LETTERS TO THE EDITORS

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On the problem of turbulent flows in pipes at very large Reynolds numbers

(reply to comment by I I Vigdorovich [*Phys. Usp.* **58** 196 (2015); *Usp. Fiz. Nauk* **185** 213 (2015)] on "Turbulent flows at very large Reynolds numbers: new lessons learned" [*Phys. Usp.* **57** 250 (2014); *Usp. Fiz. Nauk* **184** 265 (2014)])

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Abstract. **The problem of turbulent flow in pipes, although at first sight of purely engineering interest, has since the 1930s been the subject of much attention by mathematicians and physicists, including such outstanding figures as Th von Kármán, L Prandtl, and L D Landau. It has turned out that despite — or perhaps due to — the seemingly simple formulation of this problem, research on it has revealed new aspects of the still very mysterious phenomenon of turbulence. Reference [1] briefly summarizes our work over the last twenty years on the problem. Some of our results strongly disagree with commonly accepted views which, unsurprisingly, makes them difficult to accept. This is well exemplified by letter [2], so its analysis here may hopefully be of interest to *UFN*'s (*Physics – Uspekhi*) readers.**

Keywords: turbulence, intermediate asymptotic laws, asymptotic at high Reynolds numbers, turbulent flow in pipes, scaling laws, universal logarithmic law

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1. We briefly recapitulate the notions needed for what follows. For a statistically averaged mean velocity u of a flow at a distance y from a pipe wall, the formula (the von Kármán–Prandtl universal logarithmic law)

$$u_+ = \frac{1}{\kappa} \ln y_+ + B\,, \quad u_+ = \frac{u}{u_*}\,, \quad y_+ = \frac{y u_*}{\nu} \tag{1}$$

was proposed.

For the reader's convenience, we change the notation from that used in our paper [1] to that of letter [2], so that $u_* = (\tau/\rho)^{1/2}$ is the 'dynamic velocity', τ is the shear stress at the pipe wall, $\mathrm{Re} = Ud/\nu$, U is the section-mean velocity, i.e., the transport rate divided over the cross section area, d is the pipe diameter, and ρ and ν, respectively, are the density and kinematic viscosity of fluid. We stress the *intermediate-asymptotic* character of formula (1): it is considered valid for an intermediate range of distances from the wall, outside the thin 'viscous sublayer', where the contributions from turbulent and viscous stresses are comparable, and a narrow vicinity of the pipe axis. The constants κ (the von Kármán constant) and B are assumed to be universal, i.e., in principle, the same for all experiments in all laboratories. This was, in particular, stressed by L D Landau, who proposed in 1944 [3, 4] a simple derivation of relationship (1) based on the hypothesis explicitly formulated by von Kármán:

"On the basis of these experimentally well-established facts, we make the assumption that away from the close vicinity of the wall, the velocity distribution of the mean flow is viscosity independent."

Indeed, dimensional analysis [5, 6] gives

$$\partial_y u = \frac{u_*}{y}\, \Phi\left(\mathrm{Re}, \frac{y u_*}{\nu}\right),$$

or

$$\partial_{y_+} u_+ = \frac{1}{y_+}\, \Phi(\mathrm{Re}, y_+)\,.$$

The independence from viscosity, which enters both arguments of the function Φ, implies that the function Φ is a constant; it is traditionally denoted as $1/\kappa$, so that

$$\partial_{y_+} u_+ = \frac{1}{\kappa y_+}\,. \tag{2}$$

The integration of the last relationship (we stress the ordinary integration, studied in a standard calculus course) leads to the von Kàrmàn–Prandtl universal law (1).

2. Let us turn now to letter [2]. On page 197 we read:

"*Reference* [1] *correctly mentions that problems arise in attempts to determine the empiric constants* κ *and* B *from the results of velocity profile measurements. These problems can be associated with different causes. First, they are certainly linked to the measurement errors and also to the difference in the interpretation of the results obtained and their processing technique. So Ref.* [4] (*it is Ref.* [7] *here*) *mentioned in Ref.* [1] *proposes the values of* $\kappa = 0.44$ *and* $B = 6.3$, *while in the more recent study* [5], *carried out with the same setup* [4], *gives* $\kappa = 0.425 \pm 0.002$ *and* $B = 5.6 \pm 0.08$, *which is already fairly close to the pair* $\kappa = 0.41$ *and* $B = 5$ *used most frequently. Second, one may criticize formula (3) proper. Indeed, the derivation given by Landau [6] only strictly states that*

$$u_+ = \frac{1}{\kappa} \ln y_+ + o(\ln y_+) \,, \quad y_+ \to \infty \,. \tag{5}$$

That the second term of asymptotic form (5) *turns out to be constant is an additional assumption which should possibly be abandoned. The problem here, however, is the lack of theoretical arguments for constraining the form of the second term in expansion* (5)" [italics are ours — *the Authors*].

