*Physics – Uspekhi* **58** (12) 1220 – 1222 (2015) 

***LETTERS TO THE EDITORS***

PACS numbers: **05.45.–a**, **45.40.–f**

# Notes on new friction models and nonholonomic mechanics

A V Borisov, I S Mamaev

DOI: 10.3367/UFNe.0185.201512g.1339

**Abstract.** **This is a reply to the comment by V F Zhuravlev (see *Usp. Fiz. Nauk* 185 1337 (2015) [*Phys. Usp.* 58 1218 (2015)]) on the inadequacy of the nonholonomic model when applied to the rolling of rigid bodies. The model of nonholonomic mechanics is discussed. Using recent results as examples, it is shown that the validity and potential of the nonholonomic model are not inferior to those of other dynamics and friction models.**

**Keywords:** nonholonomic model, dry friction, rattleback, rolling motion of a rigid body

This article has to do with our reaction to the comments by V F Zhuravlev [1] and presents, in essence, the continuation of the discussion on the plausibility of the models of friction opened several years ago on the pages of *Nonlinear Dynamics* (see, in particular, Ref. [2]).

From the first lines of the newly initiated polemics, it becomes clear that V F Zhuravlev still adheres to his principal position on the inconsistency of the nonholonomic model of rolling and intends to replace it by his own model of polycomponent dry friction based on the Contensou–Erismann theory [3]. The style of Zhuravlev's exposition remains without changes, and the comments [1] contain statements that are too bold and incorrect formulations of the results of work on nonholonomic mechanics, based on distortion of the chronology of events and the essence of research.

We will present our impressions and counter-arguments in due course below.

(1) Immediately in the first paragraph, V F Zhuravlev begins with a chronological blunder, almost all but granting himself the laurel wreath for demonstrating to the international scientific community (namely, French mathematicians in 1995) paradoxical effects in rattleback dynamics. In actual fact these effects have been known, without exaggeration, for a very long time, since the time of the ancient Celts (hence, its name 'Celtic stone'). In all probability, the first rigorous scientific work on rattleback dynamics was presented by G T Walker as early as 1896 (nearly a century prior to the talk given by Dr. Zhuravlev in France) to the UK mathematics community and, undoubtedly, fostered great interest even then. The description of the system and analysis of its dynamics where included in the course by Routh [5] in 1905 (with a reference to paper [4]). In 1979 (16 years before Zhuravlev's talk), *Scientific American* published a popular scientific article by J Walker [6] on the surprising ability of the rattleback to change spontaneously its rotation direction.

Rattlebacks are known today everywhere in the world, and it is not difficult to purchase their commercial models. However, the problem with their dynamics has not been fully solved yet; it awaits solution and continues to attract attention from different physics and mathematics standpoints. Most well-known are the nonholonomic model of the system (see, for example, Refs [7, 8]) and the model with friction (see, for example, Ref. [9]). The recent development of the methods of nonlinear dynamics (a comprehensive review can be found in Ref. [10]) advances research on rattleback effects in the nonholonomic formulation to a new level. One gets a chance to detect and analyze in detail the regimes of motion characteristic for dissipative systems: stable equilibria, strange chaotic attractors, limit cycles [7], and others.

(2) The statement by V F Zhuravlev on the elasticity of the cut ellipsoid and half-plane in the model of rattlebacks, that he considered in Ref. [9] deserves a special critical remark. In reality, the model used there corresponds to a rigid body with an artificially introduced contact area, borrowed from the elastic interaction model. Exploring a system combining a model of elastic bodies and contact interaction possessing an infinite number of degrees of freedom is presently at the edge of computer capabilities. Methods of such an analysis are only being developed and tested. The introduction of a contact area, proposed in Ref. [9], which, at first glance, simplifies the problem, also entails a set of undetermined factors. Among them is the unknown size of the contact area, which is neglected in the course of computations, and the pressure distribution over the contact area, which is borrowed from the statical theory of Hertz stresses (for a moving body!). We also note that the Hertz theory applies to the problem of local spherical contact, in which one of the spheres turns into a plane in the limit case of its radius going to infinity. The form of contact area for the ellipsoid considered in Ref. [9] is highly different from the contact between spheres considered by Hertz.

