LETTERS TO THE EDITORS

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Does the power formula describe turbulent velocity profiles in tubes?

(comment on "Turbulent flows at very large Reynolds numbers: new lessons learned" [*Phys. Usp.* **57** 250 (2014); *Usp. Fiz. Nauk* **184** 265 (2014)])

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Abstract. The power formula for turbulent velocity profiles in a circular tube offered by G I Barenblatt, A J Chorin, and V M Prostokishin in their paper "Turbulent flows at very large Reynolds numbers: new lessons learned" (*Phys. Usp.* 57 250 (2014); *Usp. Fiz. Nauk* 184 265 (2014)) is discussed.

Keywords: turbulence, turbulent flow in pipes, scaling laws, log law, power formula for the velocity profile

In recently published review article [1], which, as far as we can judge, is the result of more than a quarter-century of work by its authors on the problem being discussed, it is argued that the von Kármán–Prandtl universal logarithmic law for the velocity profile of wall flows "cannot be recognized as correct and cannot serve as the basis for teaching and engineering practice." Instead, for the flow in a circular pipe the authors propose the formula [1]

$$u_+ = \left(\frac{\sqrt{3}\ln \mathrm{Re}}{3} + \frac{5}{2}\right) \exp\left(\frac{3\ln y_+}{2\ln \mathrm{Re}}\right),$$
$$u_+ = \frac{u}{u_*}\,, \quad y_+ = \frac{y u_*}{\nu}\,, \tag{1}$$

in which y is the distance to the wall, u_* is the friction velocity (calculated from the longitudinal pressure drop), Re is the Reynolds number based on the mean velocity and the pipe diameter d, and ν is the kinematic viscosity of the fluid. The applicability region of power law (1) is the "basic intermediate region" which, as pointed out in Section 8 of Ref. [1], matches the viscous sublayer close to the wall. Thus, as argued by the authors, the region where (1) is valid is the same as for the logarithmic law.

The logarithmic law, as rightly written by the authors in their paper, is "considered to be one of the fundamental laws of engineering science," so before refuting it, it is wise to learn what the proposed alternative relation (1) represents on its own.

[1] There is a typo [2] in formula (26) of Ref. [1].

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We begin, however, by recalling traditional (we can term them classical) ideas on the behavior of the mean velocity profile in a pipe and the known similarity laws, the derivation of which can be found, for example, in monograph [3]. In a viscous sublayer, where $y_+ = O(1)$, the Prandtl law of the wall is valid, according to which all velocity profiles can be described by a single universal function $u_+(y_+)$. This function exhibits the following asymptotic behavior near the wall:

$$u_+ = y_+ + O(y_+^4)\,, \quad y_+ \to 0\,, \tag{2}$$

and far from it one has

$$u_+ = \frac{1}{\kappa}\ln y_+ + B + o(1)\,, \quad y_+ \to \infty\,. \tag{3}$$

Relationship (3) is the logarithmic law for the velocity profile in the near-wall region. Asymptotic form (2) is the exact result which follows from the Navier–Stokes equations.

In the central part of the pipe, where $1/\xi = O(1)$, $\xi = 2y/d$, the von Kármán velocity defect law holds:

$$\frac{u_{\max} - u}{u_*} = f(\xi)\,.$$

Here, $u_{\max}$ is the velocity at the pipe axis, and $f(\xi)$ is a universal function which has the following asymptotic form at the wall:

$$f = -\frac{1}{\kappa}\ln \xi + A + o(1)\,, \quad \xi \to 0\,, \tag{4}$$

where A is a constant. Relationship (4) formulates the logarithmic law for the velocity profile in the outer variables. Thus, the velocity field is described by two similarity laws in terms of the two universal functions $u_+(y_+)$ and $f(\xi)$. The region where both asymptotic laws (3) and (4) do overlap is the logarithmic sublayer. (One can note that asymptotic form (3) begins to describe the velocity profile from the value of $y_+ = 30$ [3], which is, in practice, the lower boundary of the logarithmic sublayer or the upper boundary of the viscous sublayer.)

This fairly elegant concept of two characteristic scales and two similarity laws by no means offers a place for formula (1). Asymptotic form (3), for instance, cannot be replaced by formula (1), for, were it the case, one would at least be forced to assume that at large Reynolds numbers in a very thin viscous sublayer, for $y_+ = O(1)$, the velocity profile in the wall variables is not a universal one, but depends on the pipe diameter through the Reynolds number, which, if anything, contradicts experiments.

