been found in the behavior of the particle diffusion coefficient and the Kolmogorov entropy. The main cause of these deviations is the fact that as the velocity amplitude increases, the convective channels become narrower and the rate of increase in the effective transport decreases compared with both quasilinear, $D_{\mathrm{eff}} \propto V_0^2$, and linear (Bohm), $D_{\mathrm{eff}} \propto V_0$, estimates.

On the other hand, using the result in Section 7 for stochastic layers as the seed diffusion,

$$D_0 \propto D_\psi \propto \varepsilon_\omega^2 \lambda^2 \omega_0\,, \tag{82}$$

and the scaling for convective cells, we can readily obtain an estimate for the effective diffusion coefficient:

$$D_{\mathrm{eff}}(\varepsilon_\omega) \propto \sqrt{D_0} \propto \varepsilon_\omega\sqrt{\omega_0}\,. \tag{83}$$

This type of dependence of the effective diffusivity on the perturbation amplitude has both theoretical and experimental confirmation. Solomon and Gollub considered a stream function that models regimes of periodic instability in a system of convective cells [149],

$$\Psi(x,y,t) = V_0\lambda \sin\left[\frac{2\pi}{\lambda}(x + \varepsilon_\omega\lambda\sin\omega t)\right]\sin\left(\frac{2\pi}{\lambda}\,y\right), \tag{84}$$

where λ is the characteristic spatial scale, ω is the characteristic frequency of perturbations, and $\varepsilon_\omega \propto \sqrt{\mathrm{Ra} - \mathrm{Ra}_*}$ is the scale of perturbation related to the deviation of the Rayleigh number

$$\mathrm{Ra} = \frac{\alpha_\rho g \Delta T L_0^3}{k_{\mathrm{B}}\nu_{\mathrm{f}}} \tag{85}$$

from its critical value Ra_* at which cells can develop periodic oscillations. Here, α_ρ is the thermal expansion coefficient, g is acceleration of gravity, and k_{B} is the Boltzmann constant. An important experimental result was a substantial increase in transport compared to that in the steady case. The flux of particles occurring in these regimes was proportional to the perturbation amplitude: $\varepsilon_\omega \propto (\mathrm{Ra} - \mathrm{Ra}_*)^{1/2}$.

9. Mel'nikov function and the stochastic layer

The nontrivial character of transport in the Solomon–Gollub model (see Section 8) can be explained by considering the appearance of stochastic layers in the vicinity of separatrices bounding the vortex cells, in the framework of the perturbation theory for the stream function (Hamiltonian) describing a steady cellular flow $\Psi_0(x,y)$:

$$\Psi(x,y,t) = \Psi_0(x,y) + \varepsilon_\omega\Psi_1(x,y,t)\,. \tag{86}$$

With the appearance of oscillating perturbations, the regular boundaries of cells (separatrices) are subject to deformations (Fig. 19) related to the formation of heteroclinic trajectories [76–80]. In 1963, Mel'nikov [138, 150] proposed a method for exploring oscillating perturbations of this kind, and practically simultaneously such an approach was applied by Morozov and Solov'ev [151], who considered oscillating 'lobes' of magnetic field lines to analyze the structure of eroded magnetic surfaces in plasma traps for systems of controlled fusion. In the case of a scalar, the 'lobes' of streamlines lead to stochasticity of the Lagrangian particle transport (lobe transport), which augments fluxes compared to those for molecular diffusion.

Research pertaining to the nontrivial behavior of phase trajectories of perturbed Hamiltonian systems was already initiated by Poincaré [2–5] and continued later by Birkhoff

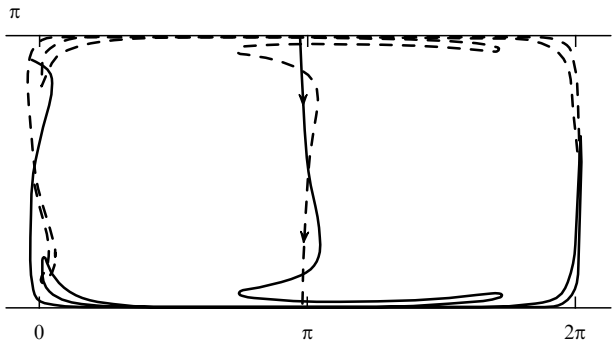

Figure 19. Splitting of separatrices in a system of convective (vortex) cells.

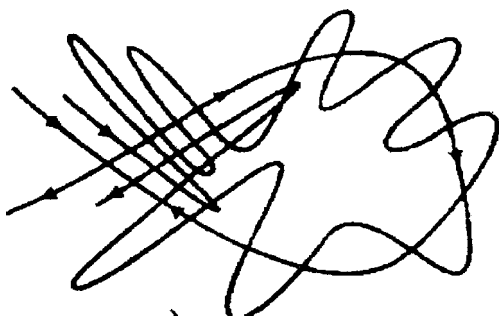

Figure 20. Phase portrait of a system with one hyperbolic point.

and Andronov. Related questions, dealing with notions of stable and unstable manifolds, are described in detail in the extensive educational literature [77–81, 102, 147, 152]. For the purposes of this review, we schematically present the main ideas of this approach using the example of a separatrix loop with a single hyperbolic point (Fig. 20). Such a situation is encountered, for example, when a particle moves in a 'cubic' potential of the form

$$U(x) = c_1 x^2 + c_2 x^3 \,, \tag{87}$$

where c_1 and c_2 are constants chosen based on physical considerations.

Close to the separatrix, the period of particle oscillations tends to infinity. That is why a small perturbation in frequency may cause a substantial phase shift. The emerging local instability of trajectories leads to splitting of the perturbed separatrix, which was first discovered by Poincaré.

Indeed, four 'branches' pass through the hyperbolic point of the separatrix being considered, whereas the separatrix itself delineates finite and infinite trajectories of particles in the selected cubic potential. We consider two branches, S_+ and S_-. Perturbations may lead to the appearance of a small gap between S_+ and S_-, and, because of the periodicity of perturbations, the branches begin to oscillate independently (Fig. 21). The oscillations grow in amplitude, but their step simultaneously decreases. For S_+, this happens in the direction of S_-, while for S_-, the oscillations grow in amplitude, and their step simultaneously decreases in moving from the hyperbolic point in the direction of S_+. This specific behavior comes from the amplification of instability with the distance from the hyperbolic point (increasing the deformation of phase elements) and from the motion deceleration on approaching the end of the loop. The branches considered here intersect transversely, leading to the occurrence of a stochastic layer in the vicinity of an unperturbed separatrix.

Figure 21. Two branches of a split separatrix in a system with one hyperbolic point.

As we have seen in Sections 2–8, the erosion of the separatrix has a pronounced effect on transport processes, allowing tracer particles to pass through the barriers separating regions of finite motions (corresponding to particle confinement in the vortex) and infinite motions in the unperturbed system of streamlines.

Mel'nikov [138, 150] proposed an analytic method allowing the computation of branch displacement in the direction normal to the unperturbed separatrix. The change of sign of this quantity for a point moving along the separatrix loop would indicate transverse intersections of stable and unstable fixed-point manifolds in the perturbed Hamiltonian system. The amplitude of the oscillating Mel'nikov function can be interpreted as the width of the stochastic layer.

The main idea lies in the fact that on an unperturbed separatrix, the vector that corresponds to the velocity of motion of a point along the phase trajectory $(\dot{x}(t), \dot{y}(t))$ is tangent to the separatrix. On the other hand, it is easy to construct an orthogonal vector $(-\dot{y}(t), \dot{x}(t))$. Then the dot product of this vector, orthogonal to the unperturbed separatrix, with the vector describing the perturbed motion of the depicting point in the phase plane, $(x_1(t), y_1(t))$, can be treated as the displacement of the perturbation along the normal to the unperturbed separatrix:

$$\dot{d}_{\mathrm{m}}(t) = \frac{\partial \Psi_1(x,y)}{\partial x}\frac{\partial \Psi_0}{\partial y} - \frac{\partial \Psi_1(x,y)}{\partial y}\frac{\partial \Psi_0}{\partial x} \,. \tag{88}$$

This quantity has two components, which correspond to the stable, $\dot{d}_{\mathrm{s}}(t)$, and unstable, $\dot{d}_{\mathrm{u}}(t)$, manifolds (Fig. 22). It

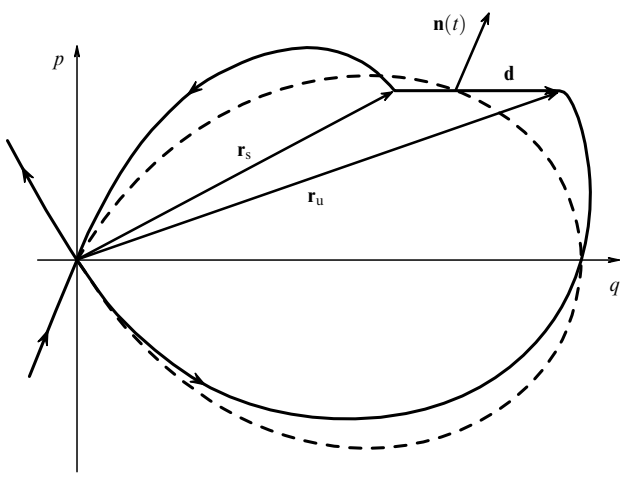

Figure 22. Schematic of the vector that defines the Mel'nikov distance; $\mathbf{n}(t)$ is the normal to the separatrix, $\mathbf{d}$ is the Mel'nikov vector, and $\mathbf{r}_{\mathrm{s}}$ and $\mathbf{r}_{\mathrm{u}}$ are the respective radius vectors of the entering and exiting branches of the perturbed separatrix.

should also be borne in mind that the stable manifold for $t \to \infty$ and unstable manifold for $t \to -\infty$ should tend to the same fixed point (in the homoclinic case, we have one hyperbolic point):

$$x_{\mathrm{u}1}(-\infty) = x_{\mathrm{s}1}(\infty)\,, \tag{89}$$

$$y_{\mathrm{u}1}(-\infty) = y_{\mathrm{s}1}(\infty)\,. \tag{90}$$

This implies that in calculating, we must use the condition that displacements of perturbed branches of the separatrix are equal:

$$d_{\mathrm{s}}(\infty) = d_{\mathrm{u}}(-\infty)\,. \tag{91}$$

Turning to integration and subtracting the displacements corresponding to stable and unstable branches, we arrive at the Mel'nikov integral expression describing the distance between perturbed separatrix branches:

$$\delta\Psi_0(t_0) = \varepsilon_\omega \int_{-\infty}^{\infty} \mathrm{d}t \left(\frac{\partial \Psi_1(x,y)}{\partial x} \frac{\partial \Psi_0}{\partial y} - \frac{\partial \Psi_1(x,y)}{\partial y} \frac{\partial \Psi_0}{\partial x} \right). \tag{92}$$

Here, we use an asymptotic representation for particle trajectories assuming that the Hamiltonian of the system is represented as a perturbation of an integrable problem, whose solution $x(t)$, $y(t)$ is assumed to be known,

$$x(t, t_0) \approx x(t_0) + \varepsilon_\omega\, x_1(t)\,, \tag{93}$$

$$y(t, t_0) \approx y(t_0) + \varepsilon_\omega\, y_1(t)\,, \tag{94}$$

where $(x(t_0), y(t_0))$ are the initial particle positions.

The Mel'nikov method is also applicable when there are several saddle (hyperbolic) points (Fig. 23). Having chosen the stream function (Hamiltonian) in the form modeling a system of regular vortex cells,

$$\Psi_0 = V_0 \lambda \sin x \sin y\,, \tag{95}$$

and the perturbing function in the special form (convenient for subsequent computations)

$$\Psi_1 = -\varepsilon_\omega \omega \left[xy \sin\left(\omega t + \frac{\pi}{4}\right) \cos\left(\omega t + \frac{\pi}{4}\right) \right], \tag{96}$$

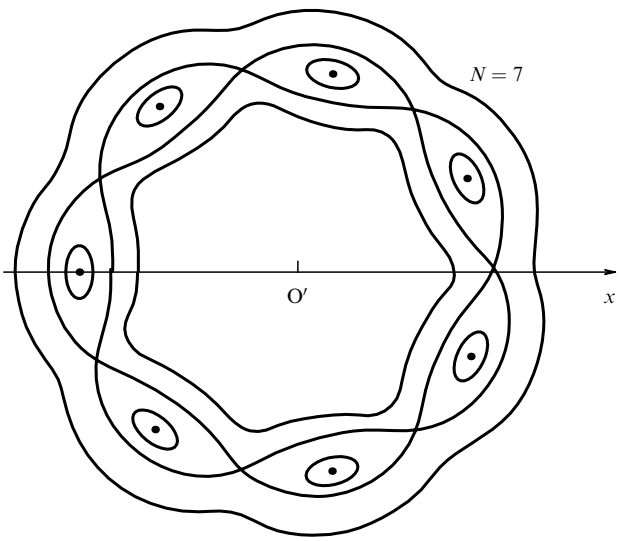

Figure 23. Schematic of separatrices in the presence of several hyperbolic points in the Morozov–Solov'ev model for a toroidal magnetic field.

it is possible to compute the Mel'nikov function by selecting a branch of the separatrix connecting the points $(0, 0)$ and $(\pi, 0)$. Following the technique described above, we then obtain [153]

$$\delta\Psi(t_0) = \varepsilon_\omega \omega \lambda^2 \pi \operatorname{sech}\left(\frac{\lambda}{V_0} \frac{\omega\pi}{2}\right) \sin\left(\omega t_0 + \frac{\pi}{4}\right). \tag{97}$$

Here, the periodic perturbation with the characteristic frequency ω also causes the splitting of the separatrix, and the width of the stochastic layer is

$$\delta\Psi \approx \varepsilon_\omega \lambda^2 \omega\pi \operatorname{sech}\left(\frac{\lambda}{V_0} \frac{\omega\pi}{2}\right), \tag{98}$$

which leads to the characteristic dependence of the effective diffusion on the amplitude of oscillating perturbations. Now, it is not difficult to obtain an estimate of transport effects that takes the erosion of separatrices bounding vortex cells into account and corresponds to the data of the Solomon–Gollub experiment,

$$D_{\mathrm{eff}}(\varepsilon_\omega) \propto \delta\Psi \propto \varepsilon_\omega\,. \tag{99}$$

We remark that the use of the simplest form of the perturbation method restricts the applicability of the Mel'nikov method to only a close vicinity of separatrices. In the domain of well-developed chaos, the renormalization group method is used, as elaborated by Zaslavsky and Chirikov [4–6, 61, 62], whereas a detailed study of the transverse structure of intersections stimulated the development of symbolic dynamics methods [61, 62].