(3) It is yet another of Zhuravlev's attempts to blame the nonholonomic model for incorrectness in the sense of Hadamard because of the jump in the body sliding time as the contact area tends to zero (see paper [11] of 1966), which leads, in his opinion, to a contradiction if one moves from a system with dry friction to a nonholonomic one. As an answer, we would like to draw Zhuravlev's attention to recent papers by V V Kozlov [12] and A P Ivanov [13] (published in 2010 in the framework of the polemic exchange mentioned above), which show that the motion of a ball tends

**A V Borisov** Udmurt State University,
ul. Universitetskaya 1, 426034 Izhevsk, Russian Federation;
National Research Nuclear University MEPhI,
Kashirskoe shosse 31, 115409 Moscow, Russian Federation
E-mail: borisov@rcd.ru
**I S Mamaev** Kalashnikov Izhevsk State Technical University,
ul. Studencheskaya 7, 426069 Izhevsk, Russian Federation
E-mail: mamaev@rcd.ru

Received 8 October 2015
*Uspekhi Fizicheskikh Nauk* **185** (12) 1339 – 1341 (2015)
DOI: 10.3367/UFNr.0185.201512g.1339
Translated by M V Tsaplina; edited by A Radzig

to nonholonomic rolling upon an unbounded reduction in the size of the contact spot and an increase in the friction coefficient for the model of uniform distribution considered in Ref. [11] (where the author did not take into account the unlimited increase in the friction coefficient), as well as for the Hertz contact model, the Contensou–Erismann model, and a more general model accounting for the dry friction of rolling [14].

(4) V F Zhuravlev's prejudiced attitude toward the nonholonomic model results in the appearance in his comments of the intriguing formulation ''the nonholonomic model of elastic contact'' in relation to the work by Contensou [15]. P Contensou explores nutational motion of a top which is a superposition of two motions — sliding and rotation — and shows that for a large top spin velocity dry friction works as a viscous one, which results in substantial simplifications in computations. The nonholonomic model does not account for friction and it is simply inapplicable for explaining these effects. The ''nonholonomic model of elastic contact'' alluded to by V F Zhuravlev just does not exist. For the same reason, nonholonomic models do not explain the turn of the Thompson top.

(5) V F Zhuravlev incorrectly argues that the collection of papers on nonholonomic dynamical systems [16] (with us as editors) ''presents an example when experimental results differ from the theoretical ones by a factor of 130/3.'' In reality, the example there does not deal with a laboratory experiment but only with a comparison of the results of numerical simulations of rattleback motion equations obtained by different groups of researchers.

(6) The model of friction proposed by V F Zhuravlev, which is based on the Contensou–Erismann theory with Padé approximations, is not supported by proofs confirming its applicability. V F Zhuravlev does not answer the questions on theoretical uncertainties underlying its construction. As a consequence, like many other dynamical theories with dry friction (see, for example, Refs [17, 18]), it bears no more than a speculative, model character and calls for experimental verification.The necessity of laboratory research on the dynamics of systems with dry friction is also backed, in particular, by recent experimental work in this area [19, 20].

(7) Nonholonomic mechanics, alongside Lagrangian and Hamiltonian mechanics and hydrodynamics of ideal fluids, belongs to the fundamental branches of science. Tasks solved in their framework fully satisfy the Kirchhoff criterion of fundamental science [21]: ''describe fully and in the simplest way motions occurring in nature.'' Hertz [22] lent support to these conditions, demanding from a fundamental law of mechanics ''that the law, being applied to a problem with approximately accurate conditions, would always give approximately correct results, but not fully inconsistent.''