Thus, letter [2] *makes an explicit statement on the inappropriateness of the universal logarithmic law* (1). *Nothing is proposed instead.*

Letter [2] reminds the reader of traditional similarity laws: the *universal* logarithmic law (3) for the velocity profile in the near-wall layer:

$$u_+ = \frac{1}{\kappa} \ln y_+ + B + o(1) \,, \quad y_+ \to \infty \tag{3}$$

(a straight line in the plane $\ln y_+$, u_+) and the *universal* law of velocity defect near the axis, the matching of which, as done by Isakson, Millikan, and von Mises, is proposed as the derivation of the *universal* (i.e., independent of the Reynolds number) logarithmic law.

Just higher we find in letter [2]:

"*Thus, the logarithmic law does not exist on its own but represents a part of a general concept used for an asymptotic description of the velocity field in a pipe. If the respected authors of paper* [1] *are willing to discard it, they then should simultaneously abandon the Prandtl and von Kármán self-similar laws and propose something instead. Put differently, if one opts for changes, then they should not be limited to just replacing the logarithmic law with the power law, but must introduce a new theory instead of the classical asymptotic theory of the velocity field in a pipe, which, as we can judge, is still lacking.*"

Here, we need to correct the respected author of the letter: such a theory has existed for a long time (see Ref. [8] and subsequent publications of the authors, in particular, book [9] published recently). Importantly, the procedure of Isakson–Millikan–von Mises, described by the author, and the self-similar laws were modified in our work with due regard for the Reynolds number effect.

3. Let us turn to experimental facts. The experiments by M V Zagarola [7] (Fig. 1) are in strict contradiction to the universality — the independence from the Reynolds number. One sees that data are split in the function of the Reynolds number, and, in particular, the Reynolds number dependence of the velocity defect near the axis (the 'domes' in Fig. 1) and straight line intervals in the plane $\ln y_+$, u_+ matching the domes.

What is said above is sufficient to challenge the correctness of the von Kármán–Prandtl law. We have realized (even before the Zagarola experiments) that this is linked to the incorrectness of the main assumption of von Kármán reproduced above, so that the influence of viscosity permeates the whole pipe section, not only the viscous sublayer. We put forward a proposition on the *incomplete self-similarity* in the parameter y_+, i.e., the invariance of the velocity distribution with respect to the group of transformations:

$$\partial_{y_+} u_+ = \frac{1}{y_+} \Phi \,, \quad \Phi = A\,(\mathrm{Re})\,(y_+)^{\alpha(\mathrm{Re})} \,.$$

This proposition includes as a particular case the complete self-similarity in the parameters y_+ and Re when $\alpha = 0$, $A = \mathrm{const}$, but is more general. We also adopted the 'vanishing viscosity hypothesis', according to which the velocity gradient tends to a finite limit as $\nu \to 0$. This hypothesis is satisfied automatically in the case of complete self-similarity in the parameter y_+ when $\Phi = \mathrm{const}$. From these two hypotheses accepted by us, we deduced rigorously the relationship

$$u_+ = (C_1 \ln \mathrm{Re} + C_2)(y_+)^{C_3/\ln \mathrm{Re}} \,.$$

The constants C_1, C_2, and C_3 featured in the last formula were obtained by us through a comparison with the results of J Nikuradse [10] (256 experiments) carried out in Prandtl's laboratory. Their values have been found to be (within experimental uncertainty):

$$C_1 = \frac{1}{\sqrt{3}} \,, \quad C_2 = \frac{5}{2} \,, \quad C_3 = \frac{3}{2} \,.$$

These constants, in contrast to individual selections for each series of experiments, which are mentioned in letter [2], were strictly fixed and kept without changes for all subsequent processing of the experimental data.

Figures 2 and 3 present a comparison of the power law proposed in our work, namely

$$u_+ = \left(\frac{1}{\sqrt{3}} \ln \mathrm{Re} + \frac{5}{2} \right) (y_+)^{3/2 \ln \mathrm{Re}} \,, \tag{4}$$

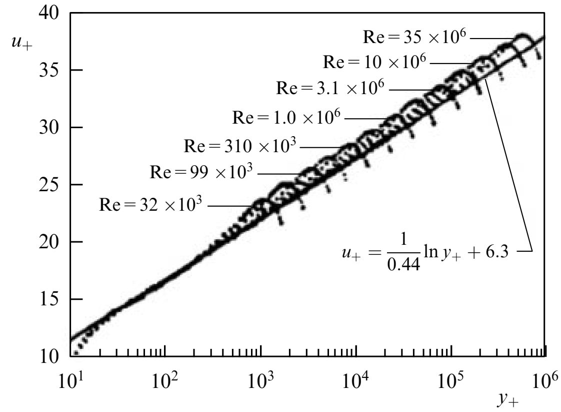

Figure 1. The universal logarithmic law is not confirmed by the data of Zagarola's experiments at the Superpipe facility (Princeton, USA).