The Lagrangian turbulence is currently a fairly independent branch of research, and we therefore limit ourselves to just general remarks. Naturally, there is special interest in models of flows in which the form of the function Ψ is constrained by the condition of being consistent with the laws of fluid dynamics. For example, in geophysical fluid dynamics, one of the first models where dynamical chaos was carefully studied was the Kida vortex flow [154]. Classical fluid dynamics has a large number of model stream functions available, which may serve as satisfactory models of ocean flows, with the dynamics of scalar transport governed by simultaneously present large-scale eddies and (meandering) jet flows. The possibility of including the effects of rotation and allowing different forms of boundaries modeling the basin shoreline is also not altogether unworthy [79–83, 152, 155].

10. Splitting of saddle points and advective transport

The appearance of stochastic layers, which we considered in Section 6, invites a question on the existence of 'transport channels' in two-dimensional flows, which may substantially influence the transport processes. An instructive example can be constructed by considering steady perturbations of a regular system of convective cells. In the case of a superposition with a periodic shear flow with the stream function $\varepsilon_\Psi \sin y$, the resultant flow field

$$\Psi(x, y) = \Psi_0(\sin x + \sin y) + \varepsilon_\Psi \sin y \tag{100}$$

then contains 'open' streamlines (Fig. 24). Here, ε_Ψ is the amplitude of the shear perturbing flow. The existence of

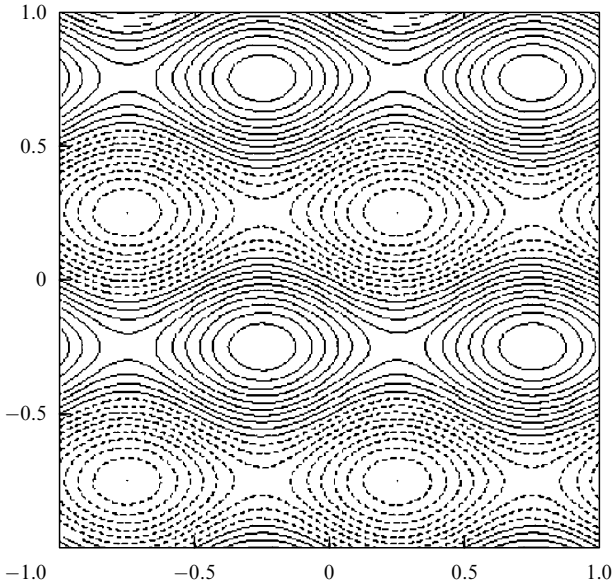

Figure 24. Appearance of open streamlines in a system of convective cells.

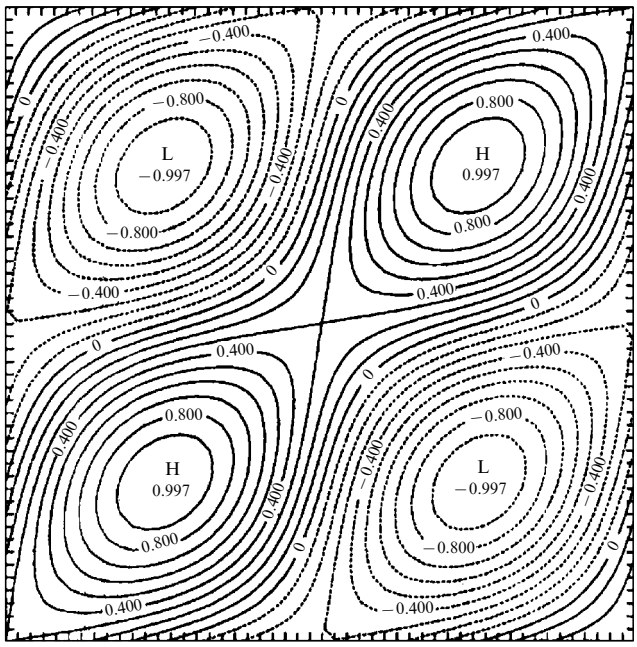

Figure 25. The cat-eye pattern in a system of convective (vortex) cells.

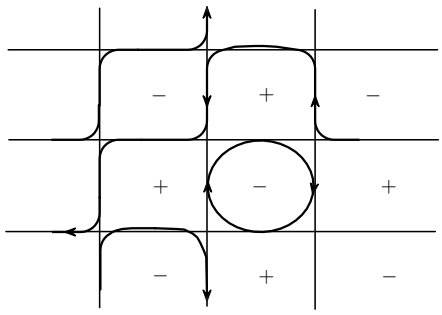

Figure 26. Random separatrix splitting in a system of convective (vortex) cells modeling a stationary chaotic two-dimensional velocity field.

streamlines piercing the entire system, obviously, leads to the dominance of convective transport over the diffusive one. Such situations occur in the physics of high-temperature magnetized plasmas, the physics of oceanic flows, etc.

The system of convective flows created by adding a small one-parameter periodic perturbation

$$\Delta\Psi(x,y) = \varepsilon_\Psi \cos x \cos y \tag{101}$$

to the system of regular convective cells (symmetric Taylor–Roberts cells) was carefully studied by numerically modeling the stream function

$$\Psi(x,y) = \Psi_0 \sin x \sin y + \varepsilon_\Psi \cos x \cos(ky)\,. \tag{102}$$

Here, ε_Ψ is the perturbation amplitude. In the case $\varepsilon_\Psi > 0$, the streamlines $\Psi = \text{const}$ form a periodic cat-eye system of vortices separated by open convective channels (Fig. 25). Here, advection is the dominant transport mechanism, and the width of convective channels for $\mathrm{Pe} \gg 1$ is estimated as [63]

$$\Delta(V_0) \propto \frac{1}{\sqrt{\mathrm{Pe}}} \propto \frac{1}{\sqrt{V_0}}\,. \tag{103}$$

Certainly, regular structures are an exception rather than the rule. A realistic but nontrivial vortex flow occurs in a system of regular convective cells if the separatrices are split randomly in the vicinity of saddle points under the action of external perturbations (Fig. 26). Such a two-dimensional steady flow with zero mean velocity is governed by an isotropic stream function $\Psi(x,y)$, oscillating on average and quasi-random in the distribution of saddle points with respect to the amplitude. In this formulation, the problem reduces to a percolation description of a topological phase transition (a model of continual percolation) (Fig. 27) in a system of streamlines [69–71]. It is rigorously proved that for any generic function $\Psi(x,y)$, there is a unique closed zero streamline (a percolation streamline) of infinite length [156, 157].

The problems of statistical topography were considered in the context of exploring two-dimensional turbulent flows from the standpoint of correlation analysis [158, 159], as well as in order to obtain effective diffusion coefficients [69, 71, 74] and to describe magnetic line reconnection in magnetic hydrodynamics [160]. As a model to analyze turbulent transport, this approach was first proposed by Kadomtsev and Pogutse [69], and detailed analysis of a monoscale steady percolation flow with

$$\Psi_0 \approx \lambda V_0\,, \qquad \lambda \approx \left|\frac{\Psi}{\nabla\Psi}\right| \tag{104}$$

was presented in [161].

A formal expression for the diffusion coefficient in the percolation limit can be written in the form generalizing a similar formula for convective cells,

$$D_{\mathrm{eff}} = \int_0^\infty \frac{\mathrm{d}\Psi_1}{\Psi_1}\, P_\infty(\Psi_1)\, \frac{a^2(\Psi_1)}{\tau(\Psi_1)}\,, \tag{105}$$

where the perturbation of the Hamiltonian is given by $\Psi_1 \approx \varepsilon_* \lambda V_0$. Here, the correlation time τ is estimated ballistically,

$$\tau \approx \tau_B \approx \frac{L(\varepsilon)}{V_0}\,, \tag{106}$$

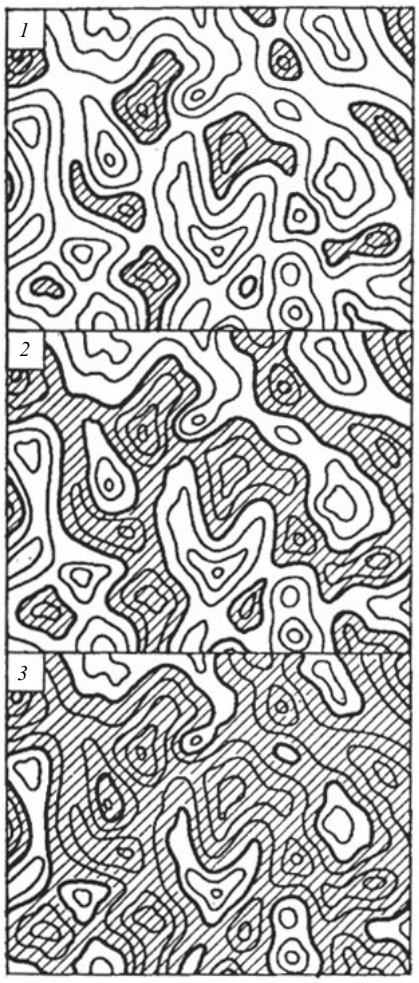

Figure 27. Percolation in a two-dimensional potential. The transition is between an 'endless' land confining isolated lakes to an 'endless' sea embracing isolated islands.

$P_\infty = L(\varepsilon)\Delta(\varepsilon)/a^2(\varepsilon)$ is the fraction of the volume occupied by percolation streamlines, Δ is the width of the percolation layer, and a is the correlation scale (the mixing length). Calculation leads to the scaling [161]

$$D_{\text{eff}}(\varepsilon) \approx \frac{a^2}{\tau}\, P_\infty \approx \frac{a^2}{\tau}\, \frac{L(\varepsilon)\Delta(\varepsilon)}{a^2} \approx V_0 \Delta(\varepsilon)\,, \tag{107}$$

which confirms the idea of the importance of considering stochastic layers in the analysis of transport under the conditions of structured turbulence. In essence, the problem of determining the turbulent diffusion coefficient in the adopted approximation reduces to computing the width of the stochastic (percolation) layer and the small parameter ε that characterizes the proximity of the system to the percolation transition.

Calculations of transport coefficients can be carried out to the end if we use scalings for the correlation scale a and the fractal streamline length L as functions of ε obtained in the continual percolation theory [156, 157],

$$a(\varepsilon) = \lambda\varepsilon^{-\nu}\,, \qquad L(\varepsilon) = \lambda\left(\frac{a}{\lambda}\right)^{D_{\text{h}}}\,, \tag{108}$$

where $\nu = 4/3$ and $D_{\text{h}} = 1 + 1/\nu$ are the percolation indices rigorously calculated for two-dimensional systems with the help of conformal field theory methods.

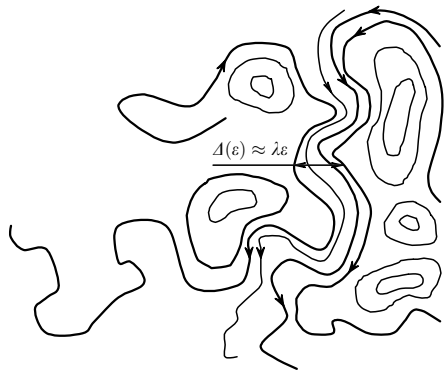

Figure 28. Internal structure of a stochastic layer containing a 'bunch' of streamlines, with Δ being the stochastic layer width, λ the characteristic spatial scale, and ε the small percolation parameter.

As it stands, the monoscale approximation of a complex two-dimensional vortex flow by a 'distorted' lattice of vortices is undoubtedly an oversimplification; however, rigorous results for percolation streamlines (the percolation hull) have been obtained only in this approximation. On the other hand, if we consider the full hierarchy of scales, it is already impressive even in this simplified case

$$L \approx \frac{a}{\varepsilon} \gg a \gg \lambda \gg \Delta \approx \lambda\varepsilon\,. \tag{109}$$

Indeed, besides the correlation scale and the 'perimeter of the shell', we have to include the width of the stochastic layer built around a percolation streamline in our consideration, because an individual line (isolated from the layer) cannot contribute to the effective transport. As mentioned above, the width of the stochastic layer is proportional to the amplitude of perturbations, $\Delta \propto \varepsilon_\omega\lambda$, which in our case can be identified with the percolation parameter $\varepsilon \approx \varepsilon_\omega$ characterizing the deviation of the system from the 'ideal' percolation transition. This ensures the finiteness of percolation streamlines in model flows. On the other hand, such a definition can be interpreted as specifying the internal metric of the percolation stochastic layer model [161, 162]. In this sense, we are dealing with a situation essentially different from that assumed in the approach by Skal–Shklovskii–de Gennes [157], where a similar internal metric was based on the macroscale $L \approx \lambda/\varepsilon$ (the length of the macrolink of the conducting cluster).

Additionally, we can introduce the characteristic scale l_{s} associated with an individual streamline because the area corresponding to the stochastic layer Δ is mush larger than the size of the 'elementary vortex cell':

$$L(\varepsilon)\Delta(\varepsilon) \approx \frac{a(\varepsilon)\lambda\varepsilon}{\varepsilon} \approx a\lambda \gg \lambda^2\,. \tag{110}$$

This is natural because the stochastic layer hosts many streamlines, not just one (Fig. 28). A suitable estimate characterizing the size of regions where the adiabatic invariance is violated (exponentially narrow layers near separatrices), l_{s}, is [163]

$$L(\varepsilon)\, l_{\text{s}}(\varepsilon) \approx \lambda^2\,. \tag{111}$$

Simple manipulations give

$$l_{\text{s}} \approx \frac{\lambda^2}{L(\varepsilon)} \approx \lambda\varepsilon^{\nu+1} \ll \Delta \approx \lambda\varepsilon\,. \tag{112}$$

In summary, the full hierarchy of spatial scales for percolation models of random two-dimensional flows,

$$\lambda\varepsilon^{\nu} \approx l_{\mathrm{s}} \ll \Delta \approx \lambda\varepsilon \ll \lambda \ll a \approx \frac{\lambda}{\varepsilon^{\nu}} \ll L \approx \frac{a}{\varepsilon} \approx \frac{\lambda}{\varepsilon^{\nu+1}}\,, \qquad (113)$$

has a sufficiently 'elaborate' form, which allows analyzing nontrivial correlation and transport effects in two-dimensional models of structured turbulence.