H R Hertz, being a founder of nonholonomic mechanics, took as a basis the motion model assuming rolling without sliding. With respect to the approximate character of this model, Hertz writes that ''rolling without sliding is in reality rolling with negligible sliding, i.e., a frictional process,'' but he considered the processes of friction on their own not to be fundamental, but manifestly empirical, for he did not see clear reasons leading to friction. Hertz correctly states that ''rolling without sliding contradicts neither the energy principle nor any law known to physics.'' This remark, together with that on the correct motion prediction, is, in all probability, the most convincing rationale underlying nonholonomic mechanics. And since in mechanics we introduce all strong constraints in a purely conceptual way, the nonholonomic constraints are not any different from the holonomic ones, which are also implemented only approximately.

(8) Both fundamental and empirical branches of science should work in the framework of questions they are suited to answer (based on principles of completeness and simplicity). V F Zhuravlev, unfortunately, never formulates questions on the nonholonomic model developed in our work nor on the model with friction proposed in his work. Our opinion in this respect stays unchanged: constructing a complex phenomenological model with friction makes sense only when the task consists in explaining the rattleback motion up to its complete stop and, possibly, some fine effects in its final dynamics. However, the methods proposed in Ref. [9] on the basis of the Contensou–Erismann theory with the Padé approximations to qualitatively analyze rattleback dynamics lead to constructing only a trajectory of body motion. At best, the authors of Ref. [9] have just repeated the results obtained in the framework of the nonholonomic model.

Besides, any results, especially those relying on phenomenological models (moreover, based on numerous additional uncertain assumptions), need to be verified against experiment.

(9) The nonholonomic model answers questions through the motion stages until the rattleback begins to stop. It is sufficiently simple, allowing substantial advancement in the analysis with the help of modern mathematical and computer methods, explains the effects of rattleback dynamics, and leads to plausible predictions. Most importantly, it unravels the true reason for reversals — the noncoincidence of geometrical and inertial axes at the contact point (dynamical and geometrical asymmetry).

Recently, the methods of nonholonomic mechanics were applied to explore the dynamics of bodies with various geometrical and inertial properties (see, for example, Ref. [23]) and to predict new effects: the effect of a rise with a turn upside down in the dynamics of an ellipsoidal rattleback [24], the reversal in the dynamics of the Chaplygin top [25], and some others. Their experimental confirmation would provide a strong argument for the relevance and legitimacy of nonholonomic mechanics.

Furthermore, the validity of the nonholonomic model is confirmed by a substantial number of experimental studies going back to S Earnshaw with a rotating plane [26] and contemporary work using high-speed video cameras [27].

(10) V F Zhuravlev once again criticizes nonholonomic mechanics purely speculatively. The attacks, unceasing through recent years, look rather strange. Nonholonomic mechanics is a branch of classical mechanics going back to studies by renowned mathematicians and specialists in mechanics, like Appel, Chaplygin, Zhukovskii, Routh, and many others. Making use of modern computational techniques and methods of nonlinear dynamics in the analysis of nonholonomic systems will help to answer new theoretical and applied questions in the future, as well.

The research was carried out under a partial financial support from the Russian Science Foundation (project 14-19-01303).

## References

1. Zhuravlev V F *Phys. Usp.* **58** 1218 (2015); *Usp. Fiz. Nauk* **185** 1337 (2015)
2. Borisov A V *Nelin. Dinamika* **6** 365 (2010)