Thus, the logarithmic law does not exist on its own but represents a part of a general concept used for an asymptotic *description of the velocity field in a pipe. If the respected authors of paper* [1] *are willing to discard it, they should then simultaneously abandon the Prandtl and von Kármán similarity laws and propose something else instead.* Put differently, if one opts for changes, they should not be limited to just replacing the logarithmic law with the power law, but must introduce a new theory instead of the classical asymptotic theory of the velocity field in a pipe, which, as we can judge, is still lacking.

Reference [1] correctly mentions that problems arise in attempts to determine the empiric constants κ and B from the results of velocity profile measurements. These problems can be associated with different causes. First, they are certainly linked to measurement errors and to the difference in the interpretation of the results obtained and their processing technique. So, Ref. [4] mentioned in Ref. [1] proposes the values of $\kappa = 0.44$ and $B = 6.3$, while the more recent study [5], carried out with the same setup [4], gives $\kappa = 0.421 \pm 0.002$ and $B = 5.6 \pm 0.08$, which is already fairly close to the pair $\kappa = 0.41$ and $B = 5$ used most frequently. Second, one may criticize formula (3) proper. Indeed, the derivation given by Landau [6] only strictly states that

$$u_+ = \frac{1}{\kappa} \ln y_+ + o(\ln y_+)\,, \quad y_+ \to \infty\,. \tag{5}$$

That the second term of asymptotic form (5) turns out to be constant is an additional assumption which should possibly be abandoned. The problem here, however, is the lack of theoretical arguments for constraining the form of the second term in expansion (5).

We consider now how formula (1) fares in describing experiments. Figure 1 displays the experimental velocity profiles [4] for $\mathrm{Re} = 2.3046 \times 10^5$ and 7.7147×10^6 in the standard semilogarithmic axes, the power-law estimates by formula (1), the straight line that corresponds to the logarithmic law for $\kappa = 0.44$ and $B = 6.3$ adopted in Ref. [4], and the linear profile (2). For the Reynolds number $\mathrm{Re} = 2.3046 \times 10^5$ (Fig. 1a), the power-law formula describes the velocity profile well over the entire logarithmic sublayer (its tentative lower boundary is drawn at $y_+ = 30$, as mentioned above), and also in some domain above it. In contrast, for the larger number $\mathrm{Re} = 7.7147 \times 10^6$ (Fig. 1b), the behavior looks rather different. Whereas for the reasons explained below the experimental points with $y_+ < 887$ are absent, it is clear that as y_+ decreases the velocity profile follows, at least approximately, the logarithmic straight line *2* and then changes to line *3*. Thus, the disparity between formula (1) and the velocity profile in the region between the two dashed lines in Fig. 1b increases in the direction toward the wall and becomes rather substantial in the vicinity of the upper boundary of the viscous sublayer. This is the manifestation of the principal flaw inherent in formula (1). For any fixed y_+ in the limit $\mathrm{Re} \to \infty$, it gives $u_+ \to \infty$. However, the value of u_+ for a fixed y_+ can only be finite.

Thus, contrary to the statement in Section 8 of review [1], in no way can the applicability region of formula (1) at large Reynolds numbers start at the boundary of the viscous sublayer. It is essentially narrower than the applicability region of the logarithmic law and, as can be seen from Fig. 1b, makes approximately an upper decade of the logarithmic sublayer. The same picture is revealed in Fig. 2, which presents data from a much earlier study [7]. *Although the conclusion that formula* (1) *for* $\mathrm{Re} \to \infty$ *is inapplicable over a large portion of the logarithmic sublayer can be drawn even*