11. Stochastic layer in the percolation limit

Gruzinov, Isichenko, and Kalda [163] considered a percolation model of transport in two-dimensional random flows. In the low-frequency case $\omega \ll V_0/\lambda$, the actual correlation scale a defining the effective transport is much less than V_0/ω, which formally describes the path length of particles along streamlines. Here, the main decorrelation mechanism is connected to topology rearrangements (for instance, the reduction in the length of streamlines because of their reconnection) (Fig. 29). We note that deriving the scalings describing vortex reconnection processes is a difficult problem, because a rigorous physical–mathematical picture of turbulence has not been created, and scaling estimates have to be used [164].

Turning to the ideas of the percolation hierarchy of spatial scales, we can consider the character of anomalous transport in the case where percolation streamlines play a defining role. The number of streamlines in the stochastic layer can be estimated as the ratio of two characteristic scales:

$$N_{\mathrm{s}}(\varepsilon) \propto \frac{\Delta}{l_{\mathrm{s}}} \propto \frac{a}{\lambda} \propto P_{\infty}(\varepsilon) \gg 1\,. \qquad (114)$$

In addition to the characteristic scale of turbulent velocity pulsations V_0, the estimate $V_{\mathrm{s}} = \lambda\omega \ll V_0$ for the velocity of the 'accompanying' motion of separatrices in the stochastic layer was proposed in [163]. The characteristic time interval for reconnection between two nearest separatrices can be estimated as

$$\tau_{\mathrm{s}} \approx \frac{l_{\mathrm{s}}(\varepsilon)}{V_{\mathrm{s}}} \approx \frac{l_{\mathrm{s}}}{\lambda\omega} \approx \frac{\lambda}{L(\varepsilon)\omega} \ll \frac{1}{\omega}\,. \qquad (115)$$

This is in fact an estimate of the stochastic instability increment $\gamma_{\mathrm{s}} \approx 1/\tau_{\mathrm{s}}$, which is noticeably different from the simplest, but widely disseminated, Batchelor approximation $\gamma_{\mathrm{s}} \approx V_0/\lambda$.

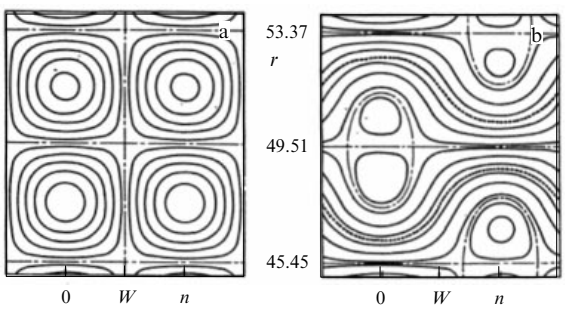

Figure 29. Intersection of streamlines in a system of convective cells. (a) System of regular convective cells. (b) Flow forming as a result of separatrix reconnection under the action of a small perturbation.

Additionally, it is possible to estimate the time of full 'mixing' in the stochastic layer (correlation time):

$$\tau(\varepsilon) \propto \frac{\Delta(\varepsilon)}{V_{\mathrm{s}}} \propto \varepsilon\,\frac{1}{\omega}\,, \qquad (116)$$

which allows a qualitative description of streamline reconnection processes in the low-frequency limit [71, 72]. It should be kept in mind that a single streamline does not contribute to the effective transport. We are dealing in reality with a nonstationary process of reconnection, which leads to the appearance of a bunch of percolation streamlines (stochastic layer). We would expect to obtain a useful result if it were possible to compute a concrete 'universal' value of the small percolation parameter $\varepsilon_* = \varepsilon_*(\omega, V_0, \lambda)$. The authors of Ref. [163] proposed that the characteristic mixing time in the stochastic layer is of the same order of magnitude as the time scalar particles move ballistically along the percolation streamline,

$$\frac{\varepsilon_*}{\omega} = \frac{L(\varepsilon_*)}{V_0}\,, \qquad (117)$$

where $L(\varepsilon) = \lambda(a/\lambda)^{D_{\mathrm{h}}}$. In this case, computations give the small percolation parameter ε_* as a function of the flow parameters ω, V_0, and λ:

$$\varepsilon_* = \left(\frac{\lambda\omega}{V_0}\right)^{1/(2+\nu)} = \left(\frac{1}{\mathrm{Ku}}\right)^{3/10}, \qquad \nu = 4/3\,. \qquad (118)$$

Here, $\varepsilon_* \ll 1$ for $\mathrm{Ku} \gg 1$. The expression for the diffusion coefficient in the percolation limit is written in the form

$$D_{\mathrm{eff}}(\Psi_1) = \int_0^{\infty} \frac{\mathrm{d}\Psi_1'}{\Psi_1'}\, P_{\infty}(\Psi_1')\, \frac{a^2(\Psi_1')}{\tau(\Psi_1')}\,, \qquad (119)$$

where $\Psi_1 \approx \varepsilon_*\lambda V_0$. Simple manipulations lead to the following expression for the effective transport characterized by D_{eff} [163]:

$$D_{\mathrm{eff}}(\varepsilon_*) \approx \lambda^2\omega\left(\frac{V_0}{\lambda\omega}\right)^{7/10} \approx \lambda^2\omega\,\mathrm{Ku}^{7/10} \propto V_0^{7/10}\omega^{3/10}\,. \qquad (120)$$

The character of percolation dependence (120) is fundamentally different from the quasilinear estimate $D_{\mathrm{eff}}(\omega) \propto V_0^2/\omega$, but agrees with simple estimates we performed previously based on the result by Chirikov and Zaslavsky for a system of convective cells, $D_{\mathrm{eff}} \propto \sqrt{V_0\omega}$. In both cases, we have an increase in the effective transport with frequency in low-frequency regimes, whereas the dependence on the amplitude of turbulent pulsations is slower than the linear one. Numerous experiments confirm the percolation scaling for regimes with $\mathrm{Ku} \gg 1$ [165–171].

If we assume that the characteristic frequency ω describing a periodic action on the system is not excessively high (the low-frequency limit), we can assume the characteristic time for stochastic instability to evolve to be comparable with the time a particle follows a percolation streamline. This allows taking the balance of characteristic times $\tau_{\mathrm{s}} \approx \tau_B$ as the basis for computing the small percolation parameter:

$$\frac{\lambda}{L(\varepsilon_*)\omega} = \frac{L(\varepsilon_*)}{V_0}\,. \qquad (121)$$

As a result of calculation, we obtain the scaling

$$\varepsilon_* \approx \left(\frac{\lambda\omega}{V_0}\right)^{1/[2(\nu+1)]} \approx \left(\frac{1}{\mathrm{Ku}}\right)^{3/14}, \qquad \nu = 4/3\,. \tag{122}$$

The expression for the stochastic instability increment γ_s in the percolation limit becomes

$$\gamma_\mathrm{s} \approx \frac{V_\mathrm{s}}{l_\mathrm{s}(\varepsilon_*)} \approx \omega\,\frac{L(\varepsilon_*)}{\lambda} \approx \omega\sqrt{\mathrm{Ku}}\,. \tag{123}$$

Expression (123) is markedly different from the quasilinear result of Kadomtsev and Pogutse, $\gamma_\mathrm{s} \approx \omega\,\mathrm{Ku}^2$ [69], and is in good agreement with the results of numerical simulations [167, 170].

Effects of stochastic instability are prominent through the stage preceding full 'mixing'. From this standpoint, it is convenient to introduce an 'evolving' (running) coefficient of diffusion,

$$D_*(t) \propto \frac{R^2(t)}{t}\,. \tag{124}$$

Here, $R^2(t)$ is the root mean square displacement of a scalar particle and t is the time. From general arguments, it is clear that for times less than the correlation time, $t \ll \tau$, particles move ballistically, $R^2 \propto V_0^2 t^2$ and $D_* \propto V_0^2 t$ (Fig. 30).

A stage then begins related to decorrelation processes in the stochastic layer forming under the action of instability. Simultaneously with the increase in the length of the percolation streamline (or the growth of the path traveled by a scalar particle along the streamline), $L(t) \propto V_0 t$, the correlation scale increases:

$$a_I(t) \approx \left(\frac{L(t)}{\lambda}\right)^{1/D_\mathrm{h}} \approx \left(\frac{V_0 t}{\lambda}\right)^{1/D_\mathrm{h}}. \tag{125}$$

We note that the increase in the correlation scale can be interpreted in the framework of the Corrsin approach, which treats the evolution of Lagrangian correlations using a model of correlation cloud growth [32, 172, 173]. This initial phase of fractal cluster growth is associated with a subdiffusive transport regime, because particles 'explore' the fractal streamline. To estimate the root mean square displacement, we use the expression

$$R^2(t) \propto a^2(t)\,P_\infty(t) \propto a(t)\lambda\,, \tag{126}$$

which leads to the scaling for the transport coefficient

$$D_*(t) \propto \frac{R^2(t)}{t} \propto \frac{1}{t^{3/7}} \tag{127}$$

and to the formula for the root mean square displacement

$$R(t) \approx \lambda\left(\frac{L(t)}{\lambda}\right)^{1/(2D_\mathrm{h})} \approx \lambda\left(\frac{V_0 t}{\lambda}\right)^{1/(2D_\mathrm{h})} \propto t^{2/7}, \tag{128}$$

found by Isichenko in studying the stochasticity of magnetic field lines [174, 175].

The last stage, $t \geqslant \tau$, has a traditional, diffusive character, and the correlation time (mixing time) can be obtained by using the linear estimate [176]

$$\Delta(t) \propto V_\mathrm{s} t \propto \lambda\omega t \tag{129}$$

and the balance of evolving correlation scales

$$\lambda\left(\frac{V_0\tau}{\lambda}\right)^{1/D_\mathrm{h}} = \lambda\left(\frac{\lambda}{\Delta(\tau)}\right)^{\nu}. \tag{130}$$

We note that the linear estimate for the evolution of the stochastic layer width is a key link between the phenomenological approximation of the correlation time in the form $\tau \propto \varepsilon T_0$ and the Zaslavsky–Filonenko scaling for the stochastic layer width in low-frequency regimes:

$$\Delta \propto \frac{\omega}{\omega_0}\,\varepsilon\lambda \propto \omega\lambda\varepsilon T_0\,.$$

Thus, as in steady random flows, the main closure of the percolation model is $\Delta \propto \varepsilon\lambda$. An estimate of the turbulent diffusion coefficient is given by the expression $D_\mathrm{eff} \approx \lambda V_0(\lambda\omega/V_0)^{3/10}$ [163], obtained previously for the nonstationary percolation model. The results given in this section are used in Section 15 for the analysis of work dealing with the transport of particles in a stochastic magnetic field of tokamaks (Fig. 31).

The Kubo number used in this section can be conveniently interpreted in terms of an adiabatic invariant if we introduce

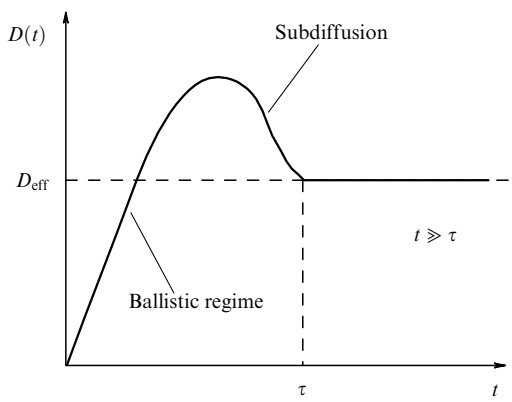

Figure 30. Evolution of the particle diffusion coefficient in the percolation model. $D(t)$ is the running coefficient of diffusion, D_eff is the effective coefficient of diffusion established as the result of evolution, and t is time.

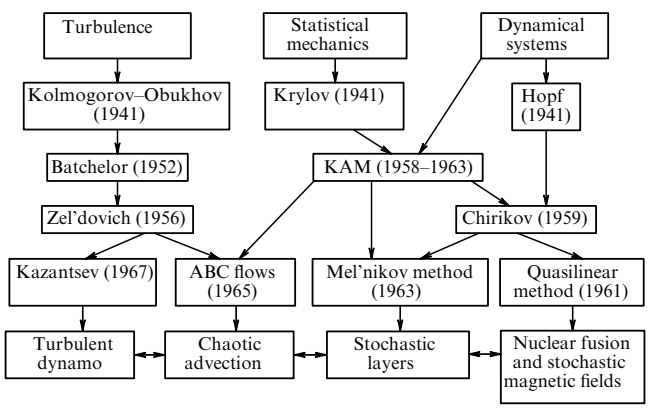

Figure 31. Flow chart of the main avenues in the development of the stochastic instability theory.

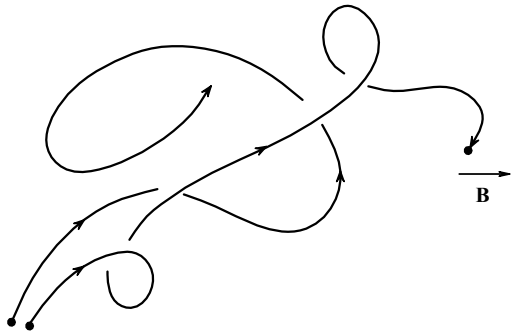

Figure 32. Divergence of the initially close field lines in a stochastic magnetic field.

the adiabaticity parameter as the ratio of the characteristic time a particle moves in a random flow to the time over which the stream function varies. For instance, when describing advection in three-dimensional flows, we do not have a universal method, but resorting to the ideas of adiabatic invariant diffusion (adiabatic chaos) [177–180] proves helpful in treating certain types of flows [181].