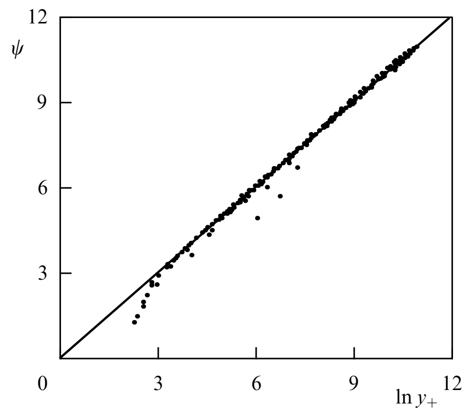

Figure 2. Nikuradse's data confirm the new power law.

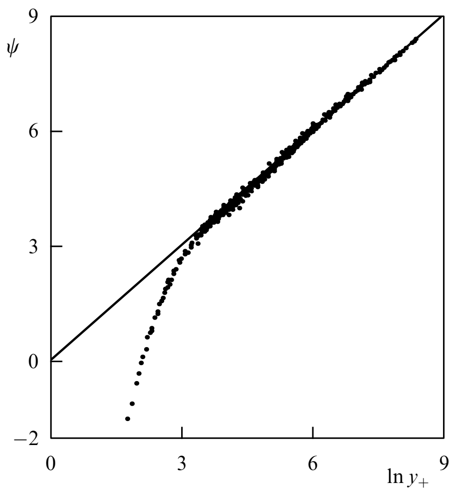

Figure 3. Zagarola's data confirm the new power law.

with the experiments by Nikuradse [10] and Zagarola [7]. We carried out the comparison using a sensitive transform: instead of u_+, we considered the function

$$\psi = \frac{1}{\alpha} \ln \frac{2\alpha u_+}{\sqrt{3} + 5\alpha}\,, \quad \alpha = \frac{3}{2 \ln \mathrm{Re}}\,.$$

This transform modifies the power law proposed by us to the simple relationship

$$\psi = \ln y_+\,.$$

The comparison shows that, beginning from the upper boundary of the viscous sublayer $y_+ \sim 30$ and up to a narrow vicinity of the axis $2y/d > 0.95$, the experimental points fit accurately the bisector of the first quadrant $\psi = \ln y_+$, which confirms the power law proposed by us.

Remarkably, for $y_+ < 30$, the experimental points fit a single curve in this plane and do not split in the Reynolds number. This fact suggests some opportunity to explore flows in the upper part of the viscous sublayer.

Concurrently, we have shown in paper [8] that Zagarola's experiments cannot be treated as performed on a setup with smooth walls beginning from $\mathrm{Re} \sim 10^6$, because the manifestations of wall roughness become visible. This conclusion was confirmed in Ref. [11].

We arrive at the following conclusions.

1. Contrary to the von Kármán assumption, the influence of viscosity in a turbulent flow in a pipe does not disappear in the entire section. The von Kármán–Prandtl universal log law cannot be accepted as a rigorous one.

2. The nonuniversal (depending on the Reynolds number) power law proposed in our work, namely

$$u_+ = \left(\frac{1}{\sqrt{3}} \ln \mathrm{Re} + \frac{5}{2} \right) (y_+)^{3/2 \ln \mathrm{Re}}\,,$$

is valid over the whole range from the upper boundary of the viscous sublayer $y_+ \sim 30$ to the narrow vicinity of the pipe axis: $2y/d > 0.95$. The experiments conducted by Nikuradse [10] confirm the power law in the full range of the Reynolds number variations up to $\mathrm{Re} = 3.24 \times 10^6$. Zagarola's experiments lend support to the power law up to 1×10^6. For larger Re number variations, wall roughness becomes noticeable at the Superpipe facility (Princeton, USA) [8, 11]. With account for corrections on roughness based on the standard procedure, the power law is confirmed for all Re in the experiments by Zagarola. Notice that attempts to use the intermediate-asymptotic power law (4) outside its applicability boundaries, present in letter [2], cannot be substantiated.

3. An additional argument in favor of the proposed power law is the outstanding agreement of the relationships based on it with the data of numerous experiments on boundary layers.

4. In our opinion, letter [2] does not affect the content of our work reflected in review [1].

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