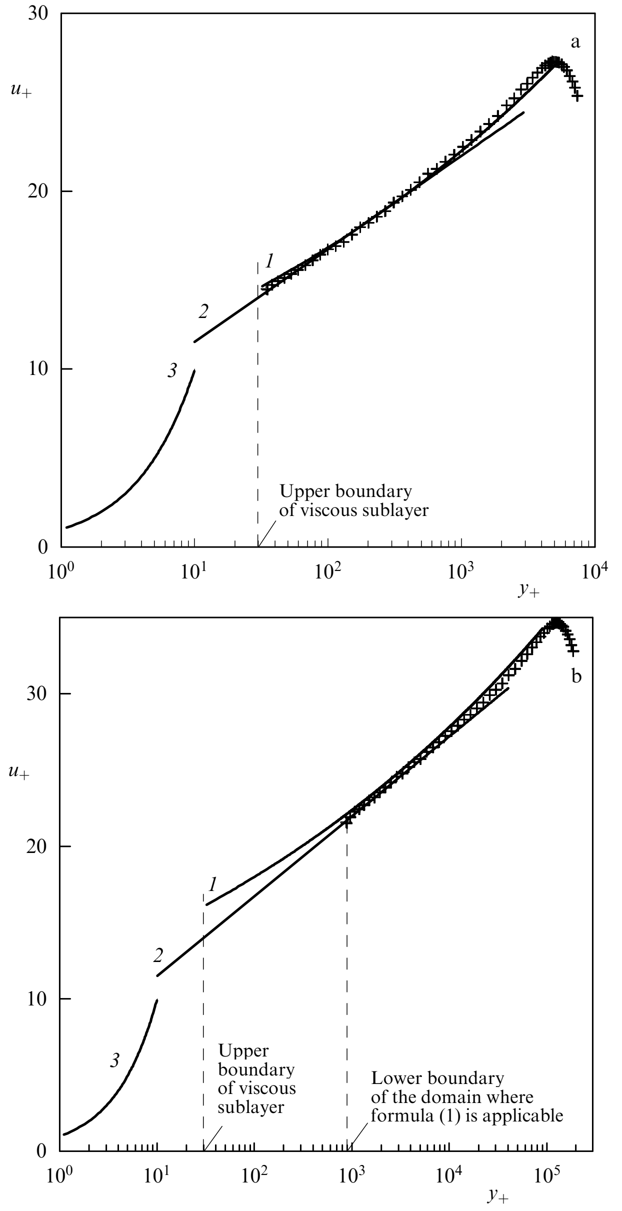

Figure 1. Experimental profiles of flow velocity in a circular pipe [4] for $\mathrm{Re} = 2.3046 \times 10^5$ (a) and 7.7147×10^6 (b); *1* formula (1), *2* the logarithmic law (3), and *3* the linear profile (2). Shown are the upper boundary of the viscous sublayer and lower boundary of the applicability region for formula (1).

without a comparison to any experimental data. It suffices simply to pay attention to the limiting value it gives for velocity as $\mathrm{Re} \to \infty$.

A question arises as to how one can explain the good performance of formula (1) against the results of measurements demonstrated by Figs 3 and 4 of Ref. [1]. The answer is that these figures only show parts of the velocity profiles that fall in the interval accessible in measurements. The point is that there exists some minimum distance to the wall where the measurements can still be conducted given that the thickness of the viscous sublayer tends to zero for $\mathrm{Re} \to \infty$. For this reason, in physical experiments with large Reynolds numbers, it is impossible to measure the velocity profile down to the boundary of the viscous sublayer. As can be seen from Figs 1b and 2b, the lower boundary of the domain where the power-law formula is still valid is approximately the distance to the