12. Stochastic magnetic field and a quasilinear estimate of the increment

The problem of divergence of initially close field lines (Fig. 32) is of great practical significance in plasma physics and astrophysics [182–184]. In this section, we consider one of the simplest estimates related to the stochastic instability of magnetic field lines in the framework of a quasilinear approach and confirm the 'universality' of the result by Kazantsev in the case of weak turbulence of magnetic field lines.

In considering stochastic magnetic fields, we need to redefine the notions of increment and of the Kolmogorov–Sinai entropy to account for the new features. Two initially close field lines diverge from each other, on average, according to the law

$$\Delta(z) = l_0 \exp\left(\frac{z}{\lambda_{\mathrm{K}}}\right), \tag{131}$$

where l_0 is the initial distance between the field lines and z is the distance traveled along the field line. The quantity $h_{\mathrm{K}} = 1/\lambda_{\mathrm{K}}$ is called the Kolmogorov entropy:

$$h_{\mathrm{K}} = \lim_{l_0 \to 0,\, z \to \infty} \left(\frac{1}{z} \ln \frac{\Delta(z)}{l_0}\right). \tag{132}$$

In the Lagrangian approach, the field line equations of motion are analogous to those describing streamlines. This suggests that the analysis results obtained previously for weakly turbulent flows can be used to describe the turbulence of magnetic field lines. Kadomtsev and Pogutse succeeded in realizing this approach in a strongly anisotropic case where a weak random field $\mathbf{B}'\,(B_x, B_y, 0)$ is superimposed on a strong constant field $\mathbf{B}(0, 0, B_0)$ aligned with the z axis,

$$\mathbf{B}(x, y, z) = B_0\, \mathbf{e}_z + \mathbf{B}'(x, y, z), \tag{133}$$

where $\operatorname{div} \mathbf{B}'(x, y, z) = 0$. The Lagrangian equations of motion for a field line are given in this approximation by the vector equation

$$\frac{\mathrm{d}\mathbf{r}_\perp}{\mathrm{d}t} = \mathbf{b}(z, \mathbf{r}_\perp), \qquad \mathbf{b} = \frac{\mathbf{B}'}{B_0}. \tag{134}$$

For estimates, we use the quantity b_0 as a characteristic relative perturbation scale. In problems related to diffusion of magnetic field lines in a high-temperature plasma, its order of magnitude is estimated as $b_0 \approx 10^{-3} - 10^{-4}$ [185, 186].

In this case, the classical Taylor expression [69, 187] for the coefficient of transverse diffusion of magnetic field lines takes the quasilinear form:

$$D_{\mathrm{m}} = \frac{1}{4} \int_{-\infty}^{\infty} \mathrm{d}z \left\langle \mathbf{b}(z, 0)\, \mathbf{b}(0, 0) \right\rangle \propto b_0^2 \lambda_z. \tag{135}$$

Here, $\langle \ldots \rangle$ denotes averaging and

$$\lambda_z = \frac{1}{b_0^2} \int_{-\infty}^{\infty} \mathrm{d}z \left\langle \mathbf{b}(z, 0)\, \mathbf{b}(0, 0) \right\rangle. \tag{136}$$

is the longitudinal correlation scale of the stochastic magnetic field. In the anisotropic case, we need to carefully analyze both longitudinal and transverse correlation effects. Here, we neglect the transverse displacement $\lambda_\perp$ in the Taylor expression for the diffusion coefficient, $\mathbf{b}(z, \lambda_\perp) \approx \mathbf{b}(z, 0)$, which is a valid approximation only if the diffusive displacement in the transverse direction is much less than the transverse correlation scale, $b_0 \lambda_z \ll \lambda_\perp$. Kadomtsev and Pogutse [69] proposed an applicability criterion for such an approach in terms of a dimensionless parameter, the magnetic Kubo number characterizing the ratio of longitudinal and transverse correlation effects:

$$\mathrm{Ku}_{\mathrm{m}} = \frac{b_0 \lambda_z}{\lambda_\perp}. \tag{137}$$

The quasilinear approximation is valid for $\mathrm{Ku}_{\mathrm{m}} < 1$.

Now, only simple manipulations are needed to derive an expression describing the initial stage of field line divergence in a stochastic magnetic field,

$$\frac{\partial}{\partial z}(r_2 - r_1) = b(z, r_2) - b(z, r_1) \approx \frac{\partial b}{\partial r}(r_2 - r_1), \tag{138}$$

in the limit of a small initial displacement $r_2 - r_1$ [69]. An estimate of the relative displacement can be obtained in the exponential form,

$$r_2(z) - r_1(z) = \Delta r \approx \Delta r\Big|_{z=0} \exp\left(\int_0^z \frac{\partial b}{\partial r}\, \mathrm{d}z\right). \tag{139}$$

For the increment of stochastic instability, scaling implies averaging this expression for $\Delta r(z)$. Assuming that the magnetic field perturbations are Gaussian, it is possible, resorting to the classical formula

$$\langle \exp A \rangle = \exp\left(\frac{1}{2} \langle A^2 \rangle\right), \tag{140}$$

to estimate the stochastic instability increment, which makes the quadratic character of the increment dependence on the magnetic field amplitude manifest:

$$\gamma_z \approx \frac{b_0^2 \lambda_z}{\lambda_\perp^2} \approx \frac{1}{\lambda_z}\, \mathrm{Ku}_{\mathrm{m}}^2. \tag{141}$$

The result in (141) can be rewritten in a somewhat different, but also informative form $\gamma_z \approx D_{\rm m}/\lambda_\perp^2$, where it is taken into account that the quasilinear limit implies the relation $D_{\rm m} \approx b_0^2\lambda_z$. The applicability limits for the expression of the stochastic instability increment coincide with those for the quasilinear approximation, $\mathrm{Ku}_{\rm m} \approx b_0\lambda_z/\lambda_\perp < 1$. We note that for a quasi-isotropic stochastic magnetic field, similar computations were carried out by Ptuskin [188].

An important role in estimates of the stochastic instability increment is played by topological details of the stochastic magnetic field. For example, in the problem of magnetic confinement of high-temperature plasma, it is important to account for the shear effect [185, 186],

$$\mathbf{B} = B_0\big(\mathbf{e}_z + q(x)\mathbf{e}_y\big) + \delta\mathbf{B}(x,y,z)\,, \tag{142}$$

where q is the magnetic field shear.

Despite a substantially increased complexity in the picture of field line behavior, the quantity γ_z is successfully redefined in this case by introducing an additional characteristic spatial scale $L_{\rm s} = (\mathrm{d}q/\mathrm{d}x)^{-1} \approx$ const for constant shear. The modified equations for the field lines take the form

$$\frac{\mathrm{d}\mathbf{r}_\perp(x,y)}{\mathrm{d}z} = \mathbf{b}(r_\perp, z) + q(x)e_y\,. \tag{143}$$

If the displacements δx and δy are small, an analog of the quasilinear approximation is the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\delta x = \frac{\partial b_x}{\partial x}\,\delta x + \frac{\partial b_x}{\partial y}\,\delta y\,, \tag{144}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\,\delta y = \left(\frac{\partial b_y}{\partial x} + \frac{1}{L_{\rm s}}\right)\delta x + \frac{\partial b_x}{\partial y}\,\delta y\,. \tag{145}$$

The qualitative approach does not work here. The equation for the evolution of $\Delta\mathbf{r} = (\langle\delta x^2\rangle, \langle\delta y^2\rangle, \langle\delta x\,\delta y\rangle)$ is written in a matrix form,

$$\frac{\mathrm{d}\Delta\mathbf{r}}{\mathrm{d}z} = \widehat{W}\Delta\mathbf{r}\,, \tag{146}$$

where eigenvalues of the matrix $\widehat{W}$ allow finding the stochastic instability increment in the case of constant shear,

$$\gamma_z = \frac{1}{(\lambda_z L_{\rm s}^2)^{1/3}}\,, \qquad L_{\rm s} < \lambda_z\,. \tag{147}$$

Unfortunately, this result, found by Kromes [94], does not allow establishing a direct link between γ_z and the Kubo number $\mathrm{Ku}_{\rm m} \approx b_0\lambda_z/\lambda_\perp$ characterizing the influence of transverse correlation effects.

13. Weak turbulence and the Kadomtsev–Pogutse scaling

If we suppose that the main decorrelation mechanism is stochastic instability, then it is natural to use the expression for its increment in estimates of transport effects. We suppose that decorrelation is attributed not to collisions between particles moving in a stochastic magnetic field, but to stochastic instability manifested in the exponential divergence of neighboring field lines. The role of seed diffusion in this case is played by the random walk of magnetized particles along and transverse to the magnetic field lines.

The simplest estimates can be derived by considering the expressions for the particle diffusion coefficient $D_\perp$ written in terms of the magnetic diffusion coefficient $D_{\rm m}$,

$$D_\perp \propto \frac{\Delta_{\rm cor}^2}{\tau} \propto D_{\rm m}\,\frac{L_{\rm cor}}{\tau}\,. \tag{148}$$

Here, $L_{\rm cor}$ is the longitudinal correlation length and τ is the correlation time. In this approach, it is assumed that the magnetic diffusion coefficient $D_{\rm m}$ and collisional coefficients of longitudinal and transverse diffusion, $\chi_\parallel$ and $\chi_\perp$, are known. The analysis is commonly carried out in terms of thermal conductivity coefficients in order to free the problem from complications brought about by plasma ambipolarity.

If the longitudinal motion of particles along the magnetic field bears a diffusive character, it is convenient to introduce a characteristic spatial scale λ_z that corresponds to the longitudinal correlation length. The quantities $L_{\rm cor}$ and τ are related to each other through the expression for the longitudinal diffusion coefficient $\chi_\parallel \approx L_{\rm cor}^2/(2\tau)$. Substituting $\tau = \tau(\chi_\parallel)$ in Eqn (148), we then obtain an estimate of the transverse diffusion coefficient:

$$D_\perp \propto D_{\rm m}(b_0)\,\frac{\chi_\parallel}{L_{\rm cor}}\,. \tag{149}$$

In terms of the correlation time τ, the expression for the transverse diffusion coefficient on scales $z > \lambda_z$ becomes

$$D_\perp(\tau) \propto D_{\rm m}(b_0)\sqrt{\frac{\chi_\parallel}{\tau}}\,. \tag{150}$$

We used the simplest model of a random walk of field lines in order to obtain expressions for particle transport effects based on simple estimates.

Kadomtsev and Pogutse proposed a new scaling for the correlation time τ, which relies on the increment of stochastic instability of field lines γ_z. Assuming that particle longitudinal motion is diffusive and resorting to dimensional arguments, it is convenient to represent the characteristic correlation time in the form [69]

$$\tau \approx \frac{1}{\gamma_z^2\chi_\parallel} \approx \frac{\lambda_\perp^4}{b_0^4\lambda_z\chi_\parallel} \approx \frac{\lambda_z^2}{\chi_\parallel}\,\mathrm{Ku}_{\rm m}^{-4}\,. \tag{151}$$

Inserting expression (151) into formula (150), we obtain the transverse particle diffusion coefficient

$$D_\perp(\mathrm{Ku}_{\rm m}) \propto D_{\rm m}\chi_\parallel\gamma_z(\mathrm{Ku}_{\rm m}) \approx D_{\rm m}\chi_\parallel\,\frac{\mathrm{Ku}_{\rm m}^2}{\lambda_z}\,. \tag{152}$$

The field line diffusion coefficient $D_{\rm m}$ here also depends on the magnetic Kubo number. Because the quasilinear approximation has been used for the stochastic instability increment, this same quasilinear approximation has to be used for the field line diffusivity, $D_{\rm m} \propto b_0^2\lambda_z$. As a result, we obtain scaling for the effective coefficient of particle transverse diffusion in a stochastic magnetic field, with a nontrivial dependence on the amplitude of turbulent field pulsations:

$$D_\perp(b_0) \propto \chi_\parallel b_0^2\mathrm{Ku}_{\rm m}^2 \approx \chi_\parallel b_0^4\left(\frac{\lambda_z}{\lambda_\perp}\right)^2\,. \tag{153}$$

The applicability condition for this regime of transport follows from the principle of fast mode selection $\tau_{\rm eff} < \tau_\perp$ [189]. Here, we assume that an alternative mechanism of decorrelation is the transverse collisional diffusion of parti-

cles (with the characteristic decorrelation time $\tau_\perp$), leading to a change in the carrier field line. Then, taking the last inequality into account, we arrive at the existence condition for regimes in which the main decorrelation mechanism is stochastic instability,

$$\frac{\lambda_\perp^2}{D_\perp(\chi_\parallel)} < \frac{\lambda_\perp^2}{\chi_\perp}\,, \tag{154}$$

or, in terms of transport coefficients,

$$D_\perp \propto \chi_\parallel b_0^4 \left(\frac{\lambda_z}{\lambda_\perp}\right)^2 > \chi_\perp\,. \tag{155}$$

It is noteworthy that in problems of high-temperature plasma confinement in tokamak installations, the coefficient of longitudinal particle diffusion is much larger that the coefficient of transverse diffusion, $\chi_\parallel \gg \chi_\perp$. This condition can easily be rewritten in terms of plasma physics:

$$\chi_\parallel \approx V_T^2 \tau_{\rm ei}\,, \tag{156}$$

$$\chi_\perp \approx \frac{r_{\rm e}^2}{\tau_{\rm ei}} \approx \frac{1}{\tau_{\rm ei}}\left(\frac{V_T}{\Omega_{\rm e}^{\rm h}}\right)^2. \tag{157}$$

Here, V_T is the thermal speed of electrons, $\nu_{\rm ei} \approx 1/\tau_{\rm ei}$ is the electron–ion collision frequency, $r_{\rm e}$ is the Larmor electron radius, and $\Omega_{\rm e}^{\rm h} \gg \nu_{\rm ei}$ is the electron gyrofrequency. Hence, we obtain the condition that longitudinal diffusion dominates, expressed in terms of characteristic time scales of the problem:

$$\frac{\chi_\parallel}{\chi_\perp} \approx (\Omega_{\rm e}^{\rm h}\tau_{\rm ei})^2 \gg 1\,. \tag{158}$$

Performing simple computations, we find the applicability region of the regime being explored, in terms of plasma physics (Fig. 33),

$$b_0 \Omega_{\rm e}^{\rm h} \tau_{\rm ei} > \frac{1}{{\rm Ku_m}} \approx \frac{\lambda_\perp}{b_0 \lambda_z}\,, \tag{159}$$

where ${\rm Ku_m} \ll 1$ because we used the quasilinear model for $D_{\rm m}$ and γ_z. We note that condition (159) can be interpreted in terms of characteristic spatial scales:

$$l_{\rm cor}(\chi_\parallel) \propto \sqrt{2\chi_\parallel \tau_{\rm eff}({\rm Ku_m})} \approx \frac{\lambda_z}{{\rm Ku_m}} \gg \lambda_z\,. \tag{160}$$

The approach proposed by Kadomtsev and Pogutse, which is based on the 'direct' use of the stochastic instability increment, does not take many details of transport in a stochastic magnetic field into account. Nevertheless, the scaling obtained is important for explaining processes occurring in magnetized plasmas and plays an important role in the analysis of correlation effects [121, 124, 125].