3. Zhuravlev V *J. Appl. Math. Mech.* **62** 705 (1998); *Priklad. Matem. Mekh.* **62** 762 (1998)
4. Walker G T *Quart. J. Pure Appl. Math.* **28** 175 (1896)
5. Routh E J *Dynamics of a System of Rigid Bodies* Vols 1, 2 (New York: Dover, 1905)
6. Walker J *Sci. Am.* **241** (4) 172 (1979)
7. Borisov A V, Kazakov A O, Kuznetsov S P *Phys. Usp.* **57** 453 (2014); *Usp. Fiz.Nauk* **184** 493 (2014)
8. Takano H *Regul. Chaotic Dyn.* **19** (1) 81 (2014)
9. Zhuravlev V Ph, Klimov D M *Mech. Solids* **43** 320 (2008); *Izv. Ross. Akad. Nauk Mekh. Tverd. Tela* (3) 8 (2008)
10. Loskutov A *Phys. Usp.* **53** 1257 (2010); *Usp. Fiz. Nauk* **180** 1305 (2010)
11. Fufaev N A *J. Appl. Math. Mech.* **30** 78 (1966); *Priklad. Matem. Mekh.* **30** 67 (1966)
12. Kozlov V V *Nelin. Dinamika* **6** 903 (2010)
13. Ivanov A P *Nelin. Dinamika* **6** 907 (2010)
14. Karapetyan A V *J. Appl. Math. Mech.* **74** 380 (2010); *Priklad. Matem. Mekh.* **74** 531 (2010)
15. Contensou P, in *Kreiselprobleme. Gyrodynamics. Intern. Symp. on Gyrodynamics, 1962, Celerina, Switzerland* (Ed. H Ziegler) (Berlin: Springer, 1963) p. 201; Translated into Russian: in *Problemy Giroskopii. Materialy Mezhdunar. Simpoziuma, 20–22 Avgusta 1962 g., Tselerina, Shveitsariya* (Ed. H Zigler) (Moscow: Mir, 1967) p. 60
16. Kilin A A, in *Negolonomnye Dinamicheskie Sistemy. Integriruemost'. Khaos. Strannye Attraktory* (Nonholonomic Dynamical Systems. Integrability. Chaos. Strange Attractors) (Eds A V Borisov, I S Mamaev) (Moscow–Izhevsk: Inst. Komp'yut. Issled., 2002) p. 321, Appendix 1
17. Mamaev I S, Ivanova T B *Regul. Chaotic Dyn.* **19** (1) 116 (2014)
18. Borisov A, Erdakova N, Ivanova T, Mamaev I *Regul. Chaotic Dyn.* **19** (6) 607 (2014)
19. Kudra G, Awrejcewicz J *Acta Mech.* **226** 2831 (2015)
20. Borisov A et al. *Regul. Chaotic Dyn.* **20** (5) 518 (2015)
21. Kirchhoff G *Vorlesungen über mathematische Physik. Mechanik* (Leipzig: Druck und Verlag von B. G. Teubner, 1876); Translated into Russian: *Mekhanika. Lektsii po Matematicheskoi Fizike* (Moscow: Izd. AN SSSR, 1962)
22. Hertz H *Prinzipien der Mechanik in neuem zusammenhange Dargestellt* (Leipzig: Johann Ambrosius Barth, 1894); Translated into English: *The Principles of Mechanics Presented in a New Form* (Mineola, N.Y.: Dover Publ., 2003); Translated into Russian: *Printsipy Mekhaniki, Izlozhennye v Novoi Svyazi* (Exec. Ed. I I Artobolevskii) (Moscow: Izd. AN SSSR, 1959)
23. Borisov A V, Mamaev I S, Bizyaev I A *Regul. Chaotic Dyn.* **18** (3) 277 (2013)
24. Borisov A V, Kilin A A, Mamaev I S *Dokl. Phys.* **51** 272 (2006); *Dokl. Ross. Akad. Nauk* **408** 192 (2006)
25. Borisov A V, Kazakov A O, Sataev I R *Regul. Chaotic Dyn.* **19** (6) 718 (2014)
26. Levy-Leblond J-M *Eur. J. Phys.* **7** 252 (1986)
27. Cross R *Am. J. Phys.* **81** 280 (2013)