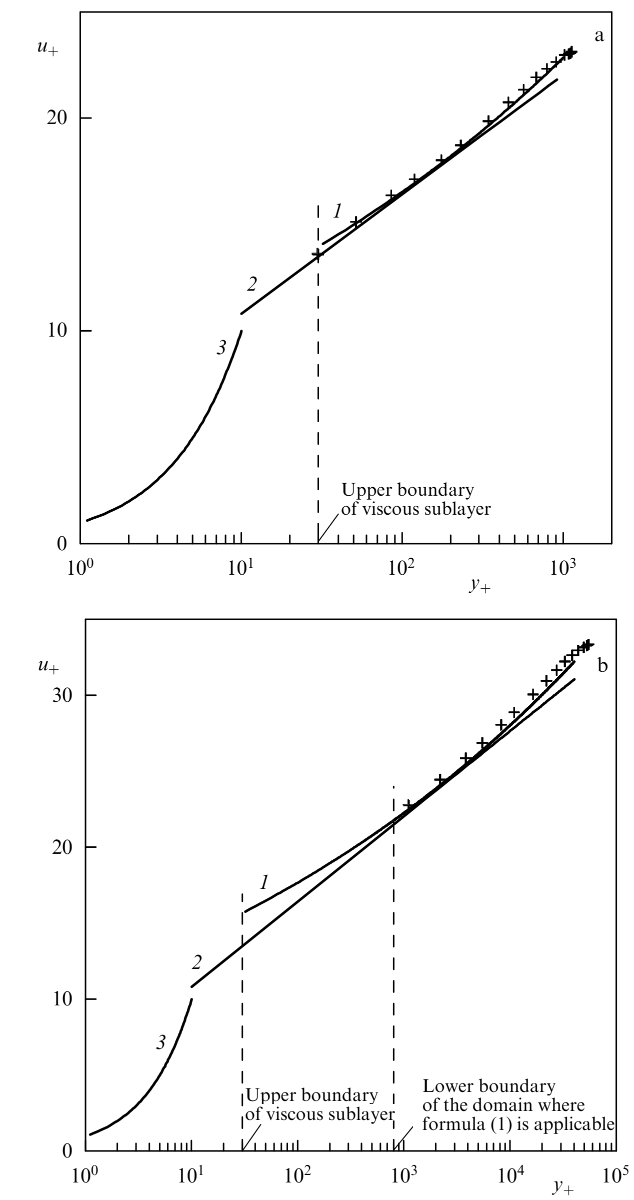

Figure 2. Experimental profiles of velocity in a circular pipe [7] for $\mathrm{Re} = 4.34 \times 10^4$ (a) and 3.24×10^6 (b). The notation is as in Fig. 1.

wall at which such measurements are still possible. Thus, had Figs 3 and 4 of Ref. [1] plotted all the data on the velocity profile in the entire range of distances to the wall (i.e., beginning from the lower boundary of the logarithmic sublayer), the emerging picture would be rather different.

The conclusions of this letter are as follows. Formula (1) cannot be an alternative to the logarithmic law simply because their applicability domains are different. The applicability region of the logarithmic law is the logarithmic sublayer. The power-law formula describes the velocity profile in one upper decade of this sublayer and in some domain above it. Over most of the logarithmic sublayer (as $\mathrm{Re} \to \infty$ this part can be arbitrarily wide in logarithmic axes), formula (1) is inapplicable since, for any fixed y_+ and $\mathrm{Re} \to \infty$, it leads to a physically incorrect result: $u_+ \to \infty$. For moderate Reynolds numbers, when the logarithmic sublayer is fairly shallow, these differences between the logarithmic and power-law formulas are hardly noticeable. Moreover, the power-law dependence exhibits a practical advantage because it well fits the part of the velocity profile resided above the logarithmic sublayer. For $\mathrm{Re} \to \infty$, the differences can be arbitrarily large, likewise the error with which formula (1) describes the real velocity profile.

It should also be noted that formula (1), like any non-universal, i.e., dependent on the Reynolds number, relationship, cannot be included in the existing concept of describing the velocity profile in terms of known similarity laws (the wall and velocity defect laws). By refuting the logarithmic law, the respected authors are in fact proposing that these laws be discarded, while not yet offering any replacement.

References

1. Barenblatt G I, Chorin A J, Prostokishin V M *Phys. Usp.* **57** 250 (2014); *Usp. Fiz. Nauk* **184** 265 (2014)
2. Barenblatt G I, Chorin A J, Prostokishin V M *Phys. Usp.* **57** 1248 (2014); *Usp. Fiz. Nauk* **184** 1371 (2014)
3. Monin A S, Yaglom A M *Statistical Fluid Mechanics*; *Mechanics of Turbulence* Vol. 1 (Cambridge, Mass.: MIT Press, 1971); Translated from Russian *Statisticheskaya Gidromekhanika* Vol. 1 (Moscow: Nauka, 1965)
4. Zagarola M V "Mean flow scaling of turbulent pipe flow", Ph.D. Thesis (Princeton: Princeton Univ., 1996)
5. McKeon B J et al. *J. Fluid Mech.* **501** 135 (2004)
6. Landau L D, Lifshitz E M *Mekhanika Sploshnykh Sred*: *Gidrodinamika i Teoriya Uprugosti* (Mechanics of Continuous Media: Fluid Dynamics and Elasticity Theory) (Theoretical Physics, Vol. 3, Ed. L D Landau) (Moscow–Leningrad: OGIZ GITTL, 1944)
7. Nikuradse J "Gesetzmäßigkeiten der turbulenten Strömung in glatten Rohren", VDI (Verein Deutscher Ingenieure)-Forschungsheft 356, Supplement to "Engineering Research" ("Forschung auf dem Gebiet des Ingenieurwesen") Edition B Vol. 3 (Berlin: VDI-Verlag, 1932); Translated into English: "Laws of turbulent flow in smooth pipes", NASA TT F-10,359 (Washington: National Aeronautics and Space Administration, 1966); Translated into Russian: "Zakonomernosti turbulentnogo dvizheniya v gladkikh trubakh", in *Problemy Turbulentnosti* (Problems of Turbulence) (Eds M A Velikanov, N T Shveikovskii) (Moscow–Leningrad: ONTI, 1936) p. 75