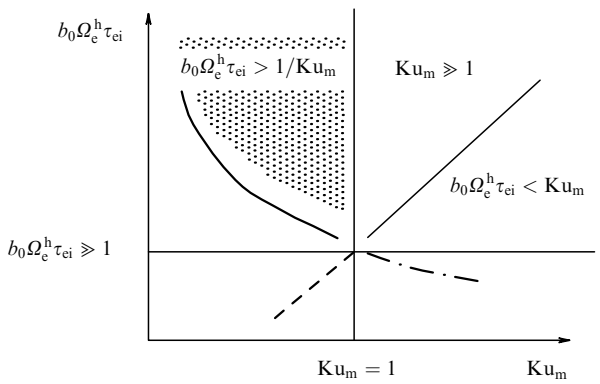

Figure 33. Diagram showing the applicability domain for the Kadomtsev–Pogutse regime. b_0 is the dimensionless amplitude of magnetic field fluctuations, $\mathrm{Ku_m}$ is the magnetic Kubo number, $\Omega_{\rm e}^{\rm h}$ is the electron rotation gyrofrequency, and $\tau_{\rm ei}$ is the characteristic time of electron–ion collisions. The Kadomtsev–Pogutse regime is applicable in the dotted domain.

14. Rochester–Rosenbluth approximation

The approach of Kadomtsev and Pogutse discussed in Section 13 was extended by them in the same Ref. [69] by an analysis of the decorrelation role of collisions of particles moving in a braided magnetic field. Decorrelations linked to collisions and stochastic instability were treated by the authors independently, and each separate mechanism led to a respective scaling [69]. It is possible to consider these effects jointly, based on ideas on the character of the behavior of a random magnetic field tube.

It is necessary to mention that the Batchelor model is well known in the theory of turbulent transport [34, 86]; there, to analyze the behavior of a tracer patch in the field of Kolmogorov turbulence, a conjecture was made on the importance of accounting for the 'competition' between the processes of exponential instability and molecular diffusion. Rochester and Rosenbluth [190] used this approach to explore the transport of particles in a stochastic magnetic field, considering the evolution of a field line tube of a stochastic magnetic field, such that the behavior of its transverse section 'copies' the already studied behavior of a 'small element of phase fluid' with the scale l_0 (Fig. 34). By virtue of the magnetic field incompressibility $\operatorname{div}\mathbf{B} = 0$, in the presence of stochastic instability, together with an exponential growth of the distance between field lines $l(z) = l_0 \exp(z/\lambda_{\rm K})$, the width of an element decreases exponentially so as to preserve the initial area,

$$\delta\big(z(t)\big) = l_0 \exp\left(-\frac{z(t)}{\lambda_{\rm K}}\right). \tag{161}$$

Here, δ is the evolving width of the magnetic tube and z is the distance traveled along the field line. In the collisionless case, $z(t) \approx V_T t$. In the case of a diffusive particle walk, it is natural to use the estimate $z^2(t) \approx 2\chi_\parallel t$, where $\chi_\parallel$ is the longitudinal coefficient of particle diffusion.

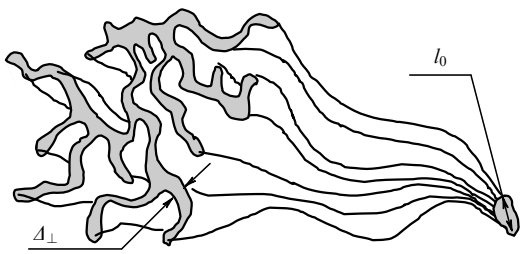

Figure 34. Cross section of a field-line tube in a braided magnetic field. $\Delta_\perp$ is the width of the stochastic layer in the tube cross section and l_0 is the initial transverse size of the cross section.

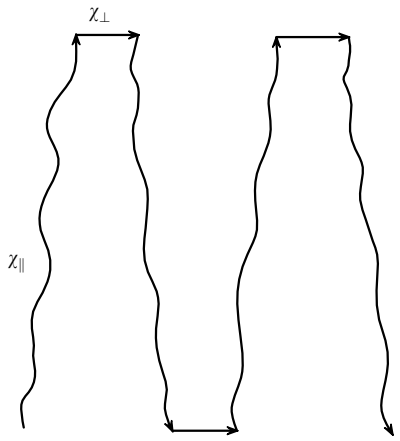

Figure 35. Schematic of the path of a charged particle in the Rochester–Rosenblut model; $\chi_\parallel$ and $\chi_\perp$ are the longitudinal and transverse heat conductivities.

However, it is necessary to include the effect of transverse diffusion processes, which increase δ. Rochester and Rosenbluth considered a collisional case, assuming that $\delta^2(z) \approx 4\chi_\perp t$, where $\chi_\perp$ is the transverse particle diffusion coefficient. After the substitution of the last expression in Eqn (161), we obtain an equation for the correlation time τ,

$$\sqrt{2\chi_\parallel \tau} \approx -\lambda_{\mathrm{K}} \ln \frac{\delta(\tau)}{l_0}\,, \tag{162}$$

which describes an equilibrium that sets in as a result of competition between the processes of exponential shrinking of the tube and its diffusive broadening. We solve transcendental equation (162) by taking $\tau \approx \lambda_{\mathrm{K}}^2/(2\chi_\parallel)$ as the first approximation. As a result, we find

$$\tau \approx \frac{\lambda_{\mathrm{K}}^2}{8\chi_\parallel} \ln^2 \left[4\left(\frac{l_0}{\lambda_{\mathrm{K}}}\right)^2 \frac{\chi_\parallel}{\chi_\perp}\right] \approx \frac{\tau_{\mathrm{K}}}{4} \ln^2 \left[4\left(\frac{l_0}{\lambda_{\mathrm{K}}}\right)^2 \frac{\chi_\parallel}{\chi_\perp}\right]. \tag{163}$$

We assume that $\chi_\parallel/\chi_\perp \approx (\Omega_{\mathrm{e}}^{\mathrm{h}}\tau_{\mathrm{ei}})^2 \gg 1$, and hence a particle performs several longitudinal diffusive steps and only then leaves the field line (decorrelates) because of the presence of transverse diffusion (Fig. 35).

For $z > \lambda_z$, we have the expression $D_\perp(\tau) \propto D_{\mathrm{m}}(\chi_\parallel/\tau)^{1/2}$ for the particle diffusion coefficient; hence, the final formula becomes

$$D_{\mathrm{RR}}(\tau) \propto 4\chi_\parallel \frac{D_{\mathrm{m}}(b_0)}{\lambda_{\mathrm{K}}} \ln^{-1} \left[\left(\frac{\lambda_\perp}{\lambda_{\mathrm{K}}}\right)^2 \frac{\chi_\parallel}{\chi_\perp}\right]. \tag{164}$$

We here used the estimates of model parameters valid for problems of magnetic confinement of high-temperature plasmas, $l_0 \approx \lambda_\perp$ and $b_0 \ll 1$. The parameters D_{m}, $\chi_\parallel$, $\chi_\perp$, $\lambda_\perp$, and λ_{K} are considered to be known, and the condition $(\lambda_\perp/\lambda_{\mathrm{K}})^2(\chi_\parallel/\chi_\perp) > 1$ is assumed to hold, which can be conveniently expressed in terms of characteristic time scales of the model (the dominance of the fast mode):

$$\tau_\perp \approx \frac{\lambda_\perp^2}{\chi_\perp} > \frac{\lambda_{\mathrm{K}}^2}{\chi_\parallel} \approx \tau_{\mathrm{K}}\,. \tag{165}$$

Additionally, we can take the magnetic Kubo number into account, which enters the condition written above if we use the plasmo-physical parameters $b_0\Omega_{\mathrm{e}}^{\mathrm{h}}\tau_{\mathrm{ei}} > \mathrm{Ku}_{\mathrm{m}}$.

The correlation time and correlation length are uniquely related. In the case of diffusive longitudinal motion considered here, the condition $\tau \approx l_{\mathrm{cor}}^2/(2\chi_\parallel)$ holds. Accordingly, we can derive a transcendental equation for the correlation length l_{cor} if we use the estimate obtained above,

$$\delta \approx \sqrt{4\chi_\perp \tau} \approx l_{\mathrm{cor}} \sqrt{2\,\frac{\chi_\perp}{\chi_\parallel}} \ll l_{\mathrm{cor}}\,, \tag{166}$$

and rearrange the initial balance for the characteristic width of the field tube in new terms:

$$l_{\mathrm{cor}} \sqrt{\frac{\chi_\perp}{\chi_\parallel}} = \lambda_\perp \exp\left(-\frac{l_{\mathrm{cor}}}{\lambda_{\mathrm{K}}}\right). \tag{167}$$

Applying the method of perturbations used above, we obtain the longitudinal correlation length

$$l_{\mathrm{cor}} \approx \lambda_{\mathrm{K}} \ln\left(\frac{\lambda_\perp}{\lambda_{\mathrm{K}}} \sqrt{\frac{\chi_\parallel}{\chi_\perp}}\right). \tag{168}$$

In addition to stationary magnetic field perturbations (braided magnetic field), its temporal fluctuations (magnetic flutter) are also of practical interest. A simple estimate of the impact of periodic perturbations of a frequency ω on the effects of stochastic instability is provided by the linear approximation for the decorrelation size evolution:

$$\frac{\mathrm{d}\delta\big(z(t),\omega\big)}{\mathrm{d}z} \approx b_0 \omega t\,. \tag{169}$$

Now, using the Taylor (quasilinear) scaling for the corresponding diffusivity [173, 174]

$$D_\omega(\omega) \approx (b_0\omega t)^2 \lambda_z \tag{170}$$

for scales greater than λ_z, it is not difficult to obtain the relation between transverse and longitudinal effects:

$$\delta^2(z,\omega) \approx D_\omega z(t) \approx D_{\mathrm{m}}(\omega t)^2 z(t)\,. \tag{171}$$

We recall that we are dealing with weak turbulence, where the coefficient of quasilinear magnetic diffusion is given by the estimate $D_{\mathrm{m}} \approx b_0^2\lambda_z$. Changing to correlation times, we suppose, as previously, that the longitudinal particle motions are of a diffusive character, $z^2(\tau) \approx 2\chi_\parallel \tau$, and hence

$$\delta^2(\tau,\omega) \approx \omega^2 D_{\mathrm{m}} \tau^{5/2} \sqrt{2\chi_\parallel}\,. \tag{172}$$

We now can use the Rochester–Rosenbluth balance equation in order to determine the characteristic correlation time τ:

$$\delta(\tau) = \lambda_\perp \exp\left(-\frac{z(t)}{\lambda_{\mathrm{K}}}\right). \tag{173}$$

By approximately solving transcendental equation (173), we obtain the correlation time

$$\tau \approx \frac{\lambda_{\mathrm{K}}^2}{2\chi_\parallel} \ln^2\left(\frac{\lambda_\perp}{\sqrt{D_{\mathrm{m}}\lambda_{\mathrm{K}}}} \frac{\chi_\parallel}{\omega\lambda_{\mathrm{K}}^2}\right) \approx \varepsilon_*^2 \tau_\parallel \ln^2\left(\frac{\lambda_\perp}{\sqrt{D_{\mathrm{m}}\lambda_{\mathrm{K}}}} \frac{\chi_\parallel}{\omega\lambda_{\mathrm{K}}^2}\right), \tag{174}$$

where $\tau_\parallel \approx \lambda_z^2/(2\chi_\parallel)$. The corresponding coefficient of particle diffusion in the field of magnetic fluctuations on scales

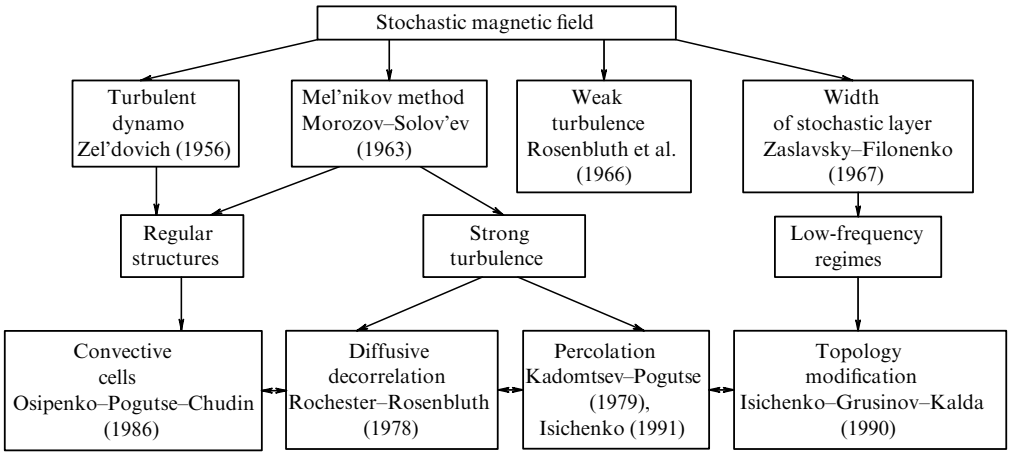

Figure 36. Flow chart of the connections between different approaches to the theoretical description of charged particle transport in a stochastic magnetic field.

$z > \lambda_z$ takes the form [175]

$$D_\perp \approx \chi_\parallel \frac{D_\mathrm{m}}{\lambda_\mathrm{K}} \ln^{-1}\left(\frac{\lambda_\perp}{\sqrt{D_\mathrm{m}\lambda_\mathrm{K}}}\frac{\chi_\parallel}{\omega\lambda_\mathrm{K}^2}\right). \tag{175}$$

It is apparent that such a weak dependence of the effective transport on the modulation frequency can only be valid in a narrow range of parameters. The criterion of applicability of estimate (175) is the obvious condition that effects of stochastic instability dominate:

$$\omega < \frac{1}{\tau_\mathrm{K}} \approx \frac{\chi_\parallel}{\lambda_\mathrm{K}^2}. \tag{176}$$

In reality, this is the low-frequency approximation, which is most interesting. It is also important to note that the models considered above are valid only for weak turbulence, $b_0 \ll 1$. The opposite case of strong turbulence, $\mathrm{Ku_m} > 1$, requiring consideration of the effects of 'long-range correlations', are addressed below based on the percolation model (Fig. 36).

15. Large-scale structures and percolation analogy

It should be kept in mind that in the strongly anisotropic case $\chi_\parallel \gg \chi_\perp$, we need to carefully analyze both longitudinal and transverse correlation effects. For this reason, neglecting the transverse displacement $\lambda_\perp$ in the Taylor (quasilinear) expression for the amplitude of magnetic perturbation $\mathbf{b}(z, \lambda_\perp) \approx \mathbf{b}(z, 0)$ is a serious drawback: this expression is valid only when the diffusive displacement in the transverse direction is much smaller that the transverse correlation scale, $b_0\lambda_z \ll \lambda_\perp$. But for problems related to strong turbulence, the case of most interest occurs when transverse correlation effects play a significant role, $b_0\lambda_z \gg \lambda_\perp$.

Kadomtsev and Pogutse [69] proposed using a new approach and formulated a criterion of its applicability in terms of a dimensionless parameter, the magnetic Kubo number $\mathrm{Ku_m}$ characterizing the ratio of longitudinal and transverse correlation effects, $\mathrm{Ku_m} = b_0\lambda_z/\lambda_\perp > 1$. The authors of Ref. [69] attributed such a regime to the percolation character of streamline behavior, which allows exploring the effects of 'long-range correlations'. It is, in fact, assumed here that the transverse decorrelation $b_0\lambda_z$ occurring in reality is larger than the formally introduced transverse scale $\lambda_\perp$.

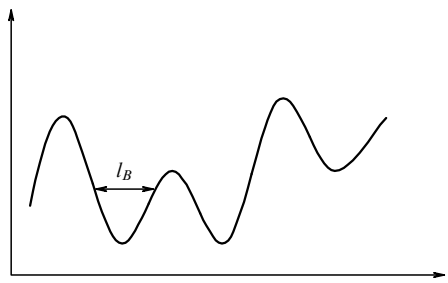

Figure 37. Percolation structure in a cross section of a magnetic stochastic field tube. $\Delta_\perp$ is the thickness of the stochastic layer, λ_z is the characteristic longitudinal spatial scale, $\lambda_\perp$ is the characteristic spatial transverse scale, ε is the small percolation parameter, b_0 is the dimensionless amplitude of magnetic field fluctuations, $L(z)$ is the perimeter of the percolation structure in the magnetic field tube at a distance z, and $r_\perp$ is the characteristic correlation scale.

We trace the stage of percolation structure formation (Fig. 37) resorting to the ballistic approximation for the perimeter $L(t)$ of a percolation cluster (shell), under the condition

$$L(z) \approx b_0 z(t) \gg \lambda_\perp. \tag{177}$$

We use expressions derived above for the main spatial percolation scales, relying on the analogy in Section 12 between the hydrodynamical and magnetic Kubo numbers [174, 175]:

$$\begin{gathered} \Delta_\perp(\varepsilon_*) \approx \varepsilon_*\lambda_\perp, \qquad a(\varepsilon_*) \approx \lambda_\perp\left(\frac{1}{\varepsilon_*}\right)^\nu, \\ L \approx \frac{a}{\varepsilon_*}, \qquad \varepsilon_*(\mathrm{Ku_m}) \approx \left(\frac{1}{\mathrm{Ku_m}}\right)^{1/(\nu+2)}, \end{gathered} \tag{178}$$

where $\nu = 4/3$ and $D_\mathrm{h} = 1 + 1/\nu = 7/4$.

The principal distinction from the hydrodynamical model is the need to introduce a hierarchy of longitudinal scales that would correspond to this hierarchy of transverse scales. It is convenient to introduce the scale λ_B from the condition $b_0\lambda_B = \lambda_\perp$. In terms of the Kubo number, we then

obtain

$$\lambda_B(\mathrm{Ku_m}) \approx \frac{\lambda_z}{\mathrm{Ku_m}} < \lambda_z \,. \tag{179}$$

On a scale smaller than λ_B, particles do not 'feel' the fractal structure of magnetic field lines. Additionally, of importance to us is the longitudinal scale (mixing length) λ_m that corresponds to the transition to the regime of complete mixing $a(\lambda_\mathrm{m}) = a(\varepsilon_*)$. Computations give

$$\lambda_\mathrm{m} \approx \varepsilon_* \lambda_z \approx \lambda_z \left(\frac{1}{\mathrm{Ku_m}}\right)^{1/(\nu+2)} \,. \tag{180}$$

To optimize the system of characteristic longitudinal scales, Isichenko suggested adopting the simplest estimate of the Kolmogorov length $\lambda_\mathrm{K} \approx \lambda_\mathrm{m} \approx \varepsilon_* \lambda_z$ [174, 175]. Eventually, we obtain a hierarchy of longitudinal scales for the percolation model

$$\lambda_B \ll \lambda_\mathrm{K} \approx \lambda_\mathrm{m} \approx \varepsilon_* \lambda_z \approx \lambda_z \left(\frac{1}{\mathrm{Ku_m}}\right)^{1/(\nu+2)} \ll \lambda_z \,. \tag{181}$$

We now modify the classical Rochester–Rosenbluth approach, preserving its basic idea on the competition between stochastic instability and diffusive decorrelation,

$$\delta(\tau) \approx l_0 \exp\left(-\frac{z(\tau)}{\lambda_\mathrm{K}}\right) . \tag{182}$$

With the above assumptions, this renormalized balance takes the form of a transcendental equation for the correlation time τ,

$$z(\tau) \approx -\varepsilon_* \lambda_z \ln \frac{\sqrt{4\chi_\perp \tau}}{\varepsilon_* \lambda_\perp} \,. \tag{183}$$

It is supposed here that the initial scale of the evolving field tube l_0 is equivalent to the width of the percolation layer $l_0 \approx \Delta_\perp(\varepsilon_*) \approx \varepsilon_* \lambda_\perp$. If the longitudinal motions of magnetized particles are diffusive, $z^2(\tau) \approx 2\chi_\parallel \tau$, on scales $z > \lambda_z$, we obtain solutions of Eqn (183):

$$\tau \approx \frac{\lambda_\mathrm{K}^2}{\chi_\parallel} \ln^2 \left[\left(\frac{b_0}{\mathrm{Ku_m}}\right)^2 \frac{\chi_\parallel}{\chi_\perp}\right] . \tag{184}$$

In the percolation limit $\mathrm{Ku_m} > 1$, $b_0 \ll 1$, and $\chi_\parallel/\chi_\perp \gg 1$, the renormalized coefficient of transverse particle diffusion in a stochastic magnetic field takes a form linear in the perturbation amplitude [174],

$$D_\perp(\tau) \approx \frac{b_0^2 \chi_\parallel}{\mathrm{Ku_m}} \ln^{-1} \left(\frac{b_0^2 \chi_\parallel}{\mathrm{Ku_m^2} \chi_\perp}\right) \propto b_0 \,, \tag{185}$$

where the estimate of the magnetic diffusion coefficient in the limit of large Kubo numbers, $D_\mathrm{m}(b_0) \approx b_0 \Delta_\perp$, is used.

Isichenko [174] also considered the stage of percolation structure formation by expressing the width of the stochastic layer through the perimeter of the percolation cluster,

$$\Delta_\perp(t) \propto \lambda_\perp \left(\frac{\lambda_\perp}{a(t)}\right)^{1/\nu} \propto \lambda_\perp \left(\frac{\lambda_\perp}{L(t)}\right)^{1/(\nu D_\mathrm{h})} . \tag{186}$$

The projection of the path traveled by a magnetic field line in the cross section of a plasma cord is given by the linear expression $L(z(t)) \approx b_0 z(t)$, and hence, assuming (as is commonly done) that longitudinal particle motions are diffusive, $z^2(t) \approx 2\chi_\parallel t$, we obtain the law for stochastic layer width evolution:

$$\Delta_\perp(t) \propto \frac{1}{(b_0\sqrt{t})^{3/7}} \,. \tag{187}$$

On the other hand, in agreement with the views of Batchelor and Rochester and Rosenbluth [187], decorrelation occurs as the result of particles hopping from one field line to another when the characteristic width of the stochastic layer $\Delta_\perp(\tau)$ becomes comparable to the characteristic transverse diffusive scale $\sqrt{4\chi_\perp \tau}$. The new equation for the correlation time takes the form

$$\lambda_\perp \left(\frac{\lambda_\perp}{b_0 z(\tau)}\right)^{3/7} \approx \sqrt{4\chi_\perp \tau} \,. \tag{188}$$

Having found the characteristic correlation time $\tau \approx (\lambda_\perp^2/\chi_\perp)(\chi_\perp/(b_0^2\chi_\parallel))^{3/10}$, we can compute the coefficient of transverse diffusion at this evolutionary stage of percolation structure formation,

$$D \propto \frac{r_\perp^2(\tau)}{\tau} \approx \frac{a^2(\tau) P_\infty(\tau)}{\tau} \approx \frac{\lambda_\perp^2}{\tau} \left(\frac{L(z)}{\lambda_\perp}\right)^{1/D_\mathrm{h}} . \tag{189}$$

As previously, $P_\infty \approx \lambda_\perp/a$ is the effective fraction of space responsible for the percolation transport and $L(z) \approx b_0 z \approx b_0\sqrt{2\chi_\parallel \tau}$. Substituting the last expression in Eqn (189), we obtain the transverse diffusion coefficient

$$D_\perp(\tau) \approx \lambda_\perp^2 \left(\frac{b_0\sqrt{\chi_\parallel}}{\lambda_\perp}\right) \tau^{-(\nu+2)/[2(\nu+1)]} \approx b_0 \sqrt{\chi_\parallel \chi_\perp} \,. \tag{190}$$

This expression is valid under the conditions

$$1 > \frac{\chi_\perp}{b_0^2 \chi_\parallel} > \frac{1}{\mathrm{Ku_m^2}} \,. \tag{191}$$

It is important to note that percolation indices dropped out from the final expression, and the result of computations coincides with the classical approximation of Kadomtsev and Pogutse for the regimes of strong turbulence, $\mathrm{Ku_m} \gg 1$ [69]. Nevertheless, the use of an ideology of this kind led to the derivation of fundamentally new scalings for the transport coefficients pertaining to magnetic flutter in the percolation limit [176].

16. Quasi-isotropic stochastic magnetic field

Problems of particle transport in a stochastic magnetic field with pronounced asymmetry are motivated by problems of high-temperature plasma confinement and trap design for controlled fusion. Quasi-isotropic stochastic magnetic fields also play an important role in astrophysical problems. For example, one encounters serious difficulties in trying to describe processes of heat conduction in a stochastic field in clusters of galaxies [191, 192], because the observed transport greatly exceeds theoretical estimates derived in the framework of the Rochester–Rosenbluth approach, even if $\chi_\parallel \approx \chi_\perp$.

An important modification of the Rochester–Rosenbluth scaling [190] was proposed in [192], where the characteristic spatial scale of the inhomogeneity of a braided magnetic field,

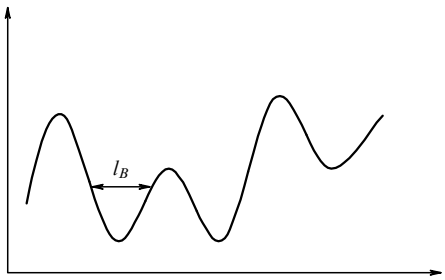

Figure 38. Magnetic field magnitude along the particle trajectory, with l_B being the characteristic spatial scale of inhomogeneity.

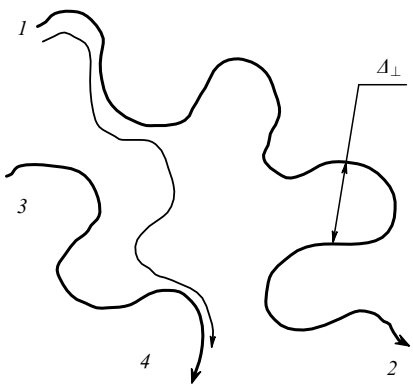

Figure 39. Magnetic field lines *1–2*, *3–4* and a particle trajectory *1–4*, with $\Delta_\perp$ being the characteristic scale on which the field is 'braided'.

l_B, is simultaneously the parameter describing the capture of electrons by magnetic traps formed due to the substantial inhomogeneity of the magnetic field in the longitudinal direction.

The expression for the longitudinal correlation length for quasi-isotropic magnetic fields can be conveniently written in the form that involves the decorrelation scale directly linked with the Larmor electron radius:

$$\rho_{\rm e} = \lambda_\perp \exp\left(-\frac{L_{\rm cor}}{\lambda_{\rm K}}\right). \tag{192}$$

The authors of Ref. [192] suppose that the transverse decorrelation of electron motion develops at distances of the order of l_B (Fig. 38), while this same scale characterizes the sizes of magnetic traps which the electrons leave by acquiring additional energy on collisions (Fig. 39). This situation is characterized by the following hierarchy of scales:

$$\rho_{\rm e} \ll l_B \approx \lambda_{\rm K} \leqslant \lambda_{\rm coll} \ll L_{\rm cor}\,. \tag{193}$$

Formally, in the case $l_B \leqslant \lambda_{\rm coll}$, where $\lambda_{\rm coll}$ is the collisional path length, the expression for the effective coefficient of magnetic diffusion can be written in a one-parametric form:

$$D_{\rm m} \approx \frac{l_B^2}{l_B} \approx l_B\,. \tag{194}$$

The corresponding formula for the longitudinal correlation length is

$$L_{\rm cor} \approx l_B \ln\frac{l_B}{\rho_{\rm e}}\,. \tag{195}$$

Scaling for the effective diffusion coefficient of electrons in a quasi-isotropic stochastic magnetic field takes the form

$$D_{\rm eff} \approx D_{\rm m}\,\frac{L_{\rm cor}}{\tau} \approx D_{\rm m}\,\frac{\chi_\parallel}{L_{\rm cor}} \approx l_B\,\frac{\chi_\parallel}{L_{\rm cor}} \approx \frac{\chi_\parallel}{\ln\left(l_B/\rho_{\rm e}\right)}\,. \tag{196}$$

The physical problem considered in [192] assumes using the Spitzer–Harm electron heat conductivity coefficient $\chi_{\rm Sp}$ [193] for $\chi_\parallel$. Then, in view of the one-dimensional character of the electron motion along field lines, we find

$$\chi_\parallel \approx \frac{\Delta_\perp^2}{2\tau} \approx \frac{\chi_{\rm Sp}}{3}\,. \tag{197}$$

On the other hand, the estimate of $\rho_{\rm e}$ under conditions that correspond to clusters of galaxies [191, 192] is $L_{\rm cor} \approx 30 l_B$, and therefore the approach in [192] gives the estimate

$$D_{\rm eff} \approx \chi_{\rm eff} \approx 10^{-2}\chi_{\rm Sp}\,. \tag{198}$$

Estimate (198) is not totally satisfactory from the standpoint of explaining the observed heat losses. Unfortunately, the main cause is the complexity of the Spitzer–Harm model [194]. This is why the approach used by Chandran and Cowly needs modification.

The monoscale quasi-isotropic model in [192] assumed an isotropic character of strong magnetohydrodynamic (MHD) turbulence, a simple model of which was considered by Iroshnikov [195] and Kraichnan [196], who arrived at the energy spectrum

$$E(k) \propto \frac{1}{k^{3/2}}\,. \tag{199}$$

However, a semi-empirical anisotropic model by Goldreich and Sridhar, proposed in 1995, allows accounting for contributions of different turbulent scales to the formation of l_B.

The authors of Ref. [200] used the model by Goldreich and Sridhar, which is based on the balance of characteristic times in the Alfvén MHD turbulence,

$$\frac{1}{\tau_{\rm A}} \approx k_\parallel V_{\rm A} \approx k_\perp V_\perp(k_\perp) \approx \frac{1}{\tau_\perp}\,, \tag{200}$$

where $1/\tau_{\rm A}$ is the Alfvén frequency, $V_{\rm A}$ is the Alfvén velocity, $k_\parallel$ is the longitudinal wave number, $k_\perp$ is the transverse wave number, $V_\perp(k_\perp)$ is the transverse velocity scale associated with the spatial scale $k_\perp$, and $\tau_\perp$ is the dimensional estimate of the nonlinear interaction time that characterizes the turbulent cascade in the direction perpendicular to the magnetic field. Certainly, expression (200) is just an approximation, but its efficiency was repeatedly confirmed by numerical modeling, which demonstrated the validity of the phenomenological scaling [201–204]

$$l_\parallel \approx \frac{1}{k_\parallel} \approx \frac{V_{\rm A} l_\perp}{V_\perp} \approx \frac{V_{\rm A} l_\perp}{(\varepsilon_{\rm K} l_\perp)^{1/3}} \approx \frac{V_{\rm A}}{\varepsilon_{\rm K}^{1/3}}\, l_\perp^{2/3}\,, \tag{201}$$

$$\varepsilon_{\rm K} \approx \frac{V_\perp^3}{l_\perp} = {\rm const}\,, \tag{202}$$

where $\varepsilon_{\rm K}$ is the spectral energy flux. The corresponding energy spectrum has the Kolmogorov form,

$$E(k_\perp) \propto \frac{1}{k_\perp^{5/3}}\,. \tag{203}$$

The link between the longitudinal and transverse scales of the form $l_\parallel \approx l_\perp^\alpha$ with $\alpha = 2/3$ corresponds to strong MHD-turbulence, and for $\alpha = 4/3$ it corresponds to an intermediate regime, which lends support to the assumption that the correlation scales characterizing transport in a stochastic magnetic field are connected by

$$\frac{L_{\rm cor}}{l_B} \approx \left(\frac{\Delta}{l_B}\right)^\alpha . \tag{204}$$

This estimate, being essentially different from the Chandran and Cowly expression $L_{\rm cor} \approx 30 l_B$, leads, for transverse displacements $\Delta \approx l_B$, to a prediction for the longitudinal correlation length:

$$L_{\rm cor} \approx \Delta \approx l_B \ll 30 l_B . \tag{205}$$

The new expression for the electron heat conductivity, of the form $\chi_{\rm eff} \approx \chi_{\rm Sp}/3$, agrees better with data from astrophysical observations.

We note that somewhat later, Goldreich and Sridhar proposed a model for weak MHD turbulence, which led to the energy spectrum [201–206]

$$E(k_\perp) \propto \frac{1}{k_\perp^2} . \tag{206}$$

But in this case, one can also use rigorous methods of weak turbulence theory, which have been successfully used in Refs [207–210] and discussed in reviews [205, 206, 211]. Our task was only to show the usefulness of the phenomenological approach for a fast qualitative assessment of effects, and we therefore urge the interested reader to consult these works.

17. Stochastic instability and the inverse cascade

In Section 16, we briefly explained how the idea of a turbulent cascade is applied to the analysis of the hierarchy of scales in problems pertaining to the description of astrophysical MHD turbulence. The idea to use the Kolmogorov phenomenology for studying the effects of stochastic instability, in relation to the analysis of scalar (passive tracer) transport, goes back to Batchelor [212]. Batchelor adopted the balance of characteristic time scales (the characteristic diffusion and dissipation times) to determine the boundary of the domain $l_{\rm Bat}$ where the scalar cascade does penetrate,

$$\tau_{\rm d} \approx \frac{l_{\rm Bat}^2}{D_0} = (\nu_{\rm f}\, \varepsilon_{\rm K})^{1/2} \approx \tau_\nu . \tag{207}$$

But this approach has not allowed deriving a consistent estimate for the stochastic instability increment that would reflect the functional dependence on the Kolmogorov spectral energy flux.

Two-dimensional turbulent flows characterized by the inverse energy cascade that maintains the formation of large-scale vortex structures [213, 214] are especially interesting from the perspective of studying the evolution of the stochastic layer. In analyzing spectra of two-dimensional turbulence, it should be borne in mind that in this case, both the kinetic energy of the fluid,

$$E = \frac{1}{2}\left\langle |\mathbf{V}|^2 \right\rangle , \tag{208}$$

and its enstrophy,

$$\Omega_V = \frac{1}{2}\left\langle \boldsymbol{\omega}_V^2 \right\rangle , \tag{209}$$

are conserved. Here, $\boldsymbol{\omega}_V = \operatorname{rot} \mathbf{V}$ is the curl of velocity.

The existence of a second conserved quantity modifies the character of cascade processes in turbulence. The transport processes are now governed by the energy dissipation rate $\varepsilon_{\rm K}$ and the enstrophy dissipation rate ε_ω [34]. If the energy and enstrophy are injected into the flow at some intermediate scale k_I, distant from the dissipation scale, they both become involved in cascade processes. The connection between spectral densities of energy and enstrophy prohibits simultaneous transfer of both quantities to small scales. For this reason, in a freely evolving flow, its spectral fluxes of energy and enstrophy have to be directed to the opposite spectral ends, with the enstrophy flux being directed to small scales and the energy flux to large scales (Fig. 40).

Therefore, two inertial ranges exist. For small scales (smaller than the energy injection scales $k > k_I$), the defining quantity is the enstrophy dissipation rate. Its dimension is $\varepsilon_\omega = [{\rm s}^{-3}]$, and hence the only possible dimensional combination gives the spectral distribution [213]

$$E(k) = C_\omega \varepsilon_\omega^{2/3} k^{-3} . \tag{210}$$

The enstrophy cascade is a direct one, i.e., the enstrophy is transferred in it from larger scales to smaller ones. At large scales (small wave numbers $k < k_I$), the cascade process is determined by the energy dissipation rate $\varepsilon_{\rm K}$ and the corresponding Kolmogorov formula

$$E(k) = C_{\rm K} \varepsilon_{\rm K}^{2/3} k^{-5/3} , \tag{211}$$

with the essential difference that energy is transferred from small to large scales in the inverse cascade. In two-dimen-

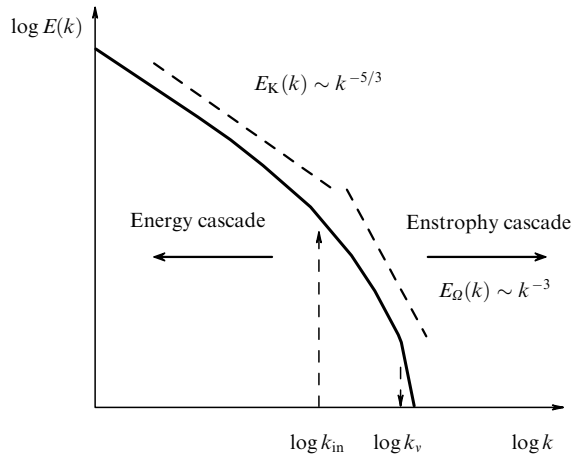

Figure 40. Schematic of the energy spectrum of two-dimensional turbulence in a double logarithmic scale. A direct cascade of enstrophy is realized in the interval of wave numbers exceeding those of injection, $k > k_I$. An inverse energy cascade exists for wave numbers that are smaller than the injection wave numbers, $k < k_I$. Energy is dissipated in the interval of wave numbers that exceed the viscous scale, $k > k_\nu$. Here, $E(k)$ is the energy spectrum, k is the wave number, $E_{\rm K}(k)$ is the Kolmogorov energy spectrum, and $E_\Omega(k)$ is the Kraichnan–Zakharov energy spectrum. The arrows indicate the cascade directions. $k_{\rm in}$ is the wave number of energy injection and k_ν is the wave number related to the dissipation scale.

sional and quasi-two-dimensional flows, this mechanism supports the formation of large-scale vortex structures because the inverse spectral energy cascade, bounded from above by the characteristic flow size, leads to the accumulation of a substantial fraction of kinetic energy in the large-scale range. In fact, these vortices, filling extended spatial regions, contribute to a nonlocal diffusion mechanism.

In this situation, the percolation method [71, 72, 215] can be applied, in which a modified Batchelor balance for characteristic times $\tau_B \approx \tau_s$ is used. The characteristic time associated with effects of stochastic instability, τ_s, should be linked to a characteristic size of the stochastic layer l_s and the Kolmogorov spectral energy flux, $\tau_s = \tau_s(l_s, \varepsilon_K)$. On the other hand, instead of the diffusion time τ_d, the characteristic time of ballistic particle motion along the percolation streamline, τ_B, is used, which in this problem statement 'is responsible for the mixing' of a scalar in the stochastic layer, $\tau_B \approx L/V_0$. Here, as previously, λ is the characteristic size of vortices in the injection range and $L(\varepsilon) \approx a(\varepsilon)/\varepsilon \approx \lambda/\varepsilon^{\nu+1}$ is the length of the percolation equipotential line. Recalling the percolation estimates discussed above, we find the width of the stochastic layer $\Delta \approx \varepsilon\lambda$ and the spatial scale $l_s \approx \lambda^2/L$ [215]:

$$\tau_s(\varepsilon_K, l_s) \approx \left(\frac{l_s^2}{\varepsilon_K}\right)^{1/3} \approx \left[\left(\frac{\lambda^2}{L(\varepsilon_*)}\right)^2 \frac{1}{\varepsilon_K}\right]^{1/3}. \tag{212}$$

To determine the small percolation parameter ε, we use the balance of characteristic times as proposed above, based on the dependence of key quantities on the small percolation parameter $\tau_s(\varepsilon) \approx \tau_B(\varepsilon)$:

$$\left[\left(\frac{\lambda^2}{L(\varepsilon_*)}\right)^2 \frac{1}{\varepsilon_K}\right]^{1/3} = \frac{L(\varepsilon_*)}{V_0}. \tag{213}$$

The increment of stochastic instability takes the form

$$\gamma_s \approx \frac{1}{\tau_s} \approx \frac{V_0}{L(\varepsilon_*)} \approx \left(\frac{V_0}{\lambda}\right)^{2/5} \left(\frac{\varepsilon_K}{\lambda^2}\right)^{1/5} \propto V_0^{2/5} \varepsilon_K^{1/5}, \tag{214}$$

where the small percolation parameter is given by the scaling

$$\varepsilon_*^{\nu+1} \approx \left(\frac{\lambda}{V_0}\right)^{3/5} \left(\frac{\varepsilon_K}{\lambda^2}\right)^{1/5}. \tag{215}$$

The dependence of the increment on the amplitude of turbulent pulsations V_0 in the percolation limit proves to be traditionally slow, $\gamma_s \propto V_0^{2/5}$. Here, the spectral energy flux ε_K is a key parameter, enabling the increment of stochastic instability to be estimated based on the quantity characterizing the scale of energy injection.

A similar method for assessing the increment of stochastic instability in two-dimensional flows with an inverse cascade can be used to describe stochastic instability in two-dimensional turbulent MHD flows, because the inverse cascade of the vector potential occurs in them [216].

An analysis of the system of two-dimensional MHD equations indicates that, as in the hydrodynamical case, there are conserved quantities. We have the conservation of the total energy,

$$E = \frac{1}{2}\int (V^2 + b^2)\, \mathrm{d}^2x = \frac{1}{2}\sum_k k^2 \left(|\Phi_k|^2 + |\Psi_k|^2\right), \tag{216}$$

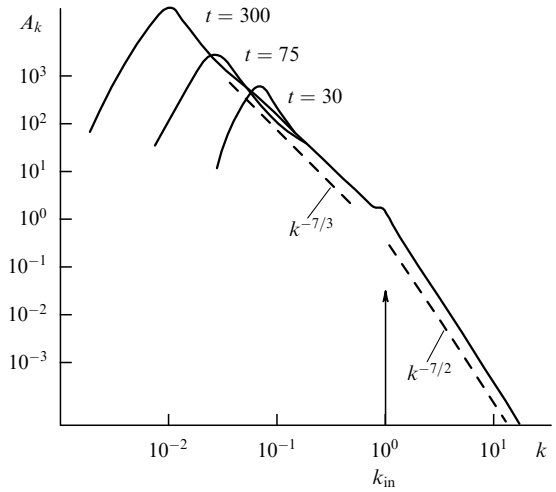

Figure 41. Inverse cascade of the vector potential; A_k is the vector potential spectrum, k is the wave number, and k_{in} is the wave number pertaining to the spatial injection scale. The arrow points to the change in the slope in the vector potential spectrum related to the presence of two cascades.

the helicity,

$$H^{\text{cross}} = \int \mathbf{V}\mathbf{b}\, \mathrm{d}^2x = \sum_k k^2 \Phi_k \Psi_{-k}, \tag{217}$$

and the mean square magnetic potential,

$$A = \int \Psi^2\, \mathrm{d}^2x = \sum_k |\Psi_k|^2. \tag{218}$$

Here, we set $\mathbf{b} = \mathbf{e}_z \times \nabla\Psi$ and $\mathbf{V} = \mathbf{e}_z \times \nabla\Phi$. It is important to mention that the energy and helicity cascade is directed to the interval of small scales, while the vector potential is carried by the inverse cascade to the interval of large scales [217]. Therefore, large-scale vortices form in the range of scales greater than the injection scale.

The spectrum of the vector potential (Fig. 41) is given by

$$A(k) \propto \varepsilon_A^{2/3} k^{-7/3}, \tag{219}$$

which is based on a conserved quantity

$$\varepsilon_A \propto \frac{\Psi^2}{\tau} = \text{const} \tag{220}$$

the spectral flux of the vector potential.

Resorting to dimensional analysis, we can express the characteristic time related to the deformation of long equipotential lines in the form that combines the characteristic spatial scale l_s and the spectral flux of the vector potential ε_A:

$$\tau_s(\varepsilon_A) \approx \left(\frac{l_s^4}{\varepsilon_A}\right)^{1/3}, \qquad l_s(\varepsilon) \approx \frac{\lambda^2}{L(\varepsilon)}. \tag{221}$$

Here, λ is the characteristic size of vortices in the injection range and L is the length of the percolation equipotential line. Anticipating the character of the scalar patch evolution in the vortex field of MHD turbulence, we write the balance of characteristic time scales as $\tau_s(\varepsilon_A) = \tau_B(V_0)$. After the

substitution, we arrive at the equation

$$\left(\frac{l_{\rm s}^4(\varepsilon_*)}{\varepsilon_{\rm A}}\right)^{1/3} = \frac{L(\varepsilon_*)}{V_0}\,. \tag{222}$$

In fact, we assume that the scalar particles manage to undergo a ballistic flight along the equipotential line they 'ride' on for the characteristic time related to the stochastic instability of the current lines. Solving Eqn (222) for the small percolation parameter ε_*, we find

$$\varepsilon_*^{\nu+1} \approx \left(\frac{\varepsilon_{\rm A}}{\lambda V_0^3}\right)^{1/7}. \tag{223}$$

Here, as above, we use the exact value of the correlation index in the two-dimensional case, $\nu = 4/3$.

The new scaling for the increment of stochastic instability in two-dimensional MHD flows with the inverse spectral cascade of the vector potential is expressed as

$$\gamma_{\rm s} \approx \frac{V_0}{\lambda}\left(\frac{\varepsilon_{\rm A}}{\lambda V_0^3}\right)^{1/7} \propto V_0^{4/7}\varepsilon_{\rm A}^{1/7}\,. \tag{224}$$

Formula (224) determines the character of the dependence of the stochastic instability increment on the amplitude of turbulent velocity pulsations V_0 and the spectral flux of the vector potential $\varepsilon_{\rm A}$.

18. Multi-scale approximations

Monoscale models of stochastic instability do not allow a detailed description of turbulent transport. Currently, the treatment of stochastic instability in the multi-scale limit is still an unsolved problem. However, there are various approximations that lead to estimates and scalings for certain effects of transport pertaining to the exponential divergence of streamlines or magnetic field lines.

One of the simplest models relies on the modification of the model differential equation of Batchelor, which describes the exponential regime. The equation describing the exponential 'separation' of initially close lines of a quasi-isotropic random magnetic field takes the form

$$\frac{{\rm d}\Delta^2}{{\rm d}l} \propto \frac{\Delta^2}{l_B}\,. \tag{225}$$

For $\Delta^2 > l_B^2$, however, the exponential regime should pass into the diffusive one:

$$\frac{{\rm d}\Delta^2}{{\rm d}l} \approx D_{\rm m} \approx \frac{l_\perp^2}{l_B}\,. \tag{226}$$

From this standpoint, it is natural to describe intermediate situations by modifying the factor before $1/l_B$, taking the increased role of large scales in intermediate regimes into account. Then, in terms of wave numbers $k \propto 2\pi/l$, it is convenient to write the model equation in the form that includes the contribution from different scales in the hierarchy,

$$\frac{{\rm d}\langle\Delta^2\rangle}{{\rm d}l} \approx \langle\Delta^2\rangle\int_{1/l_B}^{1/\Delta} k\,{\rm d}\ln k + \int_{1/\Delta}^{1/l_{\rm min}} k\,{\rm d}\ln k\,, \tag{227}$$

where $k_{\rm min} \propto 2\pi/l_{\rm min}$ is the wave number associated with the minimum spatial scale of turbulence.

Such an approach was repeatedly used to analyze the hierarchy of scales that corresponds to the model of strong Alfvén turbulence, $L_{\rm cor}/l_B \approx (\Delta/l_B)^\alpha$ [197, 200, 218], satisfying the condition

$$l_{\rm min} < l_\perp < l_B \leqslant L_{\rm cor}\,. \tag{228}$$

In terms of wave numbers $k_\perp \propto 1/l_\perp$ and $k_\parallel \propto 1/l_\parallel$, the model equation can be conveniently written in a form that takes the contributions of different scales of the hierarchy to the formation of the scale $L_{\rm K} \approx l_B$ into account:

$$\frac{{\rm d}\langle\Delta^2\rangle}{{\rm d}l} \approx \langle\Delta^2\rangle\int_{1/l_B}^{1/\Delta} k_\parallel(k_\perp)\,{\rm d}\ln k_\perp + \int_{1/\Delta}^{1/l_{\rm min}} \frac{k_\parallel(k_\perp)}{k_\perp^2}\,{\rm d}\ln k_\perp\,, \tag{229}$$

where $k_\parallel \propto k_\perp^{1/\alpha}$, and, consequently, describes the transition to regimes with $\Delta^\alpha \approx l_\perp^\alpha \approx l_\parallel$. It is important to note that despite the use of the multi-scale approach linking longitudinal and transverse motions in strong Alfvén turbulence, the characteristic scale l_B of a braided magnetic field turned out to be the universal model parameter $l_B \approx L_{\rm cor} \approx L_{\rm K} \approx \Delta$.

Another method, close in spirit, relies on the construction of approximate equation for the Richardson coefficient of relative diffusion. The exponential divergence of particles

$$l^2(t) = L_0^2\exp(\gamma_{\rm s}t) \tag{230}$$

is then associated with the differential equation

$$D_{\rm R} = \frac{\rm d}{{\rm d}t}\,l^2(t) = \gamma_{\rm s}l^2(t)\,. \tag{231}$$

Here, $\gamma_{\rm s}$ is the stochastic instability increment and L_0 is the characteristic spatial scale. On the other hand, for large times, we are dealing with the Richardson regime, which is described by the differential equation

$$D_{\rm R} = \frac{\rm d}{{\rm d}t}\,l^2(t) \propto 2D_T\,. \tag{232}$$

It is possible to construct an approximate differential equation that allows obtaining both characteristic regimes:

$$\frac{\rm d}{{\rm d}t}\,l^2(t) \propto 2D_{\rm eff}(l)\,. \tag{233}$$

An appropriate approximation can be the equation combining molecular effects with turbulent diffusion:

$$\frac{\rm d}{{\rm d}t}\,l^2(t) \propto 2D_0 + 2D_T\left(\frac{l}{\Delta_{\rm mix}}\right)^2. \tag{234}$$

Here, D_0 is the coefficient of molecular diffusion, D_T is the coefficient of turbulent diffusion, and $\Delta_{\rm mix}$ is the characteristic spatial scale of mixing. From Eqn (234), it is not difficult to obtain the characteristic time of scalar patch evolution,

$${\rm d}t = \frac{2l\,{\rm d}l}{2D_0 + 2D_T(l/\Delta_{\rm mix})^2}\,. \tag{235}$$

Computations give the following expression for the mixing time:

$$\tau_{\rm mix}({\rm Pe}) \approx \frac{\Delta_{\rm mix}^2}{D_T}\ln\frac{\rm Pe}{1 + {\rm Pe}\,L_0/\Delta_{\rm mix}}\,. \tag{236}$$

In reality, we have obtained the scaling known from the simplified monoscale approach:

$$\tau_{\mathrm{mix}}(\mathrm{Pe}) \approx \frac{1}{\gamma_{\mathrm{s}}} \ln \mathrm{Pe}\,, \qquad \mathrm{Pe} \gg 1\,. \tag{237}$$

This is not surprising because with this approximate approach we did not manage to incorporate information on the character of the turbulence field. Obviously, the development of methods for analyzing stochastic instability in conditions of developed turbulence in the multi-scale limit will require considerable efforts from the research community.

19. Conclusions

Ideas about stochastic instability and the stochastic layer have become an important tool for deriving expressions for coefficients describing turbulent transport. However, a rigorous, mathematically grounded theory of turbulence is absent thus far, which makes it necessary to use phenomenological arguments and scalings. We see that the problem of describing stochastic instability is being attacked by research from different sides. This includes estimating increments of stochastic instability, finding concrete decorrelation scales, describing various transport regimes in both hydrodynamical flows and plasmas, and attempts to construct multi-scale approximations. The main task of this review was to show the importance of determining the functional dependence of increments and transport coefficients on the amplitude of turbulent pulsations and other key parameters (characteristic frequencies of pulsations, drift velocities, spectral energy flux, and others) describing the systems being explored.

In conditions of strong structural turbulence, using formal exponential dependences for the autocorrelation function is impossible because of the existing long-range correlations. For this reason, the classical Taylor approach relying on these correlation functions often loses its meaning, even in the case of power-law approximations that allow modeling anomalous transport. The emerging difficulties lead to the need to construct phenomenological models that account for the topological features of coherent structures.

A particular focus of this review was on two-dimensional turbulent flows. The existence of the inverse energy cascade is responsible for the emergence of large-scale vortices in such systems as the result of evolution, whereas their streamlines may contribute substantially to the effective particle transport. A principal distinction of the systems of equations describing two-dimensional incompressible flows is the absence of a term responsible for the stretching of vortices. This is an essential drawback, which is considered by many researchers as a rationale for considering two-dimensional models to be oversimplified. Nevertheless, with the help of two-dimensional turbulence, we can describe important geophysical phenomena such as tropical cyclones, large-scale atmospheric motions, and oceanic flows.

We paid much attention to analyzing percolation models. This is motivated by the efficiency of such models in the analysis of two-dimensional and quasi-two-dimensional turbulent transport. Here, the small width of the stochastic layer where the adiabatic invariant is destroyed becomes a critical parameter that characterizes the proximity of the system to the phase (percolation) transition. This approach facilitated deriving scalings for the increment of stochastic instability in a random two-dimensional flow, and also transport coefficients for charged particles in a magnetic field, modeling conditions in devices for high-temperature plasma confinement. We note that in one of his last papers, Zel'dovich pointed to the percolation problem statement as an important addition to Moffat's ideas on streamline cohesion [219].

When considering problems of estimating the increment of stochastic instability in the percolation limit, we find that the concept of a scalar proves to be rather efficient. The use of scalars helps not only to visualize complex vortex structures of turbulent flows in experiments but also to find closure conditions for computing concrete physical quantities. Just the analysis of the evolution of a scalar placed in a turbulent field led to various balance equations for characteristic times.

Much consideration is given in this review to exploring plasmo-physical models describing diffusion of charged particles in a stochastic magnetic field and transport in conditions of developed drift turbulence. For example, in magnetized high-temperature plasmas, the energy flux is directed toward large scales, and therefore small perturbations may strongly influence large scales. This makes the description of turbulent transport in plasmas, which proves to be anomalous in most cases, all but trivial.

Indeed, the fundamental question on the character of the interaction and evolution of vortices in a turbulent flow remains open, and scaling continues to be the main tool for the analysis. This situation is not unique. Difficulties also persist in considering general questions of nonequilibrium statistical mechanics related to effects of stochastic instability and mixing. In spite of the fruitfulness of Boltzmann's ideas on the phase fluid evolution and fundamental work by Bogoliubov on many-particle distribution functions, we still have only limited possibilities for deriving rigorous turbulent transport equations from first principles